PSEUDODEFORMATIONS

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Mazur's deformation theory of representations of (Galois) groups played a very important role in the striking progresses that have been made in algebraic number theory in the last fifteen years (to name a few: the proof of the Taniyama-Weil conjecture, of the Serre's conjecture on the modularity of odd mod p Galois representation, and of the Sato-Tate conjecture).

The theory works as its best only when the representation $\rho : G \to \mathrm{GL}_n(\kappa)$ to be deformed is absolutely irreducible over the field κ. In this case, under some finiteness hypotheses on the group G that are satisfied in applications to Galois theory, the functor of all deformations of $\bar{\rho}$ on artinian local rings with residue field κ is pro-representable by a complete noetherian local ring R (see [M]). The tangent space (more precisely, the equal characteristic tangent space, that is the dual of $\mathfrak{m}_R/(\mathfrak{m}_R^2, \mathrm{char}\kappa)$ if $\mathfrak{m}_R$ is the maximal ideal of R) of this local ring is identified with the space $\mathrm{Ext}^1_G(\rho,\rho) = H^1(G, \mathrm{ad}\,\rho)$. Lifting questions are also solved by the construction of obstruction classes in $\mathrm{Ext}^2_G(\rho,\rho) = H^2(G, \mathrm{ad}\,\rho)$. The same results hold ore generally if, and only if, $\mathrm{End}_\kappa(\rho,\rho)$ has dimension one.

So, if ρ is not indecomposable, for example semi-simple but not irreducible, its functor of deformations is not representable. This is a serious difficulty when one tries to extend the wonderful results (such as Wiles' modularity lifting theorem and its subsequent generalisations) using Mazur's deformation theory. To overcome this difficulty, the method mainly used so far has been to replace ρ by an indecompsable representation ρ' with the same semi-simplification, and to study the deformations of ρ'. But this method has an important drawback : no such ρ' may exist, and when it does, the choice of ρ' is by no means canonical. For example, when ρ is the sum of two absolutely irreducible representations ρ_1 and ρ_2, finding an indecomposable ρ' with semi-simplification ρ amounts to choose a line either in $\mathrm{Ext}^1(\rho_1,\rho_2)$ or in $\mathrm{Ext}^1(\rho_2,\rho_1)$. This is possible only when at least one of those spaces is non zero, and leads to many possible choice of ρ' in general. Moreover, the deformation spaces of those ρ' are different, and difficult to compare. When ρ has more than two components, the existence of an indecomposable ρ' to which one could apply Mazur's theory becomes very problematic.

To avoid this drawback, it is natural to replace representations by *pseudocharacter*. Pseudocharacters of a group G over a commutative ring A are A-valued functions on G that behave like characters of representation of finite-dimensional

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representations of G over A. Thus, when A is a field (say of characteristic 0), pseudocharacters generalise the equivalence classes of semi-simple representations of G over A. Since isomorphic for representations corresponds to equality of characters, pseudocharacters are much easier to glue or to put in families. This is why Taylor introduced them in [T], building on earlier work by Procesi and Wiles (though he used the name pseudorepresentations). For the same reason, pseudocharacters have trivially a pro-representable deformation functor (though the representing complete local ring R may not be noetherian), a fact that has been noticed by Rouquier and, combined with his proof that any residually absolutely irreducible pseudocharacter comes from a representation (result proven independently by Nissen [N]), used to give a simpler proof of Mazur's representability result of the functor of deformation of an absolutely irreducible representation – cf [R].

So, if ρ is a sum of several absolutely irreducible representations, say $\rho \simeq \rho_1 \oplus \cdots \oplus \rho_r$ it is a natural idea to study the deformation ring R of the pseudocharacter $\operatorname{tr} \rho$, which is canonical, rather than the deformation ring of some non-canonical ρ'. This is the object of this article.

It is natural to call "pseudodeformation" a deformation of ρ as a pseudocharacter. This term was coined by Skinner and Wiles (in [SW, §2.4]), who were the first, and to the best of my knowledge, the only ones so far to study these objects. Their main result on this subject is an upper bound of the dimension of the tangent space of the pseudodeformation functor when ρ has dimension 2 and is the sum of two distinct characters, which, though far from being optimal as we shall see, is sufficient to prove the finiteness of this tangent space. We shall precise their result by computing exactly the tangent space in the case $r = 2$, and partly generalise this to the case of arbitrary r. Let us precise that Skinner and Wiles use pseudodeformations only tangentially in their paper [SW], using the trick of choosing an indecomposable proxy ρ' instead during most of the paper. It is the author's belief, which has still to be substantiated, that the systematic use of pseudodeformations can lead to important simplification of the argument of [SW] as well as to higher dimensional generalisation.

Let us now briefly explain our results: we assume throughout the paper that the ρ_i are pair-wise non isomorphic, in order to be able to apply the results on residually multiplicity-free pseudocharacters of [BC].

The first result is the description of a natural filtration, called the *complexity* filtration, on the tangent space $\mathfrak{t}_\rho$ of the deformation ring of $\operatorname{tr} \rho$. This filtration has length r, the number of irreducible components in ρ. We study the graded spaces of this filtration. The first one, $\mathfrak{t}_1$ corresponds to totally reducible deformations of ρ, and thus is easily described as the sum of the Mazur's tangent space of the irreducible components $\rho_1, \dots, \rho_r$ of ρ. We embed the other graded spaces $\mathfrak{t}_n/\mathfrak{t}_{n-1}$ into the direct sum over all cycles γ of length n in the set $\{1, \dots, r\}$ of the tensor product of the spaces $\operatorname{Ext}^1(\rho_\gamma(i+1), \rho_\gamma(i))$ for $i = 0, \dots, n-1$. Actually, we show that $\mathfrak{t}_n/\mathfrak{t}_{n-1}$ is even in a subspace of the latter defined by cancellation conditions on some Yoneda products. See Theorem 1 for a precise statement. This results implies the finiteness of the dimension of the tangent space of $\mathfrak{t}$ when the Ext^1 between the different components of ρ are finite dimensional. This holds in particular in the standard Galois situations (local or global), so in this case the universal ring is noetherian. We are unfortunately not able to determine precisely the image of $\mathfrak{t}_n/\mathfrak{t}_{n-1}$ in general.

We have more precise results for $n = 2$; they are explained in section 4. In this case we show that $\mathfrak{t}_2/\mathfrak{t}_1$ is exactly the kernel of the natural map

$$\oplus_{1\leq i<j\leq r}\mathrm{Ext}^1_G(\bar\rho_j,\bar\rho_i)\otimes\mathrm{Ext}^1_G(\bar\rho_i,\bar\rho_j)\longrightarrow\oplus_{k=1}^r\mathrm{Ext}^2(\bar\rho_k,\bar\rho_k).$$

So we have a complete description of the tangent space $\mathfrak{t}$ when the number of components of ρ is 2. We also partially extend Mazur's obstruction theory, constructing an obstruction that lives in the cokernel of the above map. We give an application to the lifting of Galois representation, in the spirit of [K].

In section 5, in the case $r = 2$, we characterise the pseudodeformations in $\mathfrak{t}$ that come from true representation. They corresponds to pure tensors in $\mathfrak{t}_2/\mathfrak{t}_1$. This explains the benefit we get by working with pseudocharacter instead of true representation : the "tangent space of the deformation functor of the representation ρ" is not a vector space in general.

Note for the reader: The constructions of this article rely heavily on the ones done in the chapter one of [BC]. While we have given precise references to the results of [BC] when we used them, and tried to keep this article self-contained by precisely recalling most of them, we strongly suggest the reader, for a better and easier understanding of tis paper, to read [BC, Chapter 1], at least from §1.2 to §1.4, before beginning this article.

1. Pseudodeformations

The notations below will be in use throughout the paper.

Let κ be a field, $\mathcal{C}$ the category of artinian local rings $(A,\mathfrak{m})$ together with an isomorphism $A/\mathfrak{m}\to\kappa$, G a group (or monoid) and $\bar T : G\to\kappa$ a pseudocharacter such that

$$\bar T=\sum_{i=1}^r \mathrm{tr}\,\bar\rho_i$$

where $\bar\rho_i : G\longrightarrow \mathrm{GL}_{d_i}(\kappa)$, $i = 1,\ldots,r$, are absolutely irreducible and pair-wise non-isomorphic representations. We assume that $d=\sum_{i=1}^r d_i$ is less than the characteristic of κ. Hence d is the unambiguous dimension of the pseudocharacter $\bar T$. We set $I = \{1,\ldots,r\}$

Let $PD = PD_{\bar T} : \mathcal{C}\to\mathcal{SETS}$ be the functor that associates to A the set of pseudocharacters $T : G\to A$ such that $T\otimes\kappa=\bar T$. This functor is clearly representable (see [R]).

