

THE SIGN OF GALOIS REPRESENTATIONS ATTACHED TO AUTOMORPHIC FORMS FOR UNITARY GROUPS

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1. INTRODUCTION

1.1. The sign of a representation. Let L be a field of characteristic 0 or greater than 2. Let G be a group, and $g \mapsto g^c$ an involution of G . For ρ a representation $G \rightarrow \mathrm{GL}_n(L)$, we define $\rho^\perp : G \rightarrow \mathrm{GL}_n(L)$, $g \mapsto {}^t\rho(g^c)^{-1}$. The equivalence class of the representation ρ^\perp only depends on the equivalence class of ρ .

We fix $\chi : G \rightarrow L^*$ a character such that $\chi(g) = \chi(g^c)$ for all g . This property ensures that $\rho \mapsto \rho^\perp \chi^{-1}$ is an involution. In the applications, G will be the absolute Galois group of a CM field K , c the outer automorphism defined by the non trivial element in $\mathrm{Gal}(K/F)$ where F is the maximal totally real field in K , and χ will be a power of the cyclotomic character.

Let ρ be a semi-simple representation $G \rightarrow \mathrm{GL}_n(L)$ such that

$$(1) \quad \rho^\perp \simeq \rho\chi.$$

This property is obviously stable by extension of the field of coefficients L .

We shall now attach to any *absolutely irreducible* ρ satisfying (1) an invariant, that we call its *sign*. The invariant can take the value $+1$ or -1 . By Schur's lemma there exists a unique (up to a scalar) matrix $A \in \mathrm{GL}_n(L)$ such that

$$(2) \quad \rho^\perp = A\rho A^{-1}\chi.$$

Applying this relation twice, we see that $A^t A^{-1}$ commutes with ρ^\perp , hence by Schur's lemma again is a scalar matrix λ . So ${}^t A = \lambda A$ and $\lambda = \pm 1$. This sign is called the *sign* of ρ . Note that it is necessarily 1 if n is odd, since there is no invertible antisymmetric matrix in odd dimension.

It is obvious that the sign of an absolutely irreducible representation is unchanged by arbitrary extensions of the coefficient field L .

1.2. The context: the book project on Galois representations attached to unitary groups. Let F be a totally real field and K a totally imaginary quadratic extension, $c \in \mathrm{Gal}(K/F)$ the non-trivial automorphism. Let Π be a cuspidal automorphic representation for GL_n over K , and assume that Π is polarized (i.e. the contragredient Π^\vee of Π is isomorphic to $\Pi \circ c$) and that Π_∞ is algebraic regular (see [H, General Hypotheses 2.1]).

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Under those hypotheses, the many coauthors of the *book project* ([GRFAbook]) expect to prove the existence of an attached compatible system of Galois representations (see [H, Theorem 4.2.4]):

Expected Theorem 1.1 (Book Project). *There is a number field $E(\Pi)$ and a compatible system $\rho_{\Pi,\lambda} : G_K \rightarrow GL(n, E(\Pi)_\lambda)$ of λ -adic representations, where λ runs through finite places of $E(\Pi)$ such that for all finite primes v of K of residue characteristic prime to $N_{E(\Pi)/\mathbb{Q}}(\lambda)$,*

$$\rho_{\Pi,\lambda}^{F-ss}|_{G_v} \simeq L(\Pi_v \otimes |\bullet|_v^{\frac{1-n}{2}}),$$

where G_v is a decomposition group of K at v and $L(\bullet)$ is the local Langlands correspondence.

The given property suffices to characterize uniquely $\rho_{\Pi,\lambda}$ up to isomorphism and implies that $\rho_{\pi,\lambda}$ satisfies (1); more precisely, let c be a complex conjugation in K , that is an element of $G_F - G_K$ of order 2. We set $g^c = cg c^{-1} = cgc$ for $g \in G_K$: this is an automorphism of order 2. For that automorphism, we have

$$\rho^\perp(g) = {}^t\rho(g^c)^{-1} \simeq \rho(g)\omega(g)^{n-1}$$

where ω is the cyclotomic character.

The proof of this theorem relies on a special case, which is the same theorem with stronger hypotheses on the extension K/F and on the automorphic representation Π : see [H, Expected Theorem 1.4]. This theorem is the output of the comparison of the two stabilized trace formulas, and other works, to be done in books one and two of the book project. Book one has been mainly written and most chapters are available on [GRFAbook], while the writers of book two are expected to handle their chapters by May 2008.

The derivation from [H, expected Theorem 1.4] to the theorem quoted above is carefully written in [H]. We shall use several lemmas proved by Harris during this this derivation.

The theorem also includes many other specifications on $\rho_{\Pi,\lambda}$, including the expected determination on the Hodge-Tate weights of $\rho_{\Pi,\lambda}$ at places of same residual characteristic as λ (see also §1.6 below). This description implies, since Π_∞ is cohomological, that these weights are distinct integers, hence that $\rho_{\Pi,\lambda}$ is a direct sum of non-isomorphic absolutely irreducible representations of G_K .

1.3. The result. The object of this article is to prove (admitting [H, Expected Theorem 1.4])

Theorem 1.2. *For every finite prime λ of $E(\Pi)$, every irreducible factor of $\rho_{\Pi,\lambda}$ that satisfies (1) has sign $+1$.*

It is expected that $\rho_{\Pi,\lambda}$ is absolutely irreducible (this is known if $n \leq 3$ by [BIRo1] and in many cases if $n = 4$ by an unpublished work of Ramakrishnan). If it is so, $\rho_{\Pi,\lambda}$ has only one factor and satisfies (1), and our theorem simply asserts that its sign is $+1$: this is obvious when n is odd, but new when n is even.

1.4. Historical remarks. The question of the sign of Galois representations attached to polarized automorphic representations of GL_n on a totally real or CM field is out at least since Clozel, building on the work of Kottwitz, proved their existence in many cases in the mid nineties. More recently, this question has been extensively discussed in [CIHT] where some cases of the above theorem, concerning Galois representations with some constraining properties ensuring they have a nice and workable deformation theory, are proved by a very indirect method – indeed the whole long and hard paper is written with an unknown sign ϵ and only near the end, after the Taylor-Wiles method has been adapted to unitary groups, is it shown that $\epsilon = -1$ leads to a contradiction!

Theorem 1.2 appears, without its proof, in the concluding remarks of our book [BCh] (see [BCh, theorem 9.5.1]) that was made public late 2006. We knew the proof that follows then¹, and told it to a few colleagues, but decided to wait for a more advanced version of the book project, on which it depends, before writing it.

Meanwhile, one of us, Gaëtan Chenevier, together with Laurent Clozel, has found a completely different proof of a special case of Theorem 1.2: they prove that the Galois representation attached to a polarized and cohomological automorphic representation of GL_n over a totally real field, which is square integrable at some finite place, is symplectic² when n is even. Their proof ([ChCl]) is less expensive in difficult tools than ours, using simply the new insight in the trace formula they discovered in [ChCl]. It does not seem that their method can be extended to the case of a CM field, or even to the case of an automorphic representation that does not satisfy any local square-integrability hypothesis.

Let us mention also that in a recent preprint [Gr], B. Gross introduces a general notion of *odd* Galois representations and conjectures that the expected Galois representations attached to definite reductive groups G are odd in his sense. Our

¹At least for places λ of residual characteristic p split in K .

