Calculus I Midterm #1 Solutions

February 19, 2008

* = extra credit

I. Functions (5 points each)

(a) Give the definition of a one-to-one (injective) function (from $\mathbb{R}$ to $\mathbb{R}$ is fine).

A function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one if for all $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$. Said differently, f is one-to-one if whenever x_1 and x_2 are distinct real numbers, $f(x_1)$ and $f(x_2)$ are distinct real numbers.

Since we routinely identify a function with its graph, we can also say: f is one-to-one if any horizontal line intersects its graph at most once.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a one-to-one function. Define $f^{-1} : \mathbb{R} \to \mathbb{R}$, the function inverse to f.

We define f^{-1} by the rule: $f^{-1}(x) = y \Leftrightarrow f(y) = x$. Note this rule only defines a function when f is one-to-one!

We can also say f^{-1} is the function whose graph is gotten by reflecting the graph of f through the line $y = x$.

II. Limits: compute or state the limit does not exist. Explain your work. (5 points each)

(a) $\lim\limits_{x\to 0} x^3 \sin(\dfrac{1}{\pi x})$

We have $-1 \leq \sin(\frac{1}{\pi x}) \leq 1$ for all $x \neq 0$, so we have $-x^3 \leq x^3 \sin(\frac{1}{\pi x}) \leq x^3$ for $x > 0$ and $-x^3 \geq x^3 \sin(\frac{1}{\pi x}) \geq x^3$ for $x < 0$. Let x tend to 0 in the former inequality to get $\lim\limits_{x\to 0^+} x^3 \sin(\dfrac{1}{\pi x}) = 0$, in the latter to get $\lim\limits_{x\to 0^-} x^3 \sin(\dfrac{1}{\pi x}) = 0$.

Therefore $\lim\limits_{x\to 0} x^3 \sin(\dfrac{1}{\pi x}) = 0$.

(OR just compute the left or right hand limit and observe $x^3 \sin(\frac{1}{\pi x})$ is even.)

(b) $\lim\limits_{x\to\infty} \dfrac{(x^2 + x + 1)^{1/2}}{3x + 5}$

Except at $x = 0$, $\frac{(x^2+x+1)^{1/2}}{3x+5} = \frac{(1+\frac{1}{x}+\frac{1}{x^2})^{1/2}}{3+\frac{5}{x}}$, so:

$\lim\limits_{x\to\infty} \dfrac{(x^2+x+1)^{1/2}}{3x+5} = \lim\limits_{x\to\infty} \dfrac{(1+\frac{1}{x}+\frac{1}{x^2})^{1/2}}{3+\frac{5}{x}} = \dfrac{\lim\limits_{x\to\infty}(1+\dfrac{1}{x}+\dfrac{1}{x^2})^{1/2}}{\lim\limits_{x\to\infty}(3+\dfrac{5}{x})} = \dfrac{(\lim\limits_{x\to\infty} 1+\lim\limits_{x\to\infty}\dfrac{1}{x}+\lim\limits_{x\to\infty}\dfrac{1}{x^2})^{1/2}}{\lim\limits_{x\to\infty} 3+\lim\limits_{x\to\infty}\dfrac{5}{x}} =$
$\dfrac{(1+0+0)^{1/2}}{3+0} = \dfrac{1}{3}.$

(c) $\lim\limits_{x\to 1} \arcsin(\dfrac{1-\sqrt{x}}{1-x})$

We did this in class, and it's in the book (page 125, Example 8).

First we use that applying a continuous function and taking a limit are compatible, so $\lim\limits_{x\to 1} \arcsin(\dfrac{1-\sqrt{x}}{1-x}) = \arcsin[\lim\limits_{x\to 1}(\dfrac{1-\sqrt{x}}{1-x})]$.

Now we compute $\lim\limits_{x\to 1}(\dfrac{1-\sqrt{x}}{1-x})$. We have $\frac{1-\sqrt{x}}{1-x} = \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \frac{1}{1+\sqrt{x}}$ except when $x=1$. Therefore $\lim\limits_{x\to 1}(\dfrac{1-\sqrt{x}}{1-x}) = \lim\limits_{x\to 1}\dfrac{1}{1+\sqrt{x}} = \dfrac{1}{2}$ since algebraic functions are continuous on their domains.

Putting everything together:

$\lim\limits_{x\to 1} \arcsin(\dfrac{1-\sqrt{x}}{1-x}) = \arcsin[\lim\limits_{x\to 1}(\dfrac{1-\sqrt{x}}{1-x})] = \arcsin(\dfrac{1}{2}) = \dfrac{\pi}{6}.$

(d) $\lim\limits_{x\to 3} \dfrac{|x^2-9|}{x+3}$

$\frac{|x^2-9|}{x+3}$ is continuous at $x=3$, so the limit is simply the value of the function at the point: $\lim\limits_{x\to 3} \dfrac{|x^2-9|}{x+3} = \dfrac{|3^2-9|}{3+3} = 0.$

(e) $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3}$

$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3$ except at $x=3$, so $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3} = \lim\limits_{x\to 3} x+3 = 6.$

III. Continuity and differentiability (5 points each)

(a) Let $f:\mathbb{R}\to\mathbb{R}$ be a function. What does it mean to say f is continuous at $x=a$?