We denote by $\kappa[\varepsilon]$ the ring of dual numbers over κ, generated by ε with the relation $\varepsilon^2 = 0$. The set $PD(\kappa[\varepsilon])$ has a structure of κ-vector space (see [M, page 272]). It is denoted t_{PD}.

Topological variant: Assume that κ is a complete valued field of characteristic 0, that G is a topological group, and that T is continuous. Change the definition of PD such that PD(A) is the set of *continuous* pseudocharacters $T : G \to A$ such that $T \otimes \bar{k} = \bar{T}$ – where A has its topology of a finite dimensional κ-vector space.

Then the representations $\bar{\rho}_i$ are continuous and if we agree to understand below the group $\mathrm{Ext}^l_G(\bar{\rho}_i, \bar{\rho}_j)$ as *continuous* extensions spaces, then all the results stated in this article hold in this topological situation. This follows from the same proofs as in the algebraic case, together with [BC, Prop. 1.5.9]. We leave the (easy) details to the reader.

2. The complexity filtration on the pseudodeformation functor

2.1. **Cayley-Hamilton quotients and GMA structure.** Let $A \in \mathcal{C}$, and $T \in PD(A)$ be a pseudocharacter $G \to A$ deforming $\bar{T}$. We refer the reader to [BC, 1.2.5] for the notion of *Cayley-Hamilton* quotient of $(A[G], T)$. Let us simply mention that $A[G]/\mathrm{Ker}\, T$ is an example of a Cayley-Hamilton quotient of $(A[G], T)$.

Let S be a Cayley-Hamilton quotient of $(A[G], T)$. The main result of [BC, Chapter 1] is the fact that S is a Generalised Matrix Algebra or GMA of type $d_1, \dots, d_r$. More precisely [BC, Thm 1.4.4]) constructs an isomorphism of A-algebra

$$S \simeq \begin{pmatrix} M_{d_1}(A_{1,1}) & M_{d_1,d_2}(A_{1,2}) & \dots & M_{d_1,d_r}(A_{1,r}) \\ M_{d_2,d_1}(A_{2,1}) & M_{d_2}(A_{2,2}) & \dots & M_{d_2,d_r}(A_{2,r}) \\ \vdots & \vdots & \ddots & \vdots \\ M_{d_r,d_1}(A_{r,1}) & M_{d_r,d_2}(A_{r,2}) & \dots & M_{d_r}(A_{r,r}) \end{pmatrix} \tag{1}$$

where $A_{i,j}$ are given A-modules of finite type satisfying $A_{i,i} = A$ and the structure of algebra on the right-hand side is defined by matrix multiplication using the morphisms $\phi_{i,j,k} : A_{i,j} \otimes A_{j,k} \to A_{i,k}$ to multiply the coefficients. These morphisms satisfy (cf [BC, 1.3.1]):

(UNIT) for all $i, j \in I$, $\varphi_{i,i,j} : A \otimes A_{i,j} \longrightarrow A_{i,j}$ (resp. $\varphi_{i,j,j} : A_{i,j} \otimes A \longrightarrow A_{i,j}$) is the A-module structure of $A_{i,j}$.

(ASSO) For all $i, j, k, l \in I$, the two natural maps $A_{i,j} \otimes A_{j,k} \otimes A_{k,l} \longrightarrow A_{i,l}$ coincide.

(COM) For all $i, j \in I$ and for all $x \in A_{i,j}$, $y \in A_{j,i}$, we have $\varphi_{i,j,i}(x \otimes y) = \varphi_{j,i,j}(y \otimes x)$.

Moreover, the pairings $\phi_{i,j,i}$ are non degenerate if $S = R/\mathrm{Ker}\, T$.

The composition of the map $G \to A[G] \to S$ with the isomorphism (1) sends an element $g \in G$ to a "generalised matrix"

$$\rho(g) = \begin{pmatrix} a_{1,1}(g) & a_{1,2}(g) & \dots & a_{1,r}(g) \\ a_{2,1}(g) & a_{2,2}(g) & \dots & a_{2,r}(g) \\ \vdots & \vdots & \ddots & \vdots \\ a_{r,1}(g) & a_{r,2}(g) & \dots & a_{r,r}(g) \end{pmatrix} \tag{2}$$

where $A_{i,j}(g) \in M_{d_i,d_j}(A_{i,j})$.

We have by construction, for all $i \in I$ and all $g \in G$

$$a_{i,i}(g) \equiv \rho_i(g) \pmod{\mathfrak{m}} \tag{3}$$

and the isomorphism (1) is compatible with T on the left side and the trace on the right side:

$$T(g) = \sum_{i \in I} \operatorname{tr} a_{i,i}(g). \tag{4}$$

The multiplication law on S reads:

$$a_{i,j}(gg') = \sum_{k=1}^{r} \phi_{i,k,j}(a_{i,k}(g) \otimes a_{k,j}(g')) \tag{5}$$

It is possible (see [BC, Prop. 1.3.13]) to embed all the $A_{i,j}$ in a commutative A-algebra B, in such a way that $A_{i,j}A_{j,k} \subset A_{i,k}$ for any $i, j, k \in I$ and that $\phi_{i,j,k}$ becomes the map induced by the multiplication in B. Hence it is often harmless to drop the $\phi_{i,j,k}$ from the notation. For example the multiplication law can be written

$$a_{i,j}(gg') = \sum_{k=1}^{r} a_{i,k}(g)a_{k,j}(g') \tag{6}$$

2.2. **Some graph-theoretical terminology and notations.** For $n \geq 1$ an integer, a *path of length* n is an application $\gamma : \{0, \dots, n\} \to \{1, \dots, r\}$ – that is, a path in the graph with sets of vertices I and set of edges I^2. If γ is a path of length $n \geq 2$, a *loop* is a pair $(m, m+1)$ of consecutive integers in I such that $\gamma(m) = \gamma(m+1)$. If γ and γ' are two paths of length n and m, such that $\gamma(n) = \gamma'(0)$, we define the *concatenation* $\gamma\gamma'$ as the path of length $n + m$ such that $(\gamma\gamma')(k)$ is $\gamma(k)$ if $k \leq n$ and $\gamma'(k - m)$ if $k \geq m$.

A *closed path* of length n is a path γ of length n such that $\gamma(0) = \gamma(n)$. For $i \in I$, a *closed i-path* of length n is a path γ of length n such that $\gamma(0) = \gamma(n) = i$. If γ is a closed path of length n, we denote by $\gamma[k]$ the closed path of length n : $\gamma[k](m) = \gamma(m+k \pmod n)$. The relation among closed paths $\gamma \simeq \gamma'$ if and only if $\gamma = \gamma'[k]$ for some k is an equivalence relation; we call its classes *cycles* and denote by $[\gamma]$ the cycle to which γ belongs.

A closed path γ of length n is said *simple* if γ is injective on $\{0, \dots, n-1\}$. In particular, a simple closed path has no loop. We denote by $\mathrm{SCP}(n)$ the set of simple closed path of length n. Of course γ is simple if and only if $\gamma[k]$ is simple, so it makes sense to speak of a *simple cycle*. We denote by $\mathrm{SC}(n)$ the set of all cycles of length n.

2.3. **Paths and GMA.** Let A, T and S be as in §2.1.

If γ is a path of length n, we set

$$A_\gamma := A_{\gamma(0),\gamma(1)} \dots A_{\gamma(n-1),\gamma(n)}.$$

We also define a function of n variables $g_1, \dots, g_n$ in G by

$$a_\gamma(g_1, \dots, g_n) = a_{\gamma(0),\gamma(1)}(g_1) \dots a_{\gamma(n-1),\gamma(n)}(g_n) \in M_{d_{\gamma(0)},d_{\gamma(n)}}(A_{\gamma(0)}, A_{\gamma(n)}).$$

Lemma 1. *For any closed path γ the ideal $A_\gamma \subset A$ depends only on (A, T) and not on the Cayley-Hamilton quotient S.*

Proof — Call e_i the idempotent of S corresponding to the identity of the i-th diagonal block $M_{d_i,d_i}(A)$ through the isomorphism 1. A simple application of (3) and (4) shows that we have, if γ is a closed path,

$$A_\gamma = T(e_{\gamma(0)}Se_{\gamma(1)}\dots Se_{\gamma(n)}). \tag{7}$$

The right hand side is obviously unchanged if one changes S into $S/\mathrm{Ker}\,T$. Since $S/\mathrm{Ker}\,T = R/\mathrm{Ker}\,T$ is independent of S, the result follows. □

We also note:

Lemma 2. *The modules A_γ enjoy the following properties:*

(i) *If γ and γ' are concatenable, $A_{\gamma\gamma'} = A_\gamma A'_\gamma$.*
(ii) *for any γ and k, $A_{\gamma[k]} = A_\gamma$.*

Proof — Assertion (i) is clear, while assertion (ii) follows form equation (7) and the fact that T is central. □

2.4. **The complexity of a deformation.** Let $A \in \mathcal{C}$, $T \in PD(A)$. Lemma 1 ensures that the following definition makes sense.

