NON SMOOTH CLASSICAL POINTS ON EIGENVARIETIES

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Contents

Abstract: We construct non-smooth points on unitary eigenvarieties. More precisely, we construct points such that the local ring of irreducible component of the eigenvariety through this point is non smooth, and not even a UFD. We use those points to construct geometrically several independent extensions in the relevant Selmer group.

In all this paper[1] p will be a fixed odd prime.

The *eigenvarieties* are p-adic rigid analytic spaces interpolating the Hecke eigenvalues of algebraic automorphic eigenforms for a given reductive group G over $\mathbb{Q}$, with finite slope at p, having a fixed level but an arbitrary weight. The points corresponding to the algebraic automorphic eigenforms for G on an eigenvariety X are called *classical points*. They should form a Zariski-dense subset of X when $G(\mathbb{R})$ admits discrete series. The other points parameterize more general, *p-adic*, automorphic forms.

Eigenvarieties have been subject of a broad interest in the last ten years, since the seminal paper [CM]. Still, very little is known about their geometry, either from a global, or from a local point of view. An example of a global important question is: do they have a finite numbers of irreducible components ?

In this paper, we are interested in the local question of smoothness. Although it is clear that there exist points on eigenvarieties where two or several irreducible components meet[2] and which, thus, are non smooth, the following questions, to the best of my knowledge, were not solved :

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[2]For example, the Eisenstein component $\mathcal{C}^{\text{eis}}$ of the Coleman-Mazur level 1 Eigencurve $\mathcal{C}$, which is isomorphic to the weight space, cuts the cuspidal locus $\mathcal{C}^0$ at every weight k such that $\zeta_p(k) = 0$; if p is not a regular prime, such k's do exist – see [BC2, 4.1, remarques]. Also, Calegari has an

Are the eigenvarieties smooth at classical points ?
Are the irreducible components of the eigenvarieties smooth ?

In the case of the **eigencurve** of Coleman-Mazur ([CM]) or its variant with arbitrary level, the answer to the first question is **affirmative**. This was proved in most cases in [Ki] and [BC2], and the remaining cases (mainly the case of CM points) may be handled by a similar method. We do not know the answer to the second question in this case.

In this paper, we offer a **negative answer to both** those **questions for higher dimensional eigenvarieties**. We work with any eigenvariety X of a unitary groups in 3 variables defined over $\mathbb{Q}$, in which we are able to produce many examples of classical points that are non smooth in every component in which they belong. It will appear clearly that this construction would work similarly for unitary groups with more than 3 variables, provided the needed results on Galois representations attached to their automorphic forms was known – this should be settled with the completion of the work in progress [GRFAbook].

Let us now describe more precisely our results: we work with the unitary group U(3) attached to a quadratic imaginary field E in which p is split, and which is compact at infinity.

Our starting point is a class of automorphic representations π of U(3) that were constructed (among many others) by Rogaswki in his work [R1]. Each such $\pi = \pi(f, \mu)$ depends on a classical modular form f of any weight $k > 1$ and level $\Gamma_0(N)$ and of a Grossencharacter μ of E, satisfying $\mu(z\bar{z}) = 1$ for every $z \in \mathbb{A}_E^*$.

For technical reasons, we assume that p does not divide the level N of f, and that μ is unramified at the places of E dividing p. We also assume that the type at infinity of μ is "sufficiently regular" relatively to k. Those conditions are not very restrictive: for any f, there are infinitely many possible μ.

The representation $\pi(f, \mu)$ are **tempered**, but **endoscopic of type** $(2, 1)$. This means that the Galois representation of $G_E = \mathrm{Gal}(\bar{E}/E)$ attached to $\pi(f, \mu)$ is the sum of a two-dimensional representation $\rho_0 = \rho_0(f, \mu)$ and a representation of dimension 1 (actually, the trivial character).

To put π in an eigenvariety, we have to choose a *p-refinement* $\mathcal{R}$ of the unramified representation π_p, that is an ordering between the roots of its Hecke polynomial. Among the six possible (in general) such orderings, there is always one (and sometimes two) which is suitable for our purpose, that is which is *anti-ordinary*, but whose restriction to ρ_0 is *non-critical*. We choose and fix that refinement.

We then choose *any* eigenvariety X for U(3) containing $(\pi, \mathcal{R})$. To be precise, we have to explain what kind of eigenvarieties we consider, although, as we will see, details will be completely irrelevant for our method. An eigenvariety for $G = \mathrm{U}(3)$ (or for that matter, for any reductive group G over $\mathbb{Q}$) may *a priori* depends on three things :

(i) The set of (refined) algebraic representation to be interpolated. We impose that all of them are unramified outside a finite set of prime S, but we may wish to consider various conditions at primes dividing S.

example of a non-classical point in the cuspidal locus $\mathcal{C}^0$ of the eigencurve where two irreducible components meet.

(ii) The commutative set of Hecke operators used to (locally, above the weight space) generate the ring of functions of the eigenvariety. We always take the operators in the Atkin-Lehner algebra at p, and often all the spherical Hecke operators at primes outside S (but variants are allowed, for example we may forget about the spherical Hecke algebra at a set of density 0 places).

(iii) The method used to construct it. Currently, there are at least two published methods to construct some eigenvarieties for U(3), due to Chenevier [C] and Emerton [E]; still another method has been announced by Urban.

Provided the set in (i) is big and nice enough, both Chenevier and Emerton have constructed an equidimensional of dimension 2 eigenvariety for U(3). Actually, it has been proved in [BC3] that those eigenvarieties are canonically independent of the name of their constructor. Indeed, it is proved there that a few conditions on eigenvarieties relative to a fixed data as in (i), (ii) above, and that are easily verified both in Chenevier and in Emerton's construction, are enough to characterize them uniquely : see [BC3, Definition 7.2.5 and Prop. 7.2.7]. Still, the eigenvarieties may depend of the choice of the data (i) and (ii) above. But our method to prove that the every component is non smooth actually work with any choice, since it uses only two basic properties of the eigenvarieties X: that it carries a Galois pseudocharacter T which makes X a *refined p-adic family of Galois representations* in the sense of [BC3, chapter 4], and that its local rings are topologically generated, over the weight space, by the Hecke operators chosen in (ii).

Let us now explain in more details how we prove that every component of the chosen eigenvariety X, at the point y corresponding to $(\pi, \mathcal{R})$, is non smooth. First we show that the schematic *reducibility locus* of T at y is the closed point y itself. This means that, although the pseudocharacter T is reducible at y (namely $T_y = 1 + \operatorname{tr} \rho_0$), there is no closed subscheme of X containing y on which T is reducible, except of course y. The proof of that crucial result uses, besides the two properties of X recalled above, almost all the main results we proved in the first four chapters of the book [BC3] (and in a way which is very similar to the one used in the proof of [BC3, Prop. 9.3.7]). We then study, along the lines of chapter 1 of [BC3], the algebra $\mathcal{O}_y[G_E]/\operatorname{Ker} T$, and consideration of duality implies that neither $\mathcal{O}_y$, nor any of its components, may be smooth (or even a UFD).

