Calculus I Final Solutions May 13, 2008

MATH V1101 section 6

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Every question is worth 5 points. * = extra credit. Explain your work.

Limits: compute or state the limit does not exist

(a) $\lim\limits_{\theta\to 0} \dfrac{\sin(4\theta)+\tan\theta}{2\theta}$

$= \lim\limits_{\theta\to 0} \dfrac{2\sin(2\theta)\cos(2\theta)}{2\theta} + \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{2\theta\cos\theta} = 2 + \dfrac{1}{2} = \dfrac{5}{2}$

The first equality uses $\sin 2\theta = 2\sin\theta\cos\theta$, the definition of tangent, and the limit law for sums; evaluation uses $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.

(b) $\lim\limits_{x\to\infty} \dfrac{(x^5+1)^{1/6}}{x+1}$

$= [\lim\limits_{x\to\infty} \dfrac{(x^5+1)}{(x+1)^6}]^{1/6} = 0^{1/6} = 0$

The first equality uses the fact that taking limits commutes with applying the continuous function $x^{1/6}$; the evaluation uses degree(numerator) less than degree(denominator).

(c) $\lim\limits_{x\to\infty} x^2 e^{-x}$

Apply L'Hospital's rule twice: $\lim\limits_{x\to\infty} x^2e^{-x} = \lim\limits_{x\to\infty} 2e^{-x} = 0$.

(d) $\lim\limits_{z\to 0} \dfrac{\ln(1+z+z^2)}{1+z+z^2}$

The function $\frac{\ln(1+z+z^2)}{1+z+z^2}$ is continuous at $z = 0$, so we may simply evaluate at $z = 0$. We get $\frac{\ln 1}{1} = 0$.

(e) $\lim\limits_{z\to 0^+} z^z$

$\ln[\lim\limits_{z\to 0^+} z^z]$

$= \lim\limits_{z\to 0^+} \ln(z^z)$ since $\ln x$ is continuous

$= \lim\limits_{z\to 0^+} \dfrac{\ln(z)}{\frac{1}{z}}$

$= \lim\limits_{z\to 0^+} \dfrac{\frac{1}{z}}{\frac{-1}{z^2}}$ by L'Hospital's rule

$= \lim\limits_{z\to 0^+} \dfrac{-z^2}{z} = 0$

Since $\ln[\lim_{z\to 0^+} z^z] = 0$, we conclude $\lim_{z\to 0^+} z^z = 1$.

*(f) Derive the expression of e as a limit: $\lim_{n\to\infty} (1+\frac{1}{n})^n = \lim_{x\to 0} (1+x)^{1/x} = e$.

(Hint: For $f(x) = \ln x$, use the definition of the derivative and the fact that $f'(1) = 1$.)

$1 = f'(1) = \lim_{x\to 0} \frac{\ln(1+x) - \ln(1)}{x}$ by the hint

$1 = \lim_{x\to 0} \frac{\ln(1+x) - \ln(1)}{x}$

$= \lim_{x\to 0} \frac{\ln(1+x)}{x}$ since $\ln 1 = 0$

$= \lim_{x\to 0} \ln(1+x)^{\frac{1}{x}}$

$= \ln[\lim_{x\to 0} (1+x)^{\frac{1}{x}}]$ since $\ln x$ is continuous

Applying the exponential to both sides of the equality $1 = \ln[\lim_{x\to 0} (1+x)^{\frac{1}{x}}]$ and using the fact that $\ln x$ and e^x are inverse functions, we find:

$e = \lim_{x\to 0} (1+x)^{\frac{1}{x}}$ as desired.

Continuity and Differentiability

(a) Compute $f'(x)$ if $f(x) = \arctan(\ln|x|)$.

$f'(x) = \frac{\frac{1}{x}}{1+(\ln|x|)^2}$ by the chain rule

(b) Compute $f'(x)$ if $f(x) = \frac{e^x}{x}$. Give the equation of the line tangent to the curve $y = f(x)$ at $x = 1$.