Definition 1. For $n \geq 1$ an integer, we say that T has *complexity* less or equal to n if for any simple closed path of length greater than n we have $A_\gamma = 0$.

We say that T has complexity n if it has complesity less or equal to n but not to $n-1$. Since there is no simple closed path of length $r+1$ or more, every T has a unique complexity $n \in \{1,\dots,r\}$.

Definition 2. For $n = 1,\dots,r$, we denote by PD_n the sub-functor of PD such that for all $A \in \mathcal{C}$, $PD_n(A)$ is the set of $T \in PD(A)$ that have complexity less or equal to n.

Proposition 1. *For every n, PD_n is representable by a local quotient of the ring A_{univ} representing PD.*

Proof — If $T \in PD(A)$, then $T \in PD_n(A)$ if and only if $A_\gamma = 0$ for every closed simple path γ of length greater than n. As follows from (7) the formation of A_γ commutes to any base change of the base ring A in the sense that for any map $A \to A' \in \mathcal{C}$, the corresponding module A'_γ for the pseudocharacter $T \otimes_A A'$ is $A_\gamma = A_\gamma A'$. Hence it is clear that PD_n is represented by A_{univ}/J_n where

$$J_n = \sum_{\gamma \text{ simple closed path of length } > n} (A_{\mathrm{univ}})_\gamma.$$

□

We have $PD_r = PD$. On the other hand PD_1 is the functor that classifies the "totally reducible" deformations, in the sense that $T \in PD(A)$ if and only if T is the sum of r non-trivial pseudocharacters.

3. The complexity filtration on the tangent space

Since PD_n is a representable sub-functor of PD, its tangent space is naturally a subspace $\mathfrak{t}_{D,n}$ of $\mathfrak{t}_D$. Setting $\mathfrak{t}_{D,0} = (0)$, we thus get a filtration, *the complexity filtration* on the tangent space of PD:

$$(0) = \mathfrak{t}_{D,0} \subset \mathfrak{t}_{D,1} \subset \mathfrak{t}_{D,2} \subset \cdots \subset \mathfrak{t}_{D,r} = \mathfrak{t}_D.$$

The aim of this section is to study this filtration. We state the main result of this paper:

Theorem 1. *For every $n = 1, \dots, r$, there exist natural linear applications f_n and h_n*

$$\begin{array}{c} \mathfrak{t}_{D,n}/\mathfrak{t}_{D,n-1} \\ \big\downarrow f_n \\ \bigoplus_{[\gamma]\in SC(n)} \left(\bigotimes_{m=1}^{n} Ext^1_G(\bar\rho_{\gamma(m)}, \bar\rho_{\gamma(m-1)})\right) \\ \big\downarrow h_n \\ \bigoplus_{\gamma\in SCP(n-1)} \left(Ext^2_G(\bar\rho_{\gamma(1)}, \bar\rho_{\gamma(0)}) \otimes \bigotimes_{m=2}^{n-1} Ext^1_G(\bar\rho_{\gamma(m)}, \bar\rho_{\gamma(m-1)})\right) \end{array}$$

such that f_n is injective and $h_n \circ f_n = 0$.

The map h_n is given by the Yoneda product: see §3.2. The map f_n is constructed in §3.3.

We note the obvious corollary:

Corollary 1. *If for all $i, j \in I$, $i \neq j$, we have $\dim_\kappa Ext^1_G(\bar\rho_i, \bar\rho_j) < +\infty$, then $\mathfrak{t}$ is finite dimensional and PD is pro-represented by a noetherian complete local ring.*

The corollary holds in particular in the topological variant if G the absolute Galois group of a local field, or the Galois group of the maximal extension of a global field unramified outside a finite set of primes.

3.1. **A brief reminder on Ext^1 and Ext^2-space.** Let $\bar\rho : G \to \mathrm{GL}_d(\kappa)$ and $\bar\rho' : G \to \mathrm{GL}_{d'}(\kappa)$ be two representations of G.

We define four κ-vector spaces

$$\begin{aligned} Z^1_G(\bar\rho', \bar\rho) &= \{a : G \to M_{d,d'}(\kappa), \forall g, g' \in G, a(gg') = \bar\rho(g)a(g') + a(g)\bar\rho'(g')\} \\ B^1_G(\bar\rho', \bar\rho) &= \{a : G \to M_{d,d'}(\kappa), \exists A \in M_{d,d'}(\kappa), \forall g \in G, a(g) = \bar\rho(g)A + A\bar\rho'(g)\} \\ Z^2_G(\bar\rho', \bar\rho) &= \{b : G \times G \to M_{d,d'}(\kappa), \forall g, g', g'' \in G, \\ &\qquad \bar\rho(g)b(g', g'') - b(gg', g'') + b(g, g'g'') - b(g, g')\bar\rho'(g'') = 0\} \\ B^2_G(\bar\rho', \bar\rho) &= \{b : G \times G \to M_{d,d'}(\kappa), \exists a : G \to M_{d,d'}(\kappa), \forall g, g' \in G, \\ &\qquad b(g, g') = a(g, g') - \bar\rho(g)a(g') - a(g)\bar\rho(g')\} \end{aligned}$$

Simple computations show that $B^l_G(\bar\rho', \bar\rho) \subset Z^l_G(\bar\rho', \bar\rho)$ for $l = 1, 2$. We define $\mathrm{Ext}^l_G(\bar\rho', \bar\rho)$ as $Z^l_G(\bar\rho', \bar\rho)/B^l_G(\bar\rho', \bar\rho)$.

If $\rho'' : G \to \mathrm{GL}_{d''}(\kappa)$ is another representation, it is not hard to check that the multiplication map $(a, a') \mapsto b$ with $b(g, g') = a(g)a'(g')$ sends $Z^1_G(\bar\rho', \bar\rho) \times Z^1_G(\bar\rho'', \bar\rho')$ to $Z^2_G(\bar\rho'', \bar\rho)$, and also that it sends $Z^1_G(\bar\rho', \bar\rho) \times B^1_G(\bar\rho'', \bar\rho') + B^1_G(\bar\rho', \bar\rho) \times Z^1_G(\bar\rho'', \bar\rho')$ in $B^2_G(\bar\rho'', \bar\rho)$. Hence this maps induce a bilinear map $\mathrm{Ext}^1_G(\bar\rho', \bar\rho) \times \mathrm{Ext}^1_G(\bar\rho'', \bar\rho') \to \mathrm{Ext}^2_G(\bar\rho'', \bar\rho)$. This is the well-known *Yoneda product*.

3.2. **Constinsertingruction of h_n.** We can now make explicit the map h_n of the theorem. It is defined as the sum, over all simple closed paths $\gamma \in SCP(n-1)$ and all simple cycles $[\gamma'] \in SC(n)$ of maps

$$h_{n,[\gamma'],\gamma} : \bigotimes_{m=1}^{n} \mathrm{Ext}^1_G(\bar\rho_{\gamma'(m)}, \bar\rho_{\gamma'(m-1)}) \longrightarrow \mathrm{Ext}^2_G(\bar\rho_{\gamma(1)}, \bar\rho_{\gamma(0)}) \otimes \bigotimes_{m=2}^{n-1} \mathrm{Ext}^1_G(\bar\rho_{\gamma(m)}, \bar\rho_{\gamma(m-1)})$$

If a element γ' of $[\gamma']$ may be obtained from γ by inserting a new vertices between $\gamma(0)$ and $\gamma(1)$ (explicitly: $\gamma'(0) = \gamma(0)$, $\gamma'(l) = \gamma(l-1)$ for $2 \geq l \geq n$, and $\gamma'(1)$ is any element of I different from $\gamma(0)$ and $\gamma(1)$), then this element γ' is clearly unique, and we define $h_{n,[\gamma'],\gamma}$ as the tensor product of the Yoneda product map $\mathrm{Ext}^1(\bar\rho_{\gamma'(1),\gamma'(0)}) \otimes \mathrm{Ext}^1(\bar\rho_{\gamma'(2)}, \bar\rho_{\gamma'(1)}) \to \mathrm{Ext}^2(\bar\rho_{\gamma(1)}, \bar\rho_{\gamma(0)})$ and the identity maps between the other Ext^1. Otherwise, we set $h_{n,[\gamma'],\gamma} = 0$.