²It may seem strange at first glance that this symplecticity result is a special case of our sign $+1$ result – one would naively expect -1 . Actually, a closer look shows that the two results are consistent. Let us explain why.

Chenevier and Clozel’s result is about an even dimensional absolutely irreducible representation of r of G_F , say of dimension $2n$, $r : G_F \rightarrow \mathrm{GL}_{2n}(L)$, as constructed by Clozel (the irreducibility following from works of Harris-Taylor and Taylor-Yoshida). The representation r satisfies $r^\vee \simeq r\omega^{2n-1}$ so

$${}^t r(g)^{-1} = Pr(g)P^{-1}\omega(g)^{2n-1}, \quad \forall g \in G_F,$$

where ω is again the cyclotomic character, and Chenevier-Clozel result is that P is anti-symmetric. We claim that this implies that the restriction $\rho = r|_{G_K}$, that satisfies clearly (1), has sign $+1$.

Indeed, by changing the basis, we may assume that $r(c)$ is diagonal, hence symmetric. Moreover, the relation of selfduality of r above implies

$$r(c) = -Pr(c)P^{-1}$$

since $\omega(c) = -1$. An immediate computation shows then that ${}^t \rho(g^c)^{-1} = A\rho(g)A^{-1}\omega(g)^{2n-1}$ with $A = r(c)P$. But we see that

$${}^t A = {}^t P {}^t r(c) = (-P)r(c) = r(c)P = A$$

hence the sign of ρ is $+1$.

theorem proves his conjecture when G is the a unitary group attached to a CM extension K/F , in which case it has the following meaning.

Let \tilde{G} be the semi-direct product of $\text{Gal}(K/F) = \langle c \rangle = \mathbb{Z}/2\mathbb{Z}$ by $\text{GL}_n(L) \times L^*$ with respect to the order two automorphism $(x, y) \mapsto (y^t x^{-1}, y)$ (see [CIHT, Ch.I] for similar considerations). Assume that $\rho : G_K \rightarrow \text{GL}_n(L)$ satisfies (1), is absolutely irreducible, and fix A a matrix as in (2), and $\epsilon = \pm 1$ the sign of ρ . Consider the morphism $G_K \rightarrow \text{GL}_n(L) \times L^*$ defined by $g \mapsto (\rho(g), \chi(g)^{-1})$. A simple computation shows that this map extends to a morphism $\tilde{\rho} : G_F \rightarrow \tilde{G}$ if we set $\tilde{\rho}(c) = ({}^t A^{-1}, \epsilon)c$. The map $\tilde{\rho}$ is the analog in our situation of the map denoted ρ whose existence is conjectured in [Gr, page 8] and Gross predicts that the conjugation by $\tilde{\rho}(c)$ on $\text{Lie}(\text{GL}_n)$ is a Cartan involution, that is, has the form $X \mapsto -P^t X P^{-1}$ with P a *symmetric* invertible matrix. In our situation, the conjugation by $\tilde{\rho}(c)$ on the Lie algebra is the map $X \mapsto -{}^t A^{-1} X {}^t A$. So we see that Gross' prediction amounts to “ A is symmetric”, which is exactly our theorem.³

1.5. Idea of the proof. The idea of the proof is very simple. Assume that we know that the representation $\rho_{\Pi, \lambda}$ is irreducible. Then there is nothing to prove if n is odd. When n is even, we can reduce to the odd case, as follows: descend Π to a unitary group in n variables, transfer the result to an automorphic representation π of a unitary groups in $n+1$ variables which is compact at infinity, using a special case of endoscopic transfer proved by Clozel, Harris and Labesse. Use eigenvarieties to deform π into a family of automorphic forms whose Galois representations are generically irreducible. For those Galois representations, the sign is $+1$ since their dimension is odd. Specialize this result to deduce that the components of the representation attached to π , including $\rho_{\Pi, \lambda}$, have sign $+1$.

There are several technical difficulties that make the proof a little bit more indirect: in the current state of science, we do not know that $\rho_{\Pi, \lambda}$ is (absolutely) irreducible, and we cannot descend Π to $\text{U}(n)$ or transfer it to $\text{U}(n+1)$ without supplementary assumptions on K/F and Π . Moreover, we cannot always deform a representation π in a family whose Galois representation is generically irreducible. But this is not a big issue, since, as was already observed in [BCh, §7.7], we can actually do so in two steps, deforming π in a family whose generic members can themselves be deformed irreducibly. Similarly the obstacle posed by the conditions on descent and endoscopic transfer can be solved by base change techniques inspired by the ones used in [H].

1.6. Notations and conventions. Our general convention will be that the local Langlands correspondence is normalized so that geometric Frobeniuses correspond to uniformizers (and as in [HT]). If π is an unramified complex representation of $\text{GL}_n(E)$ with E a p -adic local field, or more generally an irreducible smooth representation with a nontrivial vector fixed by a Iwahori subgroup, we shall often denote by $L(\pi)$ the semisimple conjugacy class in $\text{GL}_n(\mathbb{C})$ of the geometric Frobenius in the L -parameter of π .

³When ρ is not assumed to be irreducible anymore, note that theorem 1.2 still implies that we may find some symmetric A such that (1) holds, hence a $\tilde{\rho}$ as above satisfying Gross' conjecture.

If K is a field, we shall denote by G_K its absolute Galois group $\text{Gal}(\overline{K}/K)$; when K is a number field and v a place of K we also write G_v for G_{K_v} .

We shall use the following notions of p -adic Hodge theory. Let us fix E a finite extension of \mathbb{Q}_p , $\overline{\mathbb{Q}_p}$ an algebraic closure of \mathbb{Q}_p , and let V be a p -adic representation of G_E of dimension n over $\overline{\mathbb{Q}_p}$. To such a representation Sen attaches a monic polynomial $P_{\text{sen}}(T) \in (\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} E)[T]$ of degree n , whose roots will be called the Hodge-Tate weights of V (even when they are not natural integers). Our normalization of the Sen polynomial is the one such that the Hodge-Tate weight of the cyclotomic character $\overline{\mathbb{Q}_p}(1)$ is $-1 \in \overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} E$. Under the natural identification $\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} E = \overline{\mathbb{Q}_p}^{\text{Hom}(E, \overline{\mathbb{Q}_p})}$, we shall often write them as a collection $\{k_{i,\sigma}\}$ for all $i \in \{1, \dots, n\}$ and all $\sigma \in \text{Hom}(E, \overline{\mathbb{Q}_p})$, ordered so that for each embedding σ we have

$$k_{1,\sigma} \leq k_{2,\sigma} \leq \dots \leq k_{n,\sigma}.$$

We shall need to consider various partial sums of those weights, for which the following definitions will be useful. For I a subset of $\{1, \dots, n\} \times \text{Hom}(K_w, \overline{\mathbb{Q}_p})$, we denote by k_I the sum $\sum_{(i,\sigma) \in I} k_{i,\sigma}$. When $I = \{i\} \times \text{Hom}(K_w, \overline{\mathbb{Q}_p})$, we write k_i instead of k_I . Thus $k_i = \sum_{\sigma} k_{i,\sigma}$.

