1 CHAPTER I: SYMMETRIC FUNCTIONS

1.1 Partitions

A partition is any (finite or infinite sequence

\[ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots) \]  

of non-negative integers in decreasing order:

\[ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq \ldots \]  

and containing only finitely many nonzero terms. We don’t distinguish two such sequences which differ only by a string of zeros at the end.

The nonzero \( \lambda_i \) are called the parts of \( \lambda \). The number of parts is the length of \( \lambda \), denoted by \( l(\lambda) \); And the sum of the parts is the weight of \( \lambda \), denote by \( |\lambda| = \lambda_1 + \lambda_2 + \ldots \).

If \( |\lambda| = n \) we say that \( \lambda \) is a partition of \( n \). The set of all partitions of \( n \) is denoted by \( \mathcal{P}_n \), and the set of all partitions by \( \mathcal{P} \). In particular, \( \mathcal{P}_0 \) consists of a single element, the unique partition of zero, which we denote by 0.

Sometimes it is convenient to use a notation that indicates the number of times each integer occurs as a part:

\[ \lambda = (1^{m_1}2^{m_2}\ldots m_r\ldots) \]

means that exactly \( m_i \) parts of \( \lambda \) are equal to \( i \). The number

\[ m_i = m_i(\lambda) = \# \{j : \lambda_j = i\} \]

is called the multiplicity of \( i \) in \( \lambda \).

The diagram of a partition \( \lambda \) may be formally defined as the set of points \((i, j)\) \( \in \mathbb{Z}^2 \) such that \( 1 \leq j \leq \lambda_i \). For example, the diagram of the partition \((5441)\) consists of 5 points or
nodes in the top row, 4 in the second row, 4 in the third row, and 1 in the fourth row. We shall usually denote the diagram of a partition \( \lambda \) by the same symbol \( \lambda \).

The **conjugate** of a partition \( \lambda \) is the partition \( \lambda' \) whose diagram is the transpose of the diagram \( \lambda \), i.e., the diagram obtained by the reflection in the main diagonal. Hence \( \lambda' \) is the number of nodes in the \( i \)th column of \( \lambda \), or equivalently

\[
\lambda'_i = \#\{j : \lambda_j \geq i\} \quad (1.3)
\]

In particular, \( \lambda'_1 = l(\lambda) \) and \( \lambda_1 = l(\lambda') \). Obviously \( \lambda'' = \lambda \). For example, the conjugate of \((5441)\) is \((43331)\).

From (1.2) and (1.3) we have

\[
m_i(\lambda) = \lambda'_i - \lambda'_{i+1} \quad (1.4)
\]

For each partition \( \lambda \) we define

\[
n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i = \sum_{i \geq 1} \left(\frac{\lambda'_i}{2}\right) \quad (1.5)
\]

To be the sum of the numbers obtained by attaching a zero to each node in the top row of the diagram of \( \lambda \), a 1 to each node in the second row, and so on.

Let \( \lambda \) be a partition and let \( m \geq \lambda_1, n \geq \lambda'_1 \). Then the \( m + n \) numbers

\[
\lambda_i + n - i (1 \leq i \leq n), \quad n - 1 + j - \lambda'_j (1 \leq j \leq m)
\]

are a permutation of \( \{0, 1, 2, ..., m + n - 1\} \).

If \( \lambda, \mu \) are partitions, we shall write \( \lambda \supset \mu \) to mean that the diagram of \( \lambda \) contains the diagram of \( \mu \), i.e., \( \lambda_i \geq \mu_i \) for all \( i \geq 1 \). The set-theoretic difference \( \theta = \mu - \nu \) is called a **skew diagram**.

A **path** in a skew diagram \( \theta \) is as sequence \( x_0, x_1, ..., x_m \) of squares in \( \theta \) such that \( x_{i-1} \) and \( x_i \) have a common side, for \( 1 \leq i \leq m \). A subset \( \varphi \) of \( \theta \) is said to be **connected** if any two squares in \( \varphi \) can be connected by a path in \( \varphi \). The maximal connected subsets of \( \theta \) are themselves skew diagrams, called the **connected components** of \( \theta \). In the example that \( \lambda = (5441) \) and \( \mu = (432) \), we have three connected components.

The **conjugate** of a skew diagram \( \theta = \lambda - \mu \) is \( \theta' = \lambda' - \mu' \). Let \( \theta_i = \lambda_i - \mu_i, \theta'_i = \lambda'_i - \mu'_i \), and

\[
|\theta| = \sum_i \theta_i = |\lambda| - |\mu|
\]

A skew diagram \( \theta \) is a **horizontal** \( m \)-**strip** (resp. a **vertical** \( m \)-**strip**) if \( |\theta| = m \) and \( \theta'_i \leq 1 \) (resp. \( \theta_i \leq 1 \)) for each \( i \geq 1 \). In other words, a horizontal (resp. vertical) strip has at most one square in each column (resp. row).
If $\theta = \lambda - \mu$, a necessary and sufficient condition for $\theta$ to be a horizontal (resp. vertical) strip is that the sequence $\lambda$ and $\mu$ are interlaced, in the sense that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ...$

A skew diagram $\theta$ is a border strip (also called a skew hook by some authors, and ribbon by others) if $\theta$ is a connected and contains no $2 \times 2$ block of squares so that the successive rows (or columns) of $\theta$ overlap by exactly one square. The length of a border strip $\theta$ is the total number $|\theta|$ of square it contains, and its height is defined to be one less than the number of rows it occupies. If we think of a border strip $\theta$ as a set of modes rather than squares, then by joining contiguous nodes by horizontal or vertical line segments of unit length, we obtain a sort of staircase, and the height of $\theta$ is the number of vertical line segments or "risers" in the staircase.

A (column-strict) tableau $T$ is a sequence of partitions

$$\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(r)} = \lambda$$

such that each skew diagram $\theta^{(i)} = \lambda^{(i)} - \lambda^{(i-1)} (1 \leq i \leq r)$ is a horizontal strip. Graphically, $T$ may be described by numbering each square of the skew diagram $\theta^{(i)}$ with the number $i$, for $1 \leq i \leq r$, and we shall often think of a tableau as a numbered skew diagram in this way. The numbers inserted in $\lambda - \mu$ must increase strictly down each column (which explains the adjective "column-strict") and weakly from left to right along each row. The skew diagram $\lambda - \mu$ is called the shape of the tableau $T$ and the sequence $(|\theta^{(1)}|, |\theta^{(2)}|, ..., |\theta^{(r)}|)$ is the weight of $T$. Throughout the book, the work tableau (unqualified) will mean a column-strict tableau, as defined above.

