Calculus I Midterm #2 Solutions

I. Limits: compute or state the limit does not exist. Explain your work. (5 points each)

(a) $\lim_{\theta\to 0} \frac{\sin(4\theta)}{\theta}$

Use L'Hospital's rule. (This is circular reasoning, because we need to know $\lim_{t\to 0} \frac{\sin t}{t} = 1$ to show $(\sin x)' = \cos x$. It's okay for the exam.)

Or, write $t = 4\theta$. Then $\lim_{\theta\to 0} \frac{\sin(4\theta)}{\theta} \lim_{t\to 0} \frac{\sin t}{\frac{t}{4}} = 4$ since $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

Or, use (twice) the double-angle formula for sine: $\sin(4\theta) = 2\sin(2\theta)\cos(2\theta) = 4\sin\theta\cos\theta\cos(2\theta)$, and (of course) $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

(b) $\lim_{r\to 0^+} r^r$

Since $\ln x$ is continuous, we have $\ln(\lim_{r\to 0^+} r^r) = \lim_{r\to 0^+} \ln r^r = \lim_{r\to 0^+} \frac{\ln r}{\frac{1}{r}} = \lim_{r\to 0^+} \frac{\frac{1}{r}}{\frac{-1}{r^2}} = 0$, where the penultimate equality uses L'Hospital's rule. Since $\ln(\lim_{r\to 0^+} r^r) = 0$, we have $\lim_{r\to 0^+} r^r = 1$.

(c) $\lim_{x\to\infty} x^{100}e^{-x}$

Applying L'Hospital's Rule 100 times, we get $\lim_{x\to\infty} x^{100}e^{-x} = \lim_{x\to\infty} \frac{100!}{e^x} = 0$.

(d) $\lim_{n\to\infty} (1+\frac{13}{n})^n$ (Hint: use the substitution $m = \frac{n}{13}$.)

$\lim_{n\to\infty} (1+\frac{13}{n})^n = [\lim_{m\to\infty} (1+\frac{1}{m})^m]^{13} = e^{13}$.

(e) $\lim_{x\to 0^+} \sin x \ln x$

$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-\cos x}{\sin x^2}} = \lim_{x\to 0^+} \frac{\sin x^2}{-x\cos x} = \lim_{x\to 0^+} \frac{-\sin x}{x} \cdot \lim_{x\to 0^+} \tan x = -1\cdot 0 = 0$, where the second equality uses L'Hospital's Rule.

Think of this one as extra credit if it makes you feel better.

II. Derivatives: find the derivative of the function. Make obvious simplifications to your answer, but do not worry about returning the most compact expression.

(5 points each)

(a) $f(x) = \arctan(1 + x^2)$

Using the chain rule and $(\arctan x)' = \frac{1}{1+x^2}$, we find $f'(x) = \frac{2x}{1+(1+x^2)^2}$.

(b) $f(x) = \frac{(x-1)^{1/5}}{(x^8-1)^{1/4}(x+4)^9}$

Take log, then differentiate. We obtain: $\frac{f'(x)}{f(x)} = \frac{1}{5}\frac{1}{x-1} - \frac{1}{4}\frac{8x^7}{x^8-1} - 9\frac{1}{x+4}$.

Therefore: $f'(x) = \frac{(x-1)^{1/5}}{(x^8-1)^{1/4}(x+4)^9}[\frac{1}{5(x-1)} - \frac{2x^7}{x^8-1} - \frac{9}{x+4}]$.

(c) $f(x) = \ln|\sin x|$

We proved $(\ln|x|)' = \frac{1}{x}$, so with the chain rule this gives $(\ln|f(x)|)' = \frac{f'(x)}{f(x)}$.

Therefore $f'(x) = \frac{\cos x}{\sin x} = \cot x$.

(d) $f(x) = \arcsin x$ (Hint: use implicit differentiation.)

Write $f(x) = y = \arcsin x$. Then $x = \sin y$. Apply $\frac{d}{dx}$ to both sides to obtain $1 = \cos y \frac{dy}{dx}$.