Assume now that V is crystalline in the sense of Fontaine. Let $E_0 \subset E$ be the maximal unramified extension of \mathbb{Q}_p inside E , and let $\mathbf{v} : \overline{\mathbb{Q}_p} \rightarrow \mathbb{Q}$ be the valuation *normalized so that* $\mathbf{v}(p) = e$, where e is the absolute ramification index of E . Fontaine attaches to V an E_0 -vector space $D_{\text{crys}}(V)$ with a semilinear action of the crystalline Frobenius φ (commuting with $\overline{\mathbb{Q}_p}$), and which is free of rank n over $E_0 \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$. If $f = [E_0 : \mathbb{Q}_p]$, then φ^f is $\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} E_0$ -linear and commutes with φ , so its characteristic polynomial $P_{\varphi}(T)$ actually belongs to $\overline{\mathbb{Q}_p}[T]$. This polynomial will be referred as the *characteristic polynomial* of φ , its roots are the *eigenvalues* of the crystalline Frobenius, and their valuations (with respect to \mathbf{v}) its *slopes*.⁴ With these notations, if the $k_{i,\sigma}$ are the Hodge-Tate weights of V , then the weak admissibility property of $D_{\text{crys}}(V)$ implies in particular that

$$\mathbf{v}(P_{\varphi}(0)) = \sum_{i,\sigma} k_{i,\sigma}.$$

We can now explain a bit more precisely the p -adic part the Expected theorem 1.1. Assume that w is a finite place of K with the same residual characteristic as λ , and assume that Π_w is unramified. Let $P_w(T) \in E(\Pi)[T]$ be the characteristic polynomial of $L(\Pi_w, \cdot)^{(1-n)/2}$. Then part of Expected Theorem 1.1 asserts that $\rho_{\Pi,\lambda}|_{G_w}$ is a crystalline representation (see [H]).

At present, it is not clear whether we will be able to identify the characteristic polynomial $P_{\varphi} \in E(\Pi)_{\lambda}[T]$ of the crystalline Frobenius with the image of $P_w(T)$ in this full generality, as it should be. However, we will know that $P_{\varphi} = P_w$ by construction under the following extra assumptions : assumptions (H1) and (H2) stated in §3.1 below on K/F and Π are satisfied.

⁴This definition is slightly different from the usual definition of the slopes of an isocrystal (which are ours divided by $[E : \mathbb{Q}_p]$), but it will be convenient to us.

2. SORITES ON THE SIGN

2.1. The notion of a good representation. For a representation $\rho : G_K \rightarrow \mathrm{GL}_n(L)$ that is a direct sum of absolutely irreducible and pairwise non isomorphic representations, and that satisfies (1), say that ρ is *good* if for every irreducible factor of ρ that appears with multiplicity one and satisfies (1), the sign of this factor is $+1$.

In this language, the theorem amounts to prove that $\rho_{\Pi, \lambda}$ is good, which is good.

2.2. Some trivial lemmas. In this paragraph, $\rho : G_K \rightarrow \mathrm{GL}_n(L)$ is a direct sum of absolutely irreducible representations, and satisfies (1).

Lemma 2.1. *If $\rho : G_K \rightarrow \mathrm{GL}_n(L)$ is good, so is any twist by a character.*

Proof — Note first that since ρ satisfies (1) for a character χ , then if $\psi : G_K \rightarrow L^*$ is a character, $\rho\psi$ still satisfies (1) for the character $\chi' = \chi\psi^{-1}\psi^\perp$ (which still satisfies $\chi'(g^c) = \chi'(g)$) and is a sum of absolutely irreducible pairwise non isomorphic factors, namely the $\rho_i\psi$ where the ρ_i are the factors of ρ . Now if ρ_i is an irreducible factor that satisfies (1), a matrix A that satisfies (2) for ρ_i and χ satisfies also (2) for $\rho_i\psi$ and χ' , hence the sign of ρ_i and $\rho_i\psi$ are the same. \square

Lemma 2.2. *If $\rho : G_K \rightarrow \mathrm{GL}_n(L)$ is good, and ρ' is a sub-representation of ρ that satisfies (1), then ρ' is good too.*

Proof — That one is *really* trivial. \square

Lemma 2.3. *Let F' be a totally real extension of F , and $K' = KF$. If $\rho|_{G_{K'}}$ has the same number of irreducible components as ρ , and if those components are absolutely irreducible, then $\rho|_{G_{K'}}$ is good if and only if ρ is good.*

Proof — If ρ_i is an (absolutely) irreducible factor of ρ that satisfies (1), then $\rho_i|_{G_{K'}}$ is still absolutely irreducible by hypothesis, still satisfies (1), and has obviously the same sign as ρ_i . The lemma follows. \square

2.3. A specialization result. In this paragraph, \mathcal{O} is a henselian discrete valuation domain with fraction field L and residue field k , such that $2 \in \mathcal{O}^*$. We set also $G = G_K$ and assume that the character $\chi : G_K \rightarrow L^*$ actually falls into \mathcal{O}^* , thus it makes sense to talk about condition (1) for k or L -valued representations of G (by a slight abuse of language, we shall also denote by χ the residual character $G_K \rightarrow k^*$). A simple but crucial observation for of our proof is the following:

Proposition 2.4. *Assume that $\rho : G \rightarrow \mathrm{GL}_n(\mathcal{O})$ is such that $\rho \otimes L$ and $\bar{\rho}^{ss}$ are a sum of absolutely irreducible pairwise non isomorphic representations and satisfy (1). Then if $\rho \otimes L$ is good, so is $\bar{\rho}^{ss}$.*

Moreover, the converse holds if $\bar{\rho}^{ss}$ has the same number of irreducible factors as $\rho \otimes L$.

Of course, in this statement $\bar{\rho}^{\text{ss}}$ denotes the semisimplification of the reduction $\bar{\rho} := \rho \otimes_{\mathcal{O}} k$ of ρ .

Proof — Let $\bar{\rho}_1$ be a factor of $\bar{\rho}^{\text{ss}}$ that satisfies (1). Let τ_1, \dots, τ_k be the irreducible factors of $\rho \otimes L$. For each of them we can choose a stable \mathcal{O} -lattice, and see them as representations of G over \mathcal{O} . We have $\bar{\rho}^{\text{ss}} = \bigoplus_{i=1}^k \bar{\tau}_i^{\text{ss}}$ so $\bar{\rho}_1$ appears in exactly one of the $\bar{\tau}_i^{\text{ss}}$, say $\bar{\tau}_1^{\text{ss}}$. Moreover, $\bar{\tau}_1^{\perp}$ is isomorphic to $\bar{\tau}_i \chi$ for some $i \in \{1, \dots, k\}$. But it follows that $\bar{\tau}_i^{\text{ss}}$ contains $\bar{\rho}_1$ (since $\bar{\rho}_1$ satisfies (1)), so the only possibility is that $i = 1$. In other words, τ_1 satisfies (1), and replacing ρ by τ_1 , we are reduced to prove the lemma with the supplementary assumption that $\rho \otimes L$ is absolutely irreducible. In that case, the proposition is [BCh, Lemma 1.8.8].

Since this proposition is really one of the main tool used in our proof, and since the proof of [BCh, Lemma 1.8.8] is a little bit difficult to separate from the other concerns of [BCh, §1.8], let us sketch it here for the convenience of the reader, trying to be as pedagogical as possible.