As it should be clear from above, the methods used in this paper rely heavily on the ones of [BC3]. There is, however, a small shift of perspective: when we developed the book [BC3], and, before it, the baby-case [BC2], the methods were always used to get an upper bound on the dimension of the tangent spaces of the eigenvarieties (e.g. to prove the smoothness), or, which is logically equivalent, to use a lower bound on the dimension of the tangent space to get some information (e.g. a lower bound) on Selmer groups. Here similar techniques are used upside down, and allow one to give, in a special case, a lower bound of the dimension of the tangent space that implies the non-smoothness.

A natural question that arises at this point is if we can use in turn this lower bound on the tangent space of X at y to get some lower bound on some Selmer group. Actually, the answer is yes (see 7 below), as we can prove from it that the natural Selmer-like group of ρ_0 arising in this situation has dimension (at least) 2. However, this is of no interest arithmetically, since this information may also be obtained much more easily by a simple application of Poitou-Tate. This comes as no surprise since the only geometric Galois representation for which non-trivial (meaning, not a consequence of Poitou-Tate) lower bounds on Selmer groups are expected are ones of motivic weight 1, while ρ_0 is of weight 0 (see [BC3, Remark 5.1.4]). This, in turn, is a consequence of our starting with a *tempered* representation π.

Still, it was by no means obvious that both extensions on that Selmer group should come from the eigenvariety. What happens in the case under study is, so to speak, that the eigenvariety “sees” all the extensions that exists in the relevant Selmer group, even at the cost of being non smooth.

I consider this result as a strong encouragement for the hope, expressed in [BC3, §9.5.3] that in all cases (including the much more arithmetically significant cases studied in [BC3]) the whole relevant Selmer group is seen by the eigenvariety.

It should be clear to the reader of that introduction how much this paper owes to Gaëtan Chenevier. Not only is this work strongly based on our common work [BC3], and the ideas developed here a natural application of some of that book, but I have also benefited from many conversations with him during the writing of this paper.

1. Notations and conventions

As above, let p be a fixed odd prime.

1.1. Adèles and Galois groups. Let E be an imaginary quadratic field over $\mathbb{Q}$, in which p splits. We denote by $\mathbb{A}_E$ (resp. $\mathbb{A}_E^*$) the ring of adèles (resp. the group of idèles) of E.

We denote by $\bar{\mathbb{Q}}$ an algebraic closure of E, by G_E and $G_{\mathbb{Q}}$ the Galois group of $\bar{\mathbb{Q}}/E$ and $\bar{\mathbb{Q}}/\mathbb{Q}$, by c the non trivial element of $\mathrm{Gal}(E/\mathbb{Q})$, and we choose a lift γ of c in $G_{\mathbb{Q}}$ (that is, any element of $G_{\mathbb{Q}} - G_E$).

1.2. Class field theory, local Langlands, and motivic weights. We normalize the class field theory isomorphism in such a way that the Grossencharacter $|\ |$ correspond to the p-adic *cyclotomic character* $\omega = \mathbb{Q}_p(1)$.

We use the definition of *motivic weight* such that $\omega = \mathbb{Q}_p(1)$ has motivic weight -2.

If F is a local field, and π is an irreducible admissible complex representation of $\mathrm{GL}_n(F)$, we denote by $L(\pi)$ its L-parameter associated by the local Langlands correspondence: $L(\pi)$ is an n-dimensional complex representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$ if F is non-archimedean, and of W_F of $F = \mathbb{R}$ or $\mathbb{C}$, which sends elements of W_F to semi-simple elements in $\mathrm{GL}_n(\mathbb{C})$. The normalization chosen for L is the one compatible with our choice for the class field theory.

1.3. Contragredient and Dual conjugacy. If $\rho : G_E \to \mathrm{GL}_n(L)$ is a representation (where L is a field), then we denote by ρ^* the contragredient representation $g \mapsto {}^t\rho(g)^{-1}$ and by $\rho^\perp$ the representation $g \mapsto {}^t\rho(\gamma g \gamma^{-1})^{-1}$. The isomorphism class of $\rho^\perp$ depends only on the isomorphism class of ρ, and not on the choice of γ. The same definition will be in use for representation of Weyl groups instead of Galois group, and in particular for Grossencharacters. By class field theory, if $\mu : \mathbb{A}_E^*/E^* \longrightarrow \mathbb{C}^*$ is a Grossencharacter, we have $\mu^\perp(z) = \mu(c(z))^{-1}$.

1.4. **Unitary group.** We denote by U(3) the algebraic group over $\mathbb{Q}$ such that

$$\mathrm{U}(3)(A) = \{M \in \mathrm{GL}_3(A \otimes_{\mathbb{Q}} E),\ {}^t c(M)M = \mathrm{Id}\}$$

for any $\mathbb{Q}$-algebra E.

It is well known that U(3) becomes split over E, that is $\mathrm{U}(3)_{\times}\mathrm{Spec}\, E \simeq \mathrm{GL}_3$, that $\mathrm{U}(3)(\mathbb{R})$ is compact, while $\mathrm{U}(3)(\mathbb{Q}_l)$ is quasi-split for any prime number l.

1.5. ***A*-packets.** Rogawski has constructed certain set of classes of irreducible unitary representations of $\mathrm{U}(3)(\mathbb{A}_{\mathbb{Q}})$ called *A-packets*, that behave in accordance to Arthur's conjectures. In particular, the union of all A-packets contains all the *automorphic* representations of $\mathrm{U}(3)(\mathbb{A}_{\mathbb{Q}})$, and A-packets are disjoint.

If Π is such an A-packet, it has a base change Π_E which is an irreducible admissible unitary representation of $\mathrm{GL}_3(\mathbb{A}_E)$.

1.6. **Decomposition group at** p**.** We choose an embedding $i_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ and an embedding $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$. We denote by v the place of E above p defined by i_p, and by $\bar{v}$ the other place of E above p. We denote by G_p, G_v the decomposition subgroups at p, v of $G_{\mathbb{Q}}$, G_E. We have a canonical isomorphism $G_p \simeq G_v$. If ρ is a representation of $G_{\mathbb{Q}}$ or G_E we denote by ρ_p or ρ_v its restriction to G_p or G_v.

We normalize the definition of Hodge-Tate weight in such a way that ω_p has weight -1.

To v is associated an isomorphism $\mathrm{U}(3)(\mathbb{Q}_p) \to \mathrm{GL}_3(\mathbb{Q}_p)$, well defined up to inner conjugation.

1.7. **Selmer groups.** For $\rho : G_E \to Gl_n(\bar{\mathbb{Q}}_p)$ a geometric semi-simple representation, we recall that $H^1(G_E, \rho)$ parameterizes the set of isomorphism class of extensions of continuous G_E-representations U of the trivial representation by ρ, that is of exact sequences

$$0 \to \rho \to U \to \bar{\mathbb{Q}}_p \to 0.$$

We recall that such an extension U is said to be f at a finite place w of E if

(a) If w is prime to p, and I_w the inertia subgroup of a decomposition group at w, the following exact sequence is exact:
$$0 \to \rho^{I_w} \to U^{I_w} \to \bar{\mathbb{Q}}_p \to 0.$$
(b) If w divides p, the the following exact sequence is exact:
$$0 \to D_{\mathrm{crys}}(\rho_w) \to D_{\mathrm{crys}}(U_w) \to \bar{\mathbb{Q}}_p \to 0.$$

Recall that (a) is automatically satisfied if $\rho_{|D_w}$ (here D_w is a decomposition group at w) does not contains $\bar{\mathbb{Q}}_p(1)$.