Therefore $\frac{dy}{dx} = f'(x) = \frac{1}{\cos y} = \frac{1}{(1-\sin^2 y)^{1/2}} = \frac{1}{(1-x^2)^{1/2}}$.

(e) $f(x) = x^2 e^x \arcsin x$ (Use your answer in (d).)

Use (d) and the fact that $(fgh)' = f'gh + fg'h + fgh'$.

Therefore $f'(x) = 2xe^x \arcsin x + x^2 e^x \arcsin x + \frac{x^2 e^x}{(1-x^2)^{1/2}}$.

III. Related Rates (6 points)

If the surface area of a cube is changing at a rate of 24 square centimeters per second when a side has length 1 centimeter, what is the rate of change (in cubic centimeters per second) of the volume of the cube at this instant? (The surface area of a cube with side length x is $6x^2$. The volume of a cube with side length x is x^3.)

Since $A = 6x^2$, we have $\frac{dA}{dt} = 12x\frac{dx}{dt}$. We know $\frac{dA}{dt} = 24$ when $x = 1$, so we conclude $\frac{dx}{dt} = 2$ at this instant.

Since $V = x^3$, we have $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$. At the instant $x = 1$, therefore, $\frac{dV}{dt} = 3(1)^2(2) = 6$.

IV. Linear approximation (6 points)

Find the linearization of $f(x) = x^6$ at $a = 2$. (Recall the linearization of $f(x)$ at $x = a$ is simply the linear function whose graph is the line tangent to the graph of f at a.) Then use it to estimate $(2.0001)^6$. Is this estimate greater than or less than the actual value?

In general, the linearization of $f(x)$ at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. Here $f(x) = x^6, f'(x) = 6x^5$, and $a = 2$, so the linearization is:

$L(x) = 64 + 192(x - 2)$.

Our estimate for $(2.0001)^6$ is therefore $L(2.0001) = 64 + 192(.0001) = 64 + .0192 = 64.0192$.

Since the tangent line lies below the curve (by knowing the graph of x^6, or by checking that $30x^4 \geq 0$, so f is concave up), this is an underestimate.

V. Maximum/minimum values and Optimization (6 points each)

(a) Find the (global/absolute) maximum and minimum of $f(x) = \ln(x^2 + x + 1)$ on $[-1, 1]$ and where these extrema occur.

Method 1: since $\ln x$ is differentiable and monotonically increasing, it suffices to maximize/minimize $g(x) = x^2 + x + 1$ on $[-1, 1]$. Then $g'(x) = 2x + 1 = 0$ when $x = -\frac{1}{2}$. Since g is differentiable, the global extrema must be among $g(-1) = 1, g(-\frac{1}{2}) = \frac{3}{4}, g(1) = 3$. Therefore the global maximum of f is $f(1) = \ln 3$, and the global minimum of f is $f(-\frac{1}{2}) = \ln(\frac{3}{4})$. (Alternatively, use "elementary" facts about extrema of parabolas.)

Method 2: use the Closed Interval Method without alteration. Then $f'(x) = \frac{2x+1}{x^2+x+1} = 0$ when $x = -\frac{1}{2}$. Note $x^2 + x + 1 > 0$, so f' is always defined. Then the global extrema must be among $f(-1) = \ln 1 = 0, f(-\frac{1}{2}) = \ln(\frac{3}{4}), f(1) = \ln 3$, and we reach the same conclusion.

(b) If f is differentiable on all of $\mathbb{R}$ and $f'(c) = 0$, must f have a local extremum at c? (If so, cite the appropriate theorem; if not, give an example.)

The converse statement is Fermat's theorem (in particular, the converse is true!), but this statement is not true. For example, let $f(x) = x^3$ and $c = 0$.

VI. The Mean Value Theorem (5 points each)

Recall the Mean Value Theorem: if f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(a) Show that $P(x) = x^5 + 3x + 7$ has at most one real root.