Note first that the basic point that makes the result not obvious is that there is no reason so that we can find a matrix A for $\rho \otimes L$ as in (2) with $A \in \text{GL}_n(\mathcal{O})$. A priori we just have $A \in \text{GL}_n(L)$, and it is therefore not possible to reduce (2) mod m .

There is one case, however, where a simple proof is possible: assume that $\bar{\rho}^{\text{ss}}$ is absolutely irreducible. In this case, the representations ρ^{\perp} and $\rho \chi$ over \mathcal{O} , being isomorphic over L and residually absolutely irreducible, are isomorphic over \mathcal{O} by a theorem of Serre and Carayol. In other words, we can find a matrix $A \in \text{GL}_n(\mathcal{O})$ such that (2) holds, and reducing this modulo m , we get that $\rho \otimes L$ and $\bar{\rho}$ have the same sign in this case. Note that this proves also the last assertion of the theorem (in all cases!).

The proof of the general case consists in reducing to the residually irreducible case. This is not possible, however, if we keep working with representations of groups only. We have to work in the larger world of representations of algebras instead. As we saw, we may assume that $\rho \otimes L$ is absolutely irreducible, and we set $\bar{\rho}^{\text{ss}} = \bigoplus_i \bar{\rho}_i$.

Let R be the algebra $\mathcal{O}[G]$, and $S = \rho(R) \subset M_n(\mathcal{O})$. We have $S \otimes_{\mathcal{O}} L = M_n(L)$. The algebra S is provided with a natural \mathcal{O} -algebra anti-automorphism τ , induced by the one on R defined on $g \in G$ by $g \mapsto \chi(g)^{-1}(g^c)^{-1}$. Explicitly, by (2), we have for $M \in S$,

$$(3) \quad \tau(M) = {}^t A^{-1} M {}^t A,$$

and by our sign assumption ${}^t A = A$: the involution τ is a symmetric involution of the matrix algebra $S \otimes_{\mathcal{O}} L = M_n(L)$.

On the other hand, let \bar{S} denote the image of $k[G]$ in the k -endomorphisms of the representation $\bar{\rho}^{\text{ss}} = \bigoplus_i \bar{\rho}_i$. Then $\bar{S} \simeq \prod_i M_{n_i}(k)$ ($n_i = \dim \bar{\rho}_i$, $\sum_i n_i = n$) and \bar{S} is also provided with a natural k -algebra anti-automorphism τ as above. Moreover, there is a natural surjective \mathcal{O} -algebra map $S \rightarrow \bar{S}$ which is τ -equivariant.

Let us denote by $\epsilon_i \in \bar{S}$ be the central idempotent corresponding to $\bar{\rho}_i$. It is well known that ϵ_i can be lifted as an idempotent e_i of S as \mathcal{O} is henselian and S

finite over \mathcal{O} . However, we need a more precise lifting result. Let us fix an i such that $\bar{\rho}_i$ satisfies (1), then we have $\tau(\epsilon_i) = \epsilon_i$. What we need is an idempotent e_i in S lifting ϵ_i , such that $\tau(e_i) = e_i$. The existence of such an idempotent is easy to prove: first choose any lift $x \in S$ of ϵ_i and let S_0 be the sub- \mathcal{O} -algebra generated by $\frac{1}{2}(x + \tau(x))$. Obviously, τ fixes any element of S_0 . The restriction of the natural surjection $S \rightarrow \bar{S}$ to S_0 is onto a k -subalgebra of \bar{S} that contains the image of $\frac{1}{2}(x + \tau(x))$, that is ϵ_i . Thus, defining e_i as a lift of ϵ_i in S_0 does the job. (This result is the trivial case of [BCh, Lemma 1.8.2].) As $\bar{\rho}_i$ is absolutely irreducible and has multiplicity one in $\bar{\rho}^{\text{ss}}$ it actually turns out that the rank of e_i is $n_i = \dim \bar{\rho}_i$, and that $e_i S e_i \simeq M_{n_i}(\mathcal{O})$. Replacing ρ by a conjugate if necessary, we may then assume that e_i is a diagonal idempotent of rank n_i in $M_n(L)$.

Applying (3) to $M = e_i$ we get $Ae_i = {}^t e_i {}^t A$, that is Ae_i is symmetric. In other words, τ induces a symmetric involution on $e_i S e_i \simeq M_{n_i}(\mathcal{O})$. As a consequence, τ also induces a symmetric involution on $\epsilon_i \bar{S} \epsilon_i = \text{End}_k(\bar{\rho}_i)$, which exactly means that the sign of $\bar{\rho}_i$ is $+1$, QED. \square

3. PROOF OF THE MAIN THEOREM

3.1. Proof of theorem 1.2 under special hypotheses. We shall first prove the theorem under a set of additional hypotheses on the CM extension K/F , the automorphic representation Π and the place λ .

Let us call p the residual characteristic of λ . Recall that the automorphic representation Π defines an embedding $\iota : E(\Pi) \rightarrow \mathbb{C}$. We fix once and for all algebraic closures $\bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}}_p$ of \mathbb{Q} and \mathbb{Q}_p , as well as some embeddings $\iota_\infty : \bar{\mathbb{Q}} \rightarrow \mathbb{C}$ and $\iota_p : \bar{\mathbb{Q}}_p \rightarrow \mathbb{C}$ such that the induced map $\iota_p \iota_\infty^{-1} \iota : E(\Pi) \rightarrow \bar{\mathbb{Q}}_p$ factors through $E(\Pi)_\lambda$.

3.1.1. *Some special hypotheses.*

- (H1) Special Hypotheses 2.2 of [H], that is
 - (H1a) K/F is unramified at all finite places
 - (H1b) Π_v is spherical at all non-split non-archimedean places v of K
 - (H1c) The degree $[F : \mathbb{Q}]$ is even.
 - (H1d) All primes of small residue characteristic relative to n are split in K/F .
- (H2) Special Hypothesis 2.3 of [H], that is for at least one real place σ of K , the infinitesimal character of Π_σ is sufficiently far from the walls.⁵
- (H3) There⁶ is a place v above p in F that splits in K , and for w a place of K above v , Π_w is unramified. Denote by $\{\varphi_1, \dots, \varphi_n\}$ the eigenvalues of $L(\Pi_w | \cdot |^{(1-n)/2})$. Then the Hodge-Tate weights $\{k_{i,\sigma}\}$ of $\rho_{\Pi,\lambda} | G_w$ and the slopes $\mathbf{v}(\varphi_j)$ are in sufficiently general position in the following sense: if

$$c = \max_{i \in \{1, \dots, n\}} \min_{j \in \{1, \dots, n\}} |\mathbf{v}(\varphi_i) - k_j|$$

⁵Here and everywhere in this paper, and as in [H], this will mean that the extremal weight of the associated algebraic representation of $\text{GL}_n(\mathbb{C})$ does not belong to a wall. For our purposes, we could even assume here that this holds for all archimedean places.

⁶See §1.6 for the notations used in this assumption.

then for all subsets I and J of $\{1, \dots, n\} \times \text{Hom}(K_w, \overline{\mathbb{Q}}_p)$ with $|I| = |J| < nd$, we have

$$|k_I - k_J| > (n + 1) \cdot c.$$

In (H3) above, $\mathbf{v} : \overline{\mathbb{Q}}_p \rightarrow \overline{\mathbb{Q}}$ is the valuation such that $\mathbf{v}(p)$ is the ramification index of p in K_w .