We denote by $H^1_f(G_E, \rho)$ (resp. $H^1_{f_S}$, resp. $H^1_{f^S}$, for S a set of finite places of E) the subspaces of $H^1(G_E, \rho)$ that parameterize extensions U that are f at every finite place w of E (resp. of S, resp. outside of S).

2. The endoscopic representation π

Let f be a cuspidal modular eigenform of even weight $k \geq 1$ and level $\Gamma_0(N)$. We assume that

(H1) The level N is prime to p.
(H2) The modular form f is not a CM form.

We denote by ρ_f the Galois representation attached to f, with the standard normalization (so that $\rho_f^* \simeq \rho_f \omega^{k-1} = \rho_f(k-1)$) and ρ_f has motivic weight $k-1$).

Since p does not divide N, $(\rho_f)_p$ is crystalline, of Hodge-Tate weights 0 and $k-1$. We assume that

(H3) The eigenvalues of the crystalline Frobenius on $D_{\mathrm{crys}}((\rho_f)_p)$ are distinct.

To f is attached a cuspidal automorphic representation π_f that satisfies

$$\pi_f^* \simeq \pi_f(|\det()|^{k-1}).$$

We denote by $\pi_{f,E}$ the Langlands base change of π_f to E. This is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$, that obviously satisfies the following symmetries:

$$\pi_{f,E}^* \simeq \pi_{f,E}(|\det()|^{k-1}) \tag{1}$$

$$\pi_{f,E} \simeq \pi_{f,E} \circ c, \tag{2}$$

where $c : \mathrm{GL}_n(\mathbb{A}_E) \to \mathrm{GL}_n(\mathbb{A}_E)$ is the obvious map induced by $c \in \mathrm{Gal}(E/\mathbb{Q})$.

The representation $(\pi_{f,E})_\infty$ of $\mathrm{GL}_2(\mathbb{C})$ has the following Langlands parameter $L((\pi_{f,E})_\infty) : W_{\mathbb{C}} = \mathbb{C}^* \to \mathrm{GL}_2(\mathbb{C})$:

$$z \mapsto \mathrm{diag}(z^{1-k}, \bar{z}^{1-k}).$$

We now fix a Grossencharacter $\mu : \mathbb{A}_E^*/E^* \to \mathbb{C}^*$ such that

(i) $\mu^\perp = \mu$

(ii) $\mu_\infty(z) = (z/\bar{z})^{\frac{a}{2}}$ for all $z \in \mathbb{C}^*$ and for some $a \in \mathbb{Z}$.

(iii) μ is unramified at v and $\bar{v}$.

For all $a \in \mathbb{Z}$, there exists such a Grossencharacter (and actually, there are many such μ.) We may even choose μ unramified at all unramified places of E – see [BC3, Lemma 6.9.2]).

Proposition 1 (Rogawski)**.** *Assume that a is odd, but that $a \neq k-1$ and that $a \neq 1-k$. Then there exists a (non-empty) endoscopic tempered A-packet $\Pi = \Pi(f,\mu)$ (which is also an L-packet) for $U(3)$ whose base change Π_E to E has parameter at every place w of E*

$$L((\Pi_E)_w) = L((\pi_{f,E})_w) \otimes (\mu_w | |_w^{\frac{k-1}{2}}) \oplus 1. \tag{3}$$

Here μ_w and $| |_w$ are the restriction of the characters μ and $| |$ of $\mathbb{A}_E^*$ to E_w^*, seen as a quotient of the Weyl group W_{E_w}.

Proof — For the proof, see [R1] or [R2, chapter 14].

Alternatively, it is enough, and perhaps more informative, to see how the proposition follows form Arthur's conjecture, since those conjectures have been proved by Rogawski for U(3). We refer the reader to the appendix of [BC3] for a review of those conjectures.

We consider the parameter $L(\pi_{f,E}) : L_E \to \mathrm{GL}_2(\mathbb{C})$ of $\pi_{f,E}$. Here L_E is the global Langlands group, whose existence is conjectural. We define from this conjectural parameter another one, $L(\Pi_E) : L_E \to \mathrm{GL}_3(\mathbb{C})$ as $(L(\pi_{f,E}) \otimes \mu | |^{\frac{k-1}{2}}) \oplus 1$. Then the restriction of this parameter to the local Langlands groups L_{E_w} obviously coincides for every finite place w of E to those given by (3), and we have $L(\Pi_E)^\perp \simeq L(\Pi_E)$ where $\perp$ is given the same meaning as in §1.3 using any element $\gamma \in L_{\mathbb{Q}} - L_E$. Note that by definition, the restriction to the local Langlands group at infinity $W_{\mathbb{C}} = \mathbb{C}^*$ of the parameter is

$$L((\Pi_E)_\infty) : z \mapsto \mathrm{diag}((z/\bar{z})^{\frac{a+1-k}{2}}, (z/\bar{z})^{\frac{a+k-1}{2}}, 1),$$

and that by our hypotheses the numbers $\frac{a+1-k}{2}$, $\frac{a+k-1}{2}$ and 0 are all integers, and are distinct. Hence by [BC3, Prop A.11.3], the morphism $L(\Pi_E)$ extends uniquely (up to conjugation) to a tempered A-parameter $L(\Pi) : L_{\mathbb{Q}} \to {}^L\mathrm{U}(3)$ which is moreover relevant at infinity for the compact form $\mathrm{U}(3)(\mathbb{R})$ by [BC3, Rem. A.11.8], hence globally discrete. The A-packet Π corresponding, under Arthur's correspondence, to this A-parameter is the one whose existence is asserted by the proposition. □

From now on we assume that a is as in the proposition, and we choose any element $\pi = \pi(f, \mu)$ of this A-packet $\Pi = \Pi(f, \mu)$.

3. The Galois representation attached to π

3.1. **Galois representations attached to automorphic representations of $\mathbf{U}(3)(\mathbb{A})$.** An important result of Blasius and Rogawski (see [BlR], or [BC1, 3.1] for the result in the precise form we need) attaches to every A-packet Π of automorphic forms for $\mathrm{U}(3)$ a Galois representation $\rho_\Pi : G_E \to \mathrm{GL}_3(\bar{\mathbb{Q}}_p)$. At every place w of E, above a prime $l \neq p$ unramified in E, and such that the local A-packet Π_l is unramified, the local representation $(\rho_\Pi)_w$ is unramified and its Hecke polynomial is given by $L((\Pi_E)_w)$. In particular we have

$$\rho_\Pi^\perp \simeq \rho_\Pi.$$

Moreover, if the L-parameter $L((\Pi_E)_\infty)$ is $z \mapsto \mathrm{diag}((z/\bar{z})^a, (z/\bar{z})^b, (z/\bar{z})^c)$ with a, b, c three distinct integers (the L-parameter is always of that form), then $(\rho_\Pi)_v$ is Hodge-Tate Weight of weight $-a$, $-b$ and $-c$. If Π_p is an unramified packet (that is contains an unramified representation), then $(\rho_\Pi)_v$ is crystalline and its eigenvalues of the crystalline Frobenius are given by $L((\Pi_E)_v)$.