Suppose P has distinct roots r_1 and r_2. Since polynomials are everywhere differentiable, we may apply the MVT to P on $[r_1, r_2]$. This gives a point $c \in (r_1, r_2)$ such that $P'(c) = \frac{P(r_2)-P(r_1)}{r_2-r_1} = \frac{0-0}{r_2-r_1} = 0$.

But $P'(x) = 5x^4 + 3 \geq 3 > 0$ for all real x, so no such c exists. This contradiction shows P has at most one real root.

(In fact P has at least one real root (so exactly one real root) by the Intermediate Value Theorem. Notice $P(0) > 0$ and $P(-2) < 0$.)

*(b) Let f and g be differentiable functions. Show that if $f(0) = g(0)$ and $f'(x) > g'(x)$ for all $x \geq 0$, then $f(x) > g(x)$ for all $x > 0$.

If not, there exists $b > 0$ such that $f(b) \leq g(b)$. Now apply the MVT to $f - g$ on $[0, b]$. (This is legal because the difference of differentiable functions is differentiable.)

Then there exists $c \in (0, b)$ such that $(f-g)'(c) = \frac{(f-g)(b)-(f-g)(0)}{b-0}$.

Now $f'(x) > g'(x)$ for all $x \geq 0$, in particular $f'(c) - g'(c) = (f-g)'(c) > 0$. So the LHS is positive.

On the other hand, $(f-g)(b) - (f-g)(0) = f(b) - g(b) \leq 0$, and $b - 0 = b > 0$, so the RHS is ≤ 0. This is a contradiction.

Remark. In both (a) and (b), the first step is to recognize to what function on what interval you should apply the MVT. The second step is to show the conclusion of the MVT gives a contradiction.

VII. Curve sketching (16 points total)

This problem concerns the function $f(x) = \frac{1}{1-x^2}$.

(a) (2 points) What is the domain of f?

All real x except 1 and -1.

(b) (2 points) Find any asymptotes (vertical, horizontal, slant).

As $x \to \infty$, $f(x) \to 0$. As $x \to -\infty$, $f(x) \to 0$. So $y = 0$ is a horizontal asymptote.

Likewise, $\lim_{x \to 1^+} f(x) = \lim_{x \to -1^-} f(x) = -\infty$, and $\lim_{x \to 1^-} f(x) = \lim_{x \to -1^+} f(x) = \infty$, so $x = 1$ and $x = -1$ are vertical asymptotes.

There are no slant asymptotes.

(c) (5 points) Compute $f'(x)$ and find the intervals on which f is increasing/decreasing. Find all local extrema and where they occur.

$f'(x) = \frac{2x}{(1-x^2)^2}$. The denominator is always ≥ 0, so sign changes will come from the numerator. Clearly $2x > 0 \Leftrightarrow x > 0$ and $2x < 0 \Leftrightarrow x < 0$. Therefore:

f is decreasing on $(-\infty, -1) \cup (-1, 0)$ and f is increasing on $(0, 1) \cup (1, \infty)$.

We cannot include $x = \pm 1$ in either interval because f is not defined at those points.

The critical points occur at $x = 0, \pm 1$. Since f is not defined at $x = \pm 1$, these cannot be local extrema.

$f(0) = 1$ is the unique local minimum (by the First Derivative Test, or by the Second Derivative Test once you do (d)). There are no local maxima.

(d) (4 points) Compute $f''(x)$ and find the intervals on which f is concave upward/downward.

$f''(x) = \frac{6x^2+2}{(1-x^2)^3}$. The numerator is always positive, so the sign of $f''(x)$ is determined by the sign of the denominator. Now $(1-x^2)^3 > 0 \Leftrightarrow 1 - x^2 > 0$ and $(1-x^2)^3 < 0 \Leftrightarrow 1 - x^2 < 0$. Therefore f is concave down on $(-\infty, -1) \cup (1, \infty)$ and concave up on $(-1, 1)$.

(e) (3 points) Sketch the graph of f. Label reasonably.

Use a graphing device. The fact that f is even should be reflected in your sketch.