# ON THE PATH-DEPENDENCE OF ECONOMIC GROWTH

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Abstract. The cumulated growth of an economy during a period of time depends on the whole economical history during this period and not only of the situation at the beginning and at the end of the interval. However, if some constraint is made on the set of possible economies, this phenomenon of history-dependence may disappear. We give a simple necessary and sufficient condition on the constraint for this to happen, generalizing results of Samuelson-Swamy and Hulten. Our approach is different, based on differential and symplectic geometry. We also solve the same question for a subsector of the economy.

## 1. Introduction

The real *Gross Domestic Product* (*GDP*), absolute or relative to the number of inhabitants, and its rate of growth, called the real (as opposed to nominal, that is, after taking out the inflation) *economic growth* are certainly the most popular indicators of the overall success of a nation's economy.

Still, their definitions are quite subtle, and leads to important counter-intuitive properties that will be the object of this paper.

To be sure, there are two different methodologies used to define the GDP and the economic growth: the fixed-weight methodology and the chained-weight methodology.

The first one defines the GDP of a period $t$ relatively to a base period $t_0$ as the values that would have the total output of year $t$ *if all prices had remained at their period* $t_0$ *value*; the growth is then defined as the

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rate of change of the GDP. This method has very important undesirable properties[1] and has been abandoned by the agencies in charge of computing GDP almost everywhere (since 1996 by the B.E.A. in the US). We will not mention it anymore.

The chained weight, instead, uses contemporary prices to compute the growth of any given year. The GDP, more precisely the relative GDP of period $t$ with respect of the GDP of period $t_0$, is defined by multiplying the growth for all periods between $t_0$ and $t$, hence the "chained" name. To be a little bit more precise, there are several slightly different ways to define the growth between two consecutive periods $t-1$ and $t$. To do so, we can measure the relative increase of production between $t-1$ and $t$ of each good *weighted* by their prices at time $t-1$ (getting the *Laspeyres index*), or by their prices at time $t$ (getting the *Paasche index*) or by some average between their prices at time $t-1$ and $t$; we could also average the Laspeyres and Paasche indices, as the *Fischer index*, defined as the geometric mean of the Laspeyres and Paasches index (see Fischer (1922)) which is the one actually used by the B.E.A.; still other variants, like the *Törqnvist index* (see Törqnvist (1936)) are in use or have been proposed, and an extensive literature studies the merits of those various indices in different perspectives – see e.g. Diewert (1976).

Clearly, the smaller is the period considered, the closer are those indices from each other. We shall work in continuous time, that is to say, with infinitesimal periods. Hence the precise choice of the index will be of no interest to us. The chained growths between time $t_0$ and $t$, computed using *any* of the above mentioned indexes over smaller and smaller periods converges to the same formula, known as the Divisia line integrals (Divisia

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[1] The most striking undesirable property of fixed-weight computed growth is its dependence on the base year. For example, Whelan (2000) has shown that the growth rate of the year 1998 computed with 1995 (resp. 1990, 1980, 1970) is 4.5% (resp. 6.5%, 18.8% and a ridiculous 37.4%). Clearly, the fixed-weight growth indexes are of a little use for the analysis of long term economic growth.

1922) which will be recalled in section §2 below. For a proof, see Trivedi[2] (1981).

The main merit of the new methodology is that the quotient of the GDP between any two periods does not depend on the base year. However, this methodology presents important counter-intuitive, and in some respect undesirable, tough unavoidable, features. One is well-documented in the recent academic literature the lack of additivity of the growth with respect to subsectors of the Economy. For this question, we refer the reader to Whelan (2000) and to Landerfeld and Parker (1997). The other is the main theme of this paper: it is the fact that the quotient of GDP between time $t$ and $t_0$ does not depend only of the economic situation at time $t_0$ and at time $t$, but instead depends on the path followed by the economy between $t_0$ and $t$, that is, of the whole history of economy [3] between $t_0$ and $t$. An important consequence of this fact is that we should be very careful when using the GDP as a measure of *real output* at some time $t$, since for example it is consistent (see Prop. 2.2 below) that the GDP at time $t$ be measured as twice the GDP at time $t_0$ and at the same time all prices and production quantities be the same at time $t_0$ and at time $t$. Similarly, it is not inconsistent to imagine two countries with the exact same (or

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[2] Actually Trivedi states the result for all standard indices but proves it only for the Törnqvist index. The proof for the Paasche's, Laspeyres's, and Fischer's indices is similar, using in adition the formula $\log(1+x) = 1 + x + O(x^2)$. We take the occasion to say a word about the rate of convergence of the chained growth index to the Divisia integral, when the number of subdivision of the given time period is $n$. Trivedi states that when a chained Törnqvist index is used to approximate the Divisia integral, the error is in $o(1/n^2)$. But he makes a small terminological confusion here as the error term is only $O(1/n^2)$, that is, less than a constant times $1/n^2$. His proof is correct and really gives $O(1/n^2)$. For Paasche, Fischer, and Laspeyres index, it is not very hard to prove that the error is $O(1/n)$ and that this result is optimal (i.e. the error is not $o(1/n)$, in particular is not $O(1/n^2)$ – the $O(1/n^2)$ is irremediably lost when using the Taylor approximation $\log(1 + x) = x$. Fischer's index in this respect does not better than Paasche's and Laspeyres's indices.

[3] Saussure (1913) pointed out that economics, like linguistics but unlike most other areas of knowledge, admits a dichotomy between a synchronic approach, where one try to understand the economic facts as a given time at a whole—like in the theory of the partial or global equiibrium—and a diachronic one, where one study the evolutions of variables over time. Using that terminology, one could restate the point being made by saying that the GDP at a given time is fundamentally a diachronic concept, not a synchronic one as one could think.

proportional) economy at time $t_0$ and $t$, but with different measured growth between $t_0$ and $t$ for the two countries (if their histories between $t_0$ and $t$ differ).

This path-dependency (or history-dependency) property of Divisia indexes has of course been observed by several economists: see Richter (1966), Hillinger (1970), Hulten (1973), Sato (1976) and the very lucid exposition of Samuelson and Swamy (1974) (though most of those authors were interested in occurrences of Divisia indexes in other economic contexts than global economic growth.). But at the same time, this fact and its consequences on the interpretation of long-term real growth seem to be largely ignored or overlooked. Major macro-economics graduate textbooks do not mention it, and instead emphasize the importance of long term economic growth, as in the followng examples: "The dominant macroeconomic fact in developed economies in the last two centuries is that of output growth. [...] real GNP is about 37 times larger than it was in 1874, 7 times larger than in 1919, and 3 times larger than in 1950. Extrapolating backward leads to the well-known conclusion that economic growth at these rates cannot have been taking place for more than a few centuries. " (Blanchard) "From 1869 to 1996, over 127 years, the total output [of the U.S] was increased 64-folds" (Barro, page 4). Obviously, if the readers were made aware that this assertion would not be in contradiction with a strictly identical output in 1869 and 1996 (and what is more, a stricly identical price structure), it would not have the same impact. More surprisingly, even the literature explicitly aimed at warning users of the counter-intuitive features of the chained methodology to compute growth does not mention the history-dependence of growth: see Whelan (2000) and Landefeld and Parker (1997). Finally, for an history of misconception on those points in the economic academia, see the historical account by Samuelson and Swamy (1974), pages 579 and 580.

The aim of this article is to investigate in more details that was done earlier this history-dependence phenomenon.

We begin by recalling (section §2) why the growth actually depends on history. We claim no novelty here, especially comparted to the short presentation by Samuelson and Swamy on which it is difficult to improve. However, our presentation is a little more mathematical than elsewhere in the literature. We explain in particular that this dependence is of local type, not only global, which mathematically corresponds to the fact that the growth differential form is not only not exact, but actually not closed. This implies that *any* growth between any start and end economies can be attained by a suitable history in between, a result of which we (try to) give an elementary and intuitive proof.

We then investigate (section §3) the following question: under what constraints on the set of allowed economies is the economic growth independent on the history? Again, this question has already been studied in the literature, especially Hulten (1973) (see also Sato (1976) and Samuelson and Swamy (1974)). Those authors claim that they give a necessary and sufficient condition for a Divisia line integral to be path-independent. They do so, though, under fairly restrictive hypotheses that are not relevant in the context of economic growth (which was not their primary concern), since they all assume that the prices are a given function, independent of time, of the quantities produced. (The necessary and sufficient condition is that this function is homogeneous and, up to a scalar, a gradient.) Our approach is more general: we determine the submanifolds $M$ of the set of all possible economies that have the property that, if the economy is constrained to stay on $M$, its growth is independent of history. We call such submanifolds *ahistorical*. Thus, we do not assume a priori that the prices are determined by quantities, ou quantities by prices, or any dependency of this type whatsoever. All type of relations between prices and quantities are included in the general concept of maifolds.

Maximal ahistorical submanifolds $M$ are proved to have dimension $n+1$ (if $n$ in the number of goods in the economy) and to be locally the (saturated) graph of a function that determines a set of $n$ variables among the prices and quantites of the $n$ goods in terms of the $n$ other ones, a function which as in Hulten's result has to be homogeneous, and a gradient. The resemblance of this result with Hulten's should not let one forget that we answer a much more general question: our result excludes for example that any submanifold of dimension $n+2$ might satisfy the history-independance property, which Hulten' or Samuelson-Swamy's result does not.