3.3. **Construction of f_n.** In all this §, we shall let $A \in \mathcal{C}$ be a ring such that $\mathfrak{m}^2 = 0$. We will construct a map

$$f_n : PD_n(A) \to \mathfrak{m} \otimes_\kappa \bigoplus_{[\gamma] \in \mathrm{SC}(n)} \left(\bigotimes_{m=1}^{n} \mathrm{Ext}^1_G(\bar\rho_{\gamma(m)}, \bar\rho_{\gamma(m-1)}) \right)$$

In the special case $A = \kappa[\varepsilon]$, we have $\mathfrak{m} \simeq \kappa$ canonically, $\mathrm{PD}_n(A) = \mathfrak{t}_{D,n}$ and this map f_n is the one we need to construct for the theorem 1.

More precisely, we shall define for every $\gamma \in \mathrm{SCP}(n)$ a map

$$f_{n,\gamma} : \mathrm{PD}_n(A) \to \mathfrak{m} \otimes_k \bigotimes_{m=1}^{n} \mathrm{Ext}^1_G(\bar\rho_{\gamma(m)}, \bar\rho_{\gamma(m-1)})$$

We shall also check that $f_{n,\gamma}$ does only depend on the class $[\gamma]$ of γ in $\mathrm{SC}(n)$, if we agree to identify the different arrival spaces of $f_{n,\gamma}$ and $f_{n,\gamma'}$ (with $[\gamma'] = [\gamma]$) using the natural isomorphism expressing the commutativity of tensor product.

From now on, we fix $T \in PD_n(A)$ a pseudocharacter. We set $S = A[G]/\mathrm{Ker}\, T$ and define the structural modules $A_{i,j}$ and morphism $\phi_{i,j,k}$ accordingly.

Lemma 3. *For every $i, j \in I$, $i \neq j$, we have $\mathfrak{m}A_{i,j} = 0$.*

Proof — For every $i \neq j$, $y \in A_{j,i}$ and $x \in \mathfrak{m}A_{i,j}$, say $x = \sum_l m_l x_l$ with $m_l \in \mathfrak{m}$ and $x_l \in A_{i,j}$, we have $\phi_{i,j,i}(x \otimes y) = \sum_l m_l \phi_{i,j,i}(x_l \otimes y) \in \mathfrak{m}\mathfrak{m} = 0$. Since $\phi_{i,j,i}$ is non degenerate, $x = 0$. This proves the lemma. □

Thus we may, and shall, consider the $A_{i,j}$, $i \neq j$ as $\kappa = A/\mathfrak{m}$-vector spaces.

Lemma 4. *Let γ be a closed path without loop of length at least 2. If γ is not simple then $A_\gamma = 0$.*

Proof — Since γ is not simple, say of length $l \geq 2$, there exists $m < m' \in \{0, l-1\}$ such that $\gamma(m) = \gamma(m')$ and since γ has no loop, we actually have $m+1 < m' < l-1$. It follows that $\gamma[m]$ is the concatenation $\gamma_1\gamma_2$ of two closed path of length ≥ 2 (namely of length $m' - m$ and $l - m' + m$). Therefore, using lemma 2

$$A_\gamma = A_{\gamma[m]} = A_{\gamma_1} A_{\gamma_2} \subset \mathfrak{m}^2 = 0.$$

□

To define $f_{n,\gamma}(T)$, we consider the multiplication map

$$\phi_\gamma : A_{\gamma(0),\gamma(1)} \otimes_\kappa \cdots \otimes_\kappa A_{\gamma(n-1),\gamma(n)} \longrightarrow A_\gamma \subset \mathfrak{m}.$$

Lemma 5. *The map ϕ_γ factors through*

$$\phi_\gamma : (A_{\gamma(0),\gamma(1)}/A'_{\gamma(0),\gamma(1)}) \otimes_\kappa \cdots \otimes_\kappa (A_{\gamma(n-1),\gamma(n)}/A'_{\gamma(n-1),\gamma(n)}) \longrightarrow A_\gamma \subset \mathfrak{m},$$

where as in [BC, 1.5.3] *we have set $A'_{i,j} = \sum_{k \in I - \{i,j\}} A_{i,k} A_{k,j}$ for all $i \neq j \in I$.*

Proof — We have to check that the multiplication map ϕ_γ is 0 on each subspace

$$A_{\gamma(0),\gamma(1)} \otimes_\kappa \cdots \otimes_\kappa A'_{\gamma(m),\gamma(m+1)} \cdots \otimes_\kappa A_{\gamma(n-1),\gamma(n)}$$

of

$$A_{\gamma(0),\gamma(1)} \otimes_\kappa \cdots \otimes_\kappa A_{\gamma(n-1),\gamma(n)}$$

(that is to say is the subspace where, for one $m \in \{0, \ldots, n-1\}$, the factor $A_{\gamma(m),\gamma(m+1)}$ is replaced by its subspace $A'_{\gamma(m),\gamma(m+1)}$.) Now, by definition of $A'_{\gamma(m),\gamma(m+1)}$, the image of this subspace by ϕ_γ is the sum of the ideals $A_{\gamma'}$ over all closed paths γ' of length $n+1$ which are obtained from γ by inserting a term k different from $\gamma(m)$ and $\gamma(m+1)$ between $\gamma(m)$ and $\gamma(m+1)$. It is clear that such paths γ' have no loop. We then show that those $A_{\gamma'}$ are all 0. We distinguish two cases, according to whether k is in the image of γ or not. If it is, then the new closed path γ' is a *non-simple* closed path with no loop, so $A_{\gamma'} = 0$ by lemma 4. If it is not, then γ is a simple closed path of length $n+1$, and this $A_{\gamma'}$ is zero by definition since T is of complexity less or equal to n. □

By duality, the map ϕ_γ given by the lemma may canonically be seen as an element of

$$\mathfrak{m} \otimes (A_{\gamma(0),\gamma(1)}/A'_{\gamma(0),\gamma(1)})^* \otimes_\kappa \cdots \otimes_\kappa (A_{\gamma(n-1),\gamma(n)}/A'_{\gamma(n-1),\gamma(n)})^*.$$

Recall from [BC, Thm 1.5.5] the existence of a canonical injective map

$$\iota_{i,j} : (A_{i,j}/A'_{i,j})^* \hookrightarrow \mathrm{Ext}^1(\bar{\rho}_j, \bar{\rho}_i)$$

for all $i, j \in I$, $i \neq j$.

We thus may see the element ϕ_γ as an element of $\mathrm{Ext}^1_G(\bar{\rho}_{\gamma(1)}, \bar{\rho}_{\gamma(0)}) \otimes \cdots \otimes \mathrm{Ext}^1(\bar{\rho}_{\gamma(n)}, \bar{\rho}_{\gamma(n-1)})$. We define $f_{n,\gamma}(T)$ as this element. It is now clear from property (SYM) that $f_{n,\gamma} = f_{n,\gamma[k]}$ for any k, so $f_{n,\gamma}$ depends only of the underlying cycle γ.

3.4. **Determination of $f_n^{-1}(0)$.** We continue to assume that $A \in \mathcal{C}$ is such that $\mathfrak{m}^2 = 0$ and we keep the same notations as above.

By construction $f_n(T) = 0$ is equivalent to the fact that for all $[\gamma] \in \mathrm{SC}(n)$, the multiplication map ϕ_γ is 0. This is equivalent to the fact that $A_\gamma = 0$ for all $\gamma \in \mathrm{SC}(n)$. But this is the definition of $T \in PD_{n-1}(A)$. So $f_n^{-1}(0) = \mathrm{PD}_{n-1}(A)$.

When $A = \kappa[\varepsilon]$ we thus find that $f_n^{-1}(0) = \mathfrak{t}_{D,n-1}$. When we know that f_n is linear (see below), this will imply that f_n may be seen as an *injective* map

$$\mathfrak{t}_{D,n}/\mathfrak{t}_{D,n-1} \longrightarrow \bigotimes_{m=1}^{n} \mathrm{Ext}^1_G(\bar\rho_{\gamma'(m)}, \bar\rho_{\gamma'(m-1)}).$$

3.5. **Proof that $h_n \circ f_n = 0$.** We still assume that $A \in \mathcal{C}$ is such that $\mathfrak{m}^2 = 0$ and we keep the same notations as above. We aim to prove that $h_n(f_n(T)) = 0$. By definition of the morphisms h_n and f_n, we simply have to prove that for each fixed $\gamma \in \mathrm{SCP}(n-1)$, we have

$$s := \sum_{[\gamma'] \in \mathrm{SC}(n)} h_{n,[\gamma'],\gamma}(f_{n,[\gamma']}(T)) = 0.$$

The only $[\gamma']$ that contribute to the sum are those where γ' is obtained from γ by inserting some vertex between $\gamma(0)$ and $\gamma(1)$. To simplify notations, set $\gamma(0) = i$, $\gamma(1) = j$.