Let $L_n$ denote the reverse lexicographic ordering on the set $\mathcal{P}_n$ of partitions of $n$: that is to say, $L_n$ is the subset of $\mathcal{P}_n \times \mathcal{P}_n$ consisting of all $(\lambda, \mu)$ such that either $\lambda = \mu$ or the first non-vanishing difference $\lambda_i - \mu_i$ is positive. $L_n$ is a total ordering. For example, when $n = 5$, $L_5$ arranges $\mathcal{P}_5$ in the sequence

$$(5), (41), (32), (31^2), (2^21), (21^3), (1^5)$$

Another total ordering on $\mathcal{P}_n$ is $L'_n$, the set of all $(\lambda, \mu)$ such that either $\lambda = \mu$ or else the first non-vanishing difference $\lambda_i^* - \mu_i^*$ is negative, where $\lambda_i^* = \lambda_{n+1-i}$. The orderings $L_n, L'_n$ are distinct as soon as $n \geq 6$. In fact, we have for every $\lambda, \mu \in \mathcal{P}_n$,

$$(\lambda, \mu) \in L'_n \iff (\mu', \lambda') \in L_n$$

An ordering which is more important than either $L_n$ or $L'_n$ is the natural (partial) ordering $N_n$ on $\mathcal{P}_n$ (also called the dominance partial ordering by some authors), which is defined as follows:

$$(\lambda, \mu) \in N_n \iff \lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i, \forall i \geq 1$$

As soon as $n \geq 6$, $N_n$ is not a total ordering. We shall write $\lambda \succeq \mu$ in place of $(\lambda, \mu) \in N_n$.

Let $\lambda, \mu \in \mathcal{P}_n$. Then $\lambda \succeq \mu \iff (\lambda, \mu) \in L_n \cap L'_n \iff \mu' \geq \lambda'$.
Now, let us consider the integer vectors \( a = (a_1, ..., a_n) \in \mathbb{Z}^n \). The symmetric group \( S_n \) acts on \( \mathbb{Z}^n \) by permitting the coordinates, and the set

\[
P_n = \{ b \in \mathbb{Z}^n : b_1 \geq b_2 \geq ... \geq b_n \}
\]

is a fundamental domain for this action, i.e., the \( S_n \)-orbit of each \( a \in \mathbb{Z}^n \) meets \( P_n \) in exactly one point, which we denote by \( a^+ \). Thus \( a^+ \) is obtained by rearranging \( a_1, ..., a_n \) in descending order of magnitude.

For \( a, b \in \mathbb{Z}^n \) we define \( a \geq b \) as before to mean \( a_1 + ... + a_i \geq b_1 + ... + b_i \), \( \forall 1 \leq i \leq n \). Let \( a \in \mathbb{Z}^n \). Then \( a \in P_n \Leftrightarrow a \geq wa, \forall w \in S_n \).

For each pair of integers \( i, j \), \( 1 \leq i < j \leq n \) define \( R_{ij} : \mathbb{Z}^n \to \mathbb{Z}^n \) by

\[
R_{ij}(a_1, ..., a_n) = (a_1, ..., a_i + 1, ..., a_j - 1, ..., a_n)
\]

Any product \( R = \prod_{i<j} R_{ij} \) is called a raising operator. The order of the terms in the product is immaterial since they commute with each other.

Let \( a \in \mathbb{Z}^n \) and let \( R \) be a raising operator. Then \( Ra \geq a \). Conversely, let \( a, b \in \mathbb{Z}^n \) be such that \( a \leq b \) and \( a_1 + ... + a_n = b_1 + ... + b_n \). Then there exists a raising operator \( R \) such that \( b = Ra \).

1.2 The ring of symmetric functions

Consider the ring \( \mathbb{Z}[x_1, ..., x_n] \) of polynomials in \( n \) independent variables \( x_1, ..., x_n \) with rational integer coefficients. The symmetric group \( S_n \) acts on this ring by permitting the variables, and a polynomial is symmetric if it is invariant under this action. The symmetric polynomials form a subring

\[
\Lambda_n = \mathbb{Z}[x_1, ..., x_n]^{S_n}
\]

\( \Lambda_n \) is a graded ring: We have

\[
\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k
\]

Where \( \Lambda_n^k \) consists of the homogenous symmetric polynomials of degree \( k \), together with the zero polynomial.

For each \( \alpha = (\alpha_1, ..., \alpha_n) \) we denote by \( x^\alpha \) the polynomial

\[
x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}
\]

Let \( \lambda \) be any partition. The polynomial

\[
m_\lambda(x_1, ..., x_n) = \sum x^\alpha
\]

(1.6)

summed over all distinct permutations \( \alpha \) of \( \lambda = (\lambda_1, ..., \lambda_n) \), is clearly symmetric, and the \( m_\lambda \) (as \( \lambda \) run through all the partitions of length \( \leq n \)) form a \( \mathbb{Z} \)-basis of \( \Lambda_n \). Hence the \( m_\lambda \)
such that $l(\lambda) \leq n$ and $|\lambda| = k$ form a $\mathbb{Z}$-basis of $\lambda_n^k$. In particular, as soon as $n \geq k$, the $m_\lambda$ such that $|\lambda| = k$ form a $\mathbb{Z}$-basis of $\Lambda_n^k$.

Now, let us generate the theory above in the case of countably many independent variables. Let $m \geq n$ and consider the homomorphism

$$\mathbb{Z}[x_1, \ldots, x_m] \to \mathbb{Z}[x_1, \ldots, x_n]$$

which send each of $x_{n+1}, \ldots, x_m$ to zero and the other $x_i$ to themselves. On restriction to $\Lambda_m$ this gives a homomorphism

$$\rho_{m,n} : \Lambda_m \to \Lambda_n$$

It follows that $\rho_{m,n}$ is subjective, and on restriction to $\Lambda_m^k$ we have homomorphism

$$\rho_{m,n}^k : \Lambda_m^k \to \Lambda_n^k$$

for all $k \geq 0$ and $m \geq n$, which are always subjective, and are bijective for $m \geq n \geq k$.

We now form the inverse limit

$$\Lambda^k = \varprojlim \Lambda_n^k$$

of the $\mathbb{Z}$-modules $\Lambda_n^k$ relative to the homomorphism $\rho_{m,n}^k$. This module has a $\mathbb{Z}$-basis consisting of the monomial symmetric functions $m_\lambda$ (for all partitions $\lambda$ of $k$). Therefore, $\Lambda^k$ is a free $\mathbb{Z}$-module of rank $p(k)$, the number of partitions of $k$. Now let

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k$$

so that $\Lambda$ is the free $\mathbb{Z}$-module generated by the $m_\lambda$ for all partitions $\lambda$. We have surjective homomorphisms

$$\rho_n = \bigoplus_{k \geq 0} \rho_{n}^k : \Lambda \to \Lambda_n$$

Which give $\Lambda$ a graded ring structure.