Moreover our approach and methods are different from the one of Hulten and other mentioned authors, local (that is, differential) rather than global. We observe that the set of all possible economies (after identifying economies that have the same relative prices or relative quantities) has a natural structure of a symplectic manifold, that is a manifold provided with a differential 2-form which is non-degenerate at every point. *Symplectic geometry*, the study of symplectic manifolds, took its origin in the reformulation of classical mechanics by Lagrange and Hamilton in the 19th century and is now a important domain of mathematics on its own. We characterize the (locally) ahistorical submanifolds as the Lagrangian submanifolds of the symplectic variety of all possible economies and use the ample resources of symplectic geometry to answer the question.

We next turn (section §4) to the following problem, which was also considered by Hulten (1973). Assuming that the economy is constrained to satisfy the condition for the growth to be path-independent, when does this property still hold for a given sub-sector of the economy? Hulten's answer, beyond the same lack of generality as mentioned above, is inexact. We show how our differential approach allows for a simple solution of the problem.

In the last section, we try to measure in some way the order of magnitude of the history-dependence of the economic growth for the real world. To find a meaningful way of measure it is actually theoretically difficult, and we do not solve most of those difficulties. We present the result of some "naive" computations that nevertheless show, we believe, that the history-dependence is not negligible for long-term periods, say over 30 or 40 years.

## 2. History-dependence of the growth

### 2.1. Notations and hypotheses.

In a continuous time setting, the time is represented by a real variable $t$, in some given interval $I$ which is the period under observation.

Suppose that $n$ types of goods are produced in our economy, represented by an index $i = 1, \dots, n$. Our economy is described by $2n$ functions on $I$ : the price functions $p_i(t)$ representing the price of a unit of good $i$ at time $t$, and the quantities functions $q_i(t)$ representing the number of unit of good $i$ produced by unit of time at time $t$. We assume those functions to be $\mathcal{C}^\infty$, and the prices $p_i$ to be non negative[4].

We shall denote by $P(t)$ and $Q(t)$ the column vectors of prices and quantities. We will often omit the $t$. In general, we will use capital letters for vectors and matrices, and ${}^tX$ will denote the transpose of the matrix $X$.

Obviously, the *nominal* GDP at time $t$ by unit of time is

$$\mathrm{GDP}_{\mathrm{nom}}(t) = \sum_{i=1}^{n} p_i(t) q_i(t) = {}^tPQ(t). \tag{1}$$

We shall assume for simplicity that for all $t \in I$,

$$\mathrm{GDP}_{\mathrm{nom}}(t) = {}^tPQ(t) > 0.$$

The *nominal* (instantaneous) growth $G(t)$ at time $t$ is its relative rate of change, that is, its logarithmic derivative

$$\mathrm{G}_{\mathrm{nom}}(t) = \frac{\mathrm{GDP}_{\mathrm{nom}}{}'(t)}{\mathrm{GDP}_{\mathrm{nom}}(t)} = \frac{{}^tPQ' + {}^tP'Q}{{}^tPQ}. \tag{2}$$

### 2.2. Definition of the real growth and GDP.

The *real* (instantaneous) growth, should not depend on the change in the prices $p_i'(t)$, so one define it as:

$$G(t) = \frac{\sum_{i=1}^{n} p_i(t) q_i'(t)}{\sum_{i=1}^{n} p_i(t) q_i(t)} = \frac{{}^tPQ'}{{}^tPQ}. \tag{3}$$

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[4] We allow, however, the quantities $q_i(t)$ to be negative, since it is perfectly possible that the production process results in a net destruction of some good $i$ – this would certainly happens to the lead if someone discovered the formula to change it in gold, for example.

Similarly, the (instantaneous) inflation at time $t$ is

$$I(t) = \frac{\sum_{i=1}^n p_i'(t)q_i(t)}{\sum_{i=1}^n p_i(t)q_i(t)} = \frac{{}^tP'Q}{{}^tPQ}. \tag{4}$$

We thus have

$$G(t) + I(t) = \mathrm{G}_{\mathrm{nom}}(t) \tag{5}$$

The *real* GDP at time $t$, $\mathrm{GDP}(t)$ shall satisfy, by analogy to (2), the differential equation

$$\mathrm{GDP}'(t) = G(t)\mathrm{GDP}(t). \tag{6}$$

Of course, this defines the function $\mathrm{GDP}(t)$ only up to a multiplicative constant, which is usually determined by imposing an initial condition of the form $\mathrm{GDP}(t_0) = \mathrm{GDP}_{\mathrm{nom}}(t_0)$ for some $t_0$ in $I$: we get this way the GDP at time $t$ *in time-$t_0$ dollars.*

Anyway, the quotient $\mathrm{GDP}(b)/\mathrm{GDP}(a)$ is well defined, and might be called the cumulative real growth between $a$ and $b$. We see easily from (6) that

$$\mathrm{GDP}(b)/\mathrm{GDP}(a) = \exp(\int_a^b G(t)\ dt) \tag{7}$$

This is the so-called *Divisia index* for the growth. There is also a Divisia index for the inflation, defined by replacing $G(t)$ by $I(t)$ above.

### 2.3. **Reformulation in terms of differential forms.**

Let $\mathcal{E}$ be the subset of allowed states of the economy, that is the set of vectors $(P, Q) \in \mathbb{R}^n \times \mathbb{R}^n$ ($P$ and $Q$ denoting two column vectors in $\mathbb{R}^n$) defined by the conditions $p_i \geq 0$ for all $i$ and ${}^tPQ > 0$. It is readily seen that $\mathcal{E}$ is a convex subset of $\mathbb{R}^{2n}$. Our hypotheses can be reformulated as saying that $\gamma(t) := (P(t), Q(t))$ is a $\mathcal{C}^\infty$ $\mathcal{E}$-valued function on $I$. The vector $\gamma'(t) = (P'(t), Q'(t))$ is a tangent vector to $\mathcal{E}$ at he point $\gamma(t)$.

Let us recall[5] that a *differential $k$-form* $\omega$ on a manifold $M$ is a real valued function that takes for arguments a point $x \in M$ and $k$ tangent vectors

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[5] For a good and intuitive introduction to differential geometry—manifolds, differential forms, also cohomology group that we shall need later—aimed at a non primarily mathematical public, the reader may consult *Geometry, Topology and Physics* by Nakahara (2003). Arnold (1989) is also a very good reference.

$v_1, \dots, v_k$ to $M$ at $X$. The value of $\omega$ for those arguments will be denoted $\omega(x)(v_1, \dots, v_k)$, with the $x$ sometimes dropped when it is clear from the context. It is part of the definition that this function depends smoothly on $x$, and linearly on each vector $v_1, \dots, v_k$, with the supplementary condition that if two vectors $v_i$ and $v_j$ are exchanged, the value is changed into its negative (antisymmetry condition).

We introduce two differential 1-forms on $\mathcal{E}$ :

$$\omega_G = \frac{{}^tPdQ}{{}^tPQ} \tag{8}$$

$$\omega_I = \frac{{}^t(dP)Q}{{}^tPQ} \tag{9}$$

so that

$$G(t) = \omega_G(\gamma'(t)) \tag{10}$$

$$I(t) = \omega_I(\gamma'(t)) \tag{11}$$

$$\mathrm{G}_{\mathrm{nom}}(t) = (\omega_G + \omega_I)(\gamma'(t)) \tag{12}$$

We define the *Divisia line integral* $G(\gamma)$ as

$$G(\gamma) = \int_\gamma \omega_G = \int_a^b \omega_G(\gamma'(t))\ dt$$

which depends only of the curve defined by $\gamma$ and not of its parameterization. Similar definitions can be made for $I(\gamma)$ and $\mathrm{G}_{\mathrm{nom}}(\gamma)$. The interest of this line integral is that

$$\mathrm{GDP}(b)/\mathrm{GDP}(a) = e^{G(\gamma)}$$

if $\gamma(t)$ describes the historical prices and productions vector between time $a$ and $b$.

### 2.4. The quotient GDP$(b)$/GDP$(a)$ depends on the history between $a$ and $b$.

Recall that a $k$-differential form $\omega$ on a manifold $M$ is said to be *exact* if it is the differential[6] $\omega = d\omega_0$ of some $(k-1)$-form $\omega_0$ and to be *closed* if the $(k+1)$-form $d\omega$ is 0. Since $d \circ d = 0$, an exact form

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[6] For the meaning and definition of the differential operator $d$ on differential form, see e.g. Nakahara (2003).

is closed, the converse being true only when $M$ has a trivial first cohomology group: $H^1(M, \mathbb{R}) = 0$. This last condition is implied by the simple connexity of $M$ (that is the property that any two paths with the same end-points can be continuously deformed into each other) which in turn is a consequence of being convex – so that it works for our set $\mathcal{E}$.