3.1.2. *The theorem.* We want to prove :

Theorem 3.1. *With the supplementary hypotheses (H1), (H2) and (H3), Theorem 1.2 holds.*

The rest of this subsection is entirely devoted to the proof of this theorem.

3.1.3. *Descent and transfer.* Let $m = n$ if n is odd, and $m = n + 1$ if n is even, so that m is always odd. Let us call $U(m)$ be the unitary group over F attached to K in m variables that is quasi-split at every finite places of F and compact at every infinite place. Since m is odd, such a group always exists (uniquely up to isomorphism). Actually, $U(m)$ is simply the standard unitary group attached to the hermitian form $\sum_{i=1}^m N_{K/F}(z_i)$ on K^m ([BCh, §6.2.2]).

If n is odd, that is if $n = m$, by hypothesis (H1) and Labesse's base change theorem [GRFABook, chap. 4 book 1], we can descend Π to a representation π of $U(m)$ with $\pi_v \simeq \Pi_w$ for every place w of K split over v in F (with the natural identification $U(n)(F_v) \simeq \text{GL}_n(K_w)$), and such that for each complex place w of K above a real place v of F , π_v has the same infinitesimal character as Π_w (under the natural identification $U(n)(K_w) \simeq \text{GL}_n(\mathbb{C})$).

If n is even, we use a result of endoscopic transfer due to Clozel, Harris, and Labesse (see [GRFABook] and in particular [H]). Note first that using $\iota_\infty \iota_p^{-1}$, if $v = ww^c$ is as (H3) we may identify $\text{Hom}(K_w, \overline{\mathbb{Q}}_p) = \text{Hom}(F_v, \overline{\mathbb{Q}}_p)$ with subsets Σ_v and Σ_w of $\text{Hom}(F, \mathbb{R})$ and $\text{Hom}(K, \mathbb{C})$. Let us first fix

$$\mu : K^* \backslash \mathbb{A}_K^* \rightarrow \mathbb{C}^*$$

a Hecke character such that $\mu(c(x))^{-1} = \mu(x)$, and such that for each $s \in \Sigma_v$, $\mu_s(z) = (\sigma_s(z)/\overline{\sigma_s(z)})^{\frac{1}{2}}$ where $\sigma_s \in \Sigma_w$ is associated to s as above. This last assumption implies that $\mu|_{\mathbb{A}_F^*}$ coincides with the sign of K/F , and that μ does not come by base change from a Hecke-character of $U(1)$ (see e.g. [BCh, §6.9.2]). Such a Hecke character always exists, and as K/F is unramified at all finite places, we can even assume (and we will) that it is unramified at the finite places of K which are either above p or not split above F . Let us choose another Hecke character

$$\chi : K^* \backslash \mathbb{A}_K^* \rightarrow \mathbb{C}^*$$

such that $\chi(c(z))^{-1} = \chi(z)$ but assume now that χ descends to $U(1)$, i.e. that for each real place $s \in \Sigma_v$, $\chi_s(z) = \sigma_s(z/c(z))^{-a_s}$ for some $a_s \in \mathbb{Z}$. Assume also that χ is unramified at the finite places of K which do not split over F . Under hypotheses (H1) and (H2), and if all the $|a_s|$ are big enough, by the aforementioned results of Clozel, Harris and Labesse, we can transfer Π to an automorphic representation π of $U(m)$ in such a way that at every place w of K split over a place v in F , we have

$$(4) \quad L(\pi_v) = L(\Pi_w \mu_w) \oplus L(\chi_v).$$

Moreover, for each real place v of F and each complex place w of K above v , the infinitesimal character of π_v is obtained from the one of $\Pi_w \mu_w$ in the obvious way : in terms of the associated Harish-Chandra's cocharacter, it is the direct sum of the one of $\Pi_w \mu_w$ and the one of χ_w .

In both cases (n even or odd), the aforementioned authors actually construct a π which is moreover unramified at all the finite places of K which are not split over F (we don't really need this, but this fixes ideas).

3.1.4. *Consequences of (H3).* When $n = m$ is odd, we set $\rho_\pi := \rho_{\Pi, \Lambda}$. When n is even, the G_K representation of dimension m attached to π is by definition

$$\rho_\pi := \rho_{\Pi, \Lambda}(\mu|\cdot|^{-\frac{1}{2}}) \oplus \chi|\cdot|^{(1-m)/2}.$$

Note that $\mu|\cdot|^{-\frac{1}{2}}$ and $\chi|\cdot|^{(1-m)/2}$ are both algebraic Hecke characters of K , and we identify them here with their p -adic realization given by ι_∞ and ι_p . By assumption, $\mu|\cdot|^{-\frac{1}{2}}$ is actually unramified at the place w , and $\chi|\cdot|^{(1-m)/2}$ is crystalline, and we shall denote by φ_μ and $\varphi_\chi \in \overline{\mathbb{Q}_p}^*$ their associated Frobenius eigenvalue.

If n is even, so $m = n + 1$, we set for each $\sigma \in \text{Hom}(K_w, \overline{\mathbb{Q}_p})$,

$$k_{m, \sigma} := \frac{m-1}{2} + a_\sigma$$

(where σ is viewed as an element of $\text{Hom}(F, \mathbb{R})$ as above). Thus, in any case, the $k_{i, \sigma}$ for $i = 1, \dots, m$ and $\sigma \in \text{Hom}(K_w, \overline{\mathbb{Q}_p})$ are the Hodge-Tate weights of $\rho_\pi|_{G_w}$. We shall use for this extended collection $\{k_{i, \sigma}\}$ with all $i \in \{1, \dots, m\}$, and for a subset I of $\{1, \dots, m\} \times \text{Hom}(K_w, \overline{\mathbb{Q}_p})$, the notation k_I analogous to the one in §1.6.

If n is even, we set $\varphi'_i := \varphi_i \varphi_\mu$ for $i < m$ and $\varphi'_m := \varphi_\chi$. We have $\mathbf{v}(\varphi'_i) = \mathbf{v}(\varphi_i)$ for $i < m$ and $\mathbf{v}(\varphi'_m) = k_m$. When $n = m$ is odd, we shall simply set $\varphi'_i := \varphi_i$. Thus, in both cases, $\varphi'_1, \dots, \varphi'_m$ are the Frobenius eigenvalues of $L(\pi_w|\cdot|^{(1-m)/2})$.