For each integer $r$, the $r$th elementary symmetric function $e_r$ is the sum of all products of $r$ distinct variables $x_i$, so that $e_0 = 1$ and

$$e_r = \sum_{i_1 < i_2 < \ldots < i_r} x_{i_1} x_{i_2} \ldots x_{i_r} = m(1^r)$$

for $r \geq 1$. The generating function for the $e_r$ is

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t) \quad (1.7)$$

For each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, define

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots$$

Let $\lambda$ be a partition, $\lambda'$ its conjugate. Then
where the $a_{\lambda\mu}$ are non-negative integers, and the sum is over partitions $\mu < \lambda$ in the natural ordering. Therefore, we have

$$\Lambda = \mathbb{Z}[e_1, e_2, ...]$$

and the $e_r$ are algebraically independent over $\mathbb{Z}$.

For each integer $r$, the $r$th **complete symmetric function** $h_r$ is the sum of all monomials of total degree $r$ in the variables $x_i$, so that

$$h_r = \sum_{|\lambda| = r} m_\lambda$$

In particular, $h_0 = 1$ and $h_1 = e_1$. The generating function for the $h_r$ is

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} (1 - x_i t)^{-1} \quad (1.8)$$

From (1.7) and (1.8) we have $H(t)E(-t) = 1$, are equivalently

$$\sum_{r=0}^{n} e_r h_{n-r} = 0 \quad (1.9)$$

for all $n \geq 1$. Since the $e_r$ are algebraically independent, we may define a homomorphism of graded rings

$$\omega : \Lambda \to \Lambda$$

such that $\omega(e_r) = h_r$ for all $r \geq 0$. The symmetry of the relations (1.9) between the $e$’s and the $h$’s shows that $\omega$ is an involution, i.e., $\omega^2$ is the identity map. Also, we have

$$\Lambda = \mathbb{Z}[h_1, h_2, ...]$$

and the $h_r$ are algebraically independent over $\mathbb{Z}$.

For each partition $\lambda = (\lambda_1, \lambda_2, ...)$, define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} ...$$

Then the $h_\lambda$ form a $\mathbb{Z}$-basis of $\Lambda$. Finally, if we define

$$f_\lambda = \omega(m_\lambda)$$

for each partition $\lambda$, then the $f_\lambda$, called the "forgotten" symmetric functions, together with $m_\lambda, e_\lambda, h_\lambda$, form four $\mathbb{Z}$-bases of $\Lambda$.

Let $N$ be a positive integer and consider the matrices of $N + 1$ rows and columns

$$H = (h_{i-j})_{0 \leq i, j \leq N}, \quad E = ((-1)^{i-j} e_{i-j})$$

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Where $e_r = h_r = 0$ whenever $r$ is negative. Both $H$ and $E$ are strictly lower triangular, and $HE = EH = I_{N+1}$. Let $\lambda, \mu$ be two partitions of length $\leq p$, such that $\lambda'$ and $\mu'$ have length $\leq q$, where $p + q = N + 1$. Consider the minor of $H$ with row indices $\lambda_i + p - i$ $(1 \leq i \leq p)$ and column indices $\mu_i + p - i$ $(1 \leq i \leq p)$. The complementary cofactor of $E'$ has row indices $p - 1 + j - \lambda'_j$ $(1 \leq j \leq q)$ and column indices $p - 1 + j - \mu'_j$ $(1 \leq j \leq q)$. Hence, we have

$$\det(h_{\lambda_i - \mu_j - i + j})_{1 \leq i, j \leq p} = (-1)^{|\lambda|+|\mu|} \det((-1)^{r_i - i + j} e_{\lambda'_i - \mu'_i - i + j})_{1 \leq i, j \leq q}$$

The minus signs cancel out, and therefore we have

$$\det(h_{\lambda_i - \mu_j - i + j})_{1 \leq i, j \leq p} = \det(e_{\lambda'_i - \mu'_i - i + j})_{1 \leq i, j \leq q}$$

(Later, we will see that this could also be written as $s_{\lambda/\mu}$.) In particular, taking $\mu = 0$ we have

$$\det(h_{\lambda_i - i + j}) = \det(e_{\lambda'_i - i + j})$$

And later, we will see that both sides are equal to the Schur function $s_{\lambda}$.

For each $r \geq 1$ the **power sum** is

$$p_r = \sum_i x_i^r = m_r$$

The generating function for the $p_i$ is

$$P(t) = \sum_{r \geq 1} p_r t^{r-1} = \sum_{i \geq 1} \frac{x_i}{1 - x_i t} = \sum_{i \geq 1} \frac{d}{dt} \log \frac{1}{1 - x_i t}$$

so that

$$P(t) = \frac{d}{dt} \log H(t) = H'(t)/H(t) \quad (1.10)$$

Likewise, we have

$$P(-t) = \frac{d}{dt} \log E(t) = E'(t)/E(t) \quad (1.11)$$

From (1.10) and (1.11) we obtain

$$nh_n = \sum_{r \geq 1} p_r h_{n-r}, \quad ne_n = \sum_{r=1}^n (-1)^{r-1} p_r e_{n-r} \quad (1.12)$$

This would imply that

$$\Lambda_Q = \Lambda \otimes_Z Q = Q[p_1, p_2, ...]$$

And the $p_r$ are algebraically independent over $Q$. Hence, if we define

$$p_{\lambda} = p_{\lambda_1} p_{\lambda_2}$$

for each partition $\lambda = (\lambda_1, \lambda_2, ...)$, then the $p_{\lambda}$ form a $Q$-basis of $\Lambda_Q$. But they do NOT form a $Z$-basis of $\Lambda$. 

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Since the involution $\omega$ interchanges $E(t)$ and $H(t)$, it follows from (1.10) and (1.11) that
\[
\omega(p_n) = (-1)^{n-1}p_n
\]
for all $n \geq 1$, and hence that for any partition $\lambda$ we have
\[
\omega(p_\lambda) = \epsilon_\lambda p_\lambda
\]  
(1.13)
where $\epsilon_\lambda = (-1)^{|\lambda|-l(\lambda)}$.

Finally, we shall express $h_n$ and $e_n$ as linear combinations of the $p_\lambda$. For any partition $\lambda$, define
\[
z_\lambda = \prod_{i \geq 1} i^{m_i} \cdot m_i!
\]
where $m_i = m_i(\lambda)$ is the number of parts of $\lambda$ equal to $i$. Then we have
\[
H(t) = \sum_\lambda \epsilon_\lambda z_\lambda^{-1} p_\lambda t^{\lambda_1}, \quad E(t) = \sum_\lambda \epsilon_\lambda z_\lambda^{-1} p_\lambda t^{\lambda_1}
\]  
(1.14)
or equivalently,
\[
h_n = \sum_{|\lambda|=n} \epsilon_\lambda z_\lambda^{-1} p_\lambda, \quad e_n = \sum_{|\lambda|=n} \epsilon_\lambda z_\lambda^{-1} p_\lambda.
\]

### 1.3 Schur functions

Let $x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$ be a monomial, and consider the polynomial $a_\alpha$ obtained by antisymmetrizing $x^\alpha$: that is to say,
\[
a_\alpha = a_\alpha(x_1, ..., x_n) = \sum_{\omega \in S_n} \epsilon(\omega) \cdot \omega(x^\alpha)
\]
where $\epsilon(\omega)$ is the sign of the permutation $\omega$. This polynomial $a_\alpha$ is skew-symmetric, i.e., we have $\omega(a_\alpha) = \epsilon(\omega)a_\alpha$ for every $\omega \in S_n$; in particular, therefore, $a_\alpha$ vanished unless $\alpha_1, ..., \alpha_n$ are all distinct. Hence we may as well assume that $\alpha_1 > \alpha_2 > ... > \alpha_n \geq 0$, and therefore we may write $\alpha = \lambda + \delta$, where $\lambda$ is a partition of length $\leq n$, and $\delta = (n-1, n-2, ..., 1, 0)$. Then
\[
a_\alpha = a_{\lambda+\delta} = \sum_{\omega} \epsilon(\omega) \cdot \omega(x^{\lambda+\delta})
\]
which can be written as a determinant:
\[
a_{\lambda+\delta} = \det(x^\lambda_i + n - j)_{1 \leq i, j \leq n}
\]
This determinant is divisible in $\mathbb{Z}[x_1, ..., x_n]$ by each of the differences $x_i - x_j (1 \leq i < j \leq n)$, and hence by their product, which is the **Vandermonde determinant**
\[
\prod_{1 \leq i < j \leq n} (x_i - x_j) = \det(x_i^{n-j}) = a_\delta
\]
So $a_{\lambda+\delta}$ is divisible by $a_{\delta}$ in $\mathbb{Z}[x_1, \ldots, x_n]$, and the quotient

$$s_\lambda = s_\lambda(x_1, \ldots, x_n) = a_{\lambda+\delta}/a_\delta$$

is symmetric, i.e., is in $\Lambda_n$. It is called the Schur function in the variables $x_1, \ldots, x_n$, corresponding to the partition $\lambda$ (where $l(\lambda) \leq n$), and is homogenous of degree $|\lambda|$.

The Schur functions $s_\lambda$, where $l(\lambda) \leq n$, for a $\mathbb{Z}$-basis of $\Lambda_n$; The $s_\lambda$ for a $\mathbb{Z}$-basis of $\Lambda$, and for each $k \geq 0$, the $s_\lambda$ such that $|\lambda| = k$ form a $\mathbb{Z}$-basis of $\Lambda^k$.

In fact, we can write each Schur function $s_\lambda$ as a polynomial in the elementary symmetric functions $e_r$, and as a polynomial in the complete symmetric functions $h_r$. The formulas are

$$s_\lambda = \det(h_{\lambda_i-i+j})_{1 \leq i,j \leq n} = \det(e_{\lambda_i-i+j})_{1 \leq i,j \leq m}$$

Where $n \geq l(\lambda)$, $m \geq l(\lambda')$. It follows that for all partitions $\lambda$, we have

$$\omega(s_\lambda) = s_{\lambda'}$$

Also in particular, $s(n) = h_n$, and $s(1^n) = e_n$.

### 1.4 Orthogonality

Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be two finite or infinite sequences of independent variables. We shall denote the symmetric functions of the $x$’s by $s_\lambda(x), p_\lambda(x)$, etc., and the symmetric functions of the $y$’s by $s_\lambda(y), p_\lambda(y)$, etc.

We shall give three series expansions for the product

$$\prod_{i,j}(1 - x_iy_j)^{-1}$$

The first of these is

$$\prod_{i,j}(1 - x_iy_j)^{-1} = \sum_{\lambda} z^{-1}_\lambda p_\lambda(x)p_\lambda(y)$$

Next, we have

$$\prod_{i,j}(1 - x_iy_j)^{-1} = \sum_{\lambda} h_\lambda(x)m_\lambda(y) = \sum_{\lambda} h_\lambda(y)m_\lambda(x)$$

And the third identity is

$$\prod_{i,j}(1 - x_iy_j)^{-1} = \sum_{\lambda} s_\lambda(x)s_\lambda(y)$$

The three identities are summed over all partitions $\lambda$. We now define a scalar product on $\Lambda$, i.e., a $\mathbb{Z}$-valued bilinear form $\langle u, v \rangle$, by requiring that the bases $(h_\lambda)$ and $(m_\lambda)$ should be dual to each other:
\[ \langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu} \]

for all partitions \( \lambda, \mu \), where \( \delta_{\lambda\mu} \) is the Kronecker delta. For each \( n \geq 0 \), let \( u_\lambda, v_\lambda \) be the \( Q \)-basis of \( \Lambda_n^0 \), indexed by the partitions of \( n \). Then the following conditions are equivalent:

(a) \( \langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu} \) for all \( \lambda, \mu \);

(b) \( \sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j}(1 - x_i y_j)^{-1} \).

It follows that \( \langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \), so that the \( p_\lambda \) form an orthogonal basis of \( \Lambda Q \). Likewise, we have \( \langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu} \). Also, we see that the bilinear form \( \langle u, v \rangle \) is symmetric and positive definite, and the involution \( \omega \) is an isometry, i.e., \( \langle \omega u, \omega v \rangle = \langle u, v \rangle \). Finally, we have \( \langle e_\lambda, f_\mu \rangle = \delta_{\lambda\mu} \) (here \( f_\mu = \omega(m_\mu) \)), i.e., \( (e_\lambda) \) and \( (f_\lambda) \) are dual bases of \( \Lambda \).

1.5 Skew Schur functions

Any symmetric functions \( f \in \Lambda \) is uniquely determined by its scalar products with the \( s_\lambda \); namely

\[ f = \sum_\lambda \langle f, s_\lambda \rangle s_\lambda \]

since the \( s_\lambda \) form an orthogonal basis of \( \Lambda \).

Let \( \lambda, \mu \) be partitions, and define a symmetric functions \( s_{\lambda/\mu} \) by the relations

\[ \langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle \] (1.16)

for all partitions \( \nu \). The \( s_{\lambda/\mu} \) are called the skew Schur functions, Equivalently, if \( c^\lambda_{\mu\nu} \) are the integers defined by

\[ s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \] (1.17)

then we have \( s_{\lambda/\mu} = c^\lambda_{\mu\mu} s_\nu \). In particular, it is clear that \( s_{\lambda/0} = s_\lambda \), where \( 0 \) denotes the zero partition. Also \( c^\lambda_{\mu\nu} = 0 \) unless \( |\lambda| = |\mu| + |\nu| \), so that \( s_{\lambda/\mu} \) is homogenous of degree \( |\lambda| - |\mu| \), and is zero if \( |\lambda| < |\mu| \). Later we will see that \( s_{\lambda/\mu} = 0 \) unless \( \lambda \supset \mu \).

Now let \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \) be two sets of variables. Then

\[ \sum_\lambda s_{\lambda/\mu}(x)s_\lambda(y) = \sum_{\lambda, \nu} c^\lambda_{\mu\nu} s_\nu(x)s_\lambda(y) = \sum_\nu s_\nu(x)s_\mu(y)s_\nu(y) = s_\mu(y) \sum_\nu h_\nu(x)m_\nu(y) \]

Now let us consider the case when \( y = (y_1, \ldots, y_n) \), so that the sums above are restricted to partitions of length \( \leq n \). Recall the definition of Schur polynomials, multiplying \( a_\delta \) on both sides, we have

\[ \sum_\lambda s_{\lambda/\mu}(x)a_{\lambda+\delta}(y) = \sum_\nu h_\nu(x)m_\nu(y)a_{\mu+\delta}(y) \]
Compare the coefficient of $y^{\lambda+\delta}$ on both sides, we have

$$s_{\lambda/\mu}(x) = \det(h_{\lambda_{i}-\mu_{j}+i+j})_{1 \leq i,j \leq n} = \det(e_{\lambda'_{i}-\mu'_{j}+i+j})_{1 \leq i,j \leq m}$$

Where $n \geq l(\lambda), m \geq l(\lambda')$, and therefore $\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'}$. Also, the skew Schur function $s_{\lambda/\mu}$ is zero unless $\lambda \supset \mu$, i.e., if $\lambda_i \geq \mu_i$ for every $i$, in which it depends on the on the skew diagram $\lambda - \mu$. If $\theta_i = \lambda_i - \mu_i$ are the components of $\lambda - \mu$, we have $s_{\lambda/\mu} = \prod s_{(\theta_i)}$.

If the number of variables $x_i$ is finite, we can say more: In fact, we have $s_{\lambda/\mu}(x_1, ..., x_n) = 0$ unless $0 \leq \lambda'_i - \mu'_i \leq n$ for all $i \geq 1$.

Now let $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...), z = (z_1, z_2, ...)$ be three sets of independent variables. Then we have

$$\sum_{\lambda, \mu} s_{\lambda/\mu}(x)s_\lambda(z)s_\mu(y) = \sum_{\mu} s_\mu(y)s_\mu(z) \prod_{i,k}(1 - x_iz_k)^{-1}$$

which is furthermore equal to

$$\prod_{i,k}(1 - x_iz_k)^{-1} \prod_{j,k}(1 - y_jz_k)^{-1} = \sum_{\lambda} s_\lambda(x, y)s_\lambda(z)$$

Therefore we conclude that $s_\lambda(x, y) = \sum_{\mu} s_{\lambda/\mu}(x)s_\mu(y) = \sum_{\mu, \nu} c^\lambda_{\mu, \nu}s_\mu(y)s_\nu(x)$. More generally, we have $s_{\lambda/\mu}(x, y) = \sum_\nu s_{\lambda/\nu}(x)s_{\nu/\mu}(y)$, summed over partitions $\nu$ such that $\lambda \supset \nu \supset \mu$.

We can furthermore generate this formula as follows. Let $x^{(1)}, ..., x^{(n)}$ be $n$ sets of variables, and let $\lambda, \mu$ be partitions. Then

$$s_{\lambda/\mu}(x_1, ..., x_n) = \sum_{(\nu)} \prod_{i=1}^{n} s_{\nu^{(i)}/\nu^{(i-1)}}(x^{(i)})$$

(1.18)

summed over all sequences $(\nu) = (\nu^{(0)}, ..., \nu^{(n)})$ of partitions, such that $\nu^{(0)} = \mu, \nu^{(n)} = \lambda$, and $\nu^{(0)} \subset \nu^{(1)} \subset ... \subset \nu^{(n)}$.

We shall apply the above formula to the case that each of $x^{(1)}, ..., x^{(n)}$ consists of a single variable $x_i$. For a single $x$, it follows that $s_{\lambda/\mu}(x) = 0$ unless $\lambda - \mu$ is a horizontal strip, in which case $s_{\lambda/\mu}(x) = x^{(\lambda - |\mu|)}$. Hence each of the products in the sum on the right-hand side of (1.18) is a monomial $x_1^{\alpha_1}...x_n^{\alpha_n}$, where $\alpha_i = |\nu^{(i)} - \nu^{(i-1)}|$, and hence we have $s_{\lambda/\mu}(x_1, ..., x_n)$ expressed as a sum of monomials $x^\alpha$, one for each tableau $T$ of shape $\lambda - \mu$. If the weight of $T$ is $\alpha = (\alpha_1, ..., \alpha_n)$, we shall write $x_T$ for $x^\alpha$. Then:

$$s_{\lambda/\mu} = \sum_{T} x^T$$

(1.19)

summed over all tableau $T$ of shape $\lambda - \mu$.

For each partition $\nu$ such that $|\nu| = |\lambda| - |\mu|$, let $K_{\lambda-\mu, \nu}$ denote the number of tableau of shape $\lambda - \mu$ and weight $\nu$. From (1.19) we have
\[ s_{\lambda/\mu} = \sum_{\nu} K_{\lambda-\mu,\nu} m_{\nu} \]  

(1.20)

and therefore

\[ K_{\lambda-\mu,\nu} = \langle s_{\lambda/\mu}, h_{\nu} \rangle = \langle s_\lambda, s_\mu h_\nu \rangle \]  

(1.21)

so that

\[ s_\mu h_\nu = \sum_{\lambda} K_{\lambda-\mu,\nu} s_\lambda \]  

(1.22)

In particular, suppose that \( \mu = (r) \), a partition with only one non-zero part. Then \( K_{\lambda-\mu,(r)} \) is 1 or 0 according to as \( \lambda - \mu \) is or is not a horizontal \( r \)-strip, and therefore from (1.22) we have Pieri’s formula

\[ s_\mu h_r = \sum_{\lambda} s_\lambda \]  

(1.23)

summed over all partitions \( \lambda \) such that \( \lambda - \mu \) is a horizontal \( r \)-strip. Applying the involution \( \omega \) to the equation above, we obtain

\[ s_\mu e_r = \sum_{\lambda} s_\lambda \]  

(1.24)

summed over all partitions \( \lambda \) such that \( \lambda - \mu \) is a vertical \( r \)-strip.
CHAPTER II: HALL POLYNOMIALS

2.1 Finite $\mathfrak{o}$-modules

First, let us introduce the term "discrete valuation ring". A discrete valuation ring (DVR), denoted as $\mathfrak{o}$, is a principal ideal domain (PID) with exactly one non-zero maximal ideal, denoted as $\mathfrak{p}$. This would indicate that $\mathfrak{o}$ is a integral domain, and its field of fractions $K$ is equipped with a valuation $v : K \to \mathbb{Z} \cup \{\infty\}$ such that for all $x, y \in K$,

1. \(v(xy) = v(x)v(y)\)
2. \(v(x + y) \geq \min\{v(x), v(y)\}\)
3. \(v(x) = \infty \iff x = 0\)

and also $\mathfrak{o} = \{x | v(x) \geq 0\}$, $\mathfrak{p} = \{x | v(x) \geq 1\}$

The most important example of DVR is the ring $p$-adic integers $\mathbb{Z}_p$ consisting of elements $0$ and $s = \sum_{i,k} a_ip^i = a_k p^k + a_{k+1} p^{k+1} + ...$

where $v(s) = k \geq 0$, $a_k \in \{1, 2, ..., p - 1\}$, $a_{k+1}, a_{k+2}, ... \in \{0, 1, 2, ..., p - 1\}$. Its fields of fractions is the $p$-adic field $\mathbb{Q}_p$, where we would allow $k$ to be negative.

Now for the term finite $\mathfrak{o}$-module, we mean a module $M$ with a direct sum decomposition of the form

$$M \cong \bigoplus_{i=1}^r \mathfrak{o}/\mathfrak{p}^{\lambda_i} \quad (2.1)$$

where the $\lambda_i$ are positive integers, which we may assume are arranged in descending order: $\lambda = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r > 0$. In other words, $\lambda = (\lambda_1, ..., \lambda_r)$ is a partition. On the other hand, given a finite $\mathfrak{o}$-module $M$, let $\mu_i = \dim_k (\mathfrak{p}^{i-1}M/\mathfrak{p}^iM)$. Then $\mu = (\mu_1, \mu_2, ...)$ is the conjugate of partition $\lambda$. Therefore, the partition $\lambda$ is uniquely determined by the module $M$, and we call $\lambda$ the type of $M$. Clearly two finite $\mathfrak{o}$-modules are isomorphic if and only if they have the same type, and every partition $\lambda$ occurs as a type. If $\lambda$ is the type of $M$, then $|\lambda| = \sum \lambda_i$ is the length $l(M)$ of $M$, i.e., the length of a composition series of $M$. The length is an additive function of $M$, this means that if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of finite $\mathfrak{o}$-modules, then

$$l(M') - l(M) + l(M'') = 0$$

If $N$ is a submodule of $M$, then the cotype of $N$ in $M$ is defined to be the type of $M/N$.

A finite $\mathfrak{o}$-module $M$ is cyclic, i.e., generated by one element, if and only if its type is a partition $(r)$ consisting of a single part $r = l(M)$, and $M$ is elementary, i.e., $\mathfrak{p}M = 0$ if and only if the type of $M$ is $(1^r)$. If $M$ is elementary of type $(1^r)$, then $M$ is a vector space over $k$, and $l(M) = \dim_k M = r$. 

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Let \( M \) be a finite \( \mathfrak{o} \)-module. The dual of \( M \) is defined to be

\[
\hat{M} = \text{Hom}_\mathfrak{o}(M, E)
\]

Where \( E = \lim \frac{\mathfrak{o}}{p^n} \) is the "injective envelope" of \( k \), i.e., the smallest injective \( \mathfrak{o} \)-module which contains \( k \) as a submodule. \( \hat{M} \) and \( M \) are isomorphism and have the same type. Since \( E \) is injective, an exact sequence

\[
0 \to N \to M \to M/N \to 0
\]

(2.2)
gives rise to an exact sequence

\[
0 \leftarrow \hat{N} \leftarrow \hat{M} \leftarrow \hat{M}/N \leftarrow 0
\]

(2.3)

\( N \leftrightarrow N^0 = \hat{M}/N \) is a one-to-one correspondence between the submodules of \( M, \hat{M} \) respectively, which maps the set of all \( N \subset M \) of type \( \nu \) and cotype \( \mu \) onto the set of all \( N^0 \subset \hat{M} \) of type \( \mu \) and cotype \( \mu \).

From now on, we suppose the residue field \( k = \mathfrak{o}/p \) is FINITE with order \( q < \infty \). If \( M \) is a finite \( \mathfrak{o} \)-module and \( x \) is a non-zero element of \( M \), we shall say that \( x \) has height \( r \) if \( p^r x = 0 \) and \( p^{r-1} x \neq 0 \). The zero element of \( M \) is assigned height zero. We denote by \( M_r \) the submodule of \( M \) consisting of elements of height \( \leq r \), so that \( M_r = \ker(p^r) \).

The number of automorphisms of a finite \( \mathfrak{o} \)-module \( M \) of type \( \lambda \) is

\[
a_{\lambda}(q) = q^{\mid \lambda \mid + 2n(\lambda)} \prod_{i \geq 1} \varphi_{m_j}(\lambda)(q^{-1}) = q^{\sum_{i \geq 1} \lambda^2_i} \prod_{i \geq 1} \varphi_{m_j}(\lambda)(q^{-1})
\]

(2.4)

where \( \varphi_m(t) = (1-t)(1-t^2)...(1-t^m) \). In fact, the number of automorphisms of \( M \) is equal to the number of sequences \( (x_1, ..., x_r) \) such that \( x_i \) has height \( \lambda_i(1 \leq i \leq r) \) and \( M = \oplus_i \mathfrak{o}x_i \).

### 2.2 The Hall algebra

Let \( \lambda, \mu^{(1)}, ..., \mu^{(r)} \) be partitions, and let \( M \) be a finite \( \mathfrak{o} \)-module of type \( \lambda \). We define

\[
G^\lambda_{\mu^{(1)}, ..., \mu^{(r)}}(\mathfrak{o})
\]

to be the number of chains of submodules of \( M \):

\[
M = M_0 \supset M_1 \supset M_r = 0
\]

such that \( M_{i-1}/M_i \) has type \( \mu^{(i)} \), for \( 1 \leq i \leq r \). In particular, \( G^\lambda_{\mu^{(1)}}(\mathfrak{o}) \) is the number of submodules \( N \) of \( M \) which have type \( \nu \) and cotype \( \mu \). Since \( l(M) = l(M/N) + l(N) \), it is clear that \( G^\lambda_{\mu^{(1)}}(\mathfrak{o}) = 0 \) unless \( \mid \lambda \mid = \mid \mu \mid + \mid \nu \mid \).

Let \( H = H(\mathfrak{o}) \) be a free \( \mathbb{Z} \)-module on a basis \( u_\lambda \) indexed by all partitions \( \lambda \). Define a product in \( H \) by the rule
\[ u_{\mu}u_{\nu} = \sum_{\lambda} g_{\mu\nu}^\lambda \langle \mathfrak{o} \rangle u_{\lambda} \]

The sum on the right has only finitely many non-zero terms, which makes \( H(\mathfrak{o}) \) a commutative and associative ring with identity element \( u_0 \). We call \( H(\mathfrak{o}) \) the Hall algebra of \( \mathfrak{o} \). The ring \( H(\mathfrak{o}) \) is generated by (as a \( \mathbb{Z} \)-algebra) by the elements \( u_{(r)} \) \( (r \geq 1) \), and they are algebraically independent over \( \mathbb{Z} \).

2.3 The LR-sequence of a submodule

Let \( T \) be a tableau of shape \( \lambda - \mu \) and weight \( \nu = (\nu_1, ..., \nu_r) \). Then \( T \) determines (and is determined by) a sequence of partitions

\[ S = (\lambda^{(0)}, ..., \lambda^{(r)}) \]

such that \( \lambda^{(0)} = \mu, \lambda^{(r)} = \lambda \), and \( \lambda^{(i)} \supset \lambda^{(i-1)} \) for \( 1 \leq i \leq r \), by the condition that \( \lambda^{(i)} - \lambda^{(i-1)} \) is skew diagram consisting of the square occupied by the symbol \( i \) in \( T \) (and hence is a horizontal strip, because \( T \) is a tableau).

A sequence of partitions \( S \) as above will be called a LR-sequence of type \( (\mu, \nu; \lambda) \) if

1. (LR1) \( \lambda^{(0)} = \mu, \lambda^{(r)} = \lambda \), and \( \lambda^{(i)} \supset \lambda^{(i-1)} \) for \( 1 \leq i \leq r \);
2. (LR2) \( \lambda^{(i)} - \lambda^{(i-1)} \) is a horizontal strip of length \( \nu_i \), for \( 1 \leq i \leq r \). (These two conditions ensure that \( S \) determines a tableau \( T \).)
3. (LR3) The word \( w(T) \) obtained by reading \( T \) from right to left in successive rows, starting at the top, is a lattice permutation.

For (LR3) to be satisfied, it is necessary and sufficient that, for \( i \geq 1 \) and \( k \geq 0 \), the number of symbols \( i \) in the first \( k \) rows of \( T \) should not be less than the number of symbols \( i+1 \) in the first \( k+1 \) rows of \( T \).

Every submodule \( N \) of a finite \( \mathfrak{o} \)-module \( M \) gives rise to a LR-sequence of type \( (\mu', \nu', \lambda') \), where \( \lambda, \mu, \nu \) are the types of \( M, M/N, \) and \( N \) respectively.

2.4 Hall polynomial

Denote \( G_S(\mathfrak{o}) \) the number of submodules \( N \) of \( M \) whose associated LR-sequence \( S(N) \) of \( S \). Each \( N \) has type \( \nu \) and cotype \( \mu \).

Let \( q \) denote the number of elements in the residue field of \( \mathfrak{o} \), and recall that \( n(\lambda) = \sum (i-1)\lambda_i \), for any partition \( \lambda \). Then:

For each LR-sequence \( S \) of type \( (\mu', \nu', \lambda') \), there exists a monic polynomial \( g_S(t) \in \mathbb{Z}[t] \) of degree \( n(\lambda) - n(\mu) - n(\nu) \), independent of \( \mathfrak{o} \), such that

\[ g_S(q) = G_S(\mathfrak{o}) \]  

(2.5)

In other words, \( G_S(\mathfrak{o}) \) is a polynomial in \( q \). Now define, for any three partitions \( \lambda, \mu, \nu \)
\[ g_{\mu\nu}(t) = \sum_S g_S(t) \]

summed over all LR-sequences \( S \) of type \((\mu', \nu'; \lambda')\). This polynomial is the Hall polynomial corresponding to \( \lambda, \mu, \nu \). Recall from sections 1.5 and 1.9 that \( c_{\mu\nu}^\lambda \) denotes the coefficient \( s_\lambda \) in the product \( s_\mu s_\nu \); That \( c_{\mu\nu}^\lambda = c_{\mu'\nu'}^\lambda \) is the number of LR-sequences of type \((\mu', \nu'; \lambda')\). Then it follows that

(i) If \( c_{\mu\nu}^\lambda = 0 \), the Hall polynomial \( g_{\nu\mu}(t) \) is identically zero. In particular, \( g_{\nu\mu}(t) = 0 \) unless \( |\lambda| = |\mu| + |\nu| \) and \( \mu, \nu \subset \lambda \).

(ii) If \( c_{\mu\nu}^\lambda \neq 0 \), then \( g_{\mu\nu}(t) \) has degree \( n(\lambda) - n(\mu) - n(\nu) \) and leading coefficient \( c_{\mu\nu}^\lambda \).

(iii) In either case, \( G_{\mu\nu}(o) = g_{\mu\nu}(q) \).

(iv) \( g_{\mu\nu}(t) = g_{\nu\mu}(t) \).
3 CHAPTER V: THE HECKE RING OF $GL_n$ OVER A LOCAL FIELD

3.1 Local fields

In this chapter, we assume $F$ is a non-archimedean local field, i.e.,

(i) $F$ is a finite algebraic extension of $\mathbb{Q}_p$ for some prime $p$, or

(ii) $F = \mathbb{F}_q((t))$, where $\mathbb{F}_q$ is a finite field.

Let $\mathfrak{o} = \{a \in F : |a| \leq 1\}$ be the ring of integers, and $\mathfrak{p} = \{a \in F : |a| < 1\}$. Let $k = |\mathfrak{o}/\mathfrak{p}|$ be the residue field, and let $q < \infty$ be its order. Let $\pi$ be a generator of $\mathfrak{p}$ with $|\pi| = q^{-1}$.

3.2 The Hecke ring $H(G, K)$

Let $G = GL_n(F)$ be the group of all invertible $n \times n$ matrix over $F$. Also let

$$G^+ = G \cap M_n(\mathfrak{o})$$

be the subsemigroup of $G$ consisting of all matrices $x \in G$ with entries $x_{ij} \in \mathfrak{o}$, and let

$$K = GK_n(\mathfrak{o}) = G^+ \cap (G^+)^{-1}$$

so that $K$ consisting of all $x \in G$ with entries $x_{ij} \in \mathfrak{o}$ and $\det(x)$ a unit in $\mathfrak{o}$.

Let $dx$ denote the unique Haar measure on $G$ for which $K$ has measure 1 and is both left- and right-invariant under the multiplication of $K$. Under this measure, the measure of $Kx$ and $xK$ is 1 for all non-zero $x \in G$.

Let $L(G, K)$ denote the space of all complex-valued continuous functions of compact support of $G$ (resp. $G^+$) which are bi-invariant with respect to $K$, i.e., such that

$$f(k_1xk_2) = f(x)$$

for all $x \in G$ (resp. $G^+$) and $k_1, k_2 \in K$. We may and shall regard $L(G^+, K)$ as a subspace of $L(G, K)$.

We define a multiplication on $L(G, K)$ as follows: for all $f, g \in L(G, K)$,

$$(f \ast g)(x) = \int_G f(xy^{-1})g(y)dy$$

(Since $f$ and $g$ are compactly supported, the integration is over a compact set.) This product is associative and commutative. Since $G^+$ is closed under multiplication, it follows immediately from the definition that $L(G^+, K)$ is a subring of $L(G, K)$.

Each function $f \in L(G, K)$ is constant on each double closet $KxK$ in $G$. These double cosets are compact and mutually disjoint. Since $f$ has compact support, it follows that $f$ takes non-zero values on only finitely many double cosets $KxK$, and hence can be written as
a finite linear combination of their characteristic functions. Hence the characteristic functions of the double cosets of $K$ form a $\mathbb{C}$-basis on $L(G, K)$. The characteristic function of $K$ is the identity element of $L(G, K)$.

If we vary the definition of the algebra $L(G, K)$ (resp. $L(G^+, K)$) by requiring the functions to take their values in $\mathbb{Z}$ instead of $\mathbb{C}$, the resulting ring is called the Hecke ring of $G$ (resp. $G^+$), and we denote it by $H(G, K)$ (resp, $H(G^+, K)$.) Clearly, we have

$$L(G, K) \cong H(G, K) \otimes_{\mathbb{Z}} \mathbb{C}, \quad L(G^+, K) \cong H(G^+, K) \otimes_{\mathbb{Z}} \mathbb{C}$$

We will soon discover that the Hecke ring $H(G, K)$ is closely related to the Hall algebra $H(\mathfrak{a})$ of the discrete evaluating ring $\mathfrak{a}$.

Consider a double coset $KxK$, where $x \in G$. By multiplying $x$ by a suitable power of $\pi$ we can bring $x$ to $G^+$. The theory of elementary divisors for matrices over a principal ideal domain now shows that by pre- and post- multiplying $x$ by suitable elements of $K$ we can reduce $x$ to a diagonal matrix. Multiplying further by a diagonal matrix belonging to $x$ will produce a diagonal matrix whose entries are powers of $\pi$, and finally conjugation by a permutation matrix will get the exponents in descending order. Hence, each double coset $KxK$ has a unique representative of the form

$$\pi^\lambda = (\pi^{\lambda_1}, ..., \pi^{\lambda_n})$$

where $\lambda_1 \geq ... \geq \lambda_n$. We have $\lambda_n \geq 0$ (so that $\lambda$ is a partition) if and only if $x \in G^+$.

Let $c_\lambda$ denote the characteristic function of the double coset $K\pi^\lambda K$. Then we have the $c_\lambda$ (resp. the $c_\lambda$ such that $\lambda_n \geq 0$) form a $\mathbb{Z}$-basis of $H(G, K)$ (resp. $H(G^+, K)$). The characteristic function $c_0$ of $K$ is the identity element of $H(G, K)$ and $H(G^+, K)$. Notice that

$$H(G, K) = H(G^+, K)[c^{-1}_{\lambda}]$$

This would allows us to concentrate on $H(G^+, K)$, which has a $\mathbb{Z}$-basis consisting of the characteristic functions $c_\lambda$, where $\lambda$ runs through all partitions $(\lambda_1, ..., \lambda_n)$ of length $\leq n$.

Let $\mu, \nu$ be partitions of length $\leq n$. The product $c_\mu \ast c_\nu$ will be a linear combination of the $c_\lambda$. In fact,

$$c_\mu \ast c_\nu = \sum_\lambda g^\lambda_{\mu\nu}(q)c_\lambda$$

(3.1)

 summed over all partitions $\lambda$ of length $\leq n$, where $g^\lambda_{\mu\nu}(q)$ is the "Hall polynomial" defined in Chapter II. In fact, if we write $K\pi^\mu K = \cup_j Kx_j, K\pi^\nu K = \cup_j Ky_j$ as disjoint unions of left cosets, then we have

$$(c_\mu \ast c_\nu)(\pi^\lambda) = \int_G c_\mu(\pi^\lambda y^{-1})c_\nu(y)dy = \sum_j c_\mu(\pi^\lambda y_j^{-1})$$

(3.2)

since $K$ has measure 1. This is furthermore equal to the number of parts $(i,j)$ such that

$$\pi^\lambda = k_{xy}y_j$$
for some \( k \in K \) depending on \( i, j \), and thus \( g^k_{\mu\nu}(q) \).

From (3.1), it follows that the mapping \( u_\lambda \mapsto c_\lambda \) is a homomorphism of the Hall algebra \( H(o) \) onto \( H(G^+, K) \) whose kernel is generated by the \( u_\lambda \) such that \( l(\lambda) > n \). Hence from Chapter III, we obtain a structure theorem for \( H(G^+, K) \) and \( L(G^+, K) \): Let \( \Lambda_n[q^{-1}] \) denote the ring of symmetric polynomials in \( n \) variables with coefficient in \( \mathbb{Z}[q^{-1}] \) (resp. \( \mathbb{C} \)). Then the \( \mathbb{Z} \)-linear mapping \( \theta \) of \( H(G, K) \) into \( \Lambda_n[q^{-1}] \) (resp. the \( \mathbb{C} \)-linear mapping of \( L(G^+, K) \) into \( \Lambda_n, \mathbb{C} \)) defined by

\[
\theta(c_\lambda) = q^{-n(\lambda)} P_\lambda(x_1, ..., x_n; q^{-1})
\]

for all partitions \( \lambda \) of length \( \leq n \), is an injective ring homomorphism (resp. an isomorphism of \( \mathbb{C} \)-algebras).

Finally, let us compute the measure of a double closet \( K\pi^\lambda K \). For \( f \in L(G^+, K) \), let

\[
\mu(f) = \int_G f(x) \, dx
\]

Then \( \mu : L(G^+, K) \to \mathbb{C} \) is a \( \mathbb{C} \)-algebra homomorphism, and clearly \( \mu(c_\lambda) \) is the measure of \( K\pi^\lambda K \). In view of (3.3) we may write \( \mu = \mu' \circ \theta \), where \( \mu' : \Lambda_n, \mathbb{C} \to \mathbb{C} \) is a \( \mathbb{C} \)-algebra homomorphism, hence is determined by its effect on the generators \( e_r = P_{(1^r)}(x_1, ..., x_n; q^{-1}) \). On the other hand, \( \mu(c_{(1^r)}) \) is the number of subvector spaces in \( k^n \) with dimension \( r \), which is equal to

\[
\mu(c_{(1^r)}) = \left[ \begin{array}{c} n \\ r \end{array} \right] (q)
\]

From (3.3) we have \( \mu'(e_r) = q^{r(r-1)/2} \left[ \begin{array}{c} n \\ r \end{array} \right] (q) = e_r(q^{n-1}, q^{n-2}, ..., 1) \). Hence \( \mu' \) is the mapping which takes \( x_i \) to \( q^{n-i}(1 \leq i \leq n) \). It follows that therefore from (3.2) and (3.3) that the measure of \( K\pi^\lambda K \) is \( q^{-n(\lambda)} P_\lambda(q^{n-1}, q^{n-2}, ..., 1; q^{-1}) \). Hence, we also have the measure of \( K\pi^\lambda K \) is equal to

\[
q^{\sum (n-2i+1)\lambda_i} v_n(q^{-1})/v_\lambda(q^{-1}) = q^{2(\lambda, \rho)} v_n(q^{-1})/v_\lambda(q^{-1})
\]

where \( \rho = \frac{1}{2}(n-1, n-3, ..., 1-n) \).