Let l be any linear form on

$$\mathrm{Ext}^1_G(\bar\rho_{\gamma(3)}, \bar\rho_{\gamma(2)}) \otimes \cdots \otimes \mathrm{Ext}^1(\bar\rho_{\gamma(n-1)}, \bar\rho_{\gamma(n-2)}).$$

Obviously it is enough to prove that $(\mathrm{Id}_\mathfrak{m} \otimes \mathrm{Id}_{\mathrm{Ext}^2_G(\bar\rho_i,\bar\rho_j)} \otimes l)(s) \in \mathfrak{m} \otimes \mathrm{Ext}^2_G(\bar\rho_i, \bar\rho_j)$ is 0 for all such l.

The map

$$l \circ (\iota_{\gamma(2),\gamma(3)} \otimes \cdots \otimes \iota_{\gamma(n-2),\gamma(n-1)})$$

is a linear form on

$$(A_{\gamma(2),\gamma(3)}/A'_{\gamma(2),\gamma(3)})^* \otimes_\kappa \cdots \otimes_\kappa (A_{\gamma(n-1),\gamma(n-2)}/A'_{\gamma(n-1),\gamma(n-2)})^*,$$

that is, by biduality an element of

$$(A_{\gamma(2),\gamma(3)}/A'_{\gamma(2),\gamma(3)}) \otimes_\kappa \cdots \otimes_\kappa (A_{\gamma(n-2),\gamma(n-1)}/A'_{\gamma(n-2),\gamma(n-1)}).$$

We choose an element of $A_{\gamma(2),\gamma(3)} \otimes \cdots \otimes A_{\gamma(n-2),\gamma(n-1)}$ that lift l and call x its image in $A_{\gamma(2),\gamma(n-1)} = A_{j,i}$.

Lemma 6. *The function $b(g,g') := \sum_{k \neq i,j} a_{i,k}(g) a_{k,j}(g') x$ is independent of the chosen lift x of l, lies in $\mathfrak{m} \otimes Z^2_G(\bar\rho_j, \bar\rho_i)$, and its image in $\mathfrak{m} \otimes Ext^2_G(\bar\rho_j, \bar\rho_i)$ is $(Id_\mathfrak{m} \otimes Id_{Ext^2_G(\bar\rho_i,\bar\rho_j)} \otimes l)(s)$*

Proof — The independence of $b(g,g')$ on x follows from the same argument as in the proof of Lemma 5. For every g, g', $\sum_{k \neq i,j} a_{i,k}(g) a_{k,j}(g')$ is in $M_{d_i,d_j}(A_{i,j})$, so $\sum_{k \neq i,j} a_{i,k}(g) a_{k,j}(g') x$ is in $M_{d_i,d_j}(\mathfrak{m})$. Since the multiplication by x commutes with $\bar\rho'(g')$, we see easily that $b(g,g') \in \mathfrak{m} \otimes Z^2_G(\bar\rho_j, \bar\rho_i)$. Finally, the identification of the class of b as $(\mathrm{Id} \otimes l)(s)$ is formal using the definition of ι, x, f_n and h_n. □

But actually

Lemma 7. *The cocycle $b(g,g')$ is in $B^2_G(\bar\rho_j,\bar\rho_i)$.*

Proof — Define $a(g) = -a_{i,j}(g)x \in M_{d_i,d_j}(\mathfrak{m})$. Then the coboundary of a, $\delta a(g,g')$ satisfies

$$\begin{aligned}
\delta a(g,g') &= -\bar\rho(g)a_{i,j}(g')x + a_{i,j}(gg')x - a_{i,j}(g)x\bar\rho(g') \\
&= -\bar\rho(g)a_{i,j}(g')x + \sum_{k\in I} a_{i,k}(g)a_{k,j}(g')x - a_{i,j}(g)\bar\rho(g')x \quad \text{using (6)} \\
&= -\bar\rho(g)a_{i,j}(g')x + a_{i,i}(g)a_{i,j}(g')x \\
&\quad + \sum_{k\neq i,j} a_{i,k}(g)a_{k,j}(g')x + a_{i,j}(g)a_{j,j}(g')x - a_{i,j}(g)\bar\rho(g') \\
&= \sum_{k\in I} a_{i,k}(g)a_{k,j}(g')x \quad \text{using } \bar\rho(g) \equiv a_{i,i}(g) \pmod{\mathfrak{m}} \text{ and Lemma 3} \\
&= b(g,g').
\end{aligned}$$

□

The two lemmas together proves that $h_n(f_n(T)) = 0$.

3.6. **Linearity of f_n.** We now assume that $A = \kappa[\varepsilon]$ as in the theorem. Then $PD_n(A) = \mathfrak{t}_{D,n}$ has a natural structure of κ-vector space. We want to prove that f_n is linear.

Is is not hard to check that f_n is compatible with the multiplication by a scalar $\lambda \in k$. Indeed, if T is a deformation of $\bar T$ to $\kappa[\varepsilon]$, and λT is the pseudocharacter obtained from T by the map $\kappa[\varepsilon] \to \kappa[\varepsilon]$, $\varepsilon \mapsto \lambda\varepsilon$, it is clear from the constructions of the GMAs attached to T and λT that the structural modules $A_{i,j}$ for λT are the same as those for T, as are the maps $\phi_{i,j,k}$ when $\#\{i,j,k\} = 3$, while the maps $\phi_{i,j,i}$ are multiplied by α. This makes clear that $f_n(\lambda T) = \lambda f_n(T)$.

For the additivity of f_n we need a reinterpretation of its construction.

Let us call $\bar S^{\mathrm{in}}$ the initial Cayley-Hamilton quotient of $(k[G],\bar T)$ defined in [BC, Remark 1.2.4(ii)]. By construction, every other Cayley-Hamilton quotient of $(k[G],\bar T)$ is a quotient of $\bar S^{\mathrm{in}}$. Of course $\bar S^{\mathrm{in}}$ is in a canonical way a GMA : let us call $\bar A^{\mathrm{in}}_{i,j}$ its structural modules, that is k-vector spaces. Its structural morphisms $\bar\phi^{\mathrm{in}}_{i,j,k}$ are all 0.

Now let T be a deformation of $\bar T$ to $A = \kappa[\varepsilon]$. We call S^{in} the initial Cayley-Hamilton quotient of $(A[G],T)$ and S its faithful (hence Cayley-Hamilton) quotient $A[G]/\operatorname{Ker} T$. We can choose compatible GMA structures on S^{in} and S (by choosing first a data of idempotents for S^{in} and picking its image in S) so that the surjective map $S^{\mathrm{in}} \to S$ determines surjective maps on the structural modules

$$A^{\mathrm{in}}_{i,j} \to A_{i,j}$$

and the structural maps $\phi^{\mathrm{in}}_{i,j,k}$ factor through the quotients $A_{i,j}$, giving back the structural maps $\phi_{i,j}$. Note that the kernel of the surjective map $A^{\mathrm{in}}_{i,j} \to A_{i,j}$ contains $\varepsilon A^{\mathrm{in}}_{i,j}$ by lemma 3.

On the other hand, the formation of $S^{\rm in}$ commute with arbitrary base change. Hence $\bar{S}^{\rm in} = S^{\rm in} \otimes_A \kappa = S^{\rm in}/\varepsilon S^{\rm in}$, and we have $A^{\rm in}_{i,j}/\varepsilon A_{i,j} = \bar{A}^{\rm in}_{i,j}$. The maps $\phi^{\rm in}_{i,j,k}$ define maps $\phi^{\rm in}_{i,j,k} \otimes_A k$ on the $A^{\rm in}_{i,j}$. Since they depend on T, let us denote them by $\bar{\phi}^T_{i,j,k} := \phi^{\rm in}_{i,j,k} \otimes_A k$

We thus see that the modules (actually κ-vector spaces) $A_{i,j}$ (that depend on T) are canonically quotient of the $\bar{A}^{\rm in}_{i,j}$ (that do not depend on T !) in such a way that the maps $\phi_{i,j,k}$ between the $A_{i,j}$ descend from the maps $\bar{\phi}^T_{i,j,k}$ between the $\bar{A}^{\rm in}_{i,j}$.

Using this, we can give the promised reinterpretation of the construction of f_n, as follows : *to construct $f_n(T)$, we can proceed exactly as in §3.3 using the modules $\bar{A}^{in}_{i,j}$ instead of the modules $A_{i,j}$ and the maps $\bar{\phi}^T_{i,j,k}$ instead of the $\phi_{i,j,k}$*. Indeed, it is obvious that we get the same result both ways.