Now consider an automorphic representation for $\mathrm{U}(3)$, say τ. Following Rogawski, it belongs to exactly one global A-packet, say Π. Thus we may define unambiguously ρ_τ as ρ_Π.

3.2. **The case of π.** The Grossencharacter $|\,|^{\frac{k-1}{2}}\mu$ is algebraic, so it has a p-adic realization $(|\,|^{\frac{k-1}{2}}\mu)_p$. More precisely, for $z \in \mathbb{A}_E^{f^*}$, we have

$$(|\,|^{\frac{k-1}{2}}\mu)_p(z) = \iota_p(\iota_\infty^{-1}(|z|^{\frac{k-1}{2}}\mu(z)))z_p^{-\frac{k-1+a}{2}}c(z_p)^{-\frac{k-1-a}{2}}$$

and this character factors through $\mathbb{A}_E^{f^*}/E^* = G_E^{\mathrm{ab}}$

Set

$$\rho_0 := (\rho_f)_{|G_E}(|\,|^{\frac{k-1}{2}}\mu)_p$$

Then ρ_0 is a 2-dimensional representation of motivic weight 0, satisfying $\rho^\perp = \rho$ and such that

Lemma 2. *We have $\rho_\pi = \rho_0 \oplus 1$.*

Let us call $k_1 < k_2 < k_3$ the Hodge-Tate weight of $(\rho_\pi)_v$. We have

$$\{k_1, k_2, k_3\} = \{0, \frac{k-1-a}{2}, \frac{1-k-a}{2}\}.$$

Of course, 0 is the Hodge-Tate weight of 1_v, and the two others are the Hodge-Tate weights of $(\rho_0)_v$.

Let's call α, β the Frobenius eigenvalues of $(\rho_0)_v$. We have $\alpha \neq \beta$ by hypothesis (H3), and also, up to changing α and β

$$\frac{1-k-a}{2} \leq v(\alpha) \leq v(\beta) \leq \frac{k-1-a}{2}. \tag{4}$$

The eigenvalues of $(\rho_\pi)_v$ are of course $\alpha, \beta, 1$.

4. A refinement $\mathcal{R}$ of π

A *refinement* ([BC3, §2.4.1]) $\mathcal{F}$ of $(\rho_\pi)_v$ is by definition a full ϕ-stable filtration of $D_{\text{crys}}((\rho_\pi)_v)$. When the eigenvalues of ϕ are distinct, this is obviously equivalent to the data of an ordering $\alpha_1, \alpha_2, \alpha_3$ of the eigenvalues $\alpha, \beta, 1$ of ϕ on $D_{\text{crys}}((\rho_\pi)_v)$. Since we will soon make an assumption that implies that the eigenvalues of ϕ are distinct, let us assume from now for the simplicity of exposition that the $\alpha, \beta, 1$ are distinct.

In this case, to a refinement $\mathcal{F} = (\alpha_1, \alpha_2, \alpha_3)$ is attached is attached a permutation σ of $\{1, 2, 3\}$ (see [BC3, §4.4.2]; the definition of σ is recalled in the proof below).

Proposition 3. *There is a refinement $\mathcal{F}$ of $(\rho_\pi)_v$ whose attached permutation σ is transitive if and only if $|a| > k - 1$.*

Proof — Assume $|a| < k - 1$. We thus have $\frac{1-k-a}{2} < 0 < \frac{k-1-a}{2}$. In other words $k_2 = 0$. Recall that the weights determine a partition of $\{1, 2, 3\}$ into $W_1 = \{2\}$ and $W_{\rho_0} = \{1, 3\}$. The refinement $\mathcal{F}$ also determines a partition of $\{1, 2, 3\}$ into a singleton R_1 (containing the index i of the eigenvalue 1 in the ordering $\mathcal{F} = (\alpha_1, \alpha_2, \alpha_3)$) and R_{ρ_0} (the complement of R_1). The permutation σ is defined as sending R_1 onto W_1 (thus R_{ρ_0} onto W_{ρ_0}) and as being increasing on R_{ρ_0}. For σ to be transitive, R_1 has to be different from $\{2\}$. Thus $R_1 = \{1\}$ in which case $R_{\rho_0} = \{2, 3\}$ and $\sigma(3) = 3$, or $R_1 = \{3\}$, in which case $R_{\rho_0} = \{1, 2\}$ and $\sigma(1) = 1$. Hence we have seen that if $|a| < k - 1$, there is no refinement giving a transitive σ.

We shall construct a refinement with a transitive σ and other properties in the case $|a| > k - 1$ below. Hence we postpone the proof of the converse. □

From now on, we assume that $|a| > k - 1$. Since the case $a < -(k - 1)$ is symmetric to the case $a > k - 1$ (by interchanging v and $\bar{v}$), **we will actually assume from now on that** $a > k - 1$.

In this case, we have $k_1 = \frac{1-k-a}{2} < k_2 = \frac{k-1-a}{2} < k_3 = 0$. We recall that the restriction of the refinement $\mathcal{F}$ to $(\rho_0)_v$ is the ordering of the two eigenvalues α and β of $(\rho_0)_v$ induced by the ordering $\mathcal{F}$. We refer to [BC3, Def. 2.4.5] for the notion of a *non-critical* refinement of $(\rho_0)_v$.

Proposition 4. *The refinement $\mathcal{F} = (1, \alpha, \beta)$ of ρ_{π_v} is such that $\sigma = (3, 1, 2)$ is transitive. Moreover its restriction to $(\rho_0)_v$ is non critical.*

Proof — In this case we have $W_1 = \{3\}$ and $W_{\rho_0} = \{1, 2\}$. For the proposed refinement $\mathcal{F}$, we have $R_1 = \{1\}$ and $R_{\rho_0} = \{2, 3\}$. Thus σ sends 1 to 3 and $2, 3$ to $1, 2$ respectively, and σ is a cycle.

Moreover, the restriction of the refinement $\mathcal{F}$ to $(\rho_0)_v$ is (α, β). This refinement is *numerically non-critical* (see [BC3, Remark 2.4.6(2)]) since $v(\alpha) < \frac{k-1-a}{2}$ (otherwise we would have $v(\alpha) = v(\beta) = \frac{k-1-a}{2}$ and since $v(\alpha)+v(\beta) = \frac{1-k-a}{2} + \frac{k-1-a}{2}$ by admissibility of $D_{\text{crys}}((\rho_0)_v)$, we would have $k = 1$ which contradicts our hypothesis), hence is non critical. □

Remark 5. The only other refinement such that σ is transitive is $(1, \beta, \alpha)$. However, the restriction of this refinement to $(\rho_0)_v$ may be critical. More precisely, it is critical if ρ_0 is ordinary at v (that is if f is ordinary at p): see [BC3, Remark 2.4.6(1)].

Now to this refinement $\mathcal{F}$ of the Galois representation $(\rho_\pi)_v$ corresponds a refinement $\mathcal{R}$ (see [BC3, Def. 6.4.5]) of π_p. The refinement is accessible (see [BC3, Def. 6.4.6]) because all are for π_p since π_p is unramified and tempered (see [BC3, Prop. 6.4.7]).