If n is even, we precise now our choice of χ . We assume that $k_{\sigma, m} = a_\sigma + \frac{d(m-1)}{2}$ are all big with respect to the k_i and $\mathbf{v}(\varphi'_i)$ for $i = 1, \dots, n$, and also that they are set sufficiently far apart so that any non trivial sum of the form $\sum_{\sigma \in S} \pm k_{\sigma, m}$, where $S \subset \text{Hom}(K_w, \overline{\mathbb{Q}_p})$, is big in the same sense as above. This is of course always possible. With those assumptions:

- Lemma 3.2.** (i) *The representation π_v is a fully induced unramified principal series, and the eigenvalues of $L(\pi_w|\cdot|^{(1-m)/2})$ are $\varphi'_1, \dots, \varphi'_m$.*
(ii) *We have $c = \max_{i \in \{1, \dots, m\}} \min_{j \in \{1, \dots, m\}} |\mathbf{v}(\varphi'_i) - k_j|$, and for all distinct subsets I and J of $\{1, \dots, m\} \times \text{Hom}(K_w, \overline{\mathbb{Q}_p})$ and with $|I| = |J| < md$, we have $|k_I - k_J| > m \cdot c$.*

Proof — By (H3) and, if n is even, by (4), π_v is unramified. Moreover, the eigenvalues of $L(\pi_w|\cdot|^{(1-m)/2})$ are $\varphi'_1, \dots, \varphi'_m$, and no quotient of those eigenvalues is equal to q , the cardinal of the residue field of K_w . Indeed, a well-known result of Jacquet-Shalika asserts that for $i = 1 \dots n$, the complex numbers $q^{(1-m)/2} \iota_\infty \iota_p^{-1}(\varphi'_i)$ are $< q^{1/2}$ in absolute value, and $q^{(1-m)/2} \iota_\infty \iota_p^{-1}(\varphi'_m)$ has norm 1 by construction. Hence π_v is a full unramified principal series by Zelevinski's theorem, which is (i).

For (ii), there is nothing to prove if $n = m$ is odd. Assume n even so $m = n + 1$. Let us note that for $i = m$, we have $\min_{j \in \{1, \dots, m\}} |\mathbf{v}(\varphi'_i) - k_j| = 0$ since $\mathbf{v}(\varphi'_m) = k_m$. For $i \leq n$, the minimum $\min_{j \in \{1, \dots, m\}} |\mathbf{v}(\varphi'_i) - k_j| = 0$ is not realized for $j = m$ because k_m is much too big.

Hence

$$\begin{aligned} \max_{i \in \{1, \dots, m\}} \min_{j \in \{1, \dots, m\}} |\mathbf{v}(\varphi'_i) - k_j| &= \max_{i \in \{1, \dots, n\}} \min_{j \in \{1, \dots, n\}} |\mathbf{v}(\varphi'_i) - k_j| \\ &= c. \end{aligned}$$

It remains to prove that $|k_I - k_J| > mc = (n + 1)c$. Let I_0 (resp. J_0) be the subset of I (resp. of J) of pairs (i, σ) with $i = m$. If $I_0 = J_0$, then $k_I - k_J = k_{I-I_0} - k_{J-J_0}$ and since $I - I_0, J - J_0$, are distinct subsets of same cardinality of $\{1, \dots, n\} \times \text{Hom}(K_w, \overline{\mathbb{Q}}_p)$, the desired inequality comes directly from (H3). If $I_0 \neq J_0$, $k_I - k_J$ contains, in addition of a bounded numbers of terms $\pm k_{i, \sigma}$ for $i \leq n$, a non trivial sum of the form $\sum_S \pm k_{\sigma, m}$, where $S \subset \text{Hom}(K_w, \overline{\mathbb{Q}}_p)$, hence $|k_I - k_J|$ is again greater than mc . \square

3.1.5. Eigenvarieties and their families of Galois representations. We are ready now to start the deformation argument. Let $U = \prod_v U_v \subset \text{U}(n)(\mathbb{A}_{F,f})$ be a compact open subgroup such that $\pi^U \neq 0$, and assume that U_v is a Iwahori subgroup for the place v of (H3), and that U_v is hyperspecial for all places of K that are not split over F .

From now on, we shall reserve the notation v for the place of F of hypothesis (H3), and w for one of the place of K above v . We shall denote by d the degree of the field $F_v = K_w$ over \mathbb{Q}_p . To U , the place v and (ι_p, ι_∞) , we can attach an eigenvariety $X = X_{U, v, (\iota_\infty, \iota_p)}$ for the group $\text{U}(m)/F$ which is a reduced rigid analytic space over \mathbb{Q}_p of equidimension⁷ md . The construction of this eigenvariety follows essentially verbatim from the method of [Ch] (which is only written in the case $F = \mathbb{Q}$, but see e.g. [Bu] for the setting for a general F in dimension⁸ 2. Details should appear elsewhere as part of the book project [GRFAbook]). Alternatively, we may deduce it from the work of Emerton in [E, §3.2].

By Labesse's base change theorem, if π' is any automorphic representation of $\text{U}(m)$ which is unramified outside the split finite places of K/F , then π' admits a strong base change to GL_m/K , hence a Galois representation by Expected Theorem 1.1. As explained in [BCh, Chap. 7.5] (or [Ch]), this is enough to equip X with a continuous m -dimensional pseudocharacter $T : G_K \rightarrow \mathcal{O}(X)$ of dimension m . The eigenvariety X and this T satisfy a number of properties and we will only list below the ones we shall need. If $x \in X(\overline{\mathbb{Q}}_p)$ we note T_x the evaluation of T at x and ρ_x the semi-simple representation $G_K \rightarrow \text{GL}_m(\overline{\mathbb{Q}}_p)$ of trace T_x . There is :

- (i) Zariski-dense and accumulation subsets $Z^{\text{reg}} \subset Z \subset X(\overline{\mathbb{Q}}_p)$ of *classical points*,

⁷It is not necessary here to let the weights corresponding to the other possible places of F above p move, but we could have, and the eigenvariety would then have dimension $n[K : \mathbb{Q}]$.

⁸The situation here is actually even simpler because the center $\text{U}(1)$ of $\text{U}(m)$ is anisotropic over \mathbb{R} .

- (ii) a set of dm analytic functions⁹ $\kappa_{1,\sigma}, \dots, \kappa_{m,\sigma}$ where σ runs over the embeddings $K_w \rightarrow \overline{\mathbb{Q}_p}$,
- (iii) a set of locally constant functions $s_1, \dots, s_m : X(\overline{\mathbb{Q}_p}) \rightarrow \mathbb{Q}$,

satisfying the following conditions :

- (a) if $z \in Z$, $\rho_z|_{G_w}$ is crystalline.
- (b) if $z \in Z$, the ordered Hodge-Tate weights of $\rho_z|_{G_w}$ are $\{\kappa_{i,\sigma}\}$
- (c) let C be any real number and $Z_C := \{z \in Z^{\text{reg}}, |\kappa_I(z) - \kappa_J(z)| > C \text{ for all distinct subsets } I, J \text{ of } \{1, \dots, m\} \times \text{Hom}(K_w, \overline{\mathbb{Q}_p}) \text{ and } |I| = |J| < md\}$. Then Z_C is Zariski dense and accumulation in X .

Moreover, the classical points z in Z correspond to pairs $(\pi(z), \mathcal{R}(z))$ where $\pi(z)$ is an automorphic representation of $\text{U}(m)$ such that $\pi(z)^U \neq 0$ and $\mathcal{R}(z) = (\varphi_1, \dots, \varphi_m)$ is an accessible refinement¹⁰ of $\pi(z)_w|_{\cdot|^{(1-m)/2}}$, in the following sense : ρ_z is the Galois representation attached to the base change of $\pi(z)$ to GL_m/K by Expected Theorem 1.1 and for each $i = 1, \dots, m$, $\mathbf{v}(\varphi_i) = s_i(z) + \kappa_i(z)$.

- (d) If $z \in Z$ parameterizes $(\pi(z), \mathcal{R}(z) = (\varphi_1, \dots, \varphi_m))$, then for all i we have $\mathbf{v}(\varphi_i) = s_i(z) + \kappa_i(z)$.