Now suppose that we have a second deformation T' of $\bar{T}$ to $\kappa[\varepsilon]$. We can consider the sum $T + T'$ of those two deformations in the usual way, that is we use T and T' to define a deformation of $\bar{T}$ on the ring $k[\varepsilon, \varepsilon']$ and we push it to $k[\varepsilon]$ using the map $\varepsilon + \varepsilon' \mapsto \varepsilon$. It follows clearly from the construction on $T + T'$ and the compatibility of the formation of $S^{\rm in}$ with all base change that

$$\bar{\phi}^{T+T'}_{i,j,k} = \bar{\phi}^T_{i,j,k} + \bar{\phi}^{T'}_{i,j,k}.$$

Together with the reinterpretation of $f_n(T)$ given above, this makes the additivity of f_n obvious.

4. Further study of the functor PD_2.

4.1. An exact sequence.

Theorem 2. *We have the following canonical exact sequence of κ-vector spaces*

$$0 \longrightarrow \oplus_{k=1}^r Ext^1_G(\bar{\rho}_k, \bar{\rho}_k) \longrightarrow PD_2(k[\varepsilon]) = \mathfrak{t}_{PD,2} \xrightarrow{f_2}$$
$$\oplus_{1 \leq i < j \leq r} Ext^1_G(\bar{\rho}_j, \bar{\rho}_i) \otimes Ext^1_G(\bar{\rho}_i, \bar{\rho}_j) \xrightarrow{h_2} \oplus_{k=1}^r Ext^2(\bar{\rho}_k, \bar{\rho}_k)$$

assuming that all the Ext^1-spaces appearing in that exact sequences are finite dimensional.

The current subsection is entirely devoted to the proof of this theorem

This exact sequence is the complex of theorem 1 for $n = 2$. The only result that remains to be proved is that $\operatorname{Im} f_2 = \operatorname{Ker} h_2$. So let us pick an element in $\operatorname{Ker} h_2$, that is a family of elements $e_{i,j}$ for $(i,j) \in I, i \neq j$ in $\operatorname{Ext}^1_G(\bar{\rho}_j, \bar{\rho}_i) \otimes \operatorname{Ext}^1_G(\bar{\rho}_i, \bar{\rho}_j)$ (with $e_{i,j} = e_{j,i}$ for the natural identification given by the commutativity of tensor product) such that for every $i \in I$, we have $\sum_{j \in I,\ j \neq i} h_{2,(i,j,i)}(e_{i,j}) = 0$, where $h_{2,(i,j,i)}$ is by construction the Yoneda product map $\operatorname{Ext}^1_G(\bar{\rho}_j, \bar{\rho}_i) \otimes \operatorname{Ext}^1_G(\bar{\rho}_i, \bar{\rho}_j) \longrightarrow \operatorname{Ext}^2_G(\bar{\rho}_i, \bar{\rho}_i)$.

4.1.1. *Construction of a GMA.* In this paragraph we let $A \in \mathcal{C}$ be such that $\mathfrak{m}^2 = 0$ and $\dim_\kappa \mathfrak{m} = 1$. The basic example, and the only one we will need in this section, is $A = \kappa[\varepsilon]$, but for later use it is better to be a little bit more general. We also fix a κ-isomorphism $\mathfrak{m} \simeq \kappa$ (in the case where $A = \kappa[\varepsilon]$, we choose the one given by $\varepsilon \mapsto 1$)

Let us choose for all (i,j), $i \neq j$, a supplementary vector space $E_{i,j}$ of $B^1_G(\bar\rho_j,\bar\rho_i)$ in $Z^1_G(rhob_j,\bar\rho_i)$, and let us define $A_{i,j} = E^*_{i,j}$. The natural map $E_{i,j} \to \mathrm{Ext}^1(\bar\rho_j,\bar\rho_i)$ is an isomorphism. Hence by hypothesis the $E_{i,j}$'s and thus the $A_{i,j}$'s are finite dimensional vector spaces over κ. We can see them as A-modules on which $\mathfrak{m}$ acts by 0.

Moreover, using the isomorphisms $E_{i,j} \to \mathrm{Ext}^1(\bar\rho_j,\bar\rho_i)$, the element $e_{i,j}$ may be seen as an element of $E_{i,j} \otimes E_{j,i}$ hence by the biduality isomorphism, as a map $\phi_{i,j,i} : A_{i,j} \otimes A_{j,i} \to \kappa$.

We see these maps as maps into A (with image in $\mathfrak{m}$) using the given identification $\kappa \simeq \mathfrak{m}$. They are clearly morphisms of A-modules.

We note that by definition, $\phi_{i,j,i}(x \otimes y) = \phi j,i,j(y \otimes x)$.

We complete the data of a GMA structure by defining $A_{i,i} = A$ for all i $\phi_{i,j,k}$ as 0 if $\{i,j,k\} = 3$, as the structural map if $i = j$ or $j = k$ - the case where $i = k$ is already defined.

Lemma 8. *The $A_{i,j}$ together with the $\phi_{i,j,k}$ satisfy the axioms of a GMA.*

Proof — The property (UNIT) is satisfied by definition and (COM) has already been observed. We have to check (ASSO), that is that for all $i,j,k,l \in I$, and all $x \in A_{i,j}, y \in A_{j,k}, z \in A_{k,l}$, we have

$$\phi_{i,k,l}(\phi_{i,j,k}(x \otimes y) \otimes z) = \phi_{i,j,l}(x \otimes \phi_{j,k,l}(y \otimes z)). \tag{8}$$

When i,j,k,l are all different, then by definition all the ϕ involved are 0 and both member of the equality (8) are 0.

When $\#\{i,j,k,l\} = 3$, we have to distinguish several cases. If $j = k$, or if $i = j$, a simple look shows that in both members there is a $\phi_{a,b,c}$ involved with $\#\{a,b,c\} = 3$, so both members are 0. If $i = k$, the the right hand side is 0 since it involves the 0 map $\phi_{i,j,l}$. The left hand side is also 0 but for a more subtle reason: it is by (UNIT) $\phi_{i,j,i}(x \otimes y)z$ with $\phi_{i,j,i}(x \otimes y) \in \mathfrak{m} \subset A$, and $z \in A_{l,k}$, and this product is zero by Lemma 3. The last case, $j = l$, is symmetric to the case $i = k$.

Finally, when $\#\{i,j,k,l\} \leq 2$, then (8) follows formally from (UNIT). □

By the lemma we may consider the GMA S over A of type $(d_1,\dots,d_r)$ with structural modules $A_{i,j}$ and structural morphisms $\phi_{i,j,k}$ above. The trace map $\mathrm{tr} : S \to A$ is a Cayley-Hamilton pseudocharacter of dimension d by [BC, Corollary 1.3.16]. By construction, this pseudocharacter has a complexity less or equal to 2.

4.1.2. *Construction of maps $a_{i,j}$ when $i \neq j$.* In this §, we keep the above notations. The ring $A \in \mathcal{C}$ still satisfies $\mathfrak{m}^2 = 0$ and an isomorphism $\mathfrak{m} \simeq k$ is still given.

We now want to construct a morphism of A-algebras $\rho : A[G] \to S$. It is enough by linearity to define $\rho(g) \in S$ for $g \in G$ and to check the multiplicativity of ρ, and actually it is enough to define each component $a_{i,j}(g) \in M_{d_i,d_j}(A_{i,j}) \subset S$ of $\rho(g)$ as in equation (2) and to check the multiplication formula (5).

We define $a_{i,j}(g)$ for $i \neq j$ as follows. The evaluation at g gives a linear application $Z^G_1(\bar\rho_j,\bar\rho_i) \to M_{d_i,d_j}(\kappa)$. Restricting this linear application to $E_{i,j}$, we get a linear map $E_{i,j} \to M_{d_i,d_j}(\kappa)$ which by biduality is an element of $M_{d_i,d_j}(A_{i,j})$. This is our $a_{i,j}(g)$.

Note that by construction, every linear form l on $A_{i,j}$ allows to define a map $l(a_{i,j}) : G \to M_{d_i,d_j}(\kappa)$, which is actually the element l of $Z_1^G(\bar\rho_j, \bar\rho_i)$.

4.1.3. *Construction of the maps $a_{i,i}$.* It is more delicate to define $a_{i,i}(g)$ for $g \in G$. In order to do so, we assume that $A = \kappa[\varepsilon]$.