5. The eigenvarieties containing $(\pi, \mathcal{R})$.

We fix a model of U(3) over $\mathbb{Z}$ and an associated product Haar measure μ on $\mathrm{U}(3)(\mathbb{A}_f)$. We use the standard conventions for adèles: $\mathbb{A}_S$ (resp. $\mathbb{A}_f^S$) denotes the ring of finites adèles with components in (resp. outside) the set of primes S. Moreover, we denote by $\widehat{\mathbb{Z}}_S = \prod_{l \in S} \mathbb{Z}_l$ the ring of integers of $\mathbb{A}_S$.

The definition of an eigenvariety for G depends on the choice of a commutative Hecke algebra $\mathcal{H}$ that we fix once for all as follows. The precise definition of $\mathcal{H}$ will not really matter below, so we may be quite general, and take any $\mathcal{H} = \mathcal{H}^p \otimes \mathcal{A}_p$, where $\mathcal{H}^p$ and $\mathcal{A}_p$ are as follows : we choose a finite set S_0 of primes including p and the primes of ramification of π, and we fix a compact open subgroup K^{S_0} of $G(\mathbb{A}_f^{S_0})$ which is a product of $K_l \subset G(\mathbb{Q}_l)$ for all $l \notin S_0$, where K_l is a maximal compact subgroup. Then we set $\mathcal{H}_{\text{ur}} = \mathcal{C}_c(K^{S_0} \backslash G(\mathbb{A}_f^{S_0})/K^{S_0}, \mathbb{Z})$ which is a commutative ring. We choose any *commutative* sub-algebra $\mathcal{H}_{S_0}^p$ of $\prod_{l \in S_0 - p} \mathcal{C}_c(G(\mathbb{Q}_l), \mathbb{Z})$, and we set

$$\mathcal{H}^p = \mathcal{H}_{\text{ur}} \mathcal{H}_{S_0}^p.$$

Recall from [BC3] that $\mathcal{A}_p$ is the Atkin-Lehner subring of $\mathcal{A}_p \subset \mathcal{C}_c(I \backslash G(\mathbb{Q}_p)/I, \mathbb{Z}[1/p])$ where $I \subset G(\mathbb{Q}_p) \xrightarrow{\sim}_v \mathrm{GL}_3(\mathbb{Q}_p)$ is the standard Iwahori subgroup.

We also choose an idempotent

$$e \in \mathcal{C}_c(G(\mathbb{A}_{S_0}^p), \overline{\mathbb{Q}}) \otimes 1_{\mathcal{H}_{\text{ur}}} \subset \mathcal{C}_c(G(\mathbb{A}_f^p), \overline{\mathbb{Q}}),$$

such that $e(\pi^p) \neq 0$.

To this data is attached a unique eigenvariety X ([BC3, Theorem 7.3.1]), that is equidimensional of dimension 2 ([BC3, Remark 7.3.2 iii)]): we fix the last weight κ_3 equal to $k_3 = 0$), and that comes with a Zariski-dense accumulation subset Z of points which is in bijection with all characters $\psi : \mathcal{H} \to \bar{\mathbb{Q}}_p$, which are of the form $\psi_{\tau, \mathcal{R}}$, where τ is a classical automorphic representation for U(3) such that

$(\tau^{S_0})^{K^{S_0}} \neq 0$
$\tau_p^I \neq 0$
$e(\tau_{S_0}^p) \neq 0$
and $\mathcal{R}$ is an accessible refinement of τ.

Above $\psi_{\tau,\mathcal{R}}$ is the tensor product of the character of $\mathcal{H}^p$ defined by τ^p and of the character of $\mathcal{A}_p$ defines using τ_p and the refinement $\mathcal{R}$, seen as a character valued in $\bar{\mathbb{Q}}_p$ instead of $\mathbb{C}$ using $i_p \circ i_\infty^{-1}$ (see [BC3, 7.2.2]),

Since π satisfies the above conditions, there is a point $y \in Z$ corresponding to $\psi_{\pi,\mathcal{R}}$.

We now list the data attached to X and their properties that we will need below. First, X comes with a Galois pseudo-character $T : G_E \to \mathcal{O}_X$ of dimension 3, and (X, T, Z) is part of a *refined family of Galois representations*, up to shrinking Z, but without removing the point y. More precisely, this means that, if $\bar{\rho}_x$ for $x \in X(\bar{\mathbb{Q}}_p)$ is the semi-simple representation of G_E of trace T_x, we have

(a) 3 analytic functions $\kappa_1, \kappa_2, \kappa_3 \in \mathcal{O}(X)$,
(b) 3 analytic functions $F_1, F_2, F_3 \in \mathcal{O}(X)$,

satisfying

(i) For every $x \in X$, the Hodge-Tate-Sen weights of $(\bar{\rho}_x)_v$ are, with multiplicity, $\kappa_1(x), \kappa_2(x)$ and $\kappa_3(x) = 0$.
(ii) If $z \in Z$, $(\bar{\rho}_z)_v$ is crystalline (hence its weights are integers).
(iii) If $z \in Z$, then $\kappa_1(z) < \kappa_2(z) < \kappa_3(z) = 0$.
(iv) If $z \in Z$, the eigenvalues of the crystalline Frobenius acting on $D_{\text{crys}}((\bar{\rho}_z)_v)$ are distinct and are $(p^{\kappa_1}(z)F_1(z), p^{\kappa_2(z)}F_2(z), p^{\kappa_3}(z)F_3(z))$.
(v) For C a non-negative integer, let Z_C be the set

$$\{z \in Z,\ |\kappa_i(z) - \kappa_j(z)| > C \ \ \forall i, j \in \{1, \dots, 3\},\ i \neq j.\}$$

Then Z_C accumulates at any point of Z for all C (see [BC3, §3.3.1]).
(∗) For any $i \in \{1, 2, 3\}$, there exists a continuous character $\mathbb{Z}_p^* \longrightarrow \mathcal{O}(X)^*$ whose derivative at 1 is the map κ_i and whose evaluation at any point $z \in Z$ is the elevation to the $\kappa_i(z)$-th power. (This is obvious for $i = 3$ since κ_3 is the null function.)

Note that for every z in Z, the ordering $(p^{\kappa_1}(z)F_1(z), p^{\kappa_2(z)}F_2(z), p^{\kappa_3}(z)F_3(z))$ is a refinement $\mathcal{F}_z$ of the representation $(\bar{\rho}_z)_v$.

Another important piece of data is a map

(c) $\psi : \mathcal{H} \to \mathcal{O}(X)^{\text{rig}}$

For $x \in X(\bar{\mathbb{Q}}_p)$, we call ψ_x the composition of ψ with the map $\mathcal{O}(X)^{\text{rig}} \to \bar{\mathbb{Q}}_p$ given by x. We make more precise our assertion that Z parameterizes certain characters $\psi_{\tau,\mathcal{R}}$ by recalling that

(vi) For every $z \in Z$, corresponding to a character $\psi_{\tau,\mathcal{R}}$ as above, we have $\psi_z = \psi_{\tau,\mathcal{R}}$. Consequently,
(vi') For every $z \in Z$, corresponding to a character $\psi_{\tau,\mathcal{R}}$ as above $\bar{\rho}_z \simeq \rho_\tau$ and the refinement $\mathcal{F}_z$ of ρ_z corresponds to the refinement $\mathcal{R}$ as in [BC3, §6.4].

