Lecture IV: K_0, K_1, Whitehead Torsion, and the s-cobordism Theorem

October 15, 2020

We turn now to the s-cobordism theorem, which is the non-simpy connected version of the h-cobordism theorem. Before we can state the result we need some basic definitions from algebraic K-theory.

1 K_0

Even though it is not relevant for the s-cobordism theorem, we start with a brief review of K_0. Let A be a ring. The set of finitely generated projective A-modules forms a semi-group under direct sum. We define $K_0(A)$ to be the Grothendieck group of this semi-group. That is to say we take pairs (M_0, M_1) of finitely generated projective modules and we set $[M_0, M_1] = [M_0'.M_1']$ if $M_0 \oplus M_1'$ is isomorphic to $M_1 \oplus M_0'$.

Since every finitely generated projective $\mathbb{Z}$-module is free, $K_0(\mathbb{Z}) \equiv \mathbb{Z}$ with the isomorphism being given by the identity.

Let M be a compact, smooth manifold. Let us show that a finitely generated projective module over $C^\infty(M)$ is the same thing as a smooth vector bundle over M. The correspondence sends a vector bundle to its group of global sections with its obvious module structure over $C^\infty(M)$. If P is a finitely generated projective then there is a short exact sequenc

$$0 \to K \to F \to P \to 0$$

where F is free and finitely generated. Since P is projective, this sequence splits so that $F \equiv P \oplus K$. Localizing at any point we see that $F_x = K_x \oplus P_x$ where M_x means $M \otimes_{C^\infty(M)} \mathbb{R}$ where the map $C^\infty(M) \to \mathbb{R}$ sends $f \mapsto f(x)$. The restriction to x of the sections of the trivial bundle that lie in K generate $K_x \subset F_x = \mathbb{R}^n$. Clearly, the dimension of K_x is upper semi continuous function of x. The same is true for P_x, so that these functions are both locally constant. It follows that the $K_x \subset F_x$ form a smoothly varying family of subspaces of $\mathbb{R}^n$ of locally constant dimension. It is then easy to see that the space K_x for a smooth sub-bundle of the trivial bundle whose sections are the submodule $K \subset F$. The quotient of the trivial bundle by this sub-bundle gives a smooth bundle whose sections are identiried with P.

This shows that $K_0(C^\infty(M))$ is identified with the usual smooth topological K-theory of M (the Grothendieck group of smooth, finite dimensional vector bundles over M). A free module of rank k over $C^\infty(M)$ is the space of sections of the trivial rank-k smooth vector bundle over M. In a similar vein, if X is a topological space then $K_0(C^0(X))$ is identified with the usual topological K-theory, $K_0(X)$.

The following is an easy exercise.

Claim 1.1. *A finitely generated projective A-module represents the trivial element in $K_0(A)$ if and only if it is* stably free*, i.e., its direct sum with a finitely generated free A-module is a finitely generated, free A-module.*

2 K_1 and the Whitehead group

Definition 2.1. For any ring A, we define $K_1(A)$. Let $GL(A)$ be the direct limit under the natural inclusions of $GL_n(A)$ of invertible $n \times n$ matrices over A. This is a group under matrix multiplication. The group $K_1(A)$ is defined as the quotient of $GL(A)$ by the subgroup generated by the elementary matrices, where an elementary matrix by definition has 1s down the main diagonal and only one non-diagonal entry. This turns out to be the quotient of $GL(A)$ by its commutator subgroup, so that $K_1(A)$ is the abelian quotient of $GL(A)$. It is easy to see that $K_1(\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ with the isomorphism being the determinant.

Now consider the case $A = \mathbb{Z}[G]$, the integral group ring of a group G. We have a natural inclusion $G \times \{\pm 1\} \subset GL(A)$ given by sending $(g, \pm 1)$ to the one-by-one invertible matrix $(\pm g)$ in $GL_1(\mathbb{Z}[G])$. The *Whitehead group* of G, denoted $Wh(G)$, is defined by

$$Wh(G) = K_1(\mathbb{Z}[G])/G \times \{\pm 1\}.$$

Any $M \in GL_n(\mathbb{Z}[G])$ has its Whitehead torsion $Wh(M) \in Wh(G)$. Said another way if L and L' are free $\mathbb{Z}[G]$-modules of finite rank with ordered bases and $\alpha\colon L \to L'$ is an isomorphism then its Whitenead torsion $Wh(\alpha)$, which by the definition is the Whitehead torsion of the matrix describing α in the given bases.

Proposition 2.2. *The Whitehead torsion of a matrix $M \in GL_n(\mathbb{Z}[G])$ is trivial if and only if after stabilizing M by including it in $GL_N(\mathbb{Z}[G])$ for some $N < \infty$ there is a finite sequence of elementary column operations that transform $M + \mathrm{Id}_{N-n}$ to the identity matrix where elementary column operations are:*

- *Multiply a column by $\pm g$ for any $g \in G$ and leave all others unchanged.*
- *Replace a column by its sum with an arbitrary $\mathbb{Z}[G]$-multiple of another column.*
- *Permute the columns.*

Proof. If $[M] \in Wh(G)$ is trivial, then the image of M in $GL(\mathbb{Z}[G])$ is in the subgroup generated by elementary matrices and $\pm g$. Each elementary elementary matrix lies in $GL_N(\mathbb{Z}[G])$ for some N and hence the same is true for any finite set of them. Thus, in $GL_N(\mathbb{Z}[G])$ for some $N < \infty$ the element $M \oplus \mathrm{Id}_{N-n}$ can be written as a product of elementary matrices and matrices $\pm g$. Hence, multiplying $M \oplus \mathrm{Id}_{N-n}$ by a sequence of inverses of elementary matrices and matrices $\pm g$ from GL_1 makes it the identity. But the inverse of an elementary matrix is an elementary matrix and the inverse of $\pm g$ is $\pm g^{-1}$. Thus, multiplying $M \oplus \mathrm{Id}_{N-n}$ (on the left) by a sequence of elementary matrices and $\pm g$ turns it into the identity. These multiplications do the column operations on $M \oplus \mathrm{Id}_{N-n}$ listed in the proposition. $\square$

Exedrcises. 1. Show that any finite permutation matrix represents the zero element in $Wh(G)$.
2. Show that an element T of $GL(\mathbb{Z}[G])$ represents the trivial element of $Wh(G)$ if and only if there is a sequence of consisting of elementary matrices and matrices that are the images of elements $(\pm g) \in GL_1(\mathbb{Z}[G])$ whose product is equal to T. Said another way, viewing T as an isomorphism from a finitely generated free $\mathbb{Z}[G]$ module M_1 with a given basis to another M_2, it represents the trivial element in $Wh(G)$ if and only if there is a sequence of elementary changes of bases carrying the given basis for M_1 to the image under T^{-1} of the given basis for M_2. The elementary transformations are (i) permutation of the basis elements, (ii) multiply the first basis element by $\pm g$ for any $g \in G$ and leave the other basis elements unchanged, and (iii) for some $\alpha \in \mathbb{Z}[G]$ replace the first basis element by the sum of it plus α times the second basis element and leave the rest of the basis unchanged.

Definition 2.3. Let C_* be a chain complex of free modules of finite rank over $\mathbb{Z}[G]$, only finitely many of which are non-zero, and fix ordered bases for the chain modules. Suppose that the homology of C_* is zero. Then there is a chain homotopy $h\colon C_* \to C_{*+1}$ between the identity and 0; i.e., $\partial \circ h + h \circ \partial = \mathrm{Id}$. Let $C_{\mathrm{even}} = \oplus_k C_{2k}$ and $C_{\mathrm{odd}} = \oplus_k C_{2k+1}$. These are finitely generated free $\mathbb{Z}[G]$-modules with given bases, and for any chain homotopy h as before the map

$$\partial + h\colon C_{\mathrm{even}} \to C_{\mathrm{odd}}$$

induces an isomorphism whose class in $Wh(G)$ is independent of h and is called *the Whitehead torsion of* C_*.

The following is an easy exercise.

Proposition 2.4. *Let C_* be a free, finitely generated chain complex over $\mathbb{Z}[G]$ with trivial homology with a given ordered basis for each chain group. After replacing C_* by its sum with a finite number of chain complexes of the form $\partial\colon D_{i+1} \to D_i$ where D_{i+1} and D_i are finitely generated free modules with given bases and in these bases ∂ is given by the identity matrix, the following holds. After making a finite sequence of elementary transformations of the bases of the chain groups C_i the following holds. Suppose that C_i is the non-zero only for $0 \le i \le n$ and fix $0 \le k < n$. For every $0 \le i \le n$ we can write write $C_i = A_i \oplus B_i$ where:*

- *A_i and B_i are free finitely generated $\mathbb{Z}[G]$-modules,*
- *the ordered basis for C_i is the concatenation of an ordered basis for A_i followed by an ordered basis for B_i,*
- *$\partial\colon A_{i+1} \oplus B_{i+1} \to A_i \oplus B_i$ is the composition of the projection of the domain onto B_{i+1} followed by an isomorphism $h_{i+1}\colon B_{i+1} \to A_i$ followed by the inclusion of A_i into $A_i \oplus B_i$, and*
- *for all $i \neq k$ the matrix for the map $h_{i+1}\colon B_{i+1} \to A_i$ with respect to the given bases is the identity.*

Once we have arranged all of this, the Whitehead torsion of C_* is $(-1)^{k+1}$ times the Whitehead torsion of the isomorphism $h_{k+1}\colon B_{k+1} \to A_k$.

Corollary 2.5. *Let C_* be a chain complex over $\mathbb{Z}[G]$ with only finitely many non-trivial chain groups each being a finitely generated free module with an ordered basis. If the Whitehead torsion of C_* is zero, then after replacing C_* by its sum with a finite number of two-term complexes as in the statement of the proposition and after a finite number of elementary transformations of the bases of the C_i, for every i we have $C_i = A_i \oplus B_i$ with A_i and B_i*

being generated by complementary subsets of the basis elements of C_i and $\partial\colon C_{i+1} \to C_i$ is the composition of the projection of $C_{i+1} \to B_{i+1}$ and isomorphism $B_{i+1} \to A_i$ represented by the identity matrix in the given bases and the inclusion of $A_i \to C_i$.

3 The Whitehead torsion of a chain equivalence and of a homotopy equivalence

Let A_* and B_* be chain complexes over $\mathbb{Z}[G]$. Suppose that $\varphi\colon A_* \to B_*$ is a $\mathbb{Z}[G]$ chain map. The *mapping cylinder for* φ, $M(\varphi)$ is a chain complex over $\mathbb{Z}[G]$ with $M(\varphi)_n = B_n \oplus A_{n-1}$ and differential given by

$$\begin{pmatrix} \partial_B & \varphi \\ 0 & -\partial_A \end{pmatrix}.$$

It is easy to see that there is a long exact sequence of homology

$$\cdots \to H_n(A) \xrightarrow{\varphi_*} H_n(B) \to H_n(M(\varphi)) \to H_{n-1}(A) \to \cdots .$$

Now suppose that A and B are free and finitely generated $\mathbb{Z}[G]$-modules with given (homogeneous degree) bases. Suppose also that that φ induces an isomorphism on homology. Notice that $M(\varphi)$ is a free, finitely generated $\mathbb{Z}[G]$ module with an induced basis. Also, the homology of $M(\varphi)$ is zero.

Definition 3.1. The Whitehead torsion of $M(\varphi)$ with its induced basis is *the Whitehead torsion* of the map $\varphi\colon A \to B$. It is denoted $Wh(\varphi)$.

Lemma 3.2. *Let C_*, C'_*, C''_* be free, finitely generated complexes over $\mathbb{Z}[G]$, each with an ordered basis. Suppose $\varphi\colon C_* \to C'_*$ and $\psi\colon C'_* \to C''_*$ are chain maps inducing an isomorphism on homology. Then $Wh(\psi \circ \varphi) = Wh(\psi) + Wh(\varphi)$.*

The Whitehead torsion of φ of course depends on the choice of ordered bases. As a special case of the previous lemma, we see that changing either ordered basis by an automorphism will change the Whitehead torsion of φ by adding the Whitehead torsion of the automorphism.

Fix a finite CW complex X and a base point x that is a 0-cell. For each cell c in X chose a path $\omega(c)$ from x to the central point of the cell. Also, choose an orientation for the cell. Let $C_*(X)$ be the chain complex over $\mathbb{Z}[\pi_1(X, x)]$ whose k^{th}-chain group is freely generated by these k-cells with their orientations. The boundary map is given as follows: we can deform the attaching maps for the k-cells $b_k^i\colon \partial e_k^i \to X^{k-1}$ so that they are transverse to the central points $a^j \in e_{k-1}^j$ of the $(k-1)$-cells. For each point in $y \in (b_k^j)^{-1}(a_j)$ we form a loop in X based at x

$$\omega(e_k^i) * \gamma_y * (\omega(e_{k-1}^j)^{-1}$$

where γ_y is an arc in e_k^i from a_k^i to y. It class in $\pi_1(X,x)$ is denoted $g(y)$. Then the coefficient of e_{k-1}^j in ∂e_k^i is the sum over $y \in (b_k^i)^{-1}(a_j)$ of $\pm g(y)$ where the sign compares the boundary orientation of ∂e_k^i at y with the orientation of e_{k-1}^y at a_{k-1}^j. This defines a free finitely generated chain complex over $\mathbb{Z}[\pi_1(X,x)]$, denoted $C_*(X;\mathbb{Z}[\pi_1(X,x)])$. Reordering the cells and changing the orientation of cells and changing the paths connecting the central points of the cells to the base point changes the basis by a sequence of elementary transformations.

There is another way to view this construction. Let $\widetilde{X}$ be the universal covering of X. There is an induced cell structure on $\widetilde{X}$. Fix a 0-cell $\widetilde{x}$ in $\widetilde{X}$ above x. Then for each cell c in X there is a lifting of c to a cell $\widetilde{c}$ in $\widetilde{X}$ such that the lift of $\omega(c)$ to $\widetilde{X}$ beginning at $\widetilde{x}$ ends at the midpoint of $\widetilde{c}$. The fundamental group $\pi_1(X,x)$ acts on $\widetilde{X}$ and this action preserves the cell structure of $\widetilde{X}$ and acts freely on the set of k-cells for each k. Thus, it makes the CW complex of $\widetilde{X}$ with $\mathbb{Z}$-coefficients into a free, finitely generated chain complex over $\mathbb{Z}[\pi_1(X,x)]$, and the choice of paths $\omega(c)$, or equivalently the choice of lifts $\widetilde{c}$, for each cell c determines a basis for each chain group. The CW chain complex for $\widetilde{X}$ with $\mathbb{Z}$-coefficients is identified with $C_*(X;\mathbb{Z}[\pi_1(X,x)])$. It follows that our first definition indeed defines a chain complex, i.e., that $\partial^2 = 0$.

Let X and Y be finite, connected CW complexes with given cell structures and suppose that $f\colon X \to Y$ is a homotopy equivalence. Our goal here is to define the Whitehead torsion of f. First we deform f by homotopy until it is a cellular map. We choose as base points x, a 0-cell in X, and the image $y = f(x)$, a 0-cell in Y. Since f is a homotopy equivalence it induces an isomorphism $\pi_1(X,x) \to \pi_1(Y,y)$. The map f also induces a map of the CW chain complexes associated with the cell structures on X and Y.

Definition 3.3. Now suppose that $f\colon X \to Y$ is a cellular map inducing an isomorphism on fundamental groups and on the homology on the universal coverings. (This is equivalent to supposing that f is a homotopy equivalence.) Then f induces a chain map $f_*\colon C_*(X;\mathbb{Z}[\pi_1(X,x)]) \to C_*(Y;\pi_1(Y,y)])$ which is a $\mathbb{Z}[\pi_1(X,x)]$ module map. (The group $\pi_1(X,x)$ acts on $C_*(Y)$ through the isomorphism $f_*\colon \pi_1(X,x) \to \pi_1(Y,y)$.) The

Whitehead torsion of f_* is *the Whitehead torsion* of f with respect to any ordered basis for $C_*(X; \mathbb{Z}[\pi_1(X,x)])$ and $C_*(Y; \mathbb{Z}[\pi_1(X,x)])$ coming for choices of ordering and orienting the cells of X and Y and connecting them by paths to the base points. Since changing these choices acts of the bases by a sequence of elementary transformations, it follows that the Whitehead torsion of f_* is independent of these choices. If the Whitehead torsion of f is zero, then f is said to be a *simply homotopy equivalence.*

Even more is true:

Lemma 3.4. *Suppose that $f, g\colon X \to Y$ are cellular maps between finite CW complexes that are homotopic and are homotopy equivalences. Then the Whitehead torsion of f_* equals the Whtiehead torsion of g_*.*

Proof. We can approximate the homotopy from f to g by a cellular map $H\colon X \times I \to Y$ that is a homotopy from f to g. Since the inclusions $X \times \{0\}$ and $X \times \{1\}$ into $X \times I$ are easily seen to have zero Whtiehead torsion, it follows that the Whtiehead of f_* and of g_* both agree with the Whitehead torsion of H_*. □

Definition 3.5. Let $f\colon X \to Y$ be a homotopy equivalence between finite CW complexes. Then the *Whitehead torsion of f* is defined to be the Whitehead torsion of any cellular map f' homotopic to f.

Since any smooth manifold has a smooth triangulation combinatorially unique up to subdivision, any homotopy equivalence between compact smooth manifolds has a well-defined Whitehead torsion.

4 Morse chain complex over $\mathbb{Z}[\pi_1(W)]$

Let W be a compact, connected n manifold and suppose that $\partial_- W$ is connected. Fix a base point $w_0 \in \partial_- W$, and suppose that inclusion $\partial_- W \to W$ induces an isomorphism $\pi_1(\partial_- W, w_0) \to \pi_1(W, w_0)$. Let $f\colon W \to \mathbb{R}$ be a Morse function with *df* positive on inward pointing tangent at points of $\partial_- W$ and negative on inward pointing tangent vectors along $\partial_+ W = \partial W \setminus \partial_- W$ and let χ be a gradient-like vector field. We assume that f is self-indexing and that we have fixed an ordering of the critical points of each index. For each critical point p_i we choose a path $\omega(p_i)$ from the base point x_0 to p_i. Only the homotopy class of this path relative to its endpoints will matter. Fix a gradient-like vector field χ for f and orient the descending manifold of each critical point. The Morse chain complex $C_*(f, \chi; \mathbb{Z}[\pi_1(W, w_0)]))$ is defined to be the free $\mathbb{Z}[\pi_1(W)])$-module with basis given by the critical points p_j with the given orientation of the descending manifolds and paths connecting the critical points to w_0. To define the boundary map for this chain complex, suppose, as before, that for each k the intersections of the descending spheres S_i^k in $f^{-1}((2k+1)/2)$ from the critical points q_i of index $k+1$ intersect transversely the ascending spheres S_j^{n-k-1} from the critical points p_j of index k. To each point y of intersection we associate an element in $\mathbb{Z}[G]$ of the form $\alpha(y) = \pm g(y)$ for some $g(y) \in G$. The sign is determined as before by comparing the orientation of S_i^k and the normal orientation of S_j^{n-k-1} at the point of intersection. The element $g(y)$ is the class represented by the loop

$$\omega(q_i) * \gamma(q_i, y) * \gamma(y, p_j) * \omega(p_j)^{-1}$$

based at x_0, Here, $\gamma(q_i, y)$ is any arc in the descending disk from q_i to y and $\gamma(y, p_j)$ is any arc in the ascending disk of p_i from y to p_i. It is clear that the element $g(y)$ depends only on the choice of the homotopy class of paths from x_0 to the critical points q_i and p_j and the point y in their intersection. Summing over all points of intersection $S_i^k \cap S_j^{n-k-1}$ determines an element $\alpha_{ij} \in \mathbb{Z}[G]$, called *the algebraic intersection* of S_i^k and S_j^{n-k-1}. We define $\partial(q_i) = \sum_j \alpha_{ij} p_j$. In this way we construct the Morse chain complex $(C_*(f, \chi; \mathbb{Z}[\pi_1(W, w_0)])]$. The chain groups are free and finitely generated with given ordered bases. This is the Morse chain complex.

If we change the order of the critical points of index k, we change the basis by a permutation. If we change the orientation of the descending manifold of a critical point, we replace the corresponding basis element by its negative. If we change the basis by choosing a different path from the base point to its center, we change the basis element by multiplying by an element of $\pi_1(W, w_0)$.

Now suppose that W is an h-cobordism, i.e., that the inclusions $\partial_\pm W \to W$ are homotopy equivalences. Arguments analogous to the ones above show that the chain complex $C(f, \chi; \mathbb{Z}([\pi_1(W, w_0)]))$ is identified with for the Morse chain complex for $\widetilde{f} \colon \widetilde{W} \to \mathbb{R}$ and vector field $\widetilde{\chi}$ on the universal covering $\widetilde{W}$ of W, and the choice of lifts of the critical points of f determined by the paths connecting these critical points to w_0 gives a $\mathbb{Z}[\pi_1(W, w_0)]$-basis for the $C(\widetilde{f}, \widetilde{\chi}; \mathbb{Z}[\pi_1(W, w_0)])$. Since $\widetilde{f}$ is a homotopy equivalence, the homology of this Morse complex is trivial and hence we have its Whitehead torsion. Clearly, deforming the gradient-like vector field χ does not change the Whitehead torsion of the complex. Also cancelling pairs of critical points of f by the cancellation lemmas and by handle slides does not change the Whitehead torsion of the Morse complex, so that the torsion of the Morse complex is independent of the choice of Morse function. In fact, it is not hard to see:

Proposition 4.1. *The Whitehead torsion of any Morse complex $C_*(f, \chi; \mathbb{Z}[\pi_1(W, w_0)])$ for $(W, \partial_- W)$ is identified with the Whitehead torsion of the inclusion of smooth manifolds $\partial_- W \to W$.*

Proof. (Sketch) The retraction maps of a cobordism W with a single critical point, a non-degenerate critical point of index k, onto $\partial_- W \cup D^k$ is easily seen to be a simple homotopy equivalence. Arguing by induction on the critical points of a self-indexing Morse function, one proves this result. □

Now we come to the s-cobordism theorem

Theorem 4.2. *(s-cobordism theorem) Let $(W, \partial_- W, \partial_+ W)$ be a compact h-cobordism with the dimension of W being at least 6. Then $(W, \partial_- W, \partial_+ W)$ is diffeomorphic to $(\partial_- W \times I, \partial_- W \times \{0\}, \partial_- W \times \{1\})$ if and only if the inclusion $\partial_- W \to W$ is a simply homotopy equivalence if and only if for any Morse function f the Whitenead torsion of the Morse complex for f is zero.*

Proof. (Sketch) if $(W, \partial_- W)$ is diffeomorphic to $(\partial_- W \times I, \partial_- W \times \{0\})$, then the inclusion of $\partial_- W \to W$ is a simple homotopy equivalence, and hence the Whitehead torsion of any Morse complex for $(W, \partial_- W)$ is zero. (This argument holds in all dimensions.)

We consider the converse. It is easy to see that the special arguments for removing the critical points of index $0, 1, n-1, n$ are valid in the non-simply connected case. Thus, we can assume that all critical points have indices between 2 and $n-2$. We orient the descending manifolds and for each critical point p we fix a path $\omega(p)$ from the base point in $w_0 \in \partial_- W$ to p. This produces a basis for the Morse complex over $\mathbb{Z}[\pi_1(W, w_0)]$. We can add trivial summands of the form $\partial\colon D_{i+1} \to D_i$ as in the statement of Proposition 2.4 to the Morse complex by creating births of various dimensions. Changing bases by elementary transformations is accomplished by changing the ordering of the critical points, the orientations of their descending disks, and the paths connecting w_0 to the critical points, and handle slides. In making a handle slide we need to choose an arc γ connecting the descending sphere of one critical point p index k to the ascending sphere of another critical point q index k in order to add $\pm$ the second critical point to the first. Any choice of γ determines an element $g(\gamma) \in \pi_1(W, w_0)$ given

$$\omega(p) * \alpha * \gamma * \beta * \omega(q)^{-1}$$

where α is a path in the descending manifold for p from p to the initial point of γ and β is a path in the ascending manifold for q from the final point of γ to q. Using the arc γ gives a change in the basis element associated to p by adding to it $\pm g(\gamma)$ times the basis element determined by q. By changing γ we can change $g(\gamma)$ to be an arbitrary element of $\pi_1(W, w_0)$. In this way we do an elementary transformation adding an arbitrary multiple of the form $\pm g$ of the basis element associated with q to the basis element associated with p.

Thus, we can assume that we have made choices that give bases for the Morse complex as in the conclusion of Proposition 2.4 where the exceptional degree is fixed to be $n-3$ so that for every $i < n-3$ the map $B_{i+1} \to A_i$ is given by the identity matrix and the Whitehead torsion of the Morse complex is equal to the Whitehead torsion of the boundary map $C_{n-2} \to A_{n-3}$, which is an isomorphism.

Lemma 4.3. *(non-simply connected version of the Whitney trick) Suppose that $n \geq 6$ and f has no critical points of index $0, 1, n-1, n$. Suppose that k is the smallest index of a critical point and that p has index k. Suppose x and y are two points of $S_i^k \cap S_j^{n-k-1}$. Then we can apply the Whitney trick to cancel these two points of intersection if and only if $g(x) = g(y)$ and the local intersection numbers, $\epsilon(x)$ and $\epsilon(y)$, of these points are opposite, that is to say if and only if $\epsilon(x)g(x) + \epsilon(y)g(y) = 0$ in $\mathbb{Z}[G]$.*

Proof. To perform the Whitney trick we need an embedded disk meeting each of S_i^k and S_j^{n-k-1} in an arc from x to y and otherwise disjoint from all ascending disks from critical points of index k and descending spheres from. The cancelling signs are necessary just as in the simply connected case and provided that the signs cancel there is a thin annulus meeting the spheres as required an otherwise disjoint. We need the other boundary component of the annulus to bound an embedded disk in the complement of the ascending and descending manifolds from the critical points of indices k and $k+1$, respectively, or equivalently to be homotopically trivial in this manifold. By the argument in the simply connected case, the inclusion of the complement of these spheres in $f^{-1}((2k+1)/2)$ into W induces an isomorphism on the fundamental groups. Thus, we need the other boundary of the thin annulus to be homotopically trivial in W. The boundary of the annulus is freely homotopic in W to a loop representing the element

$g(x)g(y)^{-1}$. Thus, the Whitney trick can be applied to cancel these two points, x and y, of intersection if and only if $\epsilon(x)g(x) + \epsilon(y)g(y) = 0$ in $\mathbb{Z}[\pi_1(W)]$. $\square$

Corollary 4.4. *We can perform the Whitney trick until there is only one point of intersection between S_i^k and S_j^{n-k-1}, that being a point of transverse intersection if and only if the algebraic intersection in $\mathbb{Z}[\pi_1(W)]$ of these spheres is of the form $\pm g$ for some element $g \in G$.*

Let $2 \leq k < n-3$ be the smallest index of a critical point. Then $C_k = A_k$. Let p be a critical point of index k corresponding to a basis element e for A_k. Let q the critical point of index $k+1$ that cancels it algebraically in the sense that q determines the basis element f for B_{k+1} with $\partial f = e$. Using the non-simply connected version of the Whitney trick we arrange that there is a unique flow line connecting p and q and this is a transverse flow line. The first cancellation theorem lets us cancel p and q. Continuing in this way, we cancel all critical points of index k.

By induction on k we cancel all critical points of index $\leq n-4$. We are left with critical points of index $n-3$ and $n-2$. The remaining critical points of index $n-3$ give the basis for A_{n-3} and the critical points of index $n-2$ give the basis for $C_{n-2} = B_{n-2}$. The Whitehead torsion of the boundary map $C_{n-2} \to A_{n-3}$ is the Whitehead torsion of the original Morse complex, which is trivial by assumption. Thus, we can change the basis for C_{n-2} by elementary transformations so that the matrix for the boundary map is the identity, and hence there is a sequence of operations consisting of reordering the critical points of index $n-2$, changing the orientations of their descending manifolds and choosing other paths connecting these critical points to the base point w_0, so that the non-simply connected version of the Whitney trick applies to produce a single transverse flow line connecting each critical point of index $n-3$ to a critical point of index $n-2$. We then use the first cancellation theorem to cancel these critical points in pairs leaving no critical points and proving that $W \cong \partial_- W \times I$. $\square$