We also recall that a differential form $\omega$ is exact if and only if every line integral $\int_\gamma \omega$ depends only of the end-points of the curve defined by $\gamma$. Similarly, $\omega$ is closed if and only if every line integral $\int_\gamma \omega$ is not changed when $\gamma$ is *continuously* deformed with its end points fixed.

This being recalled, the assertion on the title, which says that $G(\gamma)$ depends only of the end points of $\gamma$, can be reformulated as $\omega_G$ is not exact. Indeed, we have more:

**Lemma 2.1.** *The* 1*-form* $\omega_G$ *and* $\omega_I$ *are not closed. We have*

$$d\omega_G = -d\omega_I$$

*and*

$$d\omega_G(Q, P)$$

*is the antisymmetric bilinear form of matrix*

$$\begin{pmatrix} 0_n & A \\ -{}^tA & 0_n \end{pmatrix} \tag{13}$$

*where*

$$A = -\frac{1}{({}^tPQ)^2}(P{}^tQ - {}^tPQ\,\mathrm{Id}_n),$$

*and* $0_n$ *is the square* $n \times n$ *zero matrix,* $\mathrm{Id}_n$ *is the square identity matrix.*

*Proof* — We have $\omega_G + \omega_I = \omega_{\mathrm{G_{nom}}} = d\log(\mathrm{G_{nom}})$ so $\omega_G + \omega_I$ is exact, hence closed, and the first equality follows. For the second one, by definition, the $(i, n+j)$ coefficient of the matrix of the bilinear form $d\omega_G(Q, P)$ in the canonical basis (that is the coefficient $(i, j)$ of the matrix $A$) is the derivative with respect to $p_i$ of the coefficient of $dq_j$ in $\omega_G$ minus the derivative with respect to $q_i$ of the coefficient of $dp_j$ – which is 0. So we see that the $(i, j)$ coefficient of $A$ is $-\frac{p_j q_i}{({}^tPQ)^2}$ if $i \neq j$ and $\frac{1}{{}^tPQ} - \frac{p_i q_i}{({}^tPQ)^2}$ when $i = j$, which computes the matrix $A$. Similar computations shows that the diagonal blocks of the matrix of $d\omega_G(Q, P)$ are made of 0's, and the left-lower block is ${}^tA$ by anti-symmetry. $\square$

The following propostion is a general consequence of the local nature of the path-dependency, that is of $\omega_G$ being not closed[7]. We however give an explicit proof.

**Proposition 2.2.** *For any given economic situations $\gamma(a) = (P(a), Q(a))$ at time $a$ and $\gamma(b) = (P(b), Q(b))$ at time $b$, and for any positive real number $\alpha > 0$, it is possible to imagine one (in fact, many) economic history $\gamma$ in between so that the quotient GDP$(b)$/GDP$(a)$ is $\alpha$.*

*Proof* — If $\alpha = 1$, cut the interval $[a, b]$ into $K$ consecutive sub-intervals $[a, a_1], [a_1, a_2], \dots, [a_{n-1}, b]$. On the first one define $E(t)$ by keeping $p_i(t)$ and $q_i(t)$ constant, equal to $p_i(a)$ and $q_i(a)$, for all $i \geq 2$; while for $p_1$ and $q_1$, on the first third of the interval $[a, a_1]$ let $p_1$ goes from $p_1(a)$ to 0, keeping $q_1$ constant, then on the second third keep $p_1$ equals to 0, letting $q_1$ going form $q_1(a)$ to $q_1(b)$, then on the last third let $p_1$ goes form to $p_1(b)$ while $q_i$ remains constant. All this can be done with $p_1$ and $q_1$ $\mathcal{C}^\infty$, with derivative of all orders equals to 0 in $a$ and $a_1$. Then we play the same game on $[a_1, a_2]$ with $p_2$ and $q_2$, keeping the others variables constant, and in general on $[a_{k-1}, a_k]$ with the variables $p_k$, $q_k$. It is clear that we define this way a $\mathcal{C}^\infty$ function $\gamma(t)$ which is the prescribed $\gamma(a)$ and $\gamma(b)$ at $b$, and whose attached growth $G(t)$ is always 0, so that $\text{GDP}(b)/\text{GDP}(a) = 1$.

More generally, to get a cumulated growth $\text{GDP}(b)/\text{GDP}(a) = \alpha > 0$, cut $[a, b]$ in two sub-intervals $[a, c]$ and $[c, b]$, keep all $p_i$ constants on $[a, c]$ while $q_i$ goes from $q_i(a)$ to $q_i(c) = (\log \alpha) q_i(a)$ in such a way that the $q_i' s$ are $\mathcal{C}^\infty$ with derivatives of all orders at $c$ equals to 0. Then on $[c, b]$ chose

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[7] When the path-dependency is of global nature, that is when the differential 1-form is closed but not exact, there are strong restrictions on the possible differences between the line integrals computed with two different paths with the same end points. For example, when Phileas Fogg has completed his journey around the world, he has gained one day. Had he completed several turns, it would have been several days, but in no case could he have gained a fraction of a day. This is the typical example of a path-dependency of global nature.

a function $\gamma(t)$ of total growth 1 as above. The $\gamma(t)$ so defined is $\mathcal{C}^\infty$ and has a total growth of $\alpha$. $\square$

## 3. Conditions for an history-independent growth

3.1. **Position of the problem.** Suppose that we deduce from theoretical analysis, or from empirical evidence, or that we simply make the hypothesis that the economy vector $(P(t), Q(t))$ always stays in a submanifold $M$ of $\mathcal{E}$. This limits the set of possible histories $\gamma$, and it is possible that the Divisia line integral (hence the growth) $G(\gamma)$ does depend only on the endpoints of $\gamma$ if $\gamma$ is constrained to stay in $M$ all along. We call such an $M$ an *ahistorical* submanifold. By the reminder made in §2.4, a submanifold $M$ is ahistorical if and only if the restriction of the 1-form $\omega_G$ to $M$ is *exact*.

In this section, we will try determine which submanifolds $M$ are ahistorical. This problems may be cut in two part, a local problem, and a global, topological problem.

The local problem is to determine which submanifolds $M$ are *locally ahistorical*, that is, have the property that for every $x \in M$, there is a neighborhood $V$ of $x$ in $M$ such that $V$ is ahistorical. This properties is equivalent to the fact that $G(\gamma)$ depends only on the end points of $\gamma$ as long as $\gamma$ stays in $M$ and is continuously deformed. It is also equivalent to the fact that the restriction of $\omega_G$ to $M$ is closed. The global one is to determine which locally ahistorical submanifold are (globally) ahistorical.

In this section, we will solve entirely the local problem and give some sufficient conditions for the global one.

3.2. **Hulten and Samuelson-Sammy's results.** The problem of characterizing path-independence of Divisia index has already been considered by several authors. Hulten (1973) proposes a necessary and sufficient condition for such a path independence. But the problem he solves is actually less general than ours. Indeed, he implicitly assumes that the prices (or maybe the relative prices) are uniquely determined by the quantities. Translated into our language, this means he considers submanifolds $M$ of $\mathcal{E}$ that are the graph of a (differentiable) vector field

$P(Q) = (p_1(Q), \ldots, p_n(Q))$ (that is to say : $P : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable map, and for a quantity vector $Q \in \mathbb{R}^n$, $P(Q)$ is the price vectors of the corresponding goods. The graph $M$ is by definition the set of all $(Q, P(Q)) \in \mathbb{R}^{2n}$ and it is assumed to be a subset of $\mathcal{E}$).

Note that such submanifolds $M$ have dimension $n$ (or $n+1$ if we allow for an arbitrary homotethies on the prices vectors). For those manifolds, Hulten's proves the following very nice result (that we have reformulated in our language):

**Theorem 3.1** (Hulten)**.** *The graph $M$ of a differentiable vector field $P(Q) = (p_1(Q), \ldots, p_n(Q))$ is ahistorical if and only if there is a linear homogeneous real-valued function $f(Q)$ on $\mathbb{R}^n$ such that for all $i = 1, \ldots, n$ and all $Q \in \mathbb{R}^n$*

$$\frac{\partial f}{\partial q_i}(Q) = \lambda(Q) p_i(Q) \tag{14}$$

*for some multiplier $\lambda$ independent of $i$ (but dependent on $Q$)*

**Remark 3.2.**

- (i) Condition (14) says that the vector field $P(Q)$ is proportional to the gradient of the function $f$ (if we identify in the obvious way the spaces of quantities $\mathbb{R}^n$ where the gradient lives, with the space of prices $\mathbb{R}^n$). In other words, it says that $M$ is the graph of the gradient of $f$.
- (ii) In the necessary and sufficient condition given by the theorem, linear homogeneous may be replaced by homogeneous of degree $\alpha$ for $\alpha$ an arbitrary positive number. Simply replace $f$ by $f^\alpha$, which does not change condition (14).

A similar result, with similar hypotheses, is also proved in Samuelson and Swamy (1974), pages 578-579.

The problem with those results are the hypothesis that prices are given functions of quantities. Such an hypothesis makes sense in certain microeconomic contexts, but in a real-world economy, with productions and utility functions of all agents changing over time, it is very restrictive to assume a priori that prices are fixed functions of the quantities.