Moreover we have, by [BC3, Theorem 7.3.1(v)] and by [BC3, Def 7.2.5]

(vii) Let $\kappa = (\kappa_1, \kappa_2) : X \to \mathbb{A}^2$. Then X is admissibly covered by affinoid subdomains such that $\kappa(\Omega)$ is an open affinoid of $\mathbb{A}^2$, $\kappa : \Omega \to \kappa(\Omega)$ is finite, and surjective on every irreducible component of Ω.
(viii) For all $x \in X$, lying in an Ω as above, the natural map (induced by ψ)

$$\mathcal{H} \otimes_{\mathbb{Z}} \mathcal{O}_{\kappa(\Omega),\kappa(x)} \to \mathcal{O}_x$$

is surjective.

Since $\rho^{\perp} \simeq \rho$ for a representation attached to an automorphic form for U(3), we have

(ix) For all g in G_E, we have $T(\gamma g^{-1}\gamma^{-1}) = T(g)$.

6. Non smoothness at $(\pi, \mathcal{R})$

Theorem 6. *Let $\mathfrak{p}$ be any minimal prime ideal of the rigid analytic local ring $\mathcal{O}_y$ of X at y. Then the local domain $\mathcal{O}_y/\mathfrak{p}$ is not a unique factorization domain. In particular, it is not a regular local ring, and the irreducible component of X through y corresponding to $\mathfrak{p}$ is not smooth at y.*

This section is devoted to the proof of this theorem. We denote by $T_{\mathcal{O}_y}$ the restriction of T to $\mathcal{O}_y$, in other words the composition $T_{\mathcal{O}_y} : G_E \stackrel{T}{\to} \mathcal{O}(X) \to \mathcal{O}_y$. Let us call m_y the maximal ideal of this local ring and $k(y)$ its residue field. Then $\bar{T}_y = T \otimes_{\mathcal{O}_y} k(y) : G_E \stackrel{T}{\to} \mathcal{O}_y \to \mathcal{O}_y/m_y = k(y)$ is the trace of the representation $\bar{\rho}_y = \rho_\pi = \rho_0 \oplus 1$ and the refinement $\mathcal{F}_y$ is $\mathcal{F}$.

Hence T is a residually multiplicity free pseudocharacter (see [BC3, §1.4.1]), and thus there is a smallest ideal $I \subset \mathcal{O}_y$ such that $T_{\mathcal{O}_y} \otimes \mathcal{O}_y/I$ is the sum of two pseudocharacters of G_E over $\mathcal{O}_y/I$ of dimensions 2 and 1. The ideal I is called the *(total) reducibility ideal of* $T_{\mathcal{O}_y}$ (see [BC3, Prop. 1.5.1 and Def 1.5.2]). Since $T_{\mathcal{O}_y} \otimes k(y) = \operatorname{tr} \rho_0 + 1$, we obviously have $I \subset m_y$. The following proposition is a key step in the proof.

Proposition 7. *We have $I = m_y$.*

Proof — Since the proof is very similar to the one of [BC3, Prop. 9.3.7], we only sketch the main steps.

Step 1: Remember that we have a subgroup G_v of G_E which is isomorphic to $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Since T is a refined family, we are in position to apply Theorem 4.4.4 of [BC3] to T, if we verify its hypotheses. They are the following :

(REG) : The refinement $\mathcal{F}_y = \mathcal{F}$ of $\bar{\rho}_y = \rho_\pi$ is regular (see [BC3, §4.4.1]). this is clear.

(NCR) The restriction of $\mathcal{F}$ to 1 and ρ_0 are non critical (see [BC3, §4.4.1]). The first one is obvious, the second is Prop 4.

(MF') (see [BC3, §4.4.1]). There is nothing to verify.

(INT) (see [BC3, §4.4.4]). We have to check that R_ρ is a subinterval for the refinement $\mathcal{F}$. But indeed $R_\rho = \{2, 3\}$.

Thus we know that the conclusions of Theorem 4.4.4 of [BC3] holds. They are two-folds :

(1) $\kappa_i - \kappa_j$ is a constant (an element of $\mathbb{Z}$) in the ring $\mathcal{O}_y/I$ for all $i, j \in \{1, 2, 3\}$. Since κ_3 is the constant 0, we see that κ_1 and κ_2 are also constant on $\mathcal{O}_y/I$.

(2) By [BC3, §1.5.2], there exist unique deformations $\tilde{\rho}_0$ and $\tilde{\chi}$ of the Galois representation ρ_0 and 1 such that $T \otimes \mathcal{O}_z/I = \operatorname{tr} \tilde{\rho}_0 + \operatorname{tr} \tilde{\chi}$. By [BC3, Theorem 4.4.4], we know that $\tilde{\rho}_0$ and $\tilde{\chi}$ are *trianguline* (see [BC3, §2.3.2]).

Step 2: We claim that the deformation $\tilde{\rho}_0$ and $\tilde{\chi}$ of ρ and 1 over the artinian (by (1) above and property (vii)) ring $\mathcal{O}_y/I$ are constant. By a standard argument it is enough to see that they are constant over every quotient of the form $k(y)[\epsilon]$ (ring of dual numbers over $k(y)$) of $\mathcal{O}_y/I$, so we may assume that $\mathcal{O}_y/I \simeq k(y)[\epsilon]$.

First we note that those deformation are Hodge-Tate by (1) and trianguline by (2), hence are crystalline at v by [BC3, Theorem 2.5.1]. Moreover, by (ix), they satisfy $\tilde{\chi}^\perp = \tilde{\chi}$ and $\tilde{\rho}^\perp = \tilde{\rho}$. Hence they are also crystalline at $\bar{v}$.

Finally, we have to see that there is no non-trivial such deformations. Those deformations define classes in $H^1_{f_{v,\bar{v}}}(G_E, \mathrm{ad}\rho_0)$ and $H^1_{f_{v,\bar{v}}}(G_E, 1)$. Since neither 1 nor $\mathrm{ad}\rho_0 = \mathrm{ad}\rho_{f,E}$ contains the cyclotomic character $\bar{\mathbb{Q}}_p(1)$, those Selmer groups are equals to $H^1_f(G_E, \mathrm{ad}\rho_{f,E})$ and $H^1_f(G_E, 1)$. The latter is well known to be trivial; By [BC3, Prop. 5.2.6] (recall that f is assumed to be non CM, and note that hypothesis (i) there holds since f has trivial nebentypus), the class of $\tilde{\rho}_0$ in the former is also trivial. This complete the proof of step 2.

Step 3: From the fact that $\tilde{\rho}_i$ and $\tilde{\chi}$ is constant, we see that T is constant on $\mathcal{O}_z/I$, and so is $T(\mathrm{Frob}_w)$ for any finite place w of E not dividing p or a prime of S_0. Similarly, we see using [BC3, Prop. 9.3.6] that the F_i are also constant on $\mathcal{O}_z/I$. As noted earlier, the same results holds for the κ_i.