The subset $Z^{\text{reg}} \subset Z$ parameterizes refined automorphic representations (π, \mathcal{R}) satisfying some additional properties, and for our concerns here we shall simply assume that they are those (π, \mathcal{R}) such that π_v unramified and such that for each real place s , the infinitesimal character of π_s is sufficiently far from the walls. Under this latter condition, the base change of an automorphic representation of $\text{U}(m)$ is not necessarily cuspidal, but always associated to a decomposition $m_1 + \dots + m_r = m$ and a r -tuple (π_1, \dots, π_r) of cuspidal (polarized, cohomological) automorphic representations π_i of $\text{GL}_{m_i}(\mathbb{A}_K)$; moreover each π_i satisfies property (H2) in dimension m_i and is unramified at v . In particular, for a $z \in Z^{\text{reg}}$, the characteristic polynomial of the crystalline Frobenius of $\rho_z|_{G_w}$ coincides with the polynomial $P_w(T)$ associated to $\pi_w|_{\cdot|^{(1-m)/2}}$ by Expected Theorem 1.1 (see §1.6), and we also have the following :

- (d') If $z \in Z^{\text{reg}}$, then the m slopes of the crystalline Frobenius of $\rho_z|_{G_w}$ are the $s_i(z) + \kappa_i(z)$, for $i = 1, \dots, m$.

3.1.6. *Choice of a refinement.* Going back to the representation π introduced above, if we choose an accessible refinement \mathcal{R} of π_v , there is a point $z_0 \in Z$ corresponding to (π, \mathcal{R}) .

Lemma 3.3. *There exists a refinement \mathcal{R} of π_v such that the pseudocharacter T is generically irreducible in a neighborhood of the corresponding point z_0 .*

⁹Again, we shall use for this collection $\{\kappa_{i,\sigma}\}$, and for a subset I of $\{1, \dots, m\} \times \text{Hom}(K_w, \overline{\mathbb{Q}_p})$ (resp. an $i \in \{1, \dots, m\}$), the notation κ_I (resp. κ_i) analogous to the one in §1.6.

¹⁰Recall that an *refinement* of an irreducible smooth representation ρ of $\text{GL}_m(K_w)$ such that $\rho^I \neq 0$ for I an Iwahori subgroup is an ordering $(\varphi_1, \dots, \varphi_m)$ of the eigenvalues of $L(\rho)(\text{Frob}_w)$. It is said *accessible* if ρ appears as a *sub-representation* of the induced representation $\text{Ind}_B^{\text{GL}_m(K_w)} \chi \delta_B^{1/2}$ where B is (say) the upper Borel subgroup, δ_B is modulus character, and χ the (unramified) character of the diagonal torus sending (x_1, \dots, x_m) to $\prod_{i=1}^m \varphi_i^{\mathbf{v}(x_i)}$ (see [BCh, §6.4.4]).

Proof — We shall eventually show that the conclusion holds for $T|G_w$. Note that by construction, for all $\sigma \in \text{Hom}(K_w, \overline{\mathbb{Q}}_p)$ and $i \in \{1, \dots, m\}$, $\kappa_{i,\sigma}(z_0) = k_{i,\sigma}$. Let us first renumber the $\varphi'_i \in \overline{\mathbb{Q}}_p^*$ so that $|\mathbf{v}(\varphi'_i) - k_i| = \min_j |\mathbf{v}(\varphi'_i) - k_j|$. By Lemma 3.2 (ii) there is one and only one way to do so, and this being done we have $\mathbf{v}(\varphi'_1) < \mathbf{v}(\varphi'_2) < \dots < \mathbf{v}(\varphi'_m)$ (strict inequalities). Then consider a transitive permutation σ of $\{1, \dots, m\}$. We choose the refinement

$$\mathcal{R} = (\varphi'_{\sigma(1)}, \dots, \varphi'_{\sigma(m)}).$$

Since π_v is a full unramified principal series by Lemma 3.2 (i), all the refinements of π_v are accessible, so π together with \mathcal{R} defines a point z_0 .

Before proving the irreducibility property of the lemma, let us observe a combinatorial property of this refinement. We have by definition $\kappa_i(z_0) = k_i$ and $s_i(z_0) = \mathbf{v}(\varphi'_{\sigma(i)}) - k_i$. We claim that for any non-empty proper subset $I \subset \{1, \dots, m\}$,

$$(5) \quad \sum_{i \in I} s_i(z_0) \neq 0.$$

Indeed, we compute

$$\begin{aligned} \left| \sum_{i \in I} s_i(z_0) \right| &= \left| \sum_{j \in J} \mathbf{v}(\varphi'_j) - \sum_{i \in I} k_i \right| \text{ where } J = \sigma(I) \\ &= \left| \left(\sum_{j \in J} k_j - \sum_{i \in I} k_i \right) + \sum_{j \in J} (\mathbf{v}(\varphi'_j) - k_j) \right| \\ &\geq \left| \sum_{j \in J} k_j - \sum_{i \in I} k_i \right| - \sum_{j \in J} |\mathbf{v}(\varphi'_j) - k_j| \\ &> mc - mc \text{ by Lemma 3.2(ii)} \\ &= 0. \end{aligned}$$

Let us choose now some affinoid neighborhood Ω of $z_0 \in X$ on which the s_i are constant and in which Z_C is Zariski-dense for $C = \sum_{i=1}^m |s_i(z_0)|$. We claim that for every point z of $Z_C \cap \Omega$, $\rho_z|G_w$ is irreducible. Indeed if it was not, it would have a sub-representation of dimension $0 < r < m$, and by the weak admissibility of $D_{\text{crys}}(\rho_z|G_w)$ there would exist a subset $I \subset \{1, \dots, m\}$ of cardinal r , and a subset $J \subset \{1, \dots, m\} \times \text{Hom}(K_w, \overline{\mathbb{Q}}_p)$ with $|J| = rd$, such that

$$\sum_{i \in I} (\kappa_i(z) + s_i(z)) = \kappa_J(z).$$

(here we use that $z \in Z^{\text{reg}}$ and property (d') of eigenvarieties.) Since $z \in Z_C$, we see at once that $I \times \text{Hom}(K_w, \overline{\mathbb{Q}}_p) = J$. But this implies

$$0 = \sum_{i \in I} s_i(z),$$

a contradiction with (5) as $s_i(z) = s_i(z_0)$ for all i . \square

3.1.7. *End of the proof.* Let $\Omega \subset X$ be the neighborhood defined above of the point z_0 , and let A be a complete discrete valuation ring, with a map of $\text{Spec } A$ to the spectrum of the rigid local ring \mathcal{O}_{z_0} of X at z_0 which sends the special point of $\text{Spec } A$ to z_0 and the generic point to the generic point of any irreducible component of Ω containing z_0 . Let us call L the fraction field of A and m its maximal ideal. By pulling back the pseudocharacter T over A , we get a representation $\rho : G_K \rightarrow \text{GL}_m(A)$ such that $\rho \otimes L$ is absolutely irreducible and satisfies (1) (for $\chi = \mathbb{Q}_p(m-1)$) and

$$\bar{\rho}^{\text{ss}} = \begin{cases} \rho_{\Pi, \lambda} & \text{if } m = n, \\ \rho_{\Pi, \lambda}(\mu|\cdot|^{-1/2}) \oplus (\chi|\cdot|^{(1-m)/2}) & \text{if } m = n + 1. \end{cases}$$

Since $\rho \otimes L$ is absolutely irreducible and satisfies (1) it has a sign that can only be $+1$. Hence it is good, and so is $\bar{\rho}^{\text{ss}}$ by Prop. 2.4, hence $\rho_{\Pi, \lambda}$ by Lemmas 2.2 and 2.1, QED.