Indeed, those results fail to see many cases where the growth Divisia index is path-independent. To give a trivial example, assume that for some reason, the produced quantities are fixed (infinitely inelastic production functions !) while relative prices vary in time with, say, the tastes of customers. Then, obviously, the growth Divisia index is always 0, and in particular is path-independent. But Hulten's and Samuelson-Swamy's results do not account for this example.

But more importantly, those results do not exculde the fact that less strong relations between the $p_i$ and the $q_i$ (for example, $p_1$ determined in a suitable way by the other prices and all quantities) would be enough to ensure the desirable path-independence property. In other words, they determine among submanifold of $\mathcal{E}$ that are graphs of functions giving prices in terms of quantities (and that are in particular of dimension $n$) which are ahistorical, but they say nothing about other submanifolds, in particular, submanifolds of greater dimensions.

3.3. **Results.** The three following results completely describe the local structure of ahistorical submanifold. The global structure is investigated in §3.8.

**Proposition 3.3.** *Every ahistorical (and locally ahistorical) submanifold of $\mathcal{E}$ has dimension at most $n+1$*

**Proposition 3.4.** *Every locally ahistorical submanifold is locally included in a locally ahistorical submanifold of dimension $n+1$ of $\mathcal{E}$.*

**Theorem 3.5.** *If $M$ is a submanifold of dimension $n+1$ of $\mathcal{E}$, and $x \in M$, $M$ is locally ahistorical at $x$ if and only if there exits a choice of $k$ goods ($0 \leq k \leq n$) among the goods $1, \ldots, n$ (which, after reordering, can and will be assumed to be the goods $1, \ldots, k$) a neighborhood of $x$,and a linear homogeneous function of $n$ variables $f(q_1, \ldots, q_k, p_{k+1}, \ldots, p_k)$ such that $M$ is defined in this neighborhood by the equations*

$$\frac{\partial f}{\partial q_i}(q_1, \ldots, q_k, p_{k+1}, \ldots, p_n) = \lambda p_i \quad \textit{for } i = 1, \ldots, k \tag{15}$$

$$\frac{\partial f}{\partial p_i}(q_1, \ldots, q_k, p_{k+1}, \ldots, p_n) = \lambda q_i \quad \textit{for } i = k+1, \ldots, n \tag{16}$$

*for some multiplier* $\lambda$ *independent of* $i$ *(but dependent on* $q_1, \ldots, q_k$*,*$p_{k+1}, \ldots, p_n$*)*

**Remark 3.6.** (i) The equations defining (locally) $M$ determine the $n$ *dependent* variables $p_1, \ldots, p_k, q_{k+1}, \ldots, q_n$ in terms of $n+1$ *independent variables*: $q_1, \ldots, q_k, p_{k+1}, \ldots, p_n$ and $\lambda$. It is thus easily seen that $M$ has dimension $n+1$.

(ii) The same remarks as in 3.2 apply here: the conditions of the theorem can be interpreted in terms of the gradient of $f$, and $f$ can be imposed to be homogeneous of degree $\alpha$ instead of 1.

(iii) The necessary and sufficient condition can be split in two assertions:

(a) In a neighborhood of $x$, a set of $n$ variables among $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ (that does not contain any pair $\{p_i, q_i\}$) is determined up to a scalar by the $n$ others. That is, if we agree to call $1, \ldots, k$ the goods whose the quantities $q_i$ is an independent variables, there is a map $F : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$F(q_1, \ldots, q_k, p_{k+1}, \ldots, p_n) = \lambda \ (p_1, \ldots, p_k, q_{k+1}, \ldots, q_k)$$

where $\lambda$ is a scalar function of $q_1, \ldots, q_k, p_{k+1}, \ldots, p_k$.

(b) The function $F$ is a gradient of some function $f$ and $f$ is homogeneous.

The point (b) is very similar to Hulten's theorem, except that ours is a little more general since we allow other dependences than "prices are determined by quantities". But the point (a), as a necessary condition for historicity, is absent of Hulten's or Samuelson-Swamy's work since a special form of (a) is assumed to begin with.

(iv) In the assertion (a) above, there can be more than one possible choice of the dependent variables and the independent variables. Indeed, as long as the suitable differentials are invertible at $x$, we can interchange a $p_i$ and a $q_i$ as a dependent or independent variables[8]. Hence if there are points $x$ where only one choice of set of

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[8] in exactly the same way that a line in the $(x, y)$-plane may be described by $y = ax+b$ or $x = a'y + b'$, excepted if it is vertical or horizontal, in which cases only one of those descriptions is available.

> independent variables is possible, in most points several or even all choices will be possible.
>
> It is easy to see, that even if there are several ways to chose the independent variables and the dependent ones, leading to different function $F$, condition (b) is independent of the way we select the dependent variables.
>
> Finally, note that the choice of the $n$ independant variables among $p_1, \dots, p_n$, $q_1, \dots, q_n$ is not totally arbitrary in the theorem: if $q_i$ is independant, then $p_i$ must be dependant and conversely.

The next subsections are devoted to the proofs of those results.

We recall that we have seen that $M$ is locally ahistorical if and only if $\omega_G$ is closed on $M$, that is if and only if $d\omega_G = 0$ on $M$.

### 3.4. **Study of the 2-form $d\omega_M$.**

**Lemma 3.7.** *Let $w$ and $w'$ be two (column) vectors in $\mathbb{R}^{2n}$ considered as the tangent space to $\mathcal{E}$ at a point $(Q, P)$. Write ${}^tw = ({}^tu, {}^tv)$ (resp. ${}^tw' = ({}^tu', {}^tv')$ where $u$ (resp. $u'$) is tangent to the space of quantities and $v$ (resp. $v'$) to the space of prices. Then we have*

$$\begin{aligned} d\omega_G(P,Q)(w',w) &= ({}^tPQ)^{-1}({}^tu'v - {}^tuv') \\ &+ ({}^tPQ)^{-2}({}^tu'P{}^tQv - {}^tuP{}^tQv'). \end{aligned} \tag{17}$$

*Proof* — This follows directly from Lemma 2.1. □

**Lemma 3.8.** *Let $(Q, P) \in \mathcal{E}$ be a quantities-prices vector. The kernel of the antisymmetric form $d\omega_G(Q, P)$ has dimension 2. It is generated by the vector $(Q, 0)$ and $(0, P)$.*

*Proof* — Recall that the kernel of a bilinear form $b$ is the space of vectors $w$ such that $b(w', w) = 0$ for all vectors $w$. By Lemma 2.1, it is readily seen the kernel of $d\omega(P, Q)$ is the sum of the kernel of the matrix $A$ on the tangent to price vectors and of the matrix ${}^tA$ on the tangent space to quantity vectors. Since

$$A = \frac{1}{({}^tPQ)^2}(P{}^tQ - {}^tPQ\mathrm{Id}),$$

the kernel of $A$ is the eigenspace of $P{}^tQ$ for the eigenvalues ${}^tPQ$. It is easy to see that this eigenspace is generated by $P$. A similar result holds for ${}^tA$ and the lemma follows. □

**Remark 3.9.** From the proof, we see that $-({}^tPQ)^2A$ is a projector on its image of kernel generated by $P$.

Recall that a *symplectic manifold*[9] is a manifold with a differential 2-form which is closed and non-degenerate (i.e. of trivial kernel) at every point. Here $d\omega_G$ does have a kernel but it is of constant dimension, and quite easy to describe. We shall see that by replacing $\mathcal{E}$ by a suitable quotient, we can make $d\omega_G$ non-degenerate without loss of information.

3.5. **Quotient and saturation.** The group $\mathbb{R}_+^* \times \mathbb{R}_+^*$ naturally acts on $\mathcal{E}$: $(\alpha, \beta)$ sends $(Q, P)$ on $(\alpha Q, \beta P)$ (which is still in $\mathcal{E}$ since $\alpha, \beta > 0$). This action defines an equivalence relation $\sim$ as follows: $(Q, P) \sim (Q', P')$ if $Q' = \alpha Q$, $P' = \beta P$ for some $\alpha, \beta > 0$; in other words two economies are considered equivalent if their relative prices and relative quantities of production are the same.

We shall denote $\mathcal{RE}$ ($\mathcal{R}$ here stands for "relative") the quotient of $\mathcal{E}$ by that action, that is the set of equivalence classes of $\sim$ in $\mathcal{E}$, or economically speaking the set of economies where only *relative* prices and quantities matter. This set $\mathcal{RE}$ has a natural structure of differentiable manifold of dimension $2n-2$, and the natural projection $\pi : \mathcal{E} \to \mathcal{RE}$, which sends a point in $\mathcal{E}$ to the equivalence class where it belongs, is a surjective and submersive differentiable map. Recall that the pull-back $\pi^*$ of a differential $k$-form $\omega$ by a differentiable map $\pi$ is defined as $\pi^*(\omega(x))(v_1, \ldots, v_k) := \omega(\pi(x))(D_{\pi,x}(v_1), \ldots, D_{\pi,x}(v_k)$ where $D_{\pi,x}$ is the differential of $\pi$ at $x$.

