Lie Groups: Fall, 2022
Lecture IV
Clifford Algebras and the Spin Groups

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1 Clifford Algebras

All the material in this section can be found in "Spin Geometry" by Lawson and Michelsohn, Princeton university Press, 1989. It is a much wider treatment of the subject including a classification of all real and complex Clifford algebras and the eight-fold periodicity of the former and the two-fold periodicity of the latter, as well as a discussion of the Dirac operator for spinors on manifolds.

Let K be a field of dharacteristic $\neq 2$.. A *quadratic form* on a finite dimensional K-vector space V is a function $Q\colon V \to K$ that is given as a finite sum

$$Q(x) = \sum_i L_i(x)M_i(x)$$

where the L_i and M_i are linear functions $V \to K$. Choosing a K-basis for V and letting $\{x^1, \ldots, x^k\}$ be the resulting coordinate functions, a quadratic form is simply a homogeneous polynomial of degree 2 in the x^i. Associated to a quadratic form is a symmetric blinear form $B\colon V \otimes_K V \to K$ defined by

$$B(x, y) = Q(x + y) - Q(x) - Q(y).$$

If $Q(x) = \sum_i L_i(x)M_i(x)$ as above, the $B(x, y) = \sum_i L_i(x)M_i(y) + L_i(y)M_i(x)$, which is clearly a symmetric bilinear from. Notice that $B(x, x) = 2Q(x)$. A quadratic form is *non-degerenrate* iff its associated symmetric bilinear form $B(x, y)$ is non-degenerate in the sense that its adjoint $\text{ad}(B)\colon V \to V^*$ is an isomorphism.

We define the *Clifford algebra* $CL(V, Q)$ as follows. It is an algebra over K generated by $v \in V$ with the relations

$$v^2 = -Q(v). \tag{1.1}$$

(It follows that $v \cdot w + w \cdot v = -B(v, w)$, where the product is the product in the Clifford algebra.). Thus, $CL(V, Q)$ is a quotient of the tensor algebra $\sum_{n \geq 0} \otimes^n V$ by the two-sided ideal generated by the relations given in Equation 1.1. Since the generating relations preserve the degree modulo 2 in the tensor algebra, there is an induced direct sum decomposition

$$CL(V, Q) = CL^0(V, Q) \oplus CL^1(V, Q).$$

The first factor is a subalgebra and the second is a free module on one generator on the first factor. If $e_1, \ldots, e_n$ is a basis for V then a K-basis for the vector space underlying $CL(V, Q)$ is all products $e_{i_1} \cdots e_{i_r}$ with $i_1 < i_2 < \ldots < i_r$. . In particular, the dimension of $CL(V, Q)$ over K is 2^n where $n = \dim(V)$. An K-basis for the summand $CL^\epsilon(V, Q)$ is all such products of length congruent modulo 2 to ϵ

The Clifford algebra $CL(V, Q)$ is a quotient of the tensor algebra $\sum_{n \geq 0} \otimes^n V$. As such it inherits an increasing filtration defined by setting $F_r(CL(V, Q))$ equal to the image of $\sum_{n=0}^r \otimes^n V$. then the graded vector space

$$Gr^{F_*}(Cl(V, Q)) = \oplus_{n \geq 0} F_n(CL(V, Q))/F_{n-1}(CL(V, Q))$$

is naturally a graded ring. Since the relations $v^2 = -Q(v)$ strictly decrease the filtration level they have no effect on this associated graded ring. Thus, the result is the same as for $Q = 0$ where the relations are that the generators skew-commute. Thus, the associated graded ring is the exterior algebra $\Lambda^*(V)$. Indeed if $Q = 0$, then the Clifford algebra is the exterior algebra. In general it is not isomorphic to it.

1.1 Basic relationships between various Clifford algebras

In this section for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ we denote by $K[n]$ the algebra of $n \times n$ matrices over K.

We say that a real quadratic form is *positive definite* if $Q(x) > 0$ for all $x \in V \setminus \{0\}$.

Lemma 1.1. *Let Q be a non-degenerate quadratic form on a finite dimensional real vector space V over K. Then there is a basis $\{e_1, \ldots, e_n\}$ for V such $B(e_i, e_j) = 0$ for all $i \neq j$.*

Assume now that $K = \mathbb{R}$. Then we can also arrange that $Q(e_i) = \pm 1$ for all i. Two such Q on V and Q' on V' are isomorphic if and only if the number of basis vectors on which Q takes values 1 and the number of basis vectors on which Q takes value -1 are the same as the corresponding numbers for Q'.

Proof. Since Q is non-degenerate, there is $x \in V$ such that $Q(x) \neq 0$. Denote by $x^\perp \subset V$ the linear subspace of elements $y \in V$ such $B(x, y) = 0$. For any $z \in V$ we consider

$$z' = z - \frac{B(z,x)}{2Q(x)} x.$$

Direct computation shows that $z' \in x^\perp$, so that $z = z' + \frac{B(z,x)}{2Q(x)} x$. This shows that every element in V is a sum on an element in $x^\perp$ and a multiple of x. On the other hand no non-zero multiple of x is in $x^\perp$. So we have $V = x^\perp \oplus K \cdot x$.