We fix $i \in I$, and proceed as follows. Let's consider the map $b : G \times G \to M_{d_i}(\kappa)$ defined by $b(g, g') = \sum_{j \neq i} \phi_{i,j,i}(a_{i,j}(g) \otimes a_{j,i}(g'))$. This map belongs by construction to $Z_G^2(\bar\rho_i, \bar\rho_i)$ and represents the element $\sum_{j \neq i} h_{2,(i,j,i)}(e_{i,j}) = 0 \in \mathrm{Ext}_G^2(\bar\rho_i, \bar\rho_i)$ by our hypothesis. Therefore, there is a map $c_i : G \to M_{d_i}(G)$ whose coboundary is b, that is

$$\sum_{j \neq i} \phi_{i,j,i}(a_{i,j}(g) \otimes a_{j,i}(g')) = \bar\rho(g)c(g') + c(gg') + c(g)\bar\rho(g') \tag{9}$$

We set

$$a_{i,i}(g) = \bar\rho_i(g) + \varepsilon c_i(g) \in M_{d_i,d_j}(A)$$

We now have to check that the multiplication formula (5)

$$a_{i,j}(gg') = \sum_{k=1}^{r} \phi_{i,k,j}(a_{i,k}(g) \otimes a_{k,j}(g'))$$

holds for the $a_{i,j}$ we have defined.

We distinguish two cases: if $i = j$ then we check easily that this equality reduces the multiplicativity of $\bar\rho_i$ and for the formula (9) for c_i. If $i \neq j$, then the only two terms that may be non zero in the RHS of (5) are the terms for $k = i$ or $k = j$, and the formula to be proved reduces to

$$a_{i,j}(gg') = a_{i,i}(g)a_{i,j}(g') + a_{i,j}(g)a_{j,j}(g')$$

Since $\varepsilon a_{i,j}(g) = \varepsilon a_{i,j}(g') = 0$ by Lemma 3, this in turns reduces to

$$a_{i,j}(gg') = \bar\rho_i(g)a_{i,j}(g') + a_{i,j}(g)\bar\rho_j(g').$$

This equality in $M_{d_i,d_j}(A_{i,j})$ may be checked after evaluation by any linear form $l \in A_{i,j}^*$. But as already observed, $l(a_{i,j})$ is in $Z_G^1(\bar\rho_j, \bar\rho_i)$ and the equality follows.

4.1.4. *End of the proof.* We now define $T : A[G] \to A$ as $\mathrm{tr} \circ \rho$. This is a pseudocharacter of G over A of dimension d and complexity less or equal to 2, and its reduction $T \otimes A/\mathfrak{m}$ is the sum of the reduction mod $\mathfrak{m}$ of the $a_{i,i}$, that is the sum of the $\bar\rho_i$. Hence $T \in PD_2(A) = \mathfrak{t}_{D,2}$.

It is clear by construction that $f_2(T)$ is the desired element, that is that $f_{2,(i,j)}(T) = e_{i,j}$.

4.2. A mod $\mathfrak{m}^2$ lifting result.

Theorem 3. *Let $A \in C$ such that $\mathfrak{m}^2 = 0$. There is a natural and canonical obstruction class $c(\bar T)$ in $\mathrm{Coker}\, h_2 \otimes_\kappa \mathfrak{m}$ such that if $c(\bar T) = 0$, then $PD_2(A)$ is not empty (and thus is an homogeneous space over $\mathrm{Ker}\, h_2 \otimes_\kappa \mathfrak{m}$).*

4.2.1. *Construction of the obstruction $c(T)$.* To the representation $\bar\rho_i$ it is possible to attach a canonical class $c_i \in \mathrm{Ext}^2_G(\bar\rho_i,\bar\rho_i)\otimes_\kappa \mathfrak{m}$ following Mazur ([M]), as follows: take any lift (not necessarily a morphism of group) $a_i : G \to \mathrm{GL}_{d_i}(A)$ of ρ_i and consider the map $b_i : G\times G \to \mathrm{Ker}\,(\mathrm{GL}_{d_i}(A)\to \mathrm{GL}_{d_i}(\kappa))$ defined by

$$b_i(g,g') = a_i(gg')a_i(g)^{-1}a_i(g')^{-1} \tag{10}$$

Since $\mathfrak{m}^2 = 0$, the group $\mathrm{Ker}\,(\mathrm{GL}_{d_i}(A)\to \mathrm{GL}_{d_i}(\kappa))$ may be naturally identified with $M_{d_i}(\mathfrak{m})$ (by the map $x\mapsto x-\mathrm{Id}$) and via this identification, we have $b_i \in Z^2(\bar\rho_i,\bar\rho_i)\otimes_\kappa \mathfrak{m}$. The class of b_i in $\mathrm{Ext}^2(\bar\rho_i,\bar\rho_i)\otimes_k\otimes\mathfrak{m}$ does not depend on the chosen lift a_i. This class is the class c_i.

We define the obstruction class $c(\bar T)$ as the image of $\oplus_{i=1}^r c_i$ in $\mathrm{Coker}\, h_2\otimes_k \mathfrak{m}$.

4.2.2. *Construction of an element of $PD_2(A)$ assuming that $c(\bar T) = 0$.* We use the notations of the above paragraph. By induction, we may assume without any restriction that $\dim_\kappa \mathfrak{m} = 1$, and we may choose an isomorphism of κ-vector spaces $\kappa\simeq\mathfrak{m}$ which allows us to drop the $\mathfrak{m}$ in the tensor product below.

If $c(T) = 0 \in \mathrm{Coker}\, h_2$, then for each $i\in I$, c_i is the image by the Yoneda product of $\sum_{j\neq i} e_{i,j}$ for some $e_{i,j}\in \mathrm{Ext}^1_G(\bar\rho_i,\bar\rho_j)\otimes \mathrm{Ext}^1_G(\bar\rho_j,\bar\rho_i)$, such that $e_{i,j}$ is the element corresponding to $e_{j,i}$ for the commutativity of tensor product.

We thus are in the situation of §4.1.1 and of §4.1.2. We have proved there the existence of a GMA S of type $(d_1,\dots,d_r)$ with structural modules $A_{i,j}$, and of maps $a_{i,j} : G\to M_{d_i,d_j}(A_{i,j})$ for all $i,j\in I$, $i\neq j$, such that for all $i\in I$ the cocycle

$$(g,g')\mapsto \sum_{j\neq i} a_{i,j}(g)a_{j,i}(g') \tag{11}$$

represents the image by the Yoneda product of $\sum_{j\neq i} e_{i,j}$ in $\mathrm{Ext}^2_G(\bar\rho_i,\bar\rho_i)$, that is, represents the elements noted c_i in §4.2.1. By construction of c_i, there is a map $a_{i,i} : G\to M_{d_i}(A)$ (which was noted a_i in §4.2.1) such that the cocycle

$$(g,g') \quad\mapsto\quad \sum_{j\neq i} a_{i,i}(g)a_{i,i}(g)^{-1}a_{i,i}(g)-\mathrm{Id} \tag{12}$$

$$= \quad a_{i,i}(gg') - \bar\rho_i(g)a_{i,i}(g') - a_{i,i}(g)\bar\rho(g') \tag{13}$$

represents c_i. Hence the two cocycles (11) and (13) are equal, up to a coboundary that can be removed by changing $a_{i,i}$. We get

$$a_{i,i}(gg') - \bar\rho_i(g)a_{i,i}(g') - a_{i,i}(g)\bar\rho_i(g') = \sum_{j\neq i} a_{i,j}(g)a_{j,i}(g').$$

We deduce that the maps $a_{i,j}$ define a morphism of group $\rho : G\to S^*$ whose trace T is the desired lift.

4.3. **A simple application to the lifting of Galois representations.** The mod l^2 lifting of irreducible mod l Galois representations of dimension 2 has been studied by Khare ([K]) in both the irreducible and reducible case. Here we want to show how our results may be used to deduce simply stronger lifting results in the reducible cases.

We fix a prime number l. We assume that $G = G_{K,S}$, the absolute Galois group of the maximal extension of a number field K that is unramified outside a finite set of prime S. We shall assume that S contains all primes above ∞ and l.

Let κ be a finite field of characteristic $l > 2$. Let A be a local field with residue field κ and maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = 0$.

Proposition 2. *Let $\bar{T} = \chi_1 + \chi_2 : G \to GL_2(k)$ be a reducible pseudocharacter with $\chi_1 \neq \chi_2$. Assume that the Leopoldt conjecture holds for K.*

Then there exists an irreducible pseudocharacters $T : G \to GL_2(A)$ that lifts $\bar{T}$ if and only if $Ext^1_G(\chi_1, \chi_2) \neq 0$ and $Ext^1_G(\chi_2, \chi_1) \neq 0$.