$\lim\limits_{x\to a} f(x) = f(a)$

(b) Draw a picture of a function such that $\lim\limits_{x\to a} f(x)$ exists, and f is defined at a, but f is *not* continuous at a.

Examples from the book:

page 119, Figure 2 ($a = 5$)

page 120, Example 2 (c) ($a = 2$)

page 167, Exercise 1 ($a = 4$)

(c) Give an example of a function which is continuous at a, but not differentiable at a. Explain (one sentence is enough) why the function is not differentiable.

$f(x) = |x|$ is continuous but not differentiable at $x = 0$ because it has a corner, i.e. the tangent line is ambiguous: from below 0 the tangent line wants to have slope -1, but from above 0 it wants to have slope 1.

*(d) Recall the intermediate value theorem: if f is continuous on $[a, b]$ and N is any number between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = N$. Use the intermediate value theorem to show $p(x) = x^3 - 2x + 1$ has a zero.

Apply the intermediate value theorem (which applies since polynomials are continuous) with $a = -2, b = 2, N = 0$. Now $p(-2) = -3 < 0$ and $p(2) = 5 > 0$, so we are guaranteed a solution to $p(c) = 0$ with $c \in (-2, 2)$.

Or observe $p(1) = 0$. (!)

IV. Derivatives (5 points each)

(a) Using the definition of the derivative, compute $f'(x)$ if $f(x) = x^3 + 1$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 + 1 - (x^3+1)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} =$$

$$\lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2$$

(b) Compute $f'(x)$ if $f(x) = 5x^2 + x^{-1} + 3e^x$. (You don't have to use the definition of the derivative, but clearly state any rules you use.) Give the equation of the line tangent to the curve $y = f(x)$ at $x = 1$.

We'll need power rule, sum rule, constant multiple rule, and $(e^x)' = e^x$. Then we find $f'(x) = 10x - x^{-2} + 3e^x$.

The tangent line at $x = 1$ passes through $(1, f(1)) = (1, 6+3e)$ and has slope $f'(1) = 9+3e$. Therefore its equation is $y - (6 + 3e) = (9 + 3e)(x - 1)$.

(c) Suppose the position of an object at time t is given by $s(t) = 4t^3 - 2t^2 + 7t - 101$. Find both its instantaneous velocity and acceleration at $t = 2$.

$v(t) = s'(t) = 12t^2 - 4t + 7$ and $a(t) = s''(t) = 24t - 4$. Then $v(2) = 47$ and $a(2) = 44$.

*(d) Let f be an even function which is differentiable. Show that f' is odd.

Proof. $f'(-a) = \lim\limits_{h \to 0} \dfrac{f(-a+h) - f(-a)}{h}$, by definition.

Let $r = -h$. Then $h \to 0$ if and only if $r \to 0$. So we can rewrite our limit:

$f'(-a) = \lim\limits_{r \to 0} \dfrac{f(-a-r) - f(-a)}{-r}$.

Since f is even, $f(-a-r) = f(-(a+r)) = f(a+r)$; likewise $f(-a) = f(a)$. Therefore:

$f'(-a) = \lim\limits_{r \to 0} \dfrac{f(a+r) - f(a)}{-r} = -\lim\limits_{r \to 0} \dfrac{f(a+r) - f(a)}{r} = -f'(a)$.

So f' is odd.

V. Epsilon-delta (5 points each)

(a) Give the $\epsilon - \delta$ definition of $\lim\limits_{x \to a} f(x) = L$.

For every $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

*(b) Using the $\epsilon - \delta$ definition, prove $\lim\limits_{x \to 2} x^2 = 4$.

(Hint: for $1 < x < 3, x + 2 = |x + 2| < 5$.)

We claim for every $\epsilon > 0$, $0 < |x-2| < min(1, \frac{\epsilon}{5})$ implies $|x^2-4| < \epsilon$. That is, $\delta = min(1, \frac{\epsilon}{5})$ works.

Proof: $0 < |x - 2| < min(1, \frac{\epsilon}{5})$ means $0 < |x - 2| < 1$ and $0 < |x - 2| < \frac{\epsilon}{5}$. The inequality $|x - 2| < 1$ implies $|x + 2| < 5$ (this is the hint). We multiply $|x + 2| < 5$ and $|x - 2| < \frac{\epsilon}{5}$ to obtain: $|x + 2||x - 2| < 5\frac{\epsilon}{5}$, which says exactly $|x^2 - 4| < \epsilon$.

*(c) Find δ such that $0 < |x - 2| < \delta$ implies $|x^2 - 4| < \frac{1}{100}$.

In the proof above we found δ as an explicit function of ϵ, namely $\delta(\epsilon) = min(1, \frac{\epsilon}{5})$. Now $\delta(\frac{1}{100}) = min(1, \frac{1}{500}) = \frac{1}{500}$. Therefore $0 < |x - 2| < \frac{1}{500}$ implies $|x^2 - 4| < \frac{1}{100}$.