Now by §3.1, we see that $\mathcal{H}$ is generated by element whose images by ψ are constant (that is, in $k(y)$) in $\mathcal{O}_y/I$. So by (viii), we see that $\mathcal{O}_y/I = k(y)$. □

We now prove the theorem: let $\mathfrak{p}$ be a minimal prime ideal of $\mathcal{O}_y$ and set $A = \mathcal{O}_y/\mathfrak{p}$. We denote by m the maximal ideal of the domain A, and K its fraction field. We denote by T_A the restriction of T to A. The pseudocharacter T_A is still a residually multiplicity free pseudocharacter (with residue pseudo-character $T_A \otimes (A/m) = T_y = \mathrm{tr}\,\rho_0 + 1$), and its reducibility ideal is m.

We set $R = A[G_E]/\mathrm{Ker}\,T_A$. Consider the anti-involution τ on $A[G_E]$ defined by $\tau(g) = \gamma g^{-1}\gamma$ and A-linearity. Then the two-sided ideal $\mathrm{Ker}\,T$ is stable by τ (this follows immediately from (ix) above) and τ induces an anti-involution on $R/\mathrm{Ker}\,\tau$ (still denoted τ).

We will now invoke some of the main results of the chapter one of [BC3] to describe the structure of the A-algebra R. By [BC3, Thm 1.4.4 and Lemma 1.8.4], R is an A-algebra of finite type that can be realized as a sub-generalized matrix algebra of type $(1,2)$ of $M_3(K)$. More precisely, there exists a data of idempotents e_1, e_{ρ_0} (that is, e_1 and e_{ρ_0} are orthogonal idempotents of sum 1, that lift the two central idempotents of $R \otimes k/\mathrm{Ker}\,(T_A \otimes k)$ attached respectively to 1 and ρ_0) such that moreover $\tau(e_1) = e_1$, $\tau(e_{\rho_0}) = e_{\rho_0}$ for R, and an injective morphism $r : R/\mathrm{Ker}\,T \hookrightarrow M_3(K)$ such that $r(e_1) = \mathrm{diag}(1,0,0)$ and $r(e_{\rho_0}) = \mathrm{diag}(0,1,1)$. We have moreover

$$R/\mathrm{Ker}\,T \simeq r(R/\mathrm{Ker}\,T) = \begin{pmatrix} A & M_{1,2}(A_{1,\rho_0}) \\ M_{2,1}(A_{\rho_0,1}) & M_2(A) \end{pmatrix} \subset M_3(K),$$

where A_{1,ρ_0} and $A_{\rho_0,1}$ are finite type A-submodules of K such that $A_{1,\rho_0}A_{\rho_0,1} \subset m \subset A$.

Actually, $A_{1,\rho_0}A_{\rho_0,1}$ is equal to the reducibility ideal of T_A by [BC3, Prop. 1.5.1], that is

$$A_{1,\rho_0}A_{\rho_0,1} = m \tag{5}$$

Since $\tau(e_1) = e_1$ and $\tau(e_{\rho_0}) = e_{\rho_0}$, and since r is adapted, we see easily (or by applying [BC3, Lemma 1.8.5]) that

$$A_{1,\rho_0} \simeq A_{\rho_0,1} \text{ as } A\text{-modules} \tag{6}$$

We are now ready to prove the theorem. By contradiction, assume that A is a UFD. This will allow us to apply the following lemma, which is also implicitly used in the proof of Prop. 1.6.1 in [BC3].

Lemma 8. *Assume that A is a UFD with fraction field K and that B and C are fractional ideals such that $BC \subset A$. Then there is an $x \in K^*$ such that $xB \subset A$ and $x^{-1}C \subset A$.*

Proof — Let V be the set of (discrete) valuations of A attached to prime ideals of height 1 of A. For a submodule $(0) \neq M$ of K, denote by $v(M)$ the infimum of the $v(y)$ for all y in $M - \{0\}$.

If B or C is (0) the result is trivial, so let us assume that $B \neq (0)$ and $C \neq (0)$. Since B is of finite type, $v(B)$ is finite for all v in V and zero for almost all v. Since A is a UFD, there is an $x \in K^*$ such that $v(x) = -v(B)$ for all $v \in V$. We now see that $v(xB) = 0$ for all v and also that $v(x^{-1}C) = v(B) + v(C) \geq 0$ since $BC \subset A$. Recall that an element y of K such that $v(y) \geq 0$ for all $v \in V$ is in A since A is an UFD, hence normal. We thus have $xB \subset A$ and $x^{-1}C \subset A$. □

Using the lemma, from $A_{1,\rho_0}A_{\rho_0,1} \subset A$ it follows that there is an $x \in K^*$ such that

$$xA_{1,\rho_0} \subset A \text{ and } x^{-1}A_{\rho_0,1} \subset A.$$

Replacing A_{1,ρ_0} by $xA_{\rho_0,1}$ and A_{1,ρ_0} by $x^{-1}A_{1,\rho_0}$, we see that (5) and (6) still hold, but with A_{1,ρ_0}, $A_{\rho_0,1}$ now *ideals of* A. But this is a contradiction!

Indeed, from (5), we see that $\{A_{1,\rho_0}, A_{\rho_0,1}\} = \{A, m\}$ but this contradicts (6) since A and m are not isomorphic as A-modules (recall that A has Krull dimension 2). This completes the proof of the theorem.

Remark 9. (1) The same proof shows that T_A is not the trace of a representation of G_E over a free A-module.

(2) In general (without assuming that A is a UFD), we see that neither A_{1,ρ_0} and $A_{\rho_0,1}$ is generated as an A-module by a single element. Otherwise both would, and so would be m, but this is absurd since A has dimension 2.

(3) In [BC2] the smoothness of the eigencurve at a critical Eisenstein point is proved. The proof there uses the same type of objects, but interstingly goes the other way around: namely the two structural modules of R there (analogous of our A_{1,ρ_0} and $A_{\rho_0,1}$ here) are shown to be principal using a Selmer groups argument, from which it follows that m is principal - which is no contradiction there, since A has dimension 1, but simply proves that A is regular.

7. Relation with Selmer groups

Since the pseudocharacter $T_{\mathcal{O}_y}$ is generically irreducible on a neighborhood of y, but such that $T_y = \rho_0 \oplus 1$, it allows us to construct non-trivial extensions of 1 by ρ_0 and of ρ_0 by 1, that is non-zero elements of $\mathrm{Ext}^1_{G_E}(1, \rho_0)$ and of $\mathrm{Ext}^1_{G_E}(\rho_0, 1)$.

More precisely, the theory developed in [BC3] attaches to T subspaces $\mathrm{Ext}^1_{G_E,T}(1,\rho_0)$ and $\mathrm{Ext}^1_{G_E,T}(\rho_0,1)$ of the above Ext^1, that parameterize all the extensions that are "seen by T", more precisely that appear as a subquotient of an $\mathcal{O}_y[G_E]$-module of finite type and torsion free over $\mathcal{O}_z$, with generic trace T.

From the theory of Chapter 1 of [BC3], we have

Lemma 10. *The operation $U \mapsto U^\perp$ induces an isomorphism from the space $Ext^1_{G_E,T}(1,\rho_0)$ onto $Ext^1_{G_E,T}(\rho_0,1)$. Moreover, these spaces have dimension ≥ 2.*

Proof — The first assertion is [BC3, Prop. 1.8.6].