Proof — We only have to check that the obstruction $\bar{c}(T)$ vanishes. Since this obstruction lives in $(\mathrm{Coker}\, h_2) \otimes \mathfrak{m}$, this would follows from the surjectivity of h_2, where h_2 is the Yoneda product $\mathrm{Ext}^1_G(\chi_1, \chi_2) \otimes \mathrm{Ext}^1_G(\chi_2, \chi_1) \to \mathrm{Ext}^2_G(\chi_1, \chi_1)$. If $\chi = chi_2 \chi_1^{-1}$, this may be rewritten as the map $H^1(G, \chi) \otimes H^1(G, \chi^{-1}) \to H^2(G, 1)$, where 1 is the trivial character of $G \to k^*$. I $H^2(G, 1) = 0$ by the Leopoldt conjecture (see [J, Example 1, page 319]) and the surjectivity is obvious.

□

Remark 1. By using the method of the next section, it is not hard to prove that the pseudocharacter T lifting $\bar{T}$ may be chosen to be the trace of a representation $\rho : G \to \mathrm{GL}_2(A)$. Moreover, if $\bar{\rho} : G \to \mathrm{GL}_2(\kappa)$ is a non semi-simple representation of trace T, we may even chose a ρ as above that lifts $\bar{\rho}$. We leave the details to the reader.

5. Characterisation of the true representations in $\mathfrak{t}_D$ in the case $r = 2$

In this section we assume $r = 2$, that is $I = \{1, 2\}$.

Lemma 9. *A pseudocharacter $T \in PD_2(\kappa[\varepsilon])$ is the trace of a representation $\rho : G \to GL_2(\kappa[\varepsilon])$ if and only if the κ-vector spaces $A_{i,j}$ for $i \neq j \in I$ attached to $R/KerT$ all have dimension 0 or 1.*

Proof — Assume $T = \mathrm{tr}\, \rho$ is the trace of a representation. Then $S^0 := A\rho(G) \subset M_2(A)$ is a Cayley-Hamilton quotient of $(A[G], T)$. By [BC, Prop. 1.3.8], it is possible to embed the structural modules $A^0_{i,j}$ of this GMA S^0 in $A = \kappa[\varepsilon]$, in such a way that the maps $\phi^0_{i,j,k}$ are given by the multiplication in A. Set $S = A[G]/\mathrm{Ker}\, T = S_0/\mathrm{Ker}\, T$, and define the $A_{i,j}$ and $\phi_{i,j,k}$ for S using the data of idempotents given by the image of the one of S^0. Then we have for any $i \neq j$

$$A_{i,j} = A^0_{i,j}/\mathrm{Ker}\,_{\mathrm{left}} \phi^0_{i,j,i}.$$

Thus the A-module $A_{i,j}$ is a quotient of a sub-module of A and, by Lemma 3, is killed by ϵ. This implies that $\dim_k A_{i,j} = 1$.

Conversely, assuming that the $A_{1,2}$ and $A_{2,1}$ of $S = R/\mathrm{Ker}\, T$ are one-dimensional vector spaces (note that they have the same dimension since $\phi_{1,2,1}$ is non degenerate, and the case where they are both 0 is trivial), we may identify both spaces to κ, and $\phi_{i,j,i}$ to the maps $k \otimes k \to \kappa \simeq \kappa\varepsilon \in A$ where the first map is the multiplication of κ. Consider the GMA

$$S' = \begin{pmatrix} M_{d_1}(A) & M_{d_1,d_2}(A) \\ M_{d_2}(\varepsilon A) & M_{d_2,d_1}(A) \end{pmatrix} \subset M_d(A).$$

Then we can lift the map $\rho : G \to S^*$ into a map $\rho' : G \to S'^*$ by defining $a'_{1,1}(g) = a_{1,1}(g)$, $a'_{2,2}(g) = a_{2,2}(g)$, $a'_{1,2}(g) = a_{1,2}(g)1_A$, $a'_{2,1}(g) = a_{2,1}(g)\varepsilon$. It is an obvious computation to check that $\rho' : G \to S^*$ is a morphism of group. But then ρ' define a true representation of trace T. □

Remark 2. (i) The *only if* part is true without any restriction on r: the same proof works.

(ii) The proof of the *if* part shows that we can even choose a ρ of trace T whose reduction $\pmod{\varepsilon}$ is an extension of $\bar\rho_2$ by $\bar\rho_1$, whose extension class is the class of the cocycle $g \mapsto a_{1,2}(g)$. A slight modification of the proof (interchanging 1 and 2) shows that we can also find a ρ which is an extension of $\bar\rho_1$ by $\bar\rho_2$ if we wish.

(iii) The *if* part is false if we do not assume that $r = 2$. Here is a counter-example with $r = 3$. Let $A = \kappa[\varepsilon]$ and define the $A_{i,j}$, $i \neq j$ as κ, the $\phi_{i,j,i}$ as the multiplication $\kappa \times \kappa \to \kappa \simeq \kappa\varepsilon \subset A$, and the $\phi_{i,j,k}$ as zero. From this data we get a GMA S – of type $(1,1,1)$ say. Set $G = S^*$, and let T be the restriction of the trace of S to G. Then by construction T is a faithful residually multiplicity free pseudocharacter of dimension 3 and of complexity 2 (since the $\phi_{i,j,k}$ are 0 when $\#\{i,j,k\} = 3$).

However, T is not the trace of a representation $\rho : G \to \mathrm{GL}_3(A)$.

To prove this, let us assume by contradiction that $T = \operatorname{tr}\rho$ and begin as in the proof of the lemma above. We can embed the structural modules $A^0_{i,j}$ of $S^0 := A(\rho(G)) \subset M_3(A)$ in A in such a way that the $\phi_{i,j,i}$ is the multiplication in A. But by construction, the $\phi_{i,j,i}$ are non-zero, meaning that for all $i,j \in I$, $i \neq j$ $A^0_{i,j}A^0_{j,i}$ is non-zero, hence is εA. Since every $A^0_{i,j}$ is an ideal of A, that is either 0, or ϵA, or A, we see that indeed for all $i \neq j$, one of $A^0_{i,j}$ and $A^0_{j,i}$ is A while the other is εA. It is an easy combinatorial task to deduce that we can find i_1, i_2, i_3 such that $\{i_1, i_2, i_3\} = \{1,2,3\}$ with $A^0_{i_1,i_2} = A^0_{i_2,i_3} = A$, but then $A^0_{i_1,i_2}A^0_{i_2,i_3}A^0_{i_3,i_2}$ is not zero, contradicting the fact that T has complexity 2.

Theorem 4. *We still assume $r = 2$. An element $T \in PD(\kappa[\varepsilon]) = \mathfrak{t}D$ is the trace of a representation if and only if $f_2(T)$ is a pure tensor.*

Proof — Let T be the trace of a representation. By the lemma, the structural modules $A_{1,2}$ and $A_{2,1}$ of $A[G]/\operatorname{Ker} T$ are κ-vector spaces of dimension 1. Therefore, any map $A_{1,2} \otimes A_{2,1} \to \kappa$, and in particular the map $\phi_{1,2,1}$, may be written $l \otimes l'$ with $l \in A^*_{1,2}$ and $l' \in A^*_{2,1}$. Thus by definition $f_{2,(1,2,1)}(T) = \iota_{1,2}(l) \otimes \iota_{2,1}(l')$ is a pure tensor.

Conversely, let us start with a family of elements $(e_{1,2}, e_{2,1})$ as in the proof of Theorem 2, that is with $e_{1,2} = e_{2,1}$ with the standard identification. Assume in addition that $e_{1,2}$ are pure tensors. Then in the GMA S constructed in §4.1.1 using the $e_{i,j}$, the map $\phi_{1,2,1} : A_{1,2} \otimes A_{2,1} \to k$ has the form $l \otimes l'$. We see easily that on S, $\operatorname{Ker} \operatorname{tr} = M_{d_1,d_2}(\operatorname{Ker} l) \oplus M_{d_2,d_1}(\operatorname{Ker} l')$. Hence, in the GMA $S^0 = S/\operatorname{Ker} \operatorname{tr}$, the structural modules $A^0_{1,2}$ and $A^0_{2,1}$ are respectively $A_{1,2}/\operatorname{Ker} l$ and $A_{2,1}/\operatorname{Ker} l'$, and they have κ-dimension at most 1. The map of algebras $\rho : A[G] \to S$ constructed in §4.1.2 induces an injective map $A[G]/\operatorname{Ker} T \hookrightarrow S/\operatorname{Ker} \operatorname{tr} = S^0$, compatible with T and tr. Thus the structural modules of $R/\operatorname{Ker} T$ have dimension at most one, and by the lemma, T is the trace of a representation. □

Remark 3. If $\operatorname{Ext}^1_G(\bar\rho_1, \bar\rho_2)$ or $\operatorname{Ext}^1_G(\bar\rho_2, \bar\rho_1)$ has dimension at most 1, then $f_2(T)$ is always a pure tensor, hence every $T \in PD(A)$ is the trace of a representation. Compare [BC, Prop 1.7.4].

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