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[9] We refer the reader to McDuff-Salamon (1998) for more information about symplectic manifolds, and their isotropic and Lagrangian submanifolds. Another good reference is Arnold (1989)

**Proposition 3.10.** *The* 2*-form* $d\omega_G$ *is the pull-back* $\pi^*\omega$ *of a unique* 2*-form* $\omega$ *on* $\mathcal{RE}$*. (Mathematicians say, in imaged terms, that* $d\omega_G$ *descends to the quotient* $\mathcal{RE}$*).*

*The* 2*-form* $\omega$ *on* $\mathcal{RE}$ *is symplectic.*

*Proof —* Differential geometry teaches us that a differential form descends to a quotient by a group if two conditions are satisfied: the differential form is invariant by the action of the group, and is trivial on the vectors that are tangent to the orbits of the group, that is on the equivalence classes for the equivalence relation defined by the action. We check that that those conditions hold in our situation.

First, the form $d\omega_G$ is invariant by the action of $(\alpha, \beta)$ since its expression is homogeneous of degree 0 in the prices, and in the quantities. Second, a tangent vector at the point $(Q, P)$ to the equivalence classes of $\sim$ is clearly of the form $(aQ, bP)$ for $a$ and $b$ two scalars, hence it is the kernel of $d\omega_G$ by Lemma 3.8.

A symplectic 2-form is a closed 2-form whose bilinear form at every point is non-degenerate. The form $\omega$ is closed since $d\omega_G$ is. The tangent space to $\mathcal{RE}$ at a point $\pi(Q, P)$ is the quotient of $\mathbb{R}^{2n}$ by the 2-dimensional spaces generated by $(Q, 0)$ and $(0, P)$. Hence $\omega(\pi(Q, P))$ is non-degenerate by Lemma 3.8. □

**Remark 3.11.** Here is another argument for the first part of the proposition, closer to the economic intuition. A differential 1-form descends to a quotient if and only if integration of this 1-form along a path depends only on the image of the path in the quotient. Hence, the 1-form $\omega_G$ descends to the quotient of $\mathcal{E}$ by the action of $\mathbb{R}_+^*$ on *prices*, since the growth depends only on relative prices and is not affected by proportional change of all prices whatsoever. (Note however that $\omega_G$ does not descend to the quotient $\mathcal{RE}$ since the growth is affected by a proportional change of all quantities). Similarly, $\omega_I$ descends to the quotient of $\mathcal{E}$ by $\mathbb{R}_+^*$ acting on *quantities*, since inflation depends only on relative quantities (and note that $\omega_I$ does not descend on $\mathcal{RE}$). Since $d\omega_G$ is also $-d\omega_I$, it is intuitive and indeed not hard to prove that it inherits the features of both its parents $\omega_G$ and $\omega_I$ and descends to the quotient on $\mathcal{E}$ by $\mathbb{R}_+^* \times \mathbb{R}_+^*$, that is on $\mathcal{RE}$.

We shall say that a subset $S$ of $\mathcal{E}$ is *saturated* if it is a reunion of equivalence classes for $\sim$. Equivalently, a set $S$ is saturated if and only if it is stable by the action of $\mathbb{R}_+^* \times \mathbb{R}_+^*$, or if and only if it is of the form $\pi^{-1}(S')$ for some $S' \subset \mathcal{RE}$, or if and only if $S = \pi^{-1}(\pi(S))$. The equivalence of all those properties is very easy and we will use any of them without further comments.

It is an easy exercise of differential geometry to see that if $M \subset \mathcal{E}$ is a saturated submanifold of dimension $k$, then $\pi(M)$ is a submanifold of $\mathcal{RE}$ of dimension $k-2$. Conversely, if $M'$ is a submanifold of $\mathcal{RE}$ of dimension $k$, $\pi^{-1}(M')$ is a submanifold of $\mathcal{E}$ of dimension $k+2$. Hence $\pi$ determines a bijection between saturated submanifolds of $\mathcal{E}$ and submanifolds of $\mathcal{RE}$.

**Lemma 3.12.** *A saturated submanifold $M$ of $\mathcal{E}$ is locally ahistorical if and only if $\pi(M)$ is an isotropic submanifold of $\mathcal{RE}$ (that is, the restriction of the symplectic form $\omega$ on $\pi(M)$ is $0$).*

*Proof —* This is a simple translation. We have already seen that $M$ is locally ahistorical if and only if $d\omega_G = 0$ on $M$. Since $d\omega_G = \pi^{-1}\omega$, that is equivalent to $\omega = 0$ on $\pi(M)$. □

There is still a minor technicality about submanifolds that are not saturated to begin with that we have to deal with. It can be ignored at first reading.

**Lemma 3.13.** *Maximal locally ahistorical submanifolds of $\mathcal{E}$ are saturated.*

*Proof —* If $M$ is an ahistorical submanifold of $\mathcal{E}$, then it is clear than the set $\pi^{-1}(\pi(M))$ is ahistorical in the sense that the growth on a smooth path included in it depends only on its end-points. But the technical problem alluded above is that this set needs not be a submanifold. Indeed $\pi(M)$ may not be smooth at points $\pi(Q,P)$ where $(Q,P)$ is a critical point of $\pi$ on $M$ (that is, the tangent space of $M$ at $(Q,P)$ had intersection with the fiber of $\pi$ greater than elsewhere). However, it is easy to see that locally near $\pi(Q,P)$ we can extend the set $\pi(M)$ to an isotropic submanifold, and the inverse image of this extension is an ahistorical manifold containing $M$, a contradiction with the maximality assumption.

Thus $\pi(M)$ and $\pi^{-1}(\pi(M))$ are submanifolds, and the latter is ahistorical. By maximality, $M = \pi^{-1}(\pi(M))$ is saturated. $\square$

3.6. **Proof of the propositions 3.3 and 3.4.** The above work took us to the well-explored country of symplectic geometry.

It is well known (see for example McDuff-Salamon (1998)) that an isotropic submanifold $M$ of a symplectic manifold $V$ satisfy $\dim M \leq (\dim V)/2$, and that it is always included (locally) in an isotropic submanifold of dimension $(\dim V)/2$ – which are called *Lagrangian* sub-manifolds.

From those facts the propositions 3.3 and 3.4 follow easily: A locally ahistorical submanifold $M$ of $\mathcal{E}$ is included in a maximal one, which is saturated by lemma 3.13. So the only thing we have to prove is that among saturated ahistorical submanifolds, the maximal ones have dimension $n+1$. If $M$ is such a variety, $\pi(M)$ is isotropic, and can be included in a isotropic submanifold $L$ of $\mathcal{RE}$ of dimension $(\dim \mathcal{RE})/2 = n-1$ (that is, a Lagrangian submanifold of $\mathcal{RE}$) Hence $M \subset \pi^{-1}(L)$, and if $M$ is maximal $M = \pi^{-1}(L)$ which has dimension $n-1+2 = n+1$.

3.7. **Proof of Theorem 3.5.** We now prove the theorem. First, we prove the sufficiency of the condition of local ahistoricity given in the theorem, and reformulated of in Remark 3.6(iii).

We have to prove that $d\omega_G$ is 0 on $M$, and it is clearly enough to prove that it is 0 of the graph of $F$. Since the form of $d\omega_G$ is invariant by interchanging $p_i$ and $q_i$, there is no restriction in assuming that the independent variables are the quantities $(q_1, \dots, q_n)$. Note by $(Q,P)$, with $P = F(Q)$ a generic point of the graph of $F$ near $x$. We call $p_1(Q), \dots, p_n(Q)$ the coordinates of $P = F(Q)$.

We have to prove that the antisymetric form $d\omega_G(Q,P)$ is null on the tangent space of the graph of $F$ at $(Q,P)$. This tangent space has dimension $n$ and is obviously generated by the following $n$ column vectors in $\mathbb{R}^{2n}$:

$$\begin{aligned} w_i &= \begin{bmatrix} u_i \\ v_i \end{bmatrix}, \\ u_i &= {}^t(0,\ldots,0,1,0\ldots,0) \text{ (1 is at the } i\text{-th position)} \\ v_i &= D_Q(F)(u_i) \quad \text{where } D_Q(F) \text{ is the differential of } F \text{ ar } Q \\ &= {}^t(\frac{\partial p_1}{\partial q_i},\ldots,\frac{\partial p_n}{\partial q_i}). \end{aligned}$$

We compute, for all $i,j=1,\ldots,n$, using (17)

$$\begin{aligned} d\omega_G(Q,P)(w_j,w_i) &= ({}^tPQ)^{-1}(\frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i}) \\ &+ ({}^tPQ)^{-2}({}^tu_jP{}^tQv_i - {}^tu_iP{}^tQv_j). \end{aligned} \tag{18}$$

The first term $({}^tPQ)^{-1}(\frac{\partial p_i}{\partial q_j} - \frac{\partial q_i}{\partial p_j})$ is 0 because $F$ is the gradient of $f$, so that $p_i = \frac{\partial f}{\partial q_i}$ and the two partial derivatives cancel out. For the second term the equations (for $i=1,\ldots,n$)

$$ {}^tQv_i = p_i \tag{19}$$

$$ {}^tu_iP = p_i \tag{20}$$

reduce it to $({}^tPQ)^{-2}(p_jp_i - p_ip_j) = 0$, so we just have to check that those equations hold. Equation (20) being obvious, let us prove (19). We have

$$\begin{aligned} {}^tQv_i &= \sum_{j=1}^n q_j \frac{\partial p_j}{\partial q_i} \\ &= \sum_{j=1}^n q_j \frac{\partial^2 f}{\partial q_i \partial q_j} \\ &= \frac{\partial\Big(\sum_{j=1}^n q_j \frac{\partial f}{\partial q_j}\Big)}{\partial q_i} \\ &= \frac{\partial f}{\partial q_i} \text{ since } f \text{ is homogenous of degree 1} \\ &= p_i. \end{aligned}$$

This proves (20) and the suffiency part of the theorem.