3.2. **Weakening of the hypothesis (H3), removal of (H2).** We consider the following hypothesis which is a weakening of (H3)

- (H4) There is a place v above p in F that splits in K , and for w a place of K above v , Π_w has a non-zero vector invariant by a Iwahori subgroup of $\text{GL}_n(K_w)$.

Theorem 3.4. *With the supplementary hypotheses (H1), and (H4), Theorem 1.2 holds.*

We shall argue by induction on $n \geq 1$, there is nothing to show if $n = 1$.

Let $U(n)$ be the n -variables unitary group over F attached to K/F that is quasi-split at every finite place and compact at every infinite place. Hasse's principle shows that this group exists, even if n is even, by condition (H1c) (see e.g. [H, Lemma 3.1]). Moreover, condition (H1) and Labesse's base change theorem also ensures that Π descends to an automorphic representation π for $U(n)$. Again, π is unramified at non split finite places of K/F and for each complex place w of K above a real place v of F , π_v has the same infinitesimal character as Π_w (under the natural identification $U(n)(K_w) \simeq \text{GL}_n(\mathbb{C})$).

Let $U = \prod_v U_v \subset U(n)(\mathbb{A}_{F,f})$ be a compact open subgroup such that $\pi^U \neq 0$, and assume that $U_v = I_v$ for the place v of (H4), and that U_v is hyperspecial for each place v of K that is not split over F . Let X be the eigenvariety attached to U , v and ι_∞, ι_p . Of course, all what we said for the eigenvarieties of $U(m)$ with m odd also applies verbatim to this X by replacing the letter m by letter n , and we shall not repeat it.

Let z_0 be the point corresponding to π together with some accessible refinement of π_v . As a general fact, there is always such a refinement and we choose it anyhow here.

Let c be the maximum of the $|s_i(z_0)|$ and $C > nc$. Let $\Omega \subset X$ be an open affinoid of X containing z_0 , in which Z_C is Zariski-dense, and over which the s_i are constant. We claim that for $z \in Z_C$, ρ_z is good. Indeed, let $\Pi(z)$ be Labesse's base change of $\pi(z)$ to $\text{GL}_n(\mathbb{A}_K)$. As $z \in Z^{\text{reg}}$, and as explained in §3.1.5, there exists

a decomposition $n = n_1 + \cdots + n_r$ and cuspidal automorphic representations Π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_K)$, satisfying (H1b), (H2) and unramified at v , such that

$$\rho_z = \bigoplus_{i=1}^r \rho_{\Pi_i, \lambda} \otimes \chi_i,$$

for some characters $\chi_i : G_K \rightarrow \overline{\mathbb{Q}_p}^*$. If $r > 1$, then ρ_z is good by induction and Lemma 2.1. If $r = 1$, then $\Pi(z)$ is cuspidal and it satisfies (H2) and (H3) by construction, so ρ_z is good by Theorem 3.1. (To check (H3), remark that for such a z , and for each $i \in \{1, \dots, n\}$, we have $\mathrm{Min}_j(s_i(z) + k_i(z) - k_j(z)) = s_i(z) = s_i(z_0)$.)

Let W be any irreducible component of Ω containing z_0 , and $\mathrm{Frac}(W)$ its associated function field. As Z_C is Zariski-dense in Ω we may find a $z \in Z_C \cap W$ such that the pseudocharacters T_z and $T \otimes_{\mathcal{O}(\Omega)} \mathrm{Frac}(W)$ have the same number of irreducible factors. Such factors are necessarily absolutely irreducible here, by [BCh, Thm. 1.4.4 (iii)]. Arguing as in the preceding section, let A be a complete discrete valuation ring with a map of $\mathrm{Spec} A$ to $\mathrm{Spec} \mathcal{O}_z$ which sends the special point of $\mathrm{Spec} A$ to z and its generic point to $\mathrm{Frac}(W)$, and let $\rho : G_K \rightarrow \mathrm{GL}_n(A)$ be a representation with trace T such that $\rho \otimes_A L$ is a direct sum of absolutely irreducible, distinct, representations (use e.g. [BCh, Prop. 1.6.1]). As we saw, $\bar{\rho} = \rho_z$ is good, hence so is $\rho \otimes_A L$ by Prop.2.4, as well as $\rho \otimes_A \mathrm{Frac}(W)$ for any irreducible component W containing z_0 . But arguing back now at the point z_0 as in the preceding section, we obtain that $\rho_{z_0} = \rho_{\Pi, \lambda}$ itself is good as a specialization of a good representation, and we are done.

3.3. Removal of Hypotheses (H1) and (H4). We now prove Theorem 1.2.

Lemma 3.5. *Let L be a finite extension of \mathbb{Q}_p and $\rho : G_K \rightarrow \mathrm{GL}_n(L)$ a continuous representation which is a direct sum of absolutely irreducible representations. There is a finite Galois extension M/K such that for every finite extension K'/K linearly disjoint from M , ρ and $\rho|_{G_{K'}}$ have the same number of irreducible factors, and the irreducible factors of $\rho|_{G_{K'}}$ are absolutely irreducible.*

Proof — We can assume without loss of generality that ρ is absolutely irreducible. In particular, there exists n^2 elements g_1, \dots, g_{n^2} such that the $\rho(g_i)$'s generate $M_n(L)$ as a L -vector space. Since G_K has a basis of neighbourhoods of 1 that are open normal subgroups, and since ρ and the determinant are continuous, there is an open normal subgroup U of G_K such that if for all $i = 1, \dots, n^2$, $g'_i \in g_i U$, then the $\rho(g'_i)$'s still generate $M_n(L)$. Set $M = \bar{K}^U$, so M is a finite Galois extension of K .

If K' is a finite extension of K which is linearly disjoint from M , so is its Galois closure. Hence we may assume that K' is Galois over K . Thus $\mathrm{Gal}(K'M/K')$ is naturally isomorphic to $\mathrm{Gal}(K/M)$. For every i , choose g'_i in $G_{K'}$ whose image in $\mathrm{Gal}(K'M/K')$ is sent to g_i by the above isomorphism. This implies that $g'_i g_i^{-1} \in U$, hence the $\rho|_{G_{K'}}(g'_i)$'s generate $M_n(L)$, and $\rho|_{G_{K'}}$ is absolutely irreducible. \square

By [H, Prop. 4.1.1 and Thm 4.2.2] for any finite extension M/K there exists a totally real solvable Galois extension F'/F such that $K' = KF'$ is linearly disjoint

to M and such that Arthur-Clozel's base change $\Pi_{K'}$ and K'/F' satisfy hypotheses (H1) and (H4). We apply this to an M as in the lemma above. By Theorem 3.4 we know that $(\rho_{\Pi,\lambda})|_{G'_K}$ is good, and by the lemma above, that it has the same number of irreducible components as $\rho_{\Pi,\Lambda}$.

Hence by Lemma 2.3 $\rho_{\Pi,\lambda}$ is good, QED.

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