1. 
(a) Are the vectors \( \langle 1, 5, 3 \rangle \) and \( \langle -3, 0, 2 \rangle \) perpendicular?
(b) Are the points \( P = (1, -1, 1), Q = (2, 1, -1), R = (0, 3, -1), S = (-1, 0, 1) \) coplanar?
(c) Find the area of the triangle with vertices \( A = (1, 0, 0), B = (0, 1, 0), C = (0, 0, 2) \).

Solution.
(a) Their dot product is
\[ -3 + 0 + 6 = 3, \]
which is not zero, therefore they are not perpendicular.

(b) We just need to check if the triple product \( |\vec{a} \cdot (\vec{b} \times \vec{c})| \) of the vectors \( \vec{PQ}, \vec{PR} \) and \( \vec{PS} \) is zero. We have \( \vec{PQ} = (1, 2, -2), \vec{PR} = (-1, 4, -2), \vec{PS} = (-2, 1, 0) \), and so \( \vec{PR} \times \vec{PS} = (2, 4, 7) \) and
\[ \vec{PQ} \cdot (\vec{PR} \times \vec{PS}) = -4. \]
Since this is not zero, the 4 points are not coplanar.

(c) The area of the triangle is half the area of the parallelogram spanned by the vectors \( \vec{AB} = \langle -1, 1, 0 \rangle \) and \( \vec{AC} = \langle -1, 0, 2 \rangle \). This area is the length of the vector
\[ \vec{AB} \times \vec{AC} = \langle 2, 2, 1 \rangle, \]
which is \( \sqrt{4 + 4 + 1} = 3 \), so the area of the triangle is \( 3/2 \).

2. 
(a) Find an equation for the plane through the points \( P = (1, 0, 1), Q = (1, 2, 2), R = (3, 1, 2) \).
(b) Find the intersection of the line
\[ \vec{r}(t) = (1, 1, 0) + t(3, 1, 1) \]
with the plane \( x + y + z = 1 \).

Solution.
(a) We need to find 2 things: a vector \( \vec{n} \) normal to the plane, and a point on the plane. To do this, we can just take \( \vec{n} = \vec{PQ} \times \vec{PR} \). We have \( \vec{PQ} = (0, 2, 1), \vec{PR} = (2, 1, 1) \), so
\[ \vec{n} = \vec{PQ} \times \vec{PR} = (1, 2, -4). \]
Since the plane passes through \( P = (1, 0, 1) \), we take this to be our point, and therefore the plane has equation
\[ (x - 1) + 2y - 4(z - 1) = 0. \]
(b) Write the line as

\[ x = 1 + 3t, \quad y = 1 + t, \quad z = t. \]

To find the intersection points we substitute these into the equation for the plane \( x + y + z = 1 \) and solve for \( t \):

\[ 1 + 3t + 1 + t + t = 1, \]

so \( t = -1/5 \), and the intersection point is

\[ (1 - 3/5, 1 - 1/5, -1/5) = (2/5, 4/5, -1/5). \]

3. Consider the following two lines:

\[ L_1: \quad x = \frac{y - 1}{2} = \frac{z - 1}{3} \]

\[ L_2: \quad 6(x - 1) = 3y + 1 = z. \]

(a) Show that \( L_1 \) and \( L_2 \) are skew.

**Solution.**

(a) We can get a parametric equation for the line \( L_1 \) by setting all the components equal to \( t \):

\[ t = x = \frac{y - 1}{2} = \frac{z - 1}{3}. \]

Solving each of these for \( t \) gives

\[ x = t \]

\[ y = 2t + 1 \]

\[ z = 3t + 1. \]

From this we see that the direction vector of \( L_1 \) is \( \vec{d}_1 = \langle 1, 2, 3 \rangle \). We can also read the direction vector from the symmetric equations. Similarly, the direction vector of \( L_2 \) is \( \vec{d}_2 = \langle 1/6, 1/3, 1 \rangle \). These vectors are not parallel, so the two lines are not parallel. We still need to check that the two lines do not intersect. The quickest way is to check if there are points \((x, y, z)\) that solve both the equations for \( L_1 \) and \( L_2 \). This would imply that

\[ x = \frac{y - 1}{2}, \quad 6 \left( \frac{y - 1}{2} - 1 \right) = 3y + 1, \quad 3y - 9 = 3y + 1, \quad -9 = 1, \]

which is nonsense, so there is no intersection point. Therefore the two lines are skew.

4. Consider the point \( P = (0, 0, 1) \) and the plane \( x + y + z = 0 \).

(a) Find the distance between \( P \) and the plane.

(b) Find the point \( Q \) on the plane which is closest to \( P \).

**Solution.**

(a) Choose a point on the plane, such as the origin \( P_0 = O = (0, 0, 0) \), and pick a normal vector \( \vec{n} \). We may take \( \vec{n} = \langle 1, 1, 1 \rangle \). Then the distance is given by

\[ \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|P_0 \cdot \vec{n}|}{||\vec{n}||} = \frac{|(0, 0, 1) \cdot (1, 1, 1)|}{||(1, 1, 1)||} = \frac{1}{\sqrt{3}}. \]
(b) (Hard!) The point $Q$ is such that $\overrightarrow{PQ}$ is perpendicular to the plane, so $\overrightarrow{PQ}$ is a multiple of $\vec{n}$. We can compute this as follows. We first project the vector $\overrightarrow{OP}$ onto $\vec{n}$; this tells us the “height” of $\overrightarrow{OP}$ (draw a picture to see what’s going on). This gives us

$$\text{proj}_{\vec{n}}(\overrightarrow{OP}) = \frac{\overrightarrow{OP} \cdot \vec{n}}{||\vec{n}||} \frac{\vec{n}}{||\vec{n}||} = \frac{\overrightarrow{OP} \cdot \vec{n}}{||\vec{n}||^2} \vec{n} = \langle 1/3, 1/3, 1/3 \rangle.$$

Now, the vector $\overrightarrow{OQ}$ (which lies in the plane) is given by $\overrightarrow{OP} - \text{proj}_{\vec{n}}(\overrightarrow{OP}) = \langle -1/3, -1/3, 2/3 \rangle$, so $Q = (-1/3, -1/3, 2/3)$.