Conversely, assume that $M$ is locally ahistorical of dimension $n+1$, hence saturated, and let $x=(Q,P)$ be a point in $M$. The image $\pi(M)$ is then a Lagrangian submanifold of $\mathcal{RE}$. Let $T$ denotes the tangent space at $\pi(x)$ of $M$. It is a subspace of the tangent space $\mathbb{R}^n/(\mathbb{R}Q)\times\mathbb{R}^n/(\mathbb{R}P)$ at $\pi(x)$ of $\mathcal{RE}$. Let $V_P$ (resp. $V_Q$) be the intersection of $V$ with $\mathbb{R}^n/(\mathbb{R}P)$ (resp. with $\mathbb{R}^n/(\mathbb{R}Q)$). Since $V$ is Lagrangian, $V_P$ and $V_Q$ are orthogonal for the bilinear form $\omega(\pi(x))$.

After those preliminaries, let us describe an algorithm to select $n$ independent variables and $n$ dependent variables among $p_1,\dots,p_n$,$q_1,\dots,q_n$ as required in part (a) of the condition to be proved.

For each $i$, consider the two images of the vector $(0,\dots,1,0,\dots,0)$ (1 in the $i$-th position) in $\mathbb{R}^n/(P\mathbb{R})$, and in $\mathbb{R}^n/(Q\mathbb{R})$. It is possible that the first vector is in $V_P$, or that the second is in $V_Q$, but not both, since $V_P$ and $V_Q$ are orthogonal. If the first vector is in $V_P$, label $p_i$ as dependent and $q_i$ as independent. If the second vector is in $V_Q$. label $q_i$ as dependent and $p_i$ as independent. Otherwise (the generic case), label either $p_i$ or $q_i$ as dependent (your choice) and the other as independent.

After renumerotation, say that the "independent" variables are $q_1,\dots,q_k$, $p_{k+1},\dots,p_n$ and that the dependent variables are the remaining ones. Deote by $D$ the vector $(q_1,\dots,q_k,p_{k+1},\dots,p_n)$ in $\mathbb{R}^n$ and by $I$ the vector $(p_1,\dots,p_k,q_{k+1},\dots,p_n)$. The tangent space $\mathbb{R}^n/(Q/\mathbb{R})\times\mathbb{R}^n/(P\mathbb{R})$ is naturally identified to $\mathbb{R}^n/(I\mathbb{R})\times\mathbb{R}^n/(D\mathbb{R})$ by the map interchanging the $q_i$ and $p_i$ for $i>k$. We see $V$ as a subspace of the latter.

Our choice of the independent and dependent variables shows that there is a unique linear map $L:\mathbb{R}^n\to\mathbb{R}^n$, such that $L(I)=D$, and whose induced map $\bar{L}:\mathbb{R}^n/(\mathbb{R}I)\to\mathbb{R}^n/(\mathbb{R}D)$ has graph $V$.

By the implicit function theorem, we therefore can, locally near $x$, find a smooth map $F:\mathbb{R}^n\to\mathbb{R}^n$, such that

(i) $M$ is the saturated graph of $F$, that is the set of state of economy such that $(p_1,\dots,p_k,q_{k+1},\dots,q_n)=\lambda F(q_1,\dots,q_k,p_{k+1},\dots,p_n)$, for some $\lambda$;

(ii) the differential at $(q_1(x),\dots,q_k(x),p_{k+1}(x),\dots,p_n(x))$ is $L$;

(iii) and moreover the differential of $F$ at any point near $x$ s ends the vector $I$ to the vector $D$.

That is to say, we have shown part (a) of the condition of the theorem (with property (iii) as a bonus).

It remains to prove part (b), that is that $F$ is a gradient of a function $f$ that is homogeneous of degree 1. This will be done by a computation which is the reverse of the one of the proof of sufficiency above. Since, as we have already noticed, the problem is invariant by the operation inetrchanging $p_i$ and $q_i$, we can assume that the independant variables are $q_1, \dots, q_n$, and we can use all notations of the proof of sufficiency above, in particular $P = F(Q) = (p_1(Q), \dots, p_n(Q))$. Since $M$ is Lagrangian, we *know* that $d\omega_G(w_j, w_i) = 0$ for all $i, j = 1, \dots, n$, so we get using (17) that

$$0 = ({}^tPQ)^{-1}(\frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i}) + ({}^tPQ)^{-2}({}^tu_jP{}^tQv_i - {}^tu_iP{}^tQv_j.)$$

Note that equation (19) and (20) still hold in our context, the second one obviously and the first one as a translation of the fact that the differential of $F$ sends the vector $I = Q$ to the vector $D = P$ (see point (iii) above). Hence the second term is 0, and we get, for all $i, j$,

$$\frac{\partial p_i}{\partial q_j} = \frac{\partial p_j}{\partial q_i}$$

which shows that $F$ is locally a gradient of a real-valued function $f(q_1, \dots, q_k)$. It is then easy to check that $f$ is homogeneous of degree 1 using the same computation as in the proof of (19) above.

This completes the proof of part (b) of the condition, and of the theorem.

### 3.8. **Global results.**

**Theorem 3.14.** *A submanifold $M$ of $\mathcal{E}$ that is locally ahistorical and such that $H^1(M, \mathbb{R}) = 0$ is (globally) ahistorical.*

*Proof* — If $M$ is locally ahistorical, $\omega_G$ is closed on $M$. If $H^1(M, \mathbb{R}) = 0$, every closed 1-form is exact, so $\omega_G$ is exact and $M$ is ahistorical. □

**Remark 3.15.**

1. The submanifolds appearing in Hulten's theorem (Theorem 3.1) are graphs of functions on $\mathbb{R}^n$, hence are homeomorphic to $\mathbb{R}^n$ and in particular have $H^1(M, \mathbb{R}) = 0$. Hence in Hulten's frame work, there is no difference between local and global ahistoricity.
2. For a locally ahistorical saturated submanifold $M$ of $\mathcal{E}$, is $H^1(M, \mathbb{R}) = 0$ a *necessary* condition of ahistoricity?

   The answer is no in general if $n \leq 3$: consider a Lagrangian submanifold $L$ of $\mathcal{RE}$, a point $y$ in $L$ and a small circle $C$ around $y$ in $L$ which is included of some disc $D \subset L$. Then $\omega$ is closed on $D$ since $D \subset L$, hence exact on $D$ since $D$ is contractible, hence exact on $C$. Now set $M = \pi^{-1}(C)$. Then $H^1(M, \mathbb{R}) = H^1(C, \mathbb{R}) = \mathbb{R}$, but $M$ is ahistorical.

   However, when $n = 2$, the answer is yes. It is also yes for any $n$ if $\pi(M)$ is assumed to be compact (an economically unnatural hypothesis), and if the famous Arnold's conjecture in symplectic topology holds for the sphere $S^{n-1}$ (see Fukaya-Seidel-Smith (2007)). Since those observations have little economic interest, we omit the proofs.

3.9. **Discussion.** Now that we have fully determined the necessary and sufficient constraint we have to impose on the set of possible states of an economy for the growth to be history-independent, it is natural to ask the question wether those conditions are satisfied in various micro-economic models.

There are two kind of models to consider, static and dynamic. In a static model, consumers and firms has utility and production functions that determine, at each time $t$, one (or several) equilibrium position of the economy $(Q^*(t), P^*(t))$. The equilibrium state may change with time as a result of changing utility or production functions (or, if they are fixed, of changing other factors, like sale taxes, if one allows them in the model) but time is only a parameter: no agent thinks beyond the instant $t$, and the equilibrium is supposed to be reached instantly. In a dynamic model, we take into account one or more of these factors : that equilibrium may take time to be reached (if it is ever reached), that production takes time, that consumer or firms may try to maximize not their utility or profit at each time $t$, but rather some average over time of their utility and production function, etc.

Consider a static model first, and to simplify let us adopt a partial-equilibrium approach (even if for the economy as a whole, only a global-equilibrium approach would be fully convincing). In this approach, at each time $t$, there is an aggregate demand function $d(P) = d(p_1, \dots, p_n)$ representing the quantity vector of the aggregated demand of all consumer with a given set of prices, and an aggregate production function $q(P) = q(p_1, \dots, p_n)$ representating the total quantities produced by all firms for a given set of prices. The equilibrium $(Q^*, P^*)$ is then by defined by the equations $Q^* = d(P^*) = q(P^*)$. The equilibrium changes with time if we allow the functions $d$ and $q$ to change with time, which would be a consequence of changing utility and production functions.