Arguing by induction on dimension, we have a basis $e_1, \ldots, e_{n-1}$ with $B(e_i, e_j) = 0$ for $i \neq j$ for $x^\perp$. Setting $e_n = x$ completes this to a basis of mutually orthogonal (under B) vectors. In the case $K = \mathbb{R}$ we multiply each vector e_i by $(\sqrt{|Q(e_i)|})^{-1}$ to obtain the required basis.

Lastly, still in the case $K = \mathbb{R}$, we show that the number of positive and negative diagonal elements of this representation is independent of the choice of basis in which B is diagonal. Let us suppose that $Q(e_i) = 1$ for $1 \leq i \leq r$ and $Q(e_i) = -1$ for $r + 1 \leq i \leq n$. Denote P and N the lineaer subspaces generated by the first r and last $n - r$ basis vectors. Let us consider any linear subspace P' on which Q is positive definite. Such a subspace must have trivial intersection with N since on N for form $-Q$ is positive definite. (We say that Q is *negative definite* on N.) This means that under the projection of $V \to P$ with kernel N the subspace P' embeds in P and consequently $\dim(P') \leq \dim(P)$. Thus, for any basis in which B is diagonal, the number of diagonal entries on which Q is positive is the maximal dimension of any linear subspace on which Q is positive definite. Similarly, the number of negative diagonal entries is the maximal dimension of any linear subspace on which Q is negative definite. □

Definition 1.2. We denote by $CL_{r,s}$ the Clifford algebra of a real vector space V with a basis $e_1, \ldots, e_{r+s}$ and a quadratic form Q for which $B(e_i, e_j) = 0$ for all $i \neq j$ and $Q(e_i) = 1$ for $i \leq r$ and $Q(e_i) - 1$ for $r + 1 \leq i \leq r + s$. That is to say (V, Q) is the standard quadratic form of type (r, s) over $\mathbb{R}$. Any such basis is called *a standard basis*. We use $\mathbb{R}^n$ to denote the quadratic space given by Euclidean n-space with quadratic form the usual positive definite form $Q(x) = \sum_{i=1}^{n}(x_i)^2$.

Lemma 1.3. *Let $\{e_1, \ldots, e_n\}$ be the usual basis for $\mathbb{R}^n$ and $\{e'_1, \ldots, e'_{n+1}\}$ be the usual basis for $\mathbb{R}^{n+1}$. Define $\varphi(e_i) = e_i e_{n+1} \in CL^0(\mathbb{R}^{n+1})$. This extends to an isomorphism of algebras $CL(\mathbb{R}^n) \to CL^0(\mathbb{R}^{n+1})$.*

Proof. It is a direct computation that φ preserves the defining relations for $CL(\mathbb{R}^n)$ and hence extends to a map of algebras. Clearly, since the generators of $CL(\mathbb{R}^n)$ map to $CL^0(\mathbb{R}^{n+1})$ the image of the map is contained is $CL^0(\mathbb{R}^{n+1})$. It is obviously an isomorphism of vector spaces and hence of algebras. □

An analogous argument shows:

Lemma 1.4. $CL^0_{r+1,s} \cong CL_{r,s}$.

Lemma 1.5. $CL_{0,r} \otimes_{\mathbb{R}} CL_{2,0} \cong CL_{r+2,0}$. *The mod 2 grading on $CL_{r+2,0}$ is induced from the sum of the two mod 2 gradings on the factors. Similarly, $CL_{r,0} \otimes CL_{0,2} \cong CL_{0,r+2}$.*

Proof. We begin by defining a map $CL(\mathbb{R}^{r+2}) \to CL_{0,r} \otimes CL_{2,0}$. Let $\{e_1, \ldots, e_r\}$ be the standard basis of Euclidean r-space but with the quadratic form $-\sum_I x_i^2$. Let f_2, f_2 be an orthonormal basis for $\mathbb{R}^2$ and let $\{e'_1, \ldots, e'_{r+2}\}$ be an orthonormal basis for $\mathbb{R}^{r+2}$ We send e'_i to $e_i \otimes f_1 f_2$ for all $1 \leq i \leq r$ and we send e'_{n+i} to $1 \otimes f_i$ for $i = 1, 2.$. Direct computation shows that this map preserves the defining relations and hence extends to a map of algebras. Clearly, it is an isomorphsm and the statement about the mod two gradings is immediate from the definition.

The other case is similar. □

Lemma 1.6. $CL_{r,s} \otimes CL_{1,1} \cong CL_{r+1,s+1}$.

Proof. Let $e_1, \ldots, e_{r+s}$ be an orthogonal basis for vector space with a quadratic form of type (r, s) where $Q(e_i) = 1$ for $1 \leq i \leq r$ and $Q(e_i) = -1$ for $r+1 \leq r+s$. Let $\{f_1, f_2\}$ be a basis for a two-dimensional vector space with quadratic form of type $(1, 1)$ with $B(f_1, f_2) = 0, Q(f_1) = 1$ and $Q(f_2) = -1$. Then the basis

$$\{e_1 \otimes f_1 f_2, \ldots, e_r \otimes f_1 f_2, 1 \otimes f_1, e_{r+1} \otimes f_1 f_2, \ldots, e_{r+s} \otimes f_1 f_2, 1 \otimes f_2\}$$

is an orthogonal basis for the tensor product with its product quadratic form. The first $r + 1$ basis elements have value 1 under the quadratic form and the last $s + 1$ have value -1 under it. Hence the tensor product with its form is a representative of $CL_{r+1,s+1}$. □

1.2 Classification of Irreducible Modules pver real Clifford Algebras

There is a general classifcation of real clifford algebras. We let $K = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ and set $j_K = \dim_{\mathbb{R}}(K)$.