5. Consider the parametric equations

$$x(t) = |3 \cos(t)|$$

$$y(t) = \frac{\sin(t)}{2},$$

with $t \in [0, 2\pi]$.

(a) Identify the curve that the equations trace out, sketch the curve, and describe how the equations trace out the curve.

(b) Suppose we change the equations to

$$x(t) = |3 \cos(2t)| = |3 \cos(-2t)|$$

$$y(t) = \frac{-\sin(2t)}{2} = \frac{\sin(-2t)}{2}.$$

How does this change the parametrization of the curve?

Solution.

(a) We have

$$\frac{x(t)^2}{3^2} + \frac{y(t)^2}{(1/2)^2} = 1,$$

so the equations all lie on the ellipse with a major axis of 3 in the $x$ direction, and a minor axis of $1/2$ in the $y$ direction. Since there are absolute value signs around the $x$ coordinate, it will always be positive, so the curve the equations trace out lies in the right half-plane. The curve begins at the point $(3, 0)$ at $t = 0$, goes up along the ellipse to $(0, \frac{1}{2})$ between $t = 0$ and $t = \frac{\pi}{2}$, then goes down along the right side of the ellipse to $(0, -\frac{1}{2})$ between $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$, and then goes back up to $(3, 0)$ from $t = \frac{3\pi}{2}$ to $t = 2\pi$.

(b) If we switch the parameter from $t$ to $-2t$, then the same curve will be traversed, but in the opposite direction (so starting at $t = 0$, the particle will move down instead of up). Additionally, the curve will be traversed twice as fast, because of the factor of 2.
6. Let \( \vec{u}(t) = (e^t, 7, t^4) \), \( \vec{v}(t) = (\ln(t), \tan(t), 0) \) for \( t > 0 \).

(a) Compute \( \vec{u}'(t), \vec{u}''(t), \vec{v}'(t), \) and \( \vec{u}(t) \times \vec{v}(t) \).

(b) Find the equation for the tangent line to the curve given by the function \( \vec{u}(t) \) at the point \( P = (e^4, 7, 16) \).

Solution.

(a) We have

\[
\vec{u}'(t) = (2te^t, 0, 4t^3)
\]
\[
\vec{u}''(t) = (2e^t + 4t^2 e^t, 0, 12t^2)
\]
\[
\vec{v}'(t) = (\frac{1}{t}, \sec(t)^2, 0)
\]
\[
\vec{u}(t) \times \vec{v}(t) = (-t^4 \tan(t), t^4 \ln(t), e^t \tan(t) - 7 \ln(t))
\]

(b) To get the equation for the tangent line, we need a point on the line and a tangent vector. Our given point is \( P = (e^4, 7, 16) \), which occurs when \( t = 2 \). The tangent vector at \( t = 2 \) is \( \vec{u}'(2) = (4e^4, 0, 32) \), and our starting vector is \( \vec{u}(2) = (e^4, 7, 16) \). Therefore, the equation for the tangent line is

\[
\vec{\ell}(t) = \vec{u}(2) + t\vec{u}'(2) = (e^4 + t4e^4, 7, 16 + t32).
\]

7. Let a curve \( C \) be parametrized by the vector-valued function \( \vec{r}(t) = (\cos(t) + t \sin(t))\vec{i} + (\sin(t) - t \cos(t))\vec{j} + (\frac{\sqrt{3}}{2} t^2)\vec{k} \).

(a) Compute \( \vec{r}'(\pi/2), \vec{r}''(t), \) and \( \vec{r}''(\pi/2) \).

(b) Find the equation for the tangent line to the curve at \( \pi/2 \).

(c) Compute the length of the curve from \( t = 0 \) to \( t = 2\pi \).

Solution.

(a) We have

\[
\vec{r}'(\pi/2) = \left( \frac{\pi}{2}, 1, \frac{\sqrt{3} \pi^2}{8} \right)
\]
\[
\vec{r}''(t) = (t \cos(t), t \sin(t), \sqrt{3}t)
\]
\[
\vec{r}''(\pi/2) = \left( 0, \frac{\pi}{2}, \frac{\sqrt{3} \pi}{2} \right)
\]

(b) We use the computation of the previous part to get

\[
\vec{\ell}(t) = \vec{r}'(\pi/2) + t\vec{r}''(\pi/2) = \left( \frac{\pi}{2}, 1 + t \frac{\pi}{2}, \frac{\sqrt{3} \pi^2}{8} + t \frac{\sqrt{3} \pi}{2} \right).
\]
(c) Using the computation of $\vec{r}'(t)$ in part (a), we find that $||\vec{r}'(t)|| = \sqrt{t^2 \cos(t)^2 + t^2 \sin(t)^2 + 3t^2} = \sqrt{4t^2} = 2t$. Therefore, the arc length is

$$\int_0^{2\pi} ||\vec{r}'(t)|| \, dt = \int_0^{2\pi} 2t \, dt = t^2 \Big|_0^{2\pi} = 4\pi^2.$$