With those general assumptions, the first part of the condition of ahistoricity is already not satisfied: if the utility or production function can change freely[10], $d$ and $q$ change with time, there is generally speaking no way to determine $n$ variables among the prices and quantities as a function of the remaining ones, even up to a sclar multiplication. In other words, the economy will not be constrained in a dimension $n$ (or $n + 1$) submanifold of $\mathcal{E}$, and the growth will depend on history.

So let us make more hypothesis, and assume than one of the functions $d$ or $q$ does not change with time. That's certainly a very strong hypothesis, but this could be to some extent justified for the demand function $d$ – it would be foolish in a study of economic growth to keep fixed the production functions, but on a limited amount fo time one can imagine that consumers' tastes do not change too much. In this case, $Q$ will be a fixed function of prices $P$: $Q = d(P)$. To check if ahistoricity holds, we know have to

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[10] Freely should not be take too strongly here. For exemple open constraints such as strict convexity do not matter here.

check part (b) of the condition of ahistoricity, which is that $d$ is up to a scalar a gradient of an homogeneous function. But here we see again that this will not happen in general if we do not make more assumtions. Indeed, an important result of Chiappori and Ekeland (1999) shows that the aggregated demand function $d$ can be (locally) any function of $P$, and has no reason to be a gradient (up to a scalar).

Only under the additional and quite unrealistic hypothesis that there is only one customer (or that all customers are clones of each other) can one ensure that $d$ is up to a scalar the gradient of some function (which would be, in that case, the indirect utility function of the customer). But even in this case, ahistoricity will need the hypothesis that the demand function is homogeneous, a very unplausible assumtion, on which we could quote the comment of Samuelson-Swamy: "if only it was as realistic as it is elegant".

Analysis with a Walrasian, complete equilibrium model would be a little bit more complicated, but would also lead to the conclusion than ahistoricity does not hold, except if we are willing to make very strong and unrealistic hypotheses.

We now turn to a dynamic model, merely to say that the question of ahistoricity in this context would take us far beyond the scope of this article. We hope to come back to this study in a forthcoming work

## 4. AGGREGATION AND PATH-INDEPENDENCE

4.1. **Position of the problem.** Assume that $M$ is a locally ahistorical submanifold, locally defined by a function $f$ as in Theorem 3.5. To simplify the notations, assume as in Hulten than $f$ is a function of the quantity variables $q_1, \dots, q_n$, but everything will carry out if the set of independent variables is otherwise (purely prices, or some mixtures of prices and quantities as in Theorem 3.5).

We consider a sub-economy $E'$ of our economy $E$ by picking out $m$ goods out of $n$ (with $0 < m < n$), say the ones with index $1, \dots, m$. Let us assume that the GDP of the subeconomy $\sum_{i=1}^{m} p_i q_i$ at $x$ is positive. Let us denote by $\bar{\mathcal{E}} \subset \mathbb{R}^{2m}$ the set of prices and quantities vectors for the sub-economy that have a positive GDP. Forgetting the variables $p_k$,$q_k$ for $k > m$ defines a map $\mathrm{pr} : \mathcal{E} \to \bar{\mathcal{E}}$ in a neighborhood of $x$.

A history of the whole economy described by a path $\gamma$ in $U$ define a history of the sub-economy $\bar{\gamma} = \mathrm{pr}(\gamma)$. The question considered by Hulten is: find a necessary and sufficient condition so that

*(*) the growth of the sub-economy along the path $\bar{\gamma}$ depends only of the end-points of $\bar{\gamma}$ for all paths $\gamma$ in $M$ (staying in a given neighborhood of $x$)*

4.2. **The incorrect solution of Hulten.** Hulten (1973) states (Corollary 1) that a necessary and sufficient condition is that $f(q_1, \dots, q_n)$ can be written $g(u(q_1, \dots, q_m), v(q_{m+1}, \dots, q_n))$, where $g$, $u$ and $v$ are differentiable functions of 2, $m$ and $n-m$ variables respectively. But the proof he gives is not perfectly rigorous, and actually, it is not hard to see that this statement is incorrect.

Consider indeed the case $m = 1$. It is of course always true that the integration on the sub-economy with one good is path-independent. So, according to Hulten's result, it should be always the case that $f(q_1, \dots, q_n)$ could be written as $f(q_1, v(q_2, \dots, q_n))$. But of course, not all functions can be aggreated this way, even homogeneous ones. For example $f(q_1, q_2, q_3) = q_1 q_2 + q_3^2$, which is homogeneous of degree 2 cannot be written $g(q_1, v(q_2, q_3))$ for differentiable $g$, $u$ and $v$. (For an example which is linear homegenous, take the square root of $f$).

4.3. **A solution to the problem.** Still, Hulten's fundamental intuition that the condition (*) is related to weak separabability questions on $f$, as considered by Leontief, is right.

**Theorem 4.1.** *A necessary and sufficient condition for (*) is that there is a a real function $g$ of $n - m + 1$ variable, and a locally homogeneous function $u$ of $m$ variables, such that in a neighborhood of* $\mathrm{pr}(x)$*,*

$$f(q_1, \dots, q_n) = g(u(q_1, \dots, q_m), q_{m+1}, \dots, q_n)$$

*Proof* — We shall denote with a $^{-}$ the operation that consists in dropping out all prices and quantities of goods of index $i > m$. For example, if

$P = (p_1, \dots, p_n) \in \mathbb{R}^n$ then $\bar P$ denotes the vector $(p_1, \dots, p_m) \in \mathbb{R}^m$. We shall use the notations of §3.7: we call $F$ the gardient of $f$, we set $P(Q) = F(Q)$ and denote by $p_1(Q), \dots, p_n(Q)$ the coordinates of $F(Q)$.

Let $M$ be the saturated graph of $F$ in $\mathbb{R}^n \times \mathbb{R}^n$. Consistently with the above, we denote by $\bar M$ the image of $M$ in $\mathbb{R}^m \times \mathbb{R}^m$. The condition (*) means that $\bar M$ is a locally ahistorical submanifold[11] near $\pi(x) = \bar x \in \bar E$.

This is equivalent to the fact that for all $i, j = 1, \dots, n$,

$$d\bar\omega_G(\bar Q, \bar P)(\bar w_i, \bar w_j) = 0,$$

which can be written using (18) as

$$0 = ({}^t\bar P\bar Q)^{-1}({}^t\bar u_j\bar v_i - {}^t\bar u_i\bar v_j) + ({}^t\bar P\bar Q)^{-2}({}^t\bar u_i\bar P{}^t\bar Q\bar v_j - {}^t\bar u_j\bar P{}^t\bar Q\bar v_i). \tag{21}$$

To analyse this equation (21), we note that $\bar u_i = 0$ as soon as $i > m$, and we consider three cases:

- $i > m$, $j > m$. In this case both $\bar u_i$ and $\bar u_j$ are 0, so the right hand side of (21) is trivially 0, and the equation become $0 = 0$.
- $i \leq m$, $j \leq m$. In this case, the first term of the right hand side of (21) is $({}^t\bar P\bar Q)^{-1}(\frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i})$ which is 0 since $F(Q) = P(Q)$ is a gradient. Hence (21) becomes

$$0 \quad = \quad p_i \sum_{k=1}^{m} q_k \frac{\partial p_k}{\partial q_j} - p_j \sum_{k=1}^{m} q_k \frac{\partial p_k}{\partial q_i} \tag{22}$$

- $i \leq m$, $j > m$. (Note that the case $i > m$, $j \leq m$ is symmetric and need not be considered). In this case, $\bar u_k = 0$, but not $\bar u_i$ and (21) becomes

$${}^t\bar P\bar Q \frac{\partial p_i}{\partial q_j} = p_i \sum_{k=1}^{m} q_k \frac{\partial p_k}{\partial q_j} \tag{23}$$

Now assume (*), that is (22) and (23).

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[11] Strictly speaking, $\bar M$ could be non smooth at some points $\bar x$, that is not a submanifold of $\bar{\mathcal{E}}$ – but this may happen only for a set of points of measure 0 (Sard's theorem) and even when this happens, the computations to be made to check the conidition (*) are the same that we make below,so we decided not to bore the reader of the main text with this technical detail.

Let's write (23) for another $i$, say $i'$ in $\{1,\dots,m\}$ but the same $j \in \{m+1,n\}$. We get, reversing the equality

$$p_{i'} \sum_{k=1}^{m} q_k \frac{\partial p_k}{\partial q_j} = {}^t\bar{P}\bar{Q} \frac{\partial p_{i'}}{\partial q_j} \tag{24}$$

Multiplying (23) and (24) and simplifing gives

$$p_{i'} \frac{\partial p_i}{\partial q_j} = p_i \frac{\partial p_{i'}}{\partial q_j},$$

or, remembering that $p_i = \frac{\partial f}{\partial q_i}$, we get, for all $i, i' \in \{1,\dots,m\}$ and $j \in \{m+1,\dots,n\}$,

$$\frac{\partial f}{\partial q_{i'}} \frac{\partial^2 f}{\partial q_i \partial q_j} = \frac{\partial f}{\partial q_i} \frac{\partial^2 f}{\partial q_{i'} \partial q_j}.$$

This relation is exactly the relation discovered by Leontief as a necessary and sufficient consition for tha variables $q_1,\dots,q_m$ to be functionally seprable in $f(q_1,\dots,q_m)$. See Leontief (1947), Prop. I and equation (II.3), page 364.