$f'(x) = \frac{xe^x - e^x}{x^2}$ by the quotient rule

$f'(1) = \frac{e-e}{1} = 0$.

$f(1) = e$

So the equation of the tangent line is $y = e$.

(c) Find y' if $x^3 + y^3 = 6xy$.

Use implicit differentiation (done in class and appears in the textbook):

$3x^2 + 3y^2y' = 6xy' + 6y$

Now solve for y'.

(d) Use linear approximation/differentials to estimate $\sqrt{99.8}$. Is this approximation an underestimate or overestimate?

We are close to $f(100)$ for $f(x) = \sqrt{x}$. Now $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(100) = \frac{1}{20}$.

Therefore the linearization of f at 100 is $L(x) = f(100) + f'(100)(x-100) = 10 + \frac{1}{20}(x-100)$.

The approximation is $L(99.8) = 10 + \frac{-0.2}{20} = 9.99$. This is an overestimate because the tangent line lies above the curve (because the curve is everywhere concave down).

(e) The height of a cylinder is changing at a rate of 1 cm/sec. If the surface area is changing at a rate of 8π square centimeters per second when $r = 2$ cm and $h = 4$ cm, what is the rate of change of the volume at this instant? [The volume of a cylinder with height h and radius r is $\pi r^2 h$. The surface area of such a cylinder is $2\pi rh + 2\pi r^2$.]

$\frac{dS}{dt} = 2\pi r\frac{dh}{dt} + 2\pi h\frac{dr}{dt} + 4\pi r\frac{dr}{dt}$

Plugging in given values $\frac{dS}{dt} = 8\pi$, $r = 2$, $h = 4$, $\frac{dh}{dt} = 1$ allows you to find $\frac{dr}{dt}$.

Now $\frac{dV}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$, so using the given information and the $\frac{dr}{dt}$ we just found, we can compute $\frac{dV}{dt}$.

*(f) Let $f(x) = \cos x$. Find a parabola $y = P(x) = Ax^2 + Bx + C$ such that $f(0) = P(0)$, $f'(0) = P'(0)$, and $f''(0) = P''(0)$. How many such parabolas are there?

$f(0) = 1$, $f'(0) = 0$, and $f''(0) = -1$

On the other hand, $P(0) = C$, $P'(0) = B$, and $P''(0) = 2A$.

Therefore the parabola is $y = 1 - \frac{x^2}{2}$. This is the only such parabola.

Maximum/Minimum Values and Optimization

(a) Find the point P on the line $y = 2x + 7$ which is closest to $(3, 3)$.

We seek to minimize distance from the point $(3, 3)$; it is equivalent to minimize the distance squared since $x \mapsto x^2$ is increasing.

$D^2(x, y) = (x-3)^2 + (y-3)^2 = (x-3)^2 + (2x+7-3)^2 =: F(x)$ when restricted to the line $y = 2x + 7$

We differentiate with respect to x:

$F'(x) = 2(x-3) + 2(2x+4)(2) = 10x + 10$

$F'(-1) = 0$, and -1 is its only zero; $F''(x) = 10$, so we found a local minimum; analyzing the sign of the first derivative shows F is strictly decreasing up to $x = -1$ and strictly increasing afterwards. Therefore $(-1, 5)$ is the closest point.

(b) If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b$ on $0 \leq x \leq 1$.

Compute $f'(x) = ax^{a-1}(1-x)^b + bx^a(1-x)^{b-1}(-1) = x^{a-1}(1-x)^{b-1}[a(1-x) - bx] = x^{a-1}(1-x)^{b-1}[a - x(a+b)] = 0$ when $x = \frac{a}{a+b}$. Since f is increasing up to this point an decreasing after, $f(\frac{a}{a+b})$ is the global maximum.

The Mean Value Theorem

(a) State the Mean Value Theorem.

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(b) Let f be a differentiable function. Show that if $f(0) = 0$ and $f'(x) > 0$ for all $x \geq 0$, then $f(x) > 0$ for all $x > 0$.

If not, there exists $c > 0$ such that $f(c) < 0$. By the Mean Value Theorem applied to f on $[0, c]$, we find a point $b \in (0, c)$ such that $f'(b) = \frac{f(c)}{c} < 0$. Contradiction.

Graphing

(a) What is the domain of the function $f(x) = 1 + \frac{1}{x} + \frac{1}{x^2}$? Find any asymptotes (horizontal, vertical, slant).

The domain is all reals except 0.

$y = 1$ is a horizontal asymptote.

$x = 0$ is a vertical asymptote.

There are no slant asymptotes.

(b) Compute $f'(x)$ and find the intervals on which f is increasing/decreasing, and all local extrema.

$f'(x) = \frac{-1}{x^2} + \frac{-2}{x^3} = \frac{-(x+2)}{x^3}$

f is increasing on $(-2, 0)$ and decreasing on $(-\infty, -2) \cup (0, \infty)$. At $x = -2$ there is a local minimum.

(c) Compute $f''(x)$ and find the intervals on which f is concave upward/downward, and any inflection points.

$f''(x) = \frac{2}{x^3} + \frac{6}{x^4} = \frac{2(x+3)}{x^4}$

f is concave up on $(-3, 0) \cup (0, \infty)$ and concave down on $(-\infty, -3)$. At $x = -3$ there is an inflection point.

(d) Sketch the graph of f.

The Fundamental Theorem of Calculus

(a) Let f be continuous on $[a, b]$. Write down an antiderivative for f on the interval $[a, b]$.

By the Fundamental Theorem of Calculus (part 1), $F(x) = \int_a^x f(t)dt$ is an antiderivative. (Any other differs from F by a constant.)

(b) Find $\frac{d}{dx} \int_x^{x^3} \sqrt{1 + \tan t}\, \mathrm{d}t$.

$= \frac{d}{dx}[\int_x^0 \sqrt{1 + \tan t}\, \mathrm{d}t + \int_0^{x^3} \sqrt{1 + \tan t}\, \mathrm{d}t]$ by one of the properties of the definite integral

$\frac{d}{dx} \int_x^0 \sqrt{1 + \tan t}\, \mathrm{d}t + \frac{d}{dx} \int_0^{x^3} \sqrt{1 + \tan t}\, \mathrm{d}t$ by the linearity of the derivative

$-\frac{d}{dx} \int_0^x \sqrt{1 + \tan t}\, \mathrm{d}t + \frac{d}{dx} \int_0^{x^3} \sqrt{1 + \tan t}\, \mathrm{d}t$ by one of the properties of the definite integral

$-\sqrt{1 + \tan x} + \sqrt{1 + \tan x^3}(3x^2)$ by the Fundamental Theorem of Calculus and the chain rule

Integrals: find an antiderivative or compute the definite integral

(a) $\int \tan x \, \mathrm{d}x$

Use substitution with $u = \cos x$ to find $\int \tan x \, \mathrm{d}x = \ln|\sec x| + C$.

(b) $\int x\sqrt{1-x} \, \mathrm{d}x$

Substitution with $u = 1 - x$.

(c) $\int_{-1}^{1} \tan x + 6x + x^2 + x^3 + \sin x \, \mathrm{d}x$

Since $\tan x + 6x + x^3 + \sin x$ is odd, $\int_{-1}^{1} \tan x + 6x + x^2 + x^3 + \sin x \, \mathrm{d}x = \int_{-1}^{1} x^2 \, \mathrm{d}x = \frac{2}{3}$.

(d) $\int \frac{1+x^2}{x^4} \mathrm{d}x$

$= \int x^{-4} + x^{-2} \mathrm{d}x = \frac{-1}{3} x^{-3} - x^{-1} + C$

(e) $\int (\sin x)^3 \cot x \, \mathrm{d}x$

$= \int (\sin x)^2 \cos x \, \mathrm{d}x = \frac{(\sin x)^3}{3} + C$ by substitution with $u = \sin x$

*(f) $\int x^2 e^x \mathrm{d}x$ (Hint: use integration by parts.)

First use $u = x^2$, $dv = e^x$ (so $du = 2xdx$ and $v = e^x$):

$\int x^2 e^x \mathrm{d}x = x^2 e^x - \int 2xe^x \mathrm{d}x$

Then use parts again with $U = x$, $dV = e^x$ (so $dU = dx$ and $V = e^x$):

$\int xe^x \mathrm{d}x = xe^x - \int e^x \mathrm{d}x = xe^x - e^x$

Therefore:

$\int x^2 e^x \mathrm{d}x = x^2 e^x - 2(xe^x - e^x) + C$

Area and Volume

(a) Find the area between the curves $y = \sin x$ and $y = \cos 2x$ between $x = 0$ and $x = \frac{\pi}{2}$. (Hint: $\cos 2x = 1 - 2\sin^2 x$.)

Use the hint to find the intersection points: $\sin x = 1 - 2\sin^2 x \Leftrightarrow 2\sin^2 x + \sin x - 1 = 0 \Leftrightarrow (2\sin x - 1)(\sin x + 1) = 0$. Now $(\sin x + 1) = 0$ has no solution on $[0, \frac{\pi}{2}]$, and $(2\sin x - 1) = 0$ holds when $x = \frac{\pi}{6}$.

By checking at a point on either side of $\frac{\pi}{6}$, for example, we find:

$\sin x < \cos 2x$ on $[0, \frac{\pi}{6}]$

$\sin x > \cos 2x$ on $[\frac{\pi}{6}, \frac{\pi}{2}]$

Therefore the area between the curves is:

$\int_0^{\frac{\pi}{2}} |\sin x - \cos 2x| dx = \int_0^{\frac{\pi}{6}} \cos 2x - \sin x dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x - \cos 2x dx$

(b) Find the volume of the solid obtained by rotating about the y-axis the area bounded by the x-axis, the curve $y = (x-1)^2(2-x)$, and the lines $x = 1$ and $x = 2$. (Hint: use shells.)

The typical radius is x, and the typical height is $(x-1)^2(2-x)$. Therefore the volume is $\int_1^2 2\pi x(x-1)^2(2-x)dx$.

(c) Describe a right circular cone (with base radius r and height h) as a solid of revolution. Compute its volume using calculus.

A right circular cone is the solid gotten by rotating about the x-axis the area underneath the line $y = \frac{r}{h}x$. Use disks. A typical disk has radius $\frac{r}{h}x$, so area $\frac{\pi r^2 x^2}{h^2}$. We have disks for every x between 0 and h. Therefore the volume is

$\int_0^h \frac{\pi r^2 x^2}{h^2} dx = \frac{\pi r^2}{h^2}[\frac{h^3}{3} - 0] = \frac{\pi r^2 h}{3}$.

Average value of a function

Compute the average value of $f(x) = x^2\sqrt{1+x^3}$ on $[0,2]$. For how many points $c \in [0,2]$ is it true that $f(c)$ equals the average value you just computed? (Just say how many and why; no need to find them.)

The average value of f on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

Therefore the average value of $x^2\sqrt{1+x^3}$ on $[0,2]$ is

$\frac{1}{2}\int_0^2 x^2\sqrt{1+x^3}dx = \frac{1}{2}\frac{2}{9}(1+x^3)^{3/2}|_{x=0}^{x=2} = \frac{1}{9}[9^{3/2}-1] = \frac{26}{9}$.

By the Mean Value Theorem for Integrals, f takes on this value at least once. Since f is increasing on $[0,2]$ (compute f' and observe every term is positive), it takes on no value more than once. Therefore f achieves its average value exactly once on this interval.

Of course solving $f(x) = \frac{26}{9}$ by hand is hopeless!