The second follows from the first and the fact that obviously, if A is $\mathcal{O}_y/\mathfrak{P}$ as in the preceding section, $\mathrm{Ext}^1_{G_E,T_A}(1,\rho_0) \subset \mathrm{Ext}^1_{G_E,T}(1,\rho_0)$, and with the notation above

$$\mathrm{Ext}^1_{G_E,T_A}(1,\rho_0) = A_{\rho_0,1}/mA_{\rho_0,1}.$$

By Remark 9, the latter space has dimension at last 2. □

Lemma 11. *We have $Ext^1_{G_E,T}(1,\rho_0) \subset H^1_{f^{\bar v}}(G_E,\rho_0)$*

Proof — If w is a finite place of E that does not divide p, then every extension U of 1 by ρ_0 is automatically f at w since $(\rho_0)_{|D_w}$ does not contain $\mathbb{Q}_p(1)$ (since it has motivic weight 0).

We have to see that if U is an extension as above in $\mathrm{Ext}^1_{G_E,T}(1,\rho_0)$, then it is f at v, that is $\dim D_{\mathrm{crys}}(U_v) = \dim D_{\mathrm{crys}}((\rho_0)_v) + 1 = 3$.

To see this, we first note that $D_{\mathrm{crys}}((\rho_0)_v)$ is a subspace of $D_{\mathrm{crys}}(U_v)$ on which ϕ acts with eigenvalues distinct of 1 (namely α and β, the second and third eigenvalues of the refinement $\mathcal{F}$.)

Second, we invoke [BC3, Thm. 4.3.6] that says that, since 1 is the first eigenvalue of the refinement $\mathcal{F}$, we have $\dim D_{\mathrm{crys}}(U_v)^{\phi=1} = 1$. [To check that the theorem actually says so, please take for the partition $\mathcal{P}$ *loc. cit.* the only non trivial one $\{\{1\},\{\rho_0\}\}$, for the ideal I *loc. cit.* the maximal ideal, and remember that the analytic functions denoted κ and F there are our κ_1 and F_1. Note also that our extension U belongs to $\mathrm{Ext}^1_{G_E,T}(1,\rho_0)$, and so by definition is in the image of $\iota_{\rho_0,1}$, so we can apply the theorem to U and thus conclude that $D_{\mathrm{crys}}(U_v(\kappa_1(y)))^{\phi=F_1(y)}$ has dimension 1. Since $\kappa_1(y) = k_1$ is a rational integer, we can twist and rewrite this as : $D_{\mathrm{crys}}(U_v)^{\phi=p^{k_1}F_1(y)}$ has dimension 1. But $p^{k_1}F_1(y) = p^{\kappa_1(y)}F_1(y)$ is precisely the first eigenvalue of the refinement $\mathcal{F}$, namely 1.]

Thus, we see that $D_{\mathrm{crys}}(U_v)$ contains a subspace of dimension 2 and a subspace of dimension 1 which have trivial intersection. Hence the results follows. □

However, as mentioned in the introduction, the lower bound (2) on the dimension of $H^1_{f^{\bar v}}(G_E,\rho_0)$ implied by the above two lemmas may be proved directly, by an easy Poitou-Tate argument. Indeed

Proposition 12. *We have $\dim H^1_{f^{\bar v}}(G_E,\rho_0) \geq 2$, and the equality holds if the conjecture of Bloch-Kato is true for ρ_0.*

Proof — If the conjecture of Bloch-Kato is true for ρ_0, then since the motivic weight of ρ_0 is 0, $H^1_f(G_E,\rho_0) = 0$. Thus the restriction map $H^1_{f^{\bar v}}(G_E,\rho_0) \to$

$H^1(G_{\bar{v}}, \rho_0)$ is injective. Since the latter space has dimension 2 by Tate's local Euler-characteristic formula, we see that, under the Bloch-Kato conjecture

$$\dim H^1_{f^{\bar{v}}}(G_E, \rho) \leq 2.$$

On the other hand, unconditionally, we have by Poitou-Tate duality and the global Euler-characteristic formula the well-known equality (see e.g. [W, Theorem 2]) :

$$\begin{aligned}\dim H^1_{G_E, f^{\bar{v}}}(\rho) \quad = \quad & \dim H^1_{f^{\bar{v}},\text{trivial at } \bar{v}}(G_E, \rho_0^*(1)) \;+ \\ & \dim H^0(G_E, \rho) - \dim H^0(G_E, \rho_0^*(1)) \;+ \\ & \sum_w [\dim H^1_{f^{\bar{v}}}(D_w, \rho_0) - \dim H^0(D_w, \rho_0)].\end{aligned}$$

In the above sum w goes among all places of E (finite or not), D_w is a decomposition group at w and $H^1_{f^{\bar{v}}}(D_w, \rho_0)$ is the subspace of $H^1(D_w, \rho)$ parameterizing extensions as D_w-representations of $\bar{\mathbb{Q}}_p$ by ρ_0 that are f if $w \neq \bar{v}$, and with no condition if $w = \bar{v}$.

We now compute all terms on the second and third lines:

- The global terms $\dim H^0(G_E, \rho_0)$ and $\dim H^0(G_E, \rho_0^*(1))$ are 0 since ρ_0 is irreducible.
- for w finite not dividing p, the term $\dim H^1_f(D_w, \rho_0) - \dim H^0(D_w, \rho_0)$ is 0 by [FPR, §3.3.11].
- For $w = v$, the term $\dim H^1_f(D_v, \rho_0) - \dim H^0(D_v, \rho_0)$ is 2 by [BC3, §3.3.11] since the *two* Hodge-Tate weights of ρ_0 are negative.
- For $w = \bar{v}$ the term $\dim H^1_f(D_{\bar{v}}, \rho_0) - \dim H^0(D_{\bar{v}}, \rho_0)$ is $\dim \rho_0 = 2$ by the local Poitou-Tate Euler-Characteristic formula and the fact that $(\rho_0)_{\bar{v}}$ does not contains $\mathbb{Q}_p(1)$ (since it has positive Hodge-Tate weights.)
- For $w = \infty$, $D_\infty = 1$, and the term is -2 since we have $\dim H^1(D_\infty, \rho_0) = 0$ and $\dim H^0(D_w, \rho_0) = \dim \rho_0 = 2$.

Summarizing, we have

$$\dim H^1_{f^{\bar{v}}}(G_E, \rho) = \dim H^1_{f^{\bar{v}},\text{trivial at } \bar{v}}(G_E, \rho_0^*(1)) + 2$$

hence

$$\dim H^1_{f^{\bar{v}}}(G_E, \rho) \geq 2.$$

□

Corollary 13. *If the Bloch-Kato conjecture holds for ρ_0, we have $Ext^1_{G_E,T}(\rho_0, 1) = H^1_{f^{\bar{v}}}(G_E, \rho_0)$.*

As we might put it : the two independent extensions of $\bar{\mathbb{Q}}_p$ by ρ_0 in the relevant Selmer group predicted by the Bloch-Kato conjecture are seen by the eigenvariety of U(3) at a point y such that $\bar{\rho}_y = \bar{\mathbb{Q}}_p \oplus \rho_0$; for that purpose, the eigenvariety X needs to be non smooth at y, and so it is.

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