[The argument of the last paragraph may be rephrased as follows, for a more intuitive, derivation. Equation (23) says that the derivative of the vector $(\frac{\partial f}{\partial q_1},\dots,\frac{\partial f}{\partial q_m}) = (p_1,\dots,p_m)$ with respect to $q_j$, $j > m$ is a vector proportional to $(p_q,\dots,p_m)$ (the constant of proprtionality depending on the variables, $q_1,\dots,q_n$). But this is one form of the condition given by Leontief (1947) for $q_1,\dots,q_m$ to be functionally separable in $f$.]

Hence we have

$$f(q_1,\dots,q_n) = g(u(q_1,\dots,q_m), q_{m+1},\dots,q_n), \tag{25}$$

for some suitable function $u$ and $g$ of $m$ and $n-m+1$ variables respectively. By differentiating (25) using the chain rule, we see that the gradient of $u$ is proportional to $(p_1,\dots,p_m)$. Hence the relation (22) says that the gradient of $u$ is proportional to its image by the operator $\sum_{i=1}^{m} q_k \frac{\partial}{\partial q_k}$ (applied coordinate by coordinate), which is equivalent to the local homogeneity of $u$. This complete the proof of the necessity of the conditions given in the theorem.

Conversely, if we assume (25) and the local homogoenity of $u$, then we have just seen that relation (22) holds. To get (23), we use Leontief's criterion (which in this sense is a simple applictaion of the chain rule) to deduce that for all $j > m$, the vector $(p_1, \dots, p_m)$ is proportional to $(\frac{\partial p_1}{\partial q_j}, \dots, \frac{\partial p_m}{\partial q_j})$ , that is to say

$$\frac{\partial p_i}{\partial q_j} = C p_i,$$

where $C$ is a function depending on all variables $q_1, \dots, q_n$ and of $j$, but not $i$. Summing the above relation over $i = 1 \dots, m$ gives $C = \frac{\sum_{k=1}^{m} q_k \frac{\partial p_k}{\partial q_j}}{{}^t\bar{P}\bar{Q}}$. Hence (23), and the theorem follows. □

## 5. An experimental measurement

In this section we want to present an experimental result we have obtained trying to test the reality and to measure the order of magnitude of the history-dependence of growth. We stress that the significance of this result is at best very modest: this is due partly to the inexperience of the author in the field of statistics and data analysis, but even more to his inability to formulate a good question to be answered by the available data.

Indeed, the conceptual problem in measuring the dependence on history of growth is very simple: the real world provide us with only one history. Mathematically speaking, this can be reformulated as: the real world provides us with a curve $\gamma(t)$ describing the economy but a curve is always locally ahistorical (for a curve is always isotropic for a symplectic structure).

A solution would be to compare the growth of two different countries, if we could find two periods of time for the two counitries, beginning and ending in the same economic states (or proportional economic states). But we have not tried to do so, due to our inability to obtain data for different countries in an easily comparable forms. An other, related, idea would be to look for a cycle in a given economy: if the economy turns outs to be at some time $b$ in the same state as it was at an earlier time $a$ (or in a sate with proportional prices, and proportional quantities), that is if the curve $\gamma(t)$ satisfies $\gamma(a) = \gamma(b)$ (or at least $\pi(\gamma(a)) = \pi(\gamma(b))$), then the growth Divisia integral should be 0 (or equal to the number $\alpha$ defined by $Q(b) = \alpha Q(a)$ in the second case) if the growth were to be path-independant. Unfortunately, we have not been able to find very convincing cycles in the U.S. economy for the period we have looked at, nor for any of its natural subsector. A more serious statistical study, of the deviations of the real growth on approximate cycles, seems necessary here.

So the only solution left to us is to compare the growth between two times $a$ and $b$ computed with the actual hisory on the one hand and with another, history of our imagination on the other hand. The problem becomes that the answer is of little interest: Proposition 2.2 shows that if we impose no constraint on the history we can always come up with an alternative history giving an arbirtraily specified growth between $a$ and $b$. Admittedly, the history constructed in the proof of Prop. 2.2 is very unrealistic, with prices and quantities varying wildly and being 0 at opportunistic times. No real history is even close to behave that way. But how to define a *realistic* history? The theoretical anlysis carried out in §3.9 provides us, so far, with no clue on this question.

The experiment we have done is the following: we compare the actual growth of an economy between time $a$ and time $b$ with the growth that would have been computed if the history had been linear, that is if all prices and quantities functions have been the linear (that is to say, affine) interpolation of their actual values at $a$ and $b$. Mathematically, if $p_i(t)$ and $q_i(t)$ denotes the actual prices and quantities functions, we define

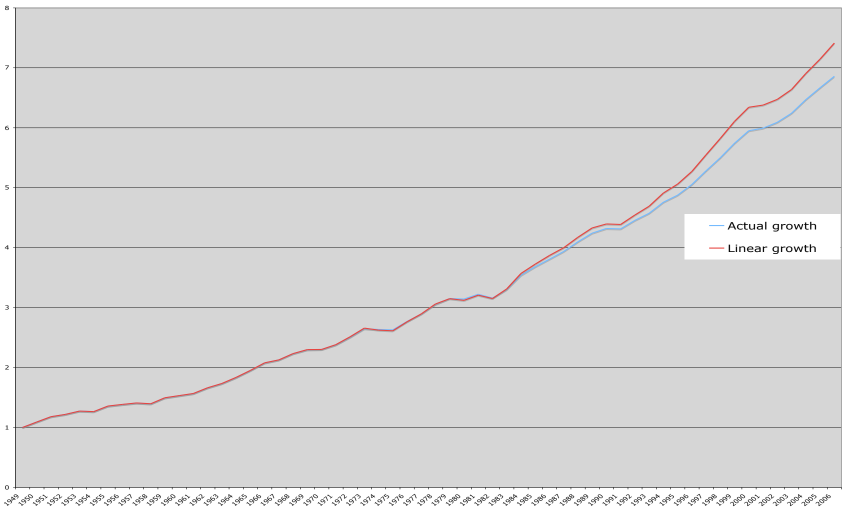
$$p_i^{\text{lin}}(t) = p_i(a) + \frac{(t-a)}{b-a} p_i(b)$$

and similarly for $q_i^{\text{lin}}$. We note $G(a, b)$ the actual growth beteen $a$ and $b$ and $LG(a, b)$ the *linear growth* which is the growth that would be computed had the economy been linear. We aim to compare $G(a, b)$ and $\mathrm{LG}(a, b)$.

The choice of the linear economy as an alternative is certainly arbitrary. The justification we can present for it are that is is a simple and an “honest” choice, in the sense that in it is not made up to make the growth change in one direction and another on purpose, and that we can think of no better alternative.

To make our comparison, we have used the data on U.S. economy provided by the Bureau of Economic Analysis. More precisely, we have used the National Income and Product Acount (NIPA) tables 1.5.4 and 1.5.5 of that Bureau, available on the web page
http://www.bea.gov/national/nipaweb
giving the prices and quantities indices for 23 sectors[12] constituing a partition of the national output. We have used annual data over the period 1949-2006 (some data were unavailable before 1949). We have used a small Java program to compute the growth and linear growth, using a chained Fischer index to approximate the Divisia line integrals (as done by the BEA). The actual growth we compute has been checked to be equal (up to a very small error) to the one given by the BEA in table 1.5.3.

Below are the graphs of the actual and linear growths between 1949 and a date $t$ moving from 1949 to 2006.



Over the period 1949–2006, the quotient GDP(2006)/GDP(1949) computed with the actual history is 6.9, while the same quotient computed

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[12] Precisely the lines 4, 5, 6, 8, 9, 10, 11, 13, 14, 17, 18, 19, 20, 23, 33, 34, 39, 40, 42, 43, 46, 49 and 52 of the B.E.A. tables.

with the linear history is 7.4 (which corresponds to respective annual pace of 3.44% and 3.57%). As seen in the graphics, the two growth seem very close for the first 30 or 35 years, after what the actual growth begins to be systematically below the linear growth. Other computations made over all subperiods of any length of 1949-2006 seem to confirm this phenomenon.

What interpretation, if any, can we give to those data? With all due reserves, this seems to show that the phenomenon of dependance of history might be ignored for short period of time (10 years, 20 years), but is not negligible for secular comparisons.

We should also emphasize that our computations are dependent of the level of disaggregation of the economy used. Ideally, we should use a minimal level of agregation with prices and produced quantities of thousands of goods, but the BEA do not provide such data, and make only public partially aggregated prices and quantities indexes, which very likely makes the observable dependence on history smaller that it really is.

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