Theorem 1.7. *For every (r,s) $CL_{r,s}$ is isomorphic to either $K[n]$ or $K[n] \oplus K[n]$ for $n^2 d_K = 2^{r+s}$ in the first case or $n^2 d_K = 2^{r+s-1}$ in the second case. In the second case $CL^0_{r,s}$ is the diagonal subalgebra.*

Theorem 1.8. *Every irreducible $CL_{r,s}$ module is a direct sum of irreducible $CL_{r,s}$ modules. For any (r,s) either there is a unique irreducible representation of $CL_{r,s}$ or there are two irreducible representations and their restriction to $CL^0_{r,s}$ are isomorphic $CL^0_{r,s}$ modules.*

Proof. Consider the subgroup of $CL^{\times}_{r,s}$ generated by $\epsilon = -1$ and the elements $e_1, \ldots, e_n$, of a standard basis. This is a group generated by the symbols $e_1 ldots, e_n, \epsilon$ where $e_i^2 = \epsilon$, for $i \leq r$ and $e_i^2 = 1$ for $r+1 \leq i \leq r+s$ and $e_i e_j = \epsilon e_j e_i$ and ϵ is in the center of the group. This is clearly a finite sub-group $F_{r,s}$ of the units of $CL_{r,s}$. The order of $F_{r,s}$ is 2^{r+s+1}.

We define an algebra homomorphism of the group algebra $\mathbb{R}[F_{r,s}] \to CL_{r,s}$. Obviously, this map is surjective and its kernel is the central subalgebra $\mathbb{R}(\epsilon+1) \subset \mathbb{R}[F_{r,s}]$. It follows that a linear representation of $CL_{r,s}$ is the same thing as a linear representation of $\mathbb{R}[F_{r,s}]$ with the property that ϵ acts by -1. Any such representation comes from a representation of $F_{r,s}$. Since $F_{r,s}$ is a finite group, such a representation is completely reducible (see the lecture on Representations of Finite Groups). This proves the complete reducibility of linear representations of $CL_{r,s}$.

As to the second, for any division algebra K, any irreducible representation of $K[n]$ is isomorphic to K^n with the action given by left matrix multiplication, so that there is only one irreducible representation of $K[n]$ up to isomorphism.

In the case of $K[n] \oplus K[n]$ there are only two irreducible representations which come by projecting the algebra to one of its factors and taking the pullback of the irreducible representation of that factor. Since $CL^0_{r,s}$ is the diagonal subalgebra these two irreducible representations become isomorphic when restricted to $CL^0_{r,s}$. □

Definition 1.9. A *spin module* for $\mathbb{R}^{r,s}$ is an irreducible representation of $CL_{r,s}$

Notice that there are either one of two spin modules for $\mathbb{R}^{r,s}$ up to isomorphism. These two are are isomorphic as $CL^0_{r,s}$ modules. Notice that we do not claim (nor is it true) that the spin modules are irreducible $CL^0_{r,s}$ modules.

1.2.1 Low Dimensional Examples

In these examples $\mathbb{R}^n$ refers to Euclidean space with its usual positive definite inner product and $\{e_1, \ldots, e_n\}$ is an orthonormal basis.
1.) $CL(R^1)$ is generated by e_1 with $e_1^2 = -1$, so that $CL(\mathbb{R}^1) \cong \mathbb{C}$ with $e_1 \mapsto i$. Its spin module is $\mathbb{C}$ with usual complex multiplication.

$CL_{0,1}$ is generated by e_1 with $e_1^2 = 1$, so that

$$CL_{0,1} = \mathbb{R}[e_1]/\{e_1^2 - 1\} = \mathbb{R} \oplus \mathbb{R}$$

where the summands are the subrings generated $(1 + e_1)/2$ and $(1 - e_1)/2$. Each is a subring of $CL_{0,1}$ and the product of an element in the first factor with an element in the second is 0. That is to say there is a ring isomorphism $CL_{0,1} \cong \mathbb{R} \oplus \mathbb{R}$. It has two spin modules, As vector spaces both are both isomorphic to $\mathbb{R}$ The two action are given by projection of $CL_{0,1}$ to one of its summands followed by usual multiplication of this factor on $\mathbb{R}$.
2.) $CL(\mathbb{R}^2)$ is generated by e_1, e_2 with $e_1^2 = e_2^2 = -1$. It follows that $(e_1e_2)^2 = -1$. There is an isomorphism $CL(\mathbb{R}^2) \cong \mathbb{H}$, where $\mathbb{H}$ is the algebra of the quaternions. The map is given by $e_1e_2 \mapsto i; e_1 \mapsto j; e_2 \mapsto k$. Notice that $CL^0(\mathbb{R}^2) = \mathbb{C}$. The spin module is the quaternions with the left multipliciation action. As a $CL^0(\mathbb{R}^2)$-module the spin module is $\mathbb{C}^2$ with the usual complex multiplication.
2'.) Let $V = \mathbb{R}^2$ and let Q be the negative of the usual positive definite inner product on $\mathbb{R}^2$. Then $CL(V, Q)$ is generated by e_1, e_2 subject to the relations that $e_1^2 = e_2^2 = 1$ and $e_1e_2 = -e_2e_1$.

$$\varphi(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \varphi(e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

One checks easily that the relations in $CL(V, Q)$ hold for these matrices. Thus, this determines an algebra map

$$CL(V, Q) \to \mathbb{R}[2],$$

which is clearly onto since $\mathrm{Id}, \varphi(e_1), \varphi(e_2)$, and $\varphi(e_1)\varphi(e_2)$ are a basis for $\mathbb{R}[2]$. Since the dimension of $\mathbb{R}$ of $CL(V, Q)$ is 4, the function φ is an algebra isomorphism $CL(V, Q) \to \mathbb{R}[2]$. Under this identification, $CL^0(V, Q) \subset CL(V, Q)$ is the subalgebra of matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

The spin module in this case is $\mathbb{R}^2$ with the action being left matrix multipliciation. $CL^0(V,Q) \cong \mathbb{C}$ and the action is the usual action of complex multiplication of $\mathbb{C}$ on $\mathbb{C}$.

It is easy to see that $CL_{1,1} \cong CL_{0,2}$. Thus, it also is isomorphic to $\mathbb{R}[2]$, again with spin module being $\mathbb{R}^2$ with left matrix multiplication. $CL^0_{1,1} \cong \mathbb{R} \oplus \mathbb{R}$ and its action on the spin module (in an appropriate basis) is the action of matrices whose non-zero entries are diagonal.

3.) In $CL(\mathbb{R}^3)$ the element $\omega = e_1e_2e_3$ squares to 1 and is in the center of the Clifford algebra. Hence, there is an isomorphism of algebras $CL(\mathbb{R}^3) = CL^0(\mathbb{R}^3)[\omega]/\{\omega^2 = 1\}$. Thus, there is an $\mathbb{R}$-algebra homomorphism

$$CL(\mathbb{R}^3) = CL^0(\mathbb{R}^3)(\frac{1+\omega}{2}) \oplus CL^0(\mathbb{R}^3)(\frac{1-\omega}{2}).$$

Each of the algebra factors is isomorphic to $CL^0(\mathbb{R}^3)$. This algebra has generators e_1e_2, e_2e_3, e_3e_1 each of which squares to -1 and with the product of the first followed by the second being the third. That is to say $CL^0(\mathbb{R}^3) \cong \mathbb{H}$ and hence $CL(\mathbb{R}^3)$ is isomorphic as an algebra to the direct sum of two copies of $\mathbb{H}$. Under this decomposition $CL^0(\mathbb{R}^3)$ is the diagonal copy of $\mathbb{H}$. The spin modules are both isomorphic to $\mathbb{H}$ with left multiplication by one of the factors and the other acting trivially.

$(2,1)$: We have $CL_{2,1} \cong CL_{1,0} \otimes CL_{1,1} \cong \mathbb{C} \otimes \mathbb{R}[2] = \mathbb{C}[2]$. The spin module is $\mathbb{C}^2$ with left matrix multipllication. $CL^0_{2,1} \cong \mathbb{R}[2]$ acting by matgrix multiplication of $\mathbb{C}^2$.

$(1,2)$: We have $CL_{1,2} \cong CL_{0,1} \otimes CL_{1,1} \cong (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{R}[2] = \mathbb{R}[2] \oplus \mathbb{R}[2]$. The spin modules are $\mathbb{R}^2$ with left multiplication from one of the factors. $CL^0_{1,2} \cong \mathbb{R}[2]$ acting by mtrix multiplication on the spin modules.

$(0,3)$: $CL_{0.3} \cong CL_{1,0} \otimes CL_{0,2} \cong \mathbb{C} \otimes \mathbb{R}[2] \cong \mathbb{C}[2]$. The spin module is $\mathbb{C}^2$ with left matrix multiplication. $CL^0_{0,3} \cong \mathbb{H}$ acting on the spin module $\mathbb{C}^2 = \mathbb{H}$ by quaternion mutllpilication.

1.3 $CL(\mathbb{R}^4, Q)$

1.3.1 The case Q is positive definitive

The discussion of $CL(\mathbb{R}^4)$ is rich enough to justify devoting a subsection to it.

Applying Lemma 1.5 for $n = 2$ we have

$$CL(\mathbb{R}^4) \cong CL_{0,2} \otimes CL_{2,0} = \mathbb{R}[2] \otimes \mathbb{H} = \mathbb{H}[2]).$$

The spin module is $\mathbb{H}^2$ with the action being matrtix multiplication.

We now have two structure results: $CL(\mathbb{R}^4) \cong \mathbb{H}[2]$ and $CL^0(\mathbb{R}^4) \cong \mathbb{H} \oplus \mathbb{H}$. As a first guess one could believe that $CL^0(\mathbb{R}^4)$ is simply the subalgebra of diagonal matrices. This is not the case; things are more interesting.

Consider the element $\omega = e_1e_2e_3e_4 \in CL^0(\mathbb{R}^4)$. It is an element of square 1. Thus, $CL(\mathbb{R}^4)$ decomposes $CL(\mathbb{R}^4) = CL^+(\mathbb{R}^4) \oplus CL^-(\mathbb{R}^4)$, the summands being the $+1$ and -1 eigenspaces of right multiplcation by $-\omega$.

Thus, intersecting with $CL^0(\mathbb{R}^4)$ gives a decomposition of this algebra as $CL^{0,+}(\mathbb{R}^4) \oplus CL^{0,-}(\mathbb{R}^4)$. This is the decomposition of $CL^0(\mathbb{R}^4)$ as $\mathbb{H} \oplus \mathbb{H}$. Notice that if we take graded ring associated to the usual filtration and take the isomorphism of this graded ring to $\Lambda^*(\mathbb{R}^2)$, the term in $F_2(CL^{0,\pm}(\mathbb{R}^4))/F_0(CL^{0,\pm}(\mathbb{R}^4))$ is the self-dual, resp. anti-self dual elements of $\Lambda^2(\mathbb{R}^4)$ under the usual Hodge star operator.

1.3.2 The case when Q is of signature $(3,1)$

This case and the next one are a crucial case for 4-dimensional relativistic physics.

According to Lemma 1.4 there is an isomorphism of algebras

$$CL_{3,1} \cong CL_{2,0} \otimes CL_{1,1} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}[2] \cong \mathbb{H}[2].$$

The spin module is $\mathbb{H}^2$ as in the case of (4.0).

By Lemma 1.4 we have $CL^0_{3,1} \cong CL_{2,1}$. Also, by Lemma 1.6

$$CL_{2,1} \cong CL_{1.0} \otimes CL_{1,1} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[2] = \mathbb{C}[2],$$

the algebra of 2×2 complex matrices. It acts by matrix multiplication on $\mathbb{H}^2$.

1.3.3 The case when Q is of signature $(1,3)$

We have

$$CL_{1,3} \cong CL_{0,2} \otimes CL_{1,1} = \mathbb{R}[2] \otimes_{\mathbb{R}} \mathbb{R}[2] \cong \mathbb{R}[4],$$

the algebra of 4×4 real matrices. Also, by Lemma 1.6 $CL^0_{1,3} \cong CL_{0,3}$. The spin module is $\mathbb{R}^4$ with matrix multiplication.

Let f_1, f_2, f_3 be the standard generators for $CL_{0,3}$, so that $f_i^2 = 1$ for all $1 \le i \le 3$ and $f_if_j = -f_jf_i$ for all $1 \le i < j \le 3$. We define a map $CL_{0,3}$ to $\mathbb{C}[2]$ be sending

$$f_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad f_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad f_3 \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

One sees that the relations in $CL_{0,3}$ hold for these matrices, so that there is induced an algebra homomorphism $CL_{0,3} \to \mathbb{C}[2]$. Direct computation shows that the eight basis elements $1, f_1, f_2, f_3, f_1f_2, f_2f_3, f_1f_3, f_1f_2f_3$ map to linearly independent matrices and hence the map is an isomorphism of algebras. This proves $CL^0_{1,3} \cong \mathbb{C}[2]$. The inclusion $CL^0_{1,3} \to CL_{1,3}$ is the natural map $\mathbb{C}[2] \to \mathbb{R}[4]$ and the action of $CL^0_{0,3}$ on $\mathbb{R}^4$ is through this embedding and matrix multiplication.

Lastly, $CL_{2,2} \cong CL_{1,1} \otimes CL_{1,1} \cong \mathbb{R}[2] \otimes \mathbb{R}[2] \cong \mathbb{R}[4]$, and again the spin module is $\mathbb{R}^4$.

1.4 Classification of Real Clifford Algebras

I will not give the entire classification but it is clear how to proceed given Lemmas 1.5 and 1.6. For example suppose that $r \geq s$. Then repeatedly using Lemma 1.6 we see that

$$CL_{r,s} = CL_{r-s,0} \otimes \mathbb{R}[2^s].$$

Now invoking Lemma 1.5 we see that if $r - s = 4k + t$ with $0 \leq t \leq 3$, then

$$CL_{r-s,0} = CL_t \otimes^k \mathbb{H}[2].$$

Since $\mathbb{H} \otimes \mathbb{H}$ is isomorphic to $\mathbb{R}[4]$ we see that if k is even then

$$\otimes^k \mathbb{H}[2] \cong \mathbb{R}[2^{2k}],$$

and

$$CL_{r,s} \cong CL_t \otimes \mathbb{R}[2^{r+s-t)/2}]$$

Finally, if $r - s - t = 4k + t$ with k odd, then

$$\otimes^k \mathbb{H}[2] \cong \mathbb{H}[2^k]$$

and

$$CL_{r,s} \cong CL_t \otimes \mathbb{H}[2^{(r+s-t)/4}].$$

In this way we can deduce the structure of all the Clifford algebras when $CL_{r,s}$ when $r \geq s$. The other case $r \leq s$ is similar.

There is one general result worth pointing out. This is eight-fold periodicity of real Clifford algebras..

Theorem 1.10. $CL_{r+8,s} \cong CL_{r,s} \otimes \mathbb{R}[16] = CL_{r,s}[16]$.

One proof of Bott periodicity for the homotopy groups of $SO(n)$, namely, $\pi_{k+8}(SO(n)) \cong \pi_k(SO(n))$ for $n >> k$ can be derived from this eight-fold periodicity for the Clifford algebras $CL(\mathbb{R}^n)$.

The complex case is even simpler: the periodicity is of order 2.

Theorem 1.11. $CL(\mathbb{C}^{n+2}) \cong CL(\mathbb{C}^n) \otimes_{\mathbb{C}} \mathbb{C}[2 = CL(\mathbb{C}^n)[2]$.

This two-fold periodicity can be used to prove Bott periodicity $\pi_k(U^n) \cong \pi_{k+2}(U^n)$ for $k << n$.

2 $Pin(V,Q)$ and $Spin(V,Q)$

2.1 Definition and Statement of Main Result

We continue with the notation: V is a finite dimensional real vector space and Q is a non-degenerate quadratic form on V. Let $CL^\times(V,Q)$ be the multiplicative group of units in $CL(V,Q)$. There is the adjoint action

$$Ad_V \colon CL^\times(V,Q) \times CL(V,Q) \to CL(V,Q); \quad Ad_V(x)(y) = xyx^{-1}.$$

If $v \in V \subset CL(V,Q)$ with $Q(v) \neq 0$ then v is a unit and $v^{-1} = \frac{v}{-Q(v)}$. Hence, if $w \in v^\perp$, then

$$Ad_V(v)(w) = \frac{-1}{Q(v)} vwv = \frac{1}{Q(v)} v^2 w = -w$$

and

$$Ad_V(v)(v) = \frac{-1}{Q(v)} vvv = v.$$

That is to say, $Ad_V(v)$ preserves $V \subset CL(V,Q)$ and acts by the identity on v and by -1 on the orthogonal complement of v in V, $v^\perp \subset V$. That is to say, $Ad_V(v) \colon V \to V$ is minus the orthogonal reflection in $v^\perp$ and, in particular, preserves Q

Of course, for any $\lambda \neq 0$, $Ad_V \lambda v = Ad_V(v)$. Thus, there is no harm in restricting to $v \in V$ with $Q(v) = \pm 1$. Let $Pin(V,Q) \subset CL^\times(V,Q)$ be the subgroup of units generated by $\{v \in V \mid Q(v) = \pm 1\}$. The adjoint action of $Pin(V,Q)$ leaves V invariant and the restriction to V determines a map

$$Ad_V \colon Pin(V,Q) \to O(V,Q),$$

where $O(V,Q)$ is the orthogonal group of Q, i.e., the group of linear isomorphisms of V preserving Q. The intersection $Pin(V,Q) \cap CL^0(V,Q)$ is defined to be $Spin(V,Q)$. It consists of those elements in that are products of an even number of elements $v_1 \cdots v_{2k}$ of elements of V with $Q(v_i) = \pm 1$. We see that the restriction of the Ad_V $Spin(V,Q)$ determines a homomorphism $Spin(V,Q) \to SO(V,Q)$, the subgroup of $O(V,Q)$ consisting of all orientation-preserving elements.

Here is the main result:

Theorem 2.1. *For (V,Q) a non-degenerate real quadratic form of type (r,s) with $r \geq 2$, the map*

$$Ad_V \colon Spin(V,Q) \to SO(V,Q)$$

is a surjective homomorphism with kernel ± 1. Furthermore, $Spin(V,Q)$ is connected, so that this map is a non-trivial double cover. In the case when $\dim(V) > 2$ and Q is positive definite, this map expresses $Spin(V,Q)$ as the universal covering of $SO(V,Q)$.

The proof of this result is given in the next subsection.

2.2 Proof of Theorem 2.1

We begin the proof by studying $O(V,Q)$.

2.2.1 Reflections generate $O(V,Q)$

Definition 2.2. By a *reflection* mean the transformation

$$w \mapsto w - \frac{B(w,v)}{Q(v)} v$$

for some element $v \in V$ with $Q(v) = \pm 1$. These are elements of $O(V,Q)$.

The reflection generated by v is the identity on $v^\perp$ and sends $v \mapsto -v$. Here is the relevant result.

Proposition 2.3. *Every element in $O(V,Q)$ is a product of reflections. Every element of $SO(V,Q)$ is a product of an even number of reflections.*

Proof. First note that the second statement follows immediately from the first since a product of reflections is in $SO(V,Q)$ if and only if it is a product of an even number. From now on we concentrate on the first statement.

The case $\dim(V) = 1$ is clear: the orthogonal group is $\{\pm 1\}$ and the non-rivial element is a reflection.

The case $\dim(V) = 2$ has three subcases: Q positive definite, Q negative definite and Q of signature $(1, 1)$. The positive and negative cases are classic: every rotation of the plane is a product of two reflections and every orientation-reversing isometry of the plane is a reflection.

Let us consider the case when there is a basis $\{e_1, e_2\}$ for V with $Q(e_1) = 1$ and $Q(e_2) = -1$ and $B(e_1, e_2) = 0$. Let e'_1 be a vector with $Q(e'_1) = 1$. Then $e'_1 = \pm(cosh(t)e_1 + \sinh(t)e_2)$ for some $t \in \mathbb{R}^*$. Since the composition of reflection in e_1 and reflection in e_2 is -1, it suffices to consider the case of positive sign. Elementary hyperbolic geometry shows that the reflection on $sinh(t/2)e_1 + cosh(t/2)e_2$ sends e_1 to $cosh(t)e_1 + sinh(t)e_2$. The case e_2 and $\pm(cosh(t)e_2 + sinh(t)e_1)$ is analogous.

Now we argue by induction on the dimension of V. Suppose, for some $n \geq 3$ that we know the result for $n - 1$ and suppose that the dimension of V is n. Let $e_1, \ldots, e_n$ be a basis of V with $Q(e_i) = \pm 1$ for all i and $B(e_i, e_j) = 0$ for all $i \neq j$. Let $A \in O(V, Q)$ and define $e'_i = A(e_i)$.

Claim 2.4. *There is a product of reflections R in $O(V, Q)$ with the property that $R(e'_1) = e_1$.*

We complete the proof assuming the claim.

We choose R as in the claim. Then $RA\colon V \to V$ is an orthogonal transformation fixing e_1. Consequently, it fixes $V' = (e_1)^{\perp}$. Since the dimension of V' is $n - 1$, induction tells us that $RA|_{V'} = R'_1 \cdots R'_k$ for reflections of V'. The elements $v'_i \in V'$ defining the $R'_i\colon V' \to V'$ also determine reflections of V (also called R'_i) that fix e_1. Then $R'_1 \cdots R'_k|_{V'} = RA|_{V'}$ and $R'_1 \cdots R'_k(e_1) = RA(e_1) = e_1$. Hence, $RR'_1 \cdots R'_k = A$ establishing the result. This completes the proof modulo the proof of the claim.

Proof. (of Claim) Given e_1 and e'_1 we must find a product of reflections of (V, Q) carrying e'_1 to e_1. Recall that $Q(e'_1) = Q(e_1) = \pm 1$. Without loss of generality we can assume that $Q(e_1) = 1$. If e'_1 and e_1 generate the same real subspace, then $e'_1 = \pm e_1$ and the required element is either the identity or the reflection generated by e_1. So we may as well assume that e_1 and e'_1 are linearly independent over $\mathbb{R}$, i.e., generate a plane V_0 in V. If the restriction of Q to V_0 is non-degenerate, then we invoke the two-dimensional case to find the required reflection.

Suppose now that $Q|_{V_0}$ is degenerate. In this case V_0 is also spanned by e_1 and some vector v with $Q(v) = 0$ and $B(e_1, v) = 0$ and $e'_1 = \pm e_1 + \beta v$ with $\beta \neq 0$.. If follows that $Q|_{e_1^{\perp}}$ is non-degenerate but is neither positive nor negative definite since $e_1^{\perp}$ contains a vector $v \neq 0$ with $Q(v) = 0$. Find a diagonal basis for $e_1^{\perp}$ with P being the span of the basis vectors on which

Q is positive and N the space of the basis vectors on which Q is negative. Let $p \in P$ and $n \in N$ be such that $v = p + n$. Then $Q(p) = -Q(n)$. We take $e_2 = p/\sqrt{Q(p)}$ and $e_3 = n/\sqrt{|Q(n)|}$. Then $B(e_2, e_3) = 0$, $Q(e_2) = 1$ and $Q(e_3) = -1$. Then $Q(e_1) = Q(e_2)$ and $B(e_2, e_3) = 0$. Notice that $v = t(e_2 + e_3)$ for $t = \sqrt{Q(p)}$. Since $B(e_2, e_1) = 0$, the restriction of b to the subspace spanned by e_1, e_2 is non-degeneratae. The restriction of B to the subspace spanned by (e_2', e_1) is also non-degenerate unless $B(e_2, e_1') = B(e_2, v) = \pm 1$. In that case we can move e_2 along the hyperboloid in the plane generated by (e_2, e_3) to e_2' satisfying $Q(e_2') = +1$ and $B(e_2', v) \neq p[m1$. Then the restriction of B to plane spanned by (e_2', e_1) and the plane spanned by (e_2', e_1') is non-degenerate.

Thus, applying the two-dimensional case to each of the 2-planes spanned by $\{e_1, e_2'\}$ and $\{e_1', e_2'\}$, we can find a reflection moving e_1' to e_2 and another reflection moving e_2 to e_1. The product of these reflections moves e_1' to e_1 as required.

□

This completes the proof of the theorem. □

2.3 Surjectivity

At this point it is convenient to twist the Ad_V to $\widetilde{Ad_V}$ be setting $\widetilde{Ad_V}(v) = -Ad_V(v)$ for every $v \in V$. Then $\widetilde{Ad_V} = Ad_V$ when restricted to units in $Spin(V, Q)$, whereas for odd elements $w \in Pin(V, Q)$ $\widetilde{Ad_V}(w) = -Ad_V(w)$. The point is that for every $v \in V$, the automoprhism of V given by $\widetilde{Ad_V}(v)$ is reflection in $v^\perp$.

Proposition 2.5. *$\widetilde{Ad_V}\colon Pin(V, Q) \to O(V, Q)$ and $Ad_V\colon Spin(V, Q) \to SO(V)$ are surjective.*

Proof. Since $\widetilde{Ad_V}(Pin(V, Q)) \subset O(V, Q)$ contains every reflection, by the Proposition 2.3 $\widetilde{Ad_V}$ is surjective. Since the restriction of $\widetilde{Ad_V}$ to $Spin(V, Q)$ is Ad_V, the second statement follows from the first, $Ad_V\colon Spin(V, Q) \to SO(V, Q)$ is surjective. □

2.4 The Kernel of $Ad_V\colon Spin(V, Q) \to SO(V, Q)$

Claim 2.6. *Let V be a vector space of dimension at least two and let Q be a non-degenerate quadratic form on V. Then -1 is an element of $Spin(V, Q)$. It is in the kernel of the map to $O(V, Q)$.*

Proof. If $-1 \in Spin(V,Q)$, then since it is in the center of $CL(V,Q)$ its adjoint action is trivial. If V contains an element e with $Q(e)=1$, then $e^2=-1 \in Spin(V,Q)$. Otherwise, V contains a pair of elements e_1, e_2 with $Q(e_1)=Q(e_2)=-1$ and $B(e_1,e_2)=0$. In this case $(e_1e_2)^2=-1 \in Spin(V,Q)$. □

Proposition 2.7. *The kernel of $Spin(V,Q) \to SO(V,Q)$ is a group of order 2. If $\dim(V)>1$ then the kernel consists of ± 1.*

Proof. First let us examine the center of $CL(V,Q)$. Fix an orthogonal basis $\{e_1,\ldots,e_n\}$ for V with $Q(e_i)=\pm 1$ for each i. It is easy to see that a standard monomial $e_{i_1}\cdots e_{i_k}$ with $i_1<\cdots<i_k$ commutes with e_j if and only if the number of $i_1,\ldots,i_k$ not equal to j is even. Otherwise, the monomial anti commutes with e_j. It now follows that the only standard monomials in the center of $CL(V,Q)$ are 1 (the empty monomial) and $e_1\cdots e_n$ if n is odd. Furthermore, these monomial generate the center of the Clifford algebra.

Lemma 2.8. *The kernel of $\widetilde{Ad_V}\colon Pin(V,Q) \to O(V,Q)$ consists of those real multiples of the identity element that are contained in $Pin(V,Q)$.*

Proof. Certainly, the real multiples of the identity element are contained in the kernel of $\widetilde{Ad_V}$. Any even element in the kernel of $\widetilde{Ad_V}$ is in the kernel of Ad_V, and hence commutes with all elements in V, and hence commutes with all elements in $CL(V,Q)$. That is to say it is an even element in the center of $CL(V,Q)$, which as we have already seen means it is a real multiple of the identity.

Any odd element in the kernel of $\widetilde{Ad_V}$ anti-commutes with every $v \in V$. Arguments analogous to the ones above show that this means that there are no such odd standard monomials, and hence no such elements. □

For any element $x \in CL(V,Q)$ left multiplication by x induces an endomorphism of $CL(V,Q)$ and there is the determinant function from the endomorphisms of any finite dimensional real vector space to $\mathbb{R}$. It is a multiplicative function. We define the norm $N(x)$ to be the determinant of the endomorphism given by left multiplication by x. This is a multiplicative function. It is easy to see that $N(e_1)=(Q(e_i)^{\dim(CL(V,Q))/2}$. Thus, except in the case when $\dim(V)=1$ and $Q(e_1)=-1$ we have $N(e_i)=1$. In the exceptional case $N(e_1)=-1$. Since N is multiplicative, it follows that any element in $Pin(V,Q)$ has norm ± 1 and any even element has norm $+1$. The only multiples of 1 that have norm 1 are ± 1. Hence, the kernel of $\widetilde{Ad_V}\colon Pin(V,Q) \to O(V,Q)$ and the kernel of $Ad_V\colon Spin(V,Q) \to SO(V,Q)$ are both $\{\pm 1\}$.

□

Lastly, let us show that for any $r \geq 2$ the element -1 is in the connected component of the identity of $Spin(r,s)$. Let $V^{r,s}$ be a vector space with a real quadratic form of type (r,s), with $r \geq 2$, Since the inclusion of $\mathbb{R}^2 \subset V^{r,s}$ induces an inclusion of $CL(\mathbb{R}^2) \to CL_{r,s}$ and hence an inclusion $Spin(2) \to Spin(r,s)$. Since the inclusion of $CL(\mathbb{R}^2) \to CL(V^{r,s})$ is the identity on the center, it sends -1 of $CL(\mathbb{R}^2)$ to -1 of $CL(V^{r,s})$. Thus, we need only establish the statement for $Spin(2)$. this statement is contained in Problem 1.

Theorem 2.9. *For any (r,s) with $r \geq 2$ the natural map $Spin(r,s) \to SO(r,s)$ is a non-trivial double covering.*

Proof. Once we know that $Spin(V,Q)$ is connected and the kernel of the map to $SO(V,Q)$ is ± 1, it follows that the Lie group homomorphism $Spin(V,Q) \to SO(V,Q)$ is a non-trivial double covering. In case $\dim(V) \geq 3$ and Q is positive definite, the fact that the fundamental group $\pi_1(SO(\dim(V)),e)$ is a group or order 2, means that there is only one non-trivial double covering of $SO(V,Q) \cong SO(\dim(V))$, that being the universal covering. □

This completes the proof of Theorem 2.1.