STOCHASTIC DIFFERENTIAL EQUATIONS AND NILPOTENT LIE ALGEBRAS (AFTER DOSS AND YAMATO) PRELIMINARY VERSION 2

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1. Doss' result on one variable autonomous stochastic differential equation

We consider a Brownian motion B_t and a stohastic differential equation

$$\begin{aligned} dX_t &= \sigma(X_t)\ dB_t + b(X_t)\ dt && (1) \\ X_0 &= x && (2) \end{aligned}$$

where X_t is a real ramdom variable, and σ, b are functions $\mathbb{R} \to \mathbb{R}$. We assume that σ and b are Lipschitz.

Theorem 1 (Doss 1977)**.** *Assume moreover that σ is C^2. There exists a deterministic $\mathbb{C}^\infty$ function $h : \mathbb{R}^2 \to \mathbb{R}$ such that the solution of (1) satisfies almost surely*

$$X_t = h(x, B_t), \forall t \geq 0 \qquad (3)$$

Moreover the function h is determined as the solution of an explicit ordinary differential equation that can be written down in term of σ and b.

I will give a proof of this theorem, but with stonger hypotheses to avoid technicalities and messy computations. For one thing, I assume that all functions are C^∞. Other assumptions will be given in the proof.

Also I won't write down the differential equation that h satisfies. which is very messy, except in a special case, the case where

$$b = \sigma\sigma'.$$

In this case, the equation (1) has the form

$$\begin{aligned} dY_t &= \tilde{\sigma}(Y_t)\ dB_t + \frac{1}{2}\tilde{\sigma}(Y_t)\tilde{\sigma}'(Y_t)\ dt && (4) \\ Y_0 &= y && (5) \end{aligned}$$

Remark 2. Equation (4) may also be written

$$dY_t = \sigma(Y_t) \circ dB_t \qquad (6)$$

where $\circ$ denotes the Stratonovich integral

So the first step is to prove that we can reduce equation (1) to equation (4)

First step : We use a method inspired by the methods of removal of the drift. The analysis goes as follows : we set $Y_t = f(X_t)$. We have by Ito's formula

$$\begin{aligned} dY_t &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)\sigma^2(X_t)\ dt \\ &= f'(X_t)\sigma(X_t)\ dB_t + (f'(X_t)b(X_t) + \frac{1}{2}f''(X_t))\sigma^2(X_t)\ dt \end{aligned}$$

Hence we are led to set $\tilde{\sigma} := f'(f^{-1})\sigma(f^{-1})$ so that the equation above become

$$dY_t = \sigma(Y_t)\ dB_t + (f'(X_t)b(X_t) + \frac{1}{2}f''(X_t)\sigma^2(X_t))\ dt$$

So what we want is for all $x, y = f(x)$,

$$\tilde{\sigma}(y)\frac{d\tilde{\sigma}(y)}{dy} = f'(x)b(x) + \frac{1}{2}f''(x)\sigma^2(x)$$

This is

$$f'(x)\sigma(x)(f'(x)\sigma'(x) + f''(x)\sigma(x))\frac{dx}{dy} = f'(x)b(x) + \frac{1}{2}f''(x)\sigma^2(x)$$

Since $\frac{dx}{dy} = \frac{1}{f'(x)}$ (isn't it?) we get

$$\frac{1}{2}f''(x)\sigma^2(x) + f'(x)(\sigma(x)\sigma'(x) - b(x)) = 0.$$

This ends our analysis. Our synthesis will be short : we make the ad hoc hypothesis and σ and b so that the differential equation above may be solved (for example, $\sigma^2 > 0$ and $|\sigma\sigma' - b|/\sigma^2$ locally integrable) so that the differential equation above has solutions. We select one solution f and we check wih Ito's formula that $Y_t = f(X_y)$ satisfies equation (4), with $y = f(x)$

We conclude by noticing that if Y_t satisfy the conclusion of the theorem, namely $Y_t = \tilde{h}(y, B_t)$, then $X_t = h(x, B_t)$ if we define h as follows : for all x, τ, $h(x,\tau) := f^{-1}(\tilde{h}(f(x), \tau))$.

Second step Now we are reduced to prove the theorem when $b = \sigma\sigma'$, that is when the equations is

$$dX_t = \sigma(X_t)\ dB_t + \sigma(X_t)\sigma'(X_t)\ dt \tag{7}$$

$$X_0 = x \tag{8}$$

We define the function $h(x,t)$ as the solution of the following ordinary differential equation :

$$\frac{\partial h}{\partial \tau}(x,\tau) = \sigma(h(x,\tau)); h(x,0) = x \tag{9}$$

Since σ is Lipschitz and bounded, such a function h exