Symmary for Symmetric Functions and Hall Polynomials

Jiahe Shen, (✉) js6157@columbia.edu[1]

[1]Department of Mathematics, Columbia University

February 6, 2024

1 CHAPTER I: SYMMETRIC FUNCTIONS

1.1 Partitions

A **partition** is any (finite or infinite sequence

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_r, ...) \tag{1.1}$$

of non-negative integers in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r \geq ...$$

and containing only finitely many nonzero terms. We don't distinguish two such sequences which differ only by a string of zeros at the end.

The nonzero λ_i are called the **parts** of λ. The number of parts is the **length** of λ, denoted by $l(\lambda)$; And the sum of the parts is the **weight** of λ, denote by $|\lambda| = \lambda_1 + \lambda_2 + ...$

If $|\lambda| = n$ we say that λ is a **partition of** n. The set of all partitions of n is denoted by $\mathscr{P}_n$, and the set of all partitions by $\mathscr{P}$. In particular, $\mathscr{P}_0$ consists of a single element, the unique partition of zero, which we denote by 0.

Sometimes it is convenient to use a notation that indicates the number of times each integer occurs as a part:

$$\lambda = (1^{m_1} 2^{m_2} ... r^{m_r} ...)$$

means that exactly m_i parts of λ are equal to i. The number

$$m_i = m_i(\lambda) = \#\{j : \lambda_j = i\} \tag{1.2}$$

is called the **multiplicity** of i in λ.

The diagram of a partition λ may be formally defined as the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$. For example, the diagram of the partition (5441) consists of 5 points or nodes in the top row, 4 in the second row, 4 in the third row, and 1 in the fourth row. We shall usually denote the diagram of a partition λ by the same symbol λ.

The **conjugate** of a partition λ is the partition λ' whose diagram is the transpose of the diagram λ, i.e., the diagram obtained by the reflection in the main diagonal. Hence λ'_i is the number of nodes in the ith column of λ, or equivalently

$$\lambda'_i = \#\{j : \lambda_j \geq i\} \tag{1.3}$$

In particular, $\lambda'_1 = l(\lambda)$ and $\lambda_1 = l(\lambda')$. Obviously $\lambda'' = \lambda$. For example, the conjugate of (5441) is (43331).

From (1.2) and (1.3) we have

$$m_i(\lambda) = \lambda'_i - \lambda'_{i+1} \tag{1.4}$$

For each partition λ we define

$$n(\lambda) = \sum_{i \geq 1}(i-1)\lambda_i = \sum_{i \geq 1}\binom{\lambda'_i}{2} \tag{1.5}$$

To be the sum of the numbers obtained by attaching a zero to each node in the top row of the diagram of λ, a 1 to each node in the second row, and so on.

Let λ be a partition and let $m \geq \lambda_1$, $n \geq \lambda'_1$. Then the $m+n$ numbers

$$\lambda_i + n - i(1 \leq i \leq n), \quad n - 1 + j - \lambda'_j(1 \leq j \leq m)$$

are a permutation of $\{0, 1, 2, ..., m+n-1\}$.

If λ, μ are partitions, we shall write $\lambda \supset \mu$ to mean that the diagram of λ contains the diagram of μ, i.e., $\lambda_i \geq \mu_i$ for all $i \geq 1$. The set-theoretic difference $\theta = \mu - \nu$ is called a **skew diagram**.

A **path** in a skew diagram θ is as sequence $x_0, x_1, ..., x_m$ of squares in θ such that x_{i-1} and x_i have a common side, for $1 \leq i \leq m$. A subset φ of θ is said to be **connected** if any two squares in φ can be connected by a path in φ. The maximal connected subsets of θ are themselves skew diagrams, called the **connected components** of θ. In the example that $\lambda = (5441)$ and $\mu = (432)$, we have three connected components.

The **conjugate** of a skew diagram $\theta = \lambda - \mu$ is $\theta' = \lambda' - \mu'$. Let $\theta_i = \lambda_i - \mu_i, \theta'_i = \lambda'_i - \mu'_i$, and

$$|\theta| = \sum_i \theta_i = |\lambda| - |\mu|$$

A skew diagram θ is a **horizontal** m**-strip** (resp. a **vertical** m**-strip**) if $|\theta| = m$ and $\theta'_i \leq 1$ (resp. $\theta_i \leq 1$) for each $i \geq 1$. In other words, a horizontal (resp. vertical) strip has at most one square in each column (resp. row).

If $\theta = \lambda - \mu$, a necessary and sufficient condition for θ to be a horizontal (resp. vertical) strip is that the sequence λ and μ are interlaced, in the sense that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ...$

A skew diagram θ is a **border strip**(also called a skew hook by some authors, and ribbon by others) if θ is a connected and contains no 2×2 block of squares so that the successive rows (or columns) of θ overlap by exactly one square. The **length** of a border strip θ is the total number $|\theta|$ of square it contains, and its height is defined to be one less than the number of rows it occupies. If we think of a border strip θ as a set of modes rather than squares, then by joining contiguous nodes by horizontal or vertical line segments of unit length, we obtain a sort of staircase, and the height of θ is the number of vertical line segments or "risers" in the staircase.

A **(column-strict) tableau** T is a sequence of partitions

$$\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset ... \subset \lambda^{(r)} = \lambda$$

such that each skew diagram $\theta^{(i)} = \lambda^{(i)} - \lambda^{(i-1)} (1 \leq i \leq r)$ is a horizontal strip. Graphically, T may be described by numbering each square of the skew diagram $\theta^{(i)}$ with the number i, for $1 \leq i \leq r$, and we shall often think of a tableau as a numbered skew diagram in this way. The numbers inserted in $\lambda - \mu$ must increase strictly down each column (which explains the adjective "column-strict") and weakly from left to right along each row. The skew diagram $\lambda - \mu$ is called the **shape** of the tableau T and the sequence $(|\theta^{(1)}|, |\theta^{(2)}|, ..., |\theta^{(r)}|)$ is the **weight** of T. Throughout the book, the work **tableau** (unqualified) will mean a column-strict tableau, as defined above.

Let L_n denote the reverse lexicographic ordering on the set $\mathscr{P}_n$ of partitions of n: that is to say, L_n is the subset of $\mathscr{P}_n \times \mathscr{P}_n$ consisting of all (λ, μ) such that either $\lambda = \mu$ or the first non-vanishing difference $\lambda_i - \mu_i$ is positive. L_n is a total ordering. For example, when $n = 5$, L_5 arranges $\mathscr{P}_5$ in the sequence

$$(5), (41), (32), (31^2), (2^2 1), (21^3), (1^5)$$

Another total ordering on $\mathscr{P}_n$ is L'_n, the set of all (λ, μ) such that either $\lambda = \mu$ or else the first non-vanishing difference $\lambda^*_i - \mu^*_i$ is negative, where $\lambda^*_i = \lambda_{n+1-i}$. The orderings L_n, L'_n are distinct as soon as $n \geq 6$. In fact, we have for every $\lambda, \mu \in \mathscr{P}_n$,

$$(\lambda, \mu) \in L'_n \Leftrightarrow (\mu', \lambda') \in L_n$$

An ordering which is more important than either L_n or L'_n is the **natural** (partial) ordering N_n on $\mathscr{P}_n$ (also called the **dominance** partial ordering by some authors), which is defined as follows:

$$(\lambda, \mu) \in N_n \Leftrightarrow \lambda_1 + ... + \lambda_i \geq \mu_1 + ... + \mu_i, \forall i \geq 1$$

As soon as $n \geq 6$, N_n is not a total ordering. We shall write $\lambda \geq \mu$ in place of $(\lambda, \mu) \in N_n$.

Let $\lambda, \mu \in \mathscr{P}_n$. Then $\lambda \geq \mu \Leftrightarrow (\lambda, \mu) \in L_n \cap L'_n \Leftrightarrow \mu' \geq \lambda'$.

Now, let us consider the integer vectors $a = (a_1, ..., a_n) \in \mathbf{Z}^n$. The symmetric group S_n acts on $\mathbf{Z}^n$ by permitting the coordinates, and the set

$$P_n = \{b \in \mathbf{Z}^n : b_1 \geq b_2 \geq ... \geq b_n\}$$

is a fundamental domain for this action, i.e., the S_n-orbit of each $a \in \mathbf{Z}^n$ meets P_n in exactly one point, which we denote by a^+. Thus a^+ is obtained by rearranging $a_1, ..., a_n$ in descending order of magnitude.

For $a, b \in \mathbf{Z}^n$ we define $a \geq b$ as before to mean $a_1 + ... + a_i \geq b_1 + ... + b_i, \forall 1 \leq i \leq n$. Let $a \in \mathbb{Z}^n$. Then $a \in P_n \Leftrightarrow a \geq wa, \forall w \in S_n$.

For each pair of integers i, j, $1 \leq i < j \leq n$ define $R_{ij} : \mathbf{Z}^n \to \mathbf{Z}^n$ by

$$R_{ij}(a_1, ..., a_n) = (a_1, ..., a_i + 1, ..., a_j - 1, ..., a_n)$$

Any product $R = \prod_{i<j} R_{ij}^{r_{ij}}$ is called a **raising operator**. The order of the terms in the product is immaterial since they commute with each other.

Let $a \in \mathbf{Z}^n$ and let R be a raising operator. Then $Ra \geq a$. Conversely, let $a, b \in \mathbf{Z}^n$ be such that $a \leq b$ and $a_1 + ... + a_n = b_1 + ... + b_n$. Then there exists a raising operator R such that $b = Ra$.

1.2 The ring of symmetric functions

Consider the ring $\mathbf{Z}[x_1, ..., x_n]$ of polynomials in n independent variables $x_1, ..., x_n$ with rational integer coefficients. The symmetric group S_n acts on this ring by permitting the variables, and a polynomial is symmetric if it is invariant under this action. The symmetric polynomials form a subring

$$\Lambda_n = \mathbf{Z}[x_1, ..., x_n]^{S_n}$$

Λ_n is a graded ring: We have

$$\Lambda_n = \oplus_{k \geq 0} \Lambda_n^k$$

Where Λ_n^k consists of the homogenous symmetric polynomials of degree k, together with the zero polynomial.

For each $\alpha = (\alpha_1, ..., \alpha_n)$ we denote by x^α the polynomial

$$x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$$

Let λ be any partition. The polynomial

$$m_\lambda(x_1, ..., x_n) = \sum x^\alpha \tag{1.6}$$

summed over all distinct permutations α of $\lambda = (\lambda_1, ..., \lambda_n)$, is clearly symmetric, and the m_λ (as λ run through all the partitions of length $\leq n$) form a $\mathbf{Z}$-basis of Λ_n. Hence the m_λ

such that $l(\lambda) \leq n$ and $|\lambda| = k$ form a **Z**-basis of λ_n^k. In particular, as soon as $n \geq k$, the m_λ such that $|\lambda| = k$ form a **Z**-basis of Λ_n^k.

Now, let us generate the theory above in the case of countably many independent variables. Let $m \geq n$ and consider the homomorphism

$$\mathbf{Z}[x_1, ..., x_m] \to \mathbf{Z}[x_1, ..., x_n]$$

which send each of $x_{n+1}, ..., x_m$ to zero and the other x_i to themselves. On restriction to Λ_m this gives a homomorphism

$$\rho_{m,n} : \Lambda_m \to \Lambda_n$$

It follows that $\rho_{m,n}$ is subjective, and on restriction to Λ_m^k we have homomorphism

$$\rho_{m,n}^k : \Lambda_m^k \to \Lambda_n^k$$

for all $k \geq 0$ and $m \geq n$, which are always subjective, and are bijective for $m \geq n \geq k$.

We now form the inverse limit

$$\Lambda^k = \varprojlim \Lambda_n^k$$

of the **Z**-modules Λ_n^k relative to the homomorphism $\rho_{m,n}^k$. This module has a **Z**-basis consisting of the monomial symmetric functions m_λ (for all partitions λ of k). Therefore, Λ^k is a free **Z**-module of rank $p(k)$, the number of partitions of k. Now let

$$\Lambda = \oplus_{k\geq 0}\Lambda^k$$

so that Λ is the free **Z**-module generated by the m_λ for all partitions λ. We have surjective homomorphisms

$$\rho_n = \oplus_{k\geq 0}\rho_n^k : \Lambda \to \Lambda_n$$

Which give Λ a graded ring structure.

For each integer r, the rth **elementary symmetric function** e_r is the sum of all products of r distinct variables x_i, so that $e_0 = 1$ and

$$e_r = \sum_{i_1<i_2<...<i_r} x_{i_1}x_{i_2}...x_{i_r} = m_{(1^r)}$$

for $r \geq 1$. The generating function for the e_r is

$$E(t) = \sum_{r\geq 0} e_r t^r = \prod_{i\geq 1}(1 + x_i t) \tag{1.7}$$

For each partition $\lambda = (\lambda_1, \lambda_2, ...)$, define

$$e_\lambda = e_{\lambda_1}e_{\lambda_2}...$$

Let λ be a partition, λ' its conjugate. Then

$$e'_\lambda = m_\lambda + \sum_\mu a_{\lambda\mu} m_\mu$$

where the $a_{\lambda\mu}$ are non-negative integers, and the sum is over partitions $\mu < \lambda$ in the natural ordering. Therefore, we have

$$\Lambda = \mathbf{Z}[e_1, e_2, ...]$$

and the e_r are algebraically independent over $\mathbf{Z}$.

For each integer r, the rth **complete symmetric function** h_r is the sum of all monomials of total degree r in the variables x_i, so that

$$h_r = \sum_{|\lambda|=r} m_\lambda$$

In particular, $h_0 = 1$ and $h_1 = e_1$. The generating function for the h_r is

$$H(t) = \sum_{r\geq 0} h_r t^r = \prod_{i\geq 1}(1 - x_i t)^{-1} \tag{1.8}$$

From (1.7) and (1.8) we have $H(t)E(-t) = 1$, are equivalently

$$\sum_{r=0}^{n} e_r h_{n-r} = 0 \tag{1.9}$$

for all $n \geq 1$. Since the e_r are algebraically independent, we may define a homomorphism of graded rings

$$\omega : \Lambda \to \Lambda$$

such that $\omega(e_r) = h_r$ for all $r \geq 0$. The symmetry of the relations (1.9) between the e's and the h's shows that ω is an involution, i.e., ω^2 is the identity map. Also, we have

$$\Lambda = \mathbf{Z}[h_1, h_2, ...]$$

and the h_r are algebraically independent over $\mathbb{Z}$.
For each partition $\lambda = (\lambda_1, \lambda_2, ...)$, define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} ...$$

Then the h_λ form a $\mathbf{Z}$-basis of Λ. Finally, if we define

$$f_\lambda = \omega(m_\lambda)$$

for each partition λ, then the f_λ, called the "forgotten" symmetric functions, together with $m_\lambda, e_\lambda, h_\lambda$, form four $\mathbb{Z}$-bases of Λ.

Let N be a positive integer and consider the matrices of $N + 1$ rows and columns

$$H = (h_{i-j})_{0\leq i,j\leq N}, \quad E = ((-1)^{i-j} e_{i-j})$$

Where $e_r = h_r = 0$ whenever r is negative. Both H and E are strictly lower triangular, and $HE = EH = I_{N+1}$. Let λ, μ be two partitions of length $\leq p$, such that λ' and μ' have length $\leq q$, where $p + q = N + 1$. Consider the minor of H with row indices $\lambda_i + p - i$ $(1 \leq i \leq p)$ and column indices $\mu_i + p - i$ $(1 \leq i \leq p)$. The complementary cofactor of E' has row indices $p - 1 + j - \lambda'_j$ $(1 \leq j \leq q)$ and column indices $p - 1 + j - \mu'_j$ $(1 \leq j \leq q)$. Hence, we have

$$\det(h_{\lambda_i - \mu_j - i + j})_{1 \leq i,j \leq p} = (-1)^{|\lambda| + |\mu|} \det((-1)^{\lambda'_i - \mu'_j - i + j} e_{\lambda'_i - \mu'_j - i + j})_{1 \leq i,j \leq q}$$

The minus signs cancel out, and therefore we have

$$\det(h_{\lambda_i - \mu_j - i + j})_{1 \leq i,j \leq p} = \det(e_{\lambda'_i - \mu'_j - i + j})_{1 \leq i,j \leq q}$$

(Later, we will see that this could also be written as $s_{\lambda/\mu}$.) In particular, taking $\mu = 0$ we have

$$\det(h_{\lambda_i - i + j}) = \det(e_{\lambda'_i - i + j})$$

And later, we will see that both sides are equal to the Schur function s_λ.

For each $r \geq 1$ the **power sum** is

$$p_r = \sum_i x_i^r = m_{(r)}$$

The generating function for the p_i is

$$P(t) = \sum_{r \geq 1} t^{r-1} = \sum_{i \geq 1} \frac{x_i}{1 - x_i t} = \sum_{i \geq 1} \frac{d}{dt} \log \frac{1}{1 - x_i t}$$

so that

$$P(t) = \frac{d}{dt} \log H(t) = H'(t)/H(t) \tag{1.10}$$

Likewise, we have

$$P(-t) = \frac{d}{dt} \log E(t) = E'(t)/E(t) \tag{1.11}$$

From (1.10) and (1.11) we obtain

$$nh_n = \sum_{r \geq 1}^{n} p_r h_{n-r}, \quad ne_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r} \tag{1.12}$$

This would imply that

$$\Lambda_{\mathbf{Q}} = \Lambda \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}[p_1, p_2, ...]$$

And the p_r are algebraically independent over $\mathbf{Q}$. Hence, if we define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2}$$

for each partition $\lambda = (\lambda_1, \lambda_2, ...)$, then the p_λ form a $\mathbf{Q}$-basis of $\Lambda_{\mathbf{Q}}$. But they do NOT form a $\mathbf{Z}$-basis of Λ.

Since the involution ω interchanges $E(t)$ and $H(t)$, it follows from (1.10) and (1.11) that

$$\omega(p_n) = (-1)^{n-1} p_n$$

for all $n \geq 1$, and hence that for any partition λ we have

$$\omega(p_\lambda) = \epsilon_\lambda p_\lambda \tag{1.13}$$

Where $\epsilon_\lambda = (-1)^{|\lambda|-l(\lambda)}$.

Finally, we shall express h_n and e_n as linear combinations of the p_λ. For any partition λ, define

$$z_\lambda = \prod_{i\geq 1} i^{m_i} \cdot m_i!$$

Where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i. Then we have

$$H(t) = \sum_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|}, \quad E(t) = \sum_\lambda \epsilon_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|} \tag{1.14}$$

or equivalently, $h_n = \sum_{|\lambda|=n} z_\lambda^{-1} p_\lambda, e_n = \sum_{|\lambda|=n} \epsilon_\lambda z_\lambda^{-1} p_\lambda$.

1.3 Schur functions

Let $x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$ be a monomial, and consider the polynomial a_α obtained by antisymmetrizing x^α: that is to say,

$$a_\alpha = a_\alpha(x_1, ..., x_n) = \sum_{\omega \in S_n} \epsilon(\omega) \cdot \omega(x^\alpha)$$

Where $\epsilon(\omega)$ is the sign of the permutation ω. This polynomial a_α is skew-symmetric, i.e., we have $\omega(a_\alpha) = \epsilon(\omega) a_\alpha$ for every $\omega \in S_n$; In particular, therefore, a_α vanished unless $\alpha_1, ..., \alpha_n$ are all distinct. Hence we may as well assume that that $\alpha_1 > \alpha_2 > ... > \alpha_n \geq 0$, and therefore we may write $\alpha = \lambda + \delta$, where λ is a partition of length $\leq n$, and $\delta = (n-1, n-2, ..., 1, 0)$. Then

$$a_\alpha = a_{\lambda+\delta} = \sum_\omega \epsilon(\omega) \cdot \omega(x^{\lambda+\delta})$$

Which can be written as a determinant:

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j + n - j})_{1 \leq i,j \leq n}$$

This determinant is divisible in $\mathbf{Z}[x_1, ..., x_n]$ by each of the differences $x_i - x_j (1 \leq i < j \leq n)$, and hence by their product, which is the **Vandermonde determinant**

$$\prod_{1 \leq i < j \leq n} (x_i - x_j) = \det(x_i^{n-j}) = a_\delta$$

So $a_{\lambda+\delta}$ is divisible by a_δ in $\mathbf{Z}[x_1,..,x_n]$, and the quotient

$$s_\lambda = s_\lambda(x_1,...,x_n) = a_{\lambda+\delta}/a_\delta$$

is symmetric, i.e., is in Λ_n. It is called the **Schur function** in the variables $x_1,...,x_n$, corresponding to the partition λ (where $l(\lambda) \leq n$), and is homogenous of degree $|\lambda|$.

The Schur functions s_λ, where $l(\lambda) \leq n$, for a $\mathbf{Z}$-basis of Λ_n; The s_λ for a $\mathbf{Z}$-basis of Λ, and for each $k \geq 0$, the s_λ such that $|\lambda| = k$ form a $\mathbf{Z}$-basis of Λ^k.

In fact, we can write each Schur function s_λ as a polynomial in the elementary symmetric functions e_r, and as a polynomial in the complete symmetric functions h_r. The formulas are

$$s_\lambda = \det(h_{\lambda_i-i+j})_{1\leq i,j\leq n} = \det(e_{\lambda'_i-i+j})_{1\leq i,j\leq m} \tag{1.15}$$

Where $n \geq l(\lambda), m \geq l(\lambda')$. It follows that for all partitions λ, we have

$$\omega(s_\lambda) = s_{\lambda'}$$

Also in particular, $s_{(n)} = h_n$, and $s_{(1^n)} = e_n$.

1.4 Orthogonality

Let $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$ be two finite or infinite sequences of independent variables. We shall denote the symmetric functions of the x's by $s_\lambda(x), p_\lambda(x)$, etc., and the symmetric functions of the y's by $s_\lambda(y), p_\lambda(y)$, etc.

We shall give three series expansions for the product

$$\prod_{i,j}(1-x_iy_j)^{-1}$$

The first of these is

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda z_\lambda^{-1} p_\lambda(x)p_\lambda(y)$$

Next, we have

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda h_\lambda(x)m_\lambda(y) = \sum_\lambda h_\lambda(y)m_\lambda(x)$$

And the third identity is

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda s_\lambda(x)s_\lambda(y)$$

The three identities are summed over all partitions λ. We now define a scalar product on Λ, i.e., a $\mathbf{Z}$-valued bilinear form $\langle u, v\rangle$, by requiring that the bases (h_λ) and (m_λ) should be dual to each other:

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$$

for all partitions λ, μ, where $\delta_{\lambda\mu}$ is the Kronecker delta. For each $n \geq 0$, let u_λ, v_λ be the $\mathbf{Q}$-basis of $\Lambda^n_{\mathbf{Q}}$, indexed by the partitions of n. Then the following conditions are equivalent:
(a)$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$ for all λ, μ;
(b)$\sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j}(1 - x_i y_j)^{-1}$.

It follows that $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$, so that the p_λ form an orthogonal basis of $\Lambda_{\mathbf{Q}}$. Likewise, we have $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. Also, we see that the bilinear form $\langle u, v \rangle$ is symmetric and positive definite, and the involution ω is an isometry, i.e., $\langle \omega u, \omega v \rangle = \langle u, v \rangle$. Finally, we have $\langle e_\lambda, f_\mu \rangle = \delta_{\lambda\mu}$ (here $f_\mu = \omega(m_\mu)$), i.e., (e_λ) and (f_λ) are dual bases of Λ.

1.5 Skew Schur functions

Any symmetric functions $f \in \Lambda$ is uniquely determined by its scalar products with the s_λ; namely

$$f = \sum_\lambda \langle f, s_\lambda \rangle s_\lambda$$

since the s_λ form an orthogonal basis of Λ.

Let λ, μ be partitions, and define a symmetric functions $s_{\lambda/\mu}$ by the relations

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle \tag{1.16}$$

for all partitions ν. The $s_{\lambda/\mu}$ are called the **skew Schur functions**, Equivalently, if $c^\lambda_{\mu\nu}$ are the integers defined by

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \tag{1.17}$$

then we have $s_{\lambda/\mu} = c^\lambda_{\mu\nu} s_\nu$. In particular, it is clear that $s_{\lambda/0} = s_\lambda$, where 0 denotes the zero partition. Also $c^\lambda_{\mu\nu} = 0$ unless $|\lambda| = |\mu| + |\nu|$, so that $s_{\lambda/\mu}$ is homogenous of degree $|\lambda| - |\mu|$, and is zero if $|\lambda| < |\mu|$. Later we will see that $s_{\lambda/\mu} = 0$ unless $\lambda \supset \mu$.

Now let $x = (x_1, x_2, ..)$ and $y = (y_1, y_2, ...)$ be two sets of variables. Then

$$\sum_\lambda s_{\lambda/\mu}(x) s_\lambda(y) = \sum_{\lambda,\nu} c^\lambda_{\mu,\nu} s_\nu(x) s_\lambda(y) = \sum_\nu s_\nu(x) s_\mu(y) s_\nu(y) = s_\mu(y) \sum_\nu h_\nu(x) m_\nu(y)$$

Now let us consider the case when $y = (y_1, ..., y_n)$, so that the sums above are restricted to partitions of length $\leq n$. Recall the definition of Schur polynomials, multiplying a_δ on both sides, we have

$$\sum_\lambda s_{\lambda/\mu}(x) a_{\lambda+\delta}(y) = \sum_\nu h_\nu(x) m_\nu(y) a_{\mu+\delta}(y)$$

Compare the coefficient of $y^{\lambda+\delta}$ on both sides, we have

$$s_{\lambda/\mu}(x) = \det(h_{\lambda_i-\mu_j-i+j})_{1\leq i,j\leq n} = \det(e_{\lambda'_i-\mu'_j-i+j})_{1\leq i,j\leq m}$$

Where $n \geq l(\lambda), m \geq l(\lambda')$, and therefore $\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'}$. Also, the skew Schur function $s_{\lambda/\mu}$ is zero unless $\lambda \supset \mu$, i.e., if $\lambda_i \geq \mu_i$ for every i, in which it depends on the on the skew diagram $\lambda - \mu$. If $\theta_i = \lambda_i - \mu_i$ are the components of $\lambda - \mu$, we have $s_{\lambda/\mu} = \prod s_{(\theta_i)}$.

If the number of variables x_i is finite, we can say more: In fact, we have $s_{\lambda/\mu}(x_1, ..., x_n) = 0$ unless $0 \leq \lambda'_i - \mu'_i \leq n$ for all $i \geq 1$.

Now let $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$, $z = (z_1, z_2, ...)$ be three sets of independent variables. Then we have

$$\sum_{\lambda,\mu} s_{\lambda/\mu}(x)s_\lambda(z)s_\mu(y) = \sum_\mu s_\mu(y)s_\mu(z)\prod_{i,k}(1 - x_iz_k)^{-1}$$

which is furthermore equal to

$$\prod_{i,k}(1 - x_iz_k)^{-1}\prod_{j,k}(1 - y_jz_k)^{-1} = \sum_\lambda s_\lambda(x, y)s_\lambda(z)$$

Therefore we conclude that $s_\lambda(x, y) = \sum_\mu s_{\lambda/\mu}(x)s_\mu(y) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu}s_\mu(y)s_\nu(x)$. More generally, we have $s_{\lambda/\mu}(x, y) = \sum_\nu s_{\lambda/\nu}(x)s_{\nu/\mu}(y)$, summed over partitions ν such that $\lambda \supset \nu \supset \mu$.

We can furthermore generate this formula as follows. Let $x^{(1)}, ..., x^{(n)}$ be n sets of variables, and let λ, μ be partitions. Then

$$s_{\lambda/\mu}(x_1, ..., x_n) = \sum_{(\nu)}\prod_{i=1}^n s_{\nu^{(i)}/\nu^{(i-1)}}(x^{(i)}) \tag{1.18}$$

summed over all sequences $(\nu) = (\nu^{(0)}, ..., \nu^{(n)})$ of partitions, such that $\nu^{(0)} = \mu, \nu^{(n)} = \lambda$, and $\nu^{(0)} \subset \nu^{(1)} \subset ... \subset \nu^{(n)}$.

We shall apply the above formula to the case that each of $x^{(1)}, ..., x^{(n)}$ consists of a single variable x_i. For a single x, it follows that $s_{\lambda/\mu}(x) = 0$ unless $\lambda - \mu$ is a horizontal strip, in which case $s_{\lambda/\mu}(x) = x^{|\lambda|-|\mu|}$. Hence each of the products in the sum on the right-hand side of (1.18) is a monomial $x_1^{\alpha_1}...x_n^{\alpha_n}$, where $\alpha_i = |\nu^{(i)} - \nu^{(i-1)}|$, and hence we have $s_{\lambda/\mu}(x_1, ..., x_n)$ expressed as a sum of monomials x^α, one for each tableau T of shape $\lambda - \mu$. If the weight of T is $\alpha = (\alpha_1, ..., \alpha_n)$, we shall write x^T for x^α. Then:

$$s_{\lambda/\mu} = \sum_T x^T \tag{1.19}$$

summed over all tableau T of shape $\lambda - \mu$.

For each partition ν such that $|\nu| = |\lambda| - |\mu|$, let $K_{\lambda-\mu,\nu}$ denote the number of tableux of shape $\lambda - \mu$ and weight ν. From (1.19) we have

$$s_{\lambda/\mu} = \sum_{v} K_{\lambda-\mu,\nu} m_\nu \tag{1.20}$$

and therefore

$$K_{\lambda-\mu,\nu} = \langle s_{\lambda/\mu}, h_\nu \rangle = \langle s_\lambda, s_\mu h_\nu \rangle \tag{1.21}$$

so that

$$s_\mu h_\nu = \sum_{\lambda} K_{\lambda-\mu,\nu} s_\lambda \tag{1.22}$$

In particular, suppose that $\mu = (r)$, a partition with only one non-zero part. Then $K_{\lambda-\mu,(r)}$ is 1 or 0 according to as $\lambda - \mu$ is or is not a horizontal r-strip, and therefore from (1.22) we have Pieri's formula

$$s_\mu h_r = \sum_{\lambda} s_\lambda \tag{1.23}$$

summed over all partitions λ such that $\lambda - \mu$ is a horizontal r-strip. Applying the involution ω to the equation above, we obtain

$$s_\mu e_r = \sum_{\lambda} s_\lambda \tag{1.24}$$

summed over all partitions λ such that $\lambda - \mu$ is a vertical r-strip.

2 CHAPTER II: HALL POLYNOMIALS

2.1 Finite $\mathfrak{o}$-modules

First, let us introduce the term "discrete valuation ring". A **discrete valuation ring (DVR)**, denoted as $\mathfrak{o}$, is a principal ideal domain (PID) with exactly one non-zero maximal ideal, denoted as $\mathfrak{p}$. This would indicate that $\mathfrak{o}$ is a integral domain, and its field of fractions K is equipped with a valuation $v : K \to \mathbb{Z} \cup \{\infty\}$ such that for all $x, y \in K$,

(i)$v(xy) = v(x)v(y)$
(ii)$v(x + y) \geq \min\{v(x), v(y)\}$
(iii)$v(x) = \infty \Leftrightarrow x = 0$
and also $\mathfrak{o} = \{x \mid v(x) \geq 0\}$, $\mathfrak{p} = \{x \mid v(x) \geq 1\}$

The most important example of DVR is the ring p-adic integers $\mathbf{Z}_p$ consisting of elements 0 and

$$s = \sum_{i=k}^{\infty} a_i p^i = a_k p^k + a_{k+1} p^{k+1} + ...$$

where $v(s) = k \geq 0$, $a_k \in \{1, 2, ..., p-1\}, a_{k+1}, a_{k+2}, ... \in \{0, 1, 2, ..., p-1\}$. Its fields of fractions is the p-adic field $\mathbf{Q}_p$, where we would allow k to be negative.

Now for the term **finite $\mathfrak{o}$-module**, we mean a module M with a direct sum decomposition of the form

$$M \cong \oplus_{i=1}^{r} \mathfrak{o}/\mathfrak{p}^{\lambda_i} \tag{2.1}$$

where the λ_i are positive integers, which we may assume are arranged in descending order: $\lambda = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r > 0$. In other words, $\lambda = (\lambda_1, ..., \lambda_r)$ is a partition. On the other hand, given a finite $\mathfrak{o}$-module M, let $\mu_i = \dim_k(\mathfrak{p}^{i-1}M/\mathfrak{p}^i M)$. Then $\mu = (\mu_1, \mu_2, ...)$ is the conjugate of partition λ. Therefore, the partition λ is uniquely determined by the module M, and we call λ the **type** λ of M. Clearly two finite $\mathfrak{o}$-modules are isomorphic if and only if they have the same type, and every partition λ occurs as a type. If λ is the type of M, then $|\lambda| = \sum_i \lambda_i$ is the length $l(M)$ of M, i.e., the length of a composition series of M. The length is an additive function of M, this means that if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of finite $\mathfrak{o}$-modules, then

$$l(M') - l(M) + l(M'') = 0$$

If N is a submodule of M, then the cotype of N in M is defined to be the type of M/N.

A finite $\mathfrak{o}$-module M is cyclic, i.e., generated by one element, if and only if its type is a partition (r) consisting of a single part $r = l(M)$, and M is elementary, i.e., $\mathfrak{p}M = 0$ if and only if the type of M is (1^r). If M is elementary of type (1^r), then M is a vector space over k, and $l(M) = \dim_k M = r$.

Let M be a finite $\mathfrak{o}$-module. The **dual** of M is defined to be

$$\hat{M} = \mathrm{Hom}_{\mathfrak{o}}(M, E)$$

Where $E = \varinjlim \mathfrak{o}/\mathfrak{p}^n$ is the "injective envelope" of k, i.e., the smallest injective $\mathfrak{o}$-module which contains k as a submodule. $\hat{M}$ and M are isomorphism and have the same type. Since E is injective, an exact sequence

$$0 \to N \to M \to M/N \to 0 \tag{2.2}$$

gives rise to an exact sequence

$$0 \leftarrow \hat{N} \leftarrow \hat{M} \leftarrow \widehat{M/N} \leftarrow 0 \tag{2.3}$$

$N \leftrightarrow N^0 = \widehat{M/N}$ is a one-to-one correspondence between the submodules of M, $\hat{M}$ respectively, which maps the set of all $N \subset M$ of type ν and cotype μ onto the set of all $N^0 \subset \hat{M}$ of type μ and cotype μ.

From now on, we suppose the residue field $k = \mathfrak{o}/\mathfrak{p}$ is FINITE with order $q < \infty$. If M is a finite $\mathfrak{o}$-module and x is a non-zero element of M, we shall say that x has height r if $\mathfrak{p}^r x = 0$ and $\mathfrak{p}^{r-1}x \neq 0$. The zero element of M is assigned height zero. We denote by M_r the submodule of M consisting of elements of height $\leq r$, so that $M_r = \ker(\mathfrak{p}^r)$.

The number of automorphisms of a finite $\mathfrak{o}$-module M of type λ is

$$a_\lambda(q) = q^{|\lambda|+2n(\lambda)} \prod_{i\geq 1} \varphi_{m_j}(\lambda)(q^{-1}) = q^{\sum_{i\geq 1}\lambda_i'^2} \prod_{i\geq 1} \varphi_{m_j}(\lambda)(q^{-1}) \tag{2.4}$$

where $\varphi_m(t) = (1-t)(1-t^2)...(1-t^m)$. In fact, the number of automorphisms of M is equal to the number of sequences $(x_1, ..., x_r)$ such that x_i has height $\lambda_i (1 \leq i \leq r)$ and $M = \oplus_i \mathfrak{o} x_i$.

2.2 The Hall algebra

Let $\lambda_1, \mu^{(1)}, ..., \mu^{(r)}$ be partitions, and let M be a finite $\mathfrak{o}$-module of type λ. We define

$$G^\lambda_{\mu^{(1)},...,\mu^{(r)}}(\mathfrak{o})$$

to be the number of chains of submodules of M:

$$M = M_0 \supset M_1 \supset M_r = 0$$

such that M_{i-1}/M_i has type $\mu^{(i)}$, for $1 \leq i \leq r$. In particular, $G^\lambda_{\mu\nu}(\mathfrak{o})$ is the number of submodules N of M which have type ν and cotype μ. Since $l(M) = l(M/N) + l(N)$, it is clear that $G^\lambda_{\mu\nu}(\mathfrak{o}) = 0$ unless $|\lambda| = |\mu| + |\nu|$.

Let $H = H(\mathfrak{o})$ be a free $\mathbf{Z}$-module on a basis u_λ indexed by all partitions λ. Define a product in H by the rule

$$u_\mu u_\nu = \sum_\lambda g^\lambda_{\mu\nu}(\mathfrak{o})u_\lambda$$

The sum on the right has only finitely many non-zero terms, which makes $H(\mathfrak{o})$ a commutative and associative ring with identity element u_0. We call $H(\mathfrak{o})$ the **Hall algebra** of $\mathfrak{o}$. The ring $H(\mathfrak{o})$ is generated by (as a **Z**-algebra) by the elements $u_{(1^r)}(r \geq 1)$, and they are algebraically independent over **Z**.

2.3 The LR-sequence of a submodule

Let T be a tableau of shape $\lambda - \mu$ and weight $\nu = (\nu_1, ..., \nu_r)$. Then T determines (and is determined by) a sequence of partitions

$$S = (\lambda^{(0)}, ..., \lambda^{(r)})$$

such that $\lambda^{(0)} = \mu, \lambda^{(r)} = \lambda$, and $\lambda^{(i)} \supset \lambda^{(i-1)}$ for $1 \leq i \leq r$, by the condition that $\lambda^{(i)} - \lambda^{(i-1)}$ is skew diagram consisting of the square occupied by the symbol i in T (and hence is a horizontal strip, because T is a tableau).

A sequence of partitions S as above will be called a LR-sequence of type $(\mu, \nu; \lambda)$ if

(LR1)$\lambda^{(0)} = \mu, \lambda^{(r)} = \lambda$, and $\lambda^{(i)} \supset \lambda^{(i-1)}$ for $1 \leq i \leq r$;

(LR2)$\lambda^{(i)} - \lambda^{(i-1)}$ is a horizontal strip of length ν_i, for $1 \leq i \leq r$. (These two conditions ensure that S determines a tableau T.)

(LR3)The word $w(T)$ obtained by reading T from right to left in successive rows, starting at the top, is a lattice permutation.

For (LR3) to be satisfied, it is necessary and sufficient that, for $i \geq 1$ and $k \geq 0$, the number of symbols i in the first k rows of T should not be less than the number of symbols $i+1$ in the first $k+1$ rows of T.

Every submodule N of a finite $\mathfrak{o}$-module M gives rise to a LR-sequence of type (μ', ν', λ'), where λ, μ, ν are the types of $M, M/N$, and N respectively.

2.4 Hall polynomial

Denote $G_S(\mathfrak{o})$ the number of submodules N of M whose associated LR-sequence $S(N)$ of S. Each N has type ν and cotype μ.

Let q denote the number of elements in the residue field of $\mathfrak{o}$, and recall that $n(\lambda) = \sum_i (i-1)\lambda_i$, for any partition λ. Then:

For each LR-sequence S of type (μ', ν', λ'), there exists a monic polynomial $g_S(t) \in \mathbf{Z}[t]$ of degree $n(\lambda) - n(\mu) - n(\nu)$, independent of $\mathfrak{o}$, such that

$$g_s(q) = G_S(\mathfrak{o}) \tag{2.5}$$

In other words, $G_S(\mathfrak{o})$ is a polynomial in $\mathfrak{q}$. Now define, for any three partitions λ, μ, ν

$$g^{\lambda}_{\mu\nu}(t) = \sum_{S} g_S(t)$$

summed over all LR-sequences S of type $(\mu', \nu'; \lambda')$. This polynomial is the **Hall polynomial** corresponding to λ, μ, ν. Recall from sections 1.5 and 1.9 that $c^{\lambda}_{\mu\nu}$ denotes the coefficient s_λ in the product $s_\mu s_\nu$; That $c^{\lambda}_{\mu\nu} = c^{\lambda'}_{\mu'\nu'}$ is the number of LR-sequences of type $(\mu', \nu'; \lambda')$. Then it follows that

(i)If $c^{\lambda}_{\mu\nu} = 0$, the Hall polynomial $g_{\nu\mu}(t)$ is identically zero. In particular, $g_{\nu\mu}(t) = 0$ unless $|\lambda| = |\mu| + |\nu|$ and $\mu, \nu \subset \lambda$.

(ii)If $c^{\lambda}_{\mu\nu} \neq 0$, then $g^{\lambda}_{\mu\nu}(t)$ has degree $n(\lambda) - n(\mu) - n(\nu)$ and leading coefficient $c^{\lambda}_{\mu\nu}$.

(iii)In either case, $G^{\lambda}_{\mu\nu}(\mathfrak{o}) = g^{\lambda}_{\mu\nu}(q)$.

(iv)$g^{\lambda}_{\mu\nu}(t) = g^{\lambda}_{\nu\mu}(t)$.

3 CHAPTER V: THE HECKE RING OF GL_n OVER A LOCAL FIELD

3.1 Local fields

In this chapter, we assume F is a **non-archimedean local field**, i.e.,

(i)F is a finite algebraic extension of $\mathbf{Q}_p$ for some prime p, or

(ii)$F = \mathbf{F}_q(t)$, where $\mathbf{F}_q$ is a finite field.

Let $\mathfrak{o} = \{a \in F : |a| \leq 1\}$ be the **ring of integers**, and $\mathfrak{p} = \{a \in F : |a| < 1\}$. Let $k = |\mathfrak{o}/\mathfrak{p}|$ be the residue field, and let $q < \infty$ be its order. Let π be a generator of $\mathfrak{p}$ with $|\pi| = q^{-1}$.

3.2 The Hecke ring $H(G, K)$

Let $G = GL_n(F)$ be the group of all invertible $n \times n$ matrix over F. Also let

$$G^+ = G \cap M_n(\mathfrak{o})$$

be the subsemigroup of G consisting of all matrices $x \in G$ with entries $x_{ij} \in \mathfrak{o}$, and let

$$K = GK_n(\mathfrak{o}) = G^+ \cap (G^+)^{-1}$$

so that K consisting of all $x \in G$ with entries $x_{ij} \in \mathfrak{o}$ and $\det(x)$ a unit in $\mathfrak{o}$.

Let dx denote the unique Haar measure on G for which K has measure 1 and is both left- and right-invariant under the multiplication of K. Under this measure, the measure of Kx and xK is 1 for all non-zero $x \in G$.

Let $L(G, K)$ denote the space of all complex-valued continuous functions of compact support of G (resp. G^+) which are bi-invariant with respect to K, i.e., such that

$$f(k_1 x k_2) = f(x)$$

for all $x \in G$ (resp, G^+) and $k_1, k_2 \in K$. We may and shall regard $L(G^+, K)$ as a subspace of $L(G, K)$.

We define a multiplication on $L(G, K)$ as follows: for all $f, g \in L(G, K)$,

$$(f * g)(x) = \int_G f(xy^{-1})g(y)dy$$

(Since f and g are compactly supported, the integration is over a compact set.) This product is associative and commutative. Since G^+ is closed under multiplication, it follows immediately from the definition that $L(G^+, K)$ is a subring of $L(G, K)$.

Each function $f \in L(G, K)$ is constant on each double closet KxK in G. These double cosets are compact and mutually disjoint. Since f has compact support, it follows that f takes non-zero values on only finitely many double cosets KxK, and hence can be written as a finite linear combination of their characteristic functions. Hence the characteristic functions of the double cosets of K form a $\mathbf{C}$-basis on $L(G, K)$. The characteristic function of K is the identity element of $L(G, K)$.

If we vary the definition of the algebra $L(G, K)$ (resp. $L(G^+, K)$) by requiring the functions to take their values in $\mathbf{Z}$ instead of C, the resulting ring is called the **Hecke ring** of G (resp, G^+), and we denote it by $H(G, K)$ (resp, $H(G^+, K)$.) Clearly, we have

$$L(G, K) \cong H(G, K) \otimes_{\mathbf{Z}} \mathbf{C}, L(G^+, K) \cong H(G^+, K) \otimes_{\mathbf{Z}} \mathbf{C}$$

We will soon discover that the Hecke ring $H(G, K)$ is closely related to the Hall algebra $H(\mathfrak{o})$ of the discrete evaluating ring $\mathfrak{o}$.

Consider a double coset KxK, where $x \in G$. By multiplying x by a suitable power of π we can bring x to G^+. The theory of elementary divisors for matrices over a principal ideal domain now shows that by pre- and post- multiplying x by suitable elements of K we can reduce x to a diagonal matrix. Multiplying further by a diagonal matrix belonging to x will produce a diagonal matrix whose entries are powers of π, and finally conjugation by a permutation matrix will get the exponents in descending order. Hence, each double coset KxK has a unique representative of the form

$$\pi^\lambda = (\pi^{\lambda_1}, ..., \pi^{\lambda_n})$$

where $\lambda_1 \geq ... \geq \lambda_n$. We have $\lambda_n \geq 0$ (so that λ is a paritition) if and only if $x \in G^+$.

Let c_λ denote the characteristic function of the double coset $K\pi^\lambda K$. Then we have the c_λ (resp. the c_λ such that $\lambda_n \geq 0$) form a $\mathbf{Z}$-basis of $H(G, K)$ (resp. $H(G^+, K)$). The characteristic function c_0 of K is the identity element of $H(G, K)$ and $H(G^+, K)$. Notice that

$$H(G, K) = H(G^+, K)[c^{-1}_{(1^n)}]$$

This would allows us to concentrate on $H(G^+, K)$, which has a $\mathbf{Z}$-basis consisting of the characteristic functions c_λ, where λ runs through all partitions $(\lambda_1, ..., \lambda_n)$ of length $\leq n$.

Let μ, ν be partitions of length $\leq n$. The product $c_\mu * c_\nu$ will be a linear combination of the c_λ. In fact,

$$c_\mu * c_\nu = \sum_\lambda g^\lambda_{\mu\nu}(q) c_\lambda \tag{3.1}$$

summed over all partitions λ of length $\leq n$, where $g^\lambda_{\mu\nu}(q)$ is the "Hall polynomial" defined in Chapter II. In fact, if we write $K\pi^\mu K = \cup_j Kx_j, K\pi^\nu K = \cup_j Ky_j$ as disjoint unions of left cosets, then we have

$$(c_\mu * c_\nu)(\pi^\lambda) = \int_G c_\mu(\pi^\lambda y^{-1}) c_\nu(y) dy = \sum_j c_\mu(\pi^\lambda y_j^{-1}) \tag{3.2}$$

since K has measure 1. This is furthermore equal to the number of parts (i, j) such that

$$\pi^\lambda = kx_iy_j$$

for some $k \in K$ depending on i, j, and thus $= g^{\lambda}_{\mu\nu}(q)$.

From (3.1), it follows that the mapping $u_\lambda \mapsto c_\lambda$ is a homomorphism of the Hall algebra $H(\mathfrak{o})$ onto $H(G^+, K)$ whose kernel is generated by the u_λ such that $l(\lambda) > n$. Hence from Chapter III, we obtain a structure theorem for $H(G^+, K)$ and $L(G^+, K)$: Let $\Lambda_n[q^{-1}]$ denote the ring of symmetric polynomials in n variables with coefficient in $\mathbf{Z}[q^{-1}]$ (resp. $\mathbf{C}$). Then the $\mathbf{Z}$-linear mapping θ of $H(G^+, K)$ into $\Lambda_n[q^{-1}]$ (resp. $\mathbf{C}$). Then the $\mathbf{Z}$-linear mapping of $H(G, K)$ into $\Lambda_n[q^{-1}]$ (resp. the $\mathbf{C}$-linear mapping of $L(G^+, K)$ into $\Lambda_{n,\mathbf{C}}$) defined by

$$\theta(c_\lambda) = q^{-n(\lambda)} P_\lambda(x_1, ..., x_n; q^{-1}) \tag{3.3}$$

for all partitions λ of length $\leq n$, is an injective ring homomorphism (resp. an isomorphism of $\mathbf{C}$-algebras).

Finally, let us compute the measure of a double closet $K\pi^\lambda K$. For $f \in L(G^+, K)$, let

$$\mu(f) = \int_G f(x)dx$$

Then $\mu : L(G^+, K) \to \mathbf{C}$ is a $\mathbf{C}$-algebra homomorphism, and clearly $\mu(c_\lambda)$ is the measure of $K\pi^\lambda K$. In view of (3.3) we may write $\mu = \mu' \circ \theta$, where $\mu' : \Lambda_{n,\mathbf{C}} \to \mathbf{C}$ is a $\mathbf{C}$-algebra homomorphism, hence is determined by its effect on the generators $e_r = P_{(1^r)}(x_1, ..., x_n; q^{-1})$. On the other hand, $\mu(c_{(1^r)})$ is the number of subvector spaces in k^n with dimension r, which is equal to

$$\mu(c_{(1^r)}) = \begin{bmatrix} n \\ r \end{bmatrix} (q)$$

From (3.3) we have $\mu'(e_r) = q^{r(r-1)/2} \begin{bmatrix} n \\ r \end{bmatrix} (q) = e_r(q^{n-1}, q^{n-2}, ..., 1)$. Hence μ' is the mapping which takes x_i to $q^{n-i}(1 \leq i \leq n)$. It follows that therefore from (3.2) and (3.3) that the measure of $K\pi^\lambda K$ is $q^{-n(\lambda)} P_\lambda(q^{n-1}, q^{n-2}, ..., 1; q^{-1})$. Hence, we also have the measure of $K\pi^\lambda K$ is equal to

$$q^{\sum(n-2i+1)\lambda_i} v_n(q^{-1})/v_\lambda(q^{-1}) = q^{2\langle\lambda,\rho\rangle} v_n(q^{-1})/v_\lambda(q^{-1}) \tag{3.4}$$

where $\rho = \frac{1}{2}(n-1, n-3, ..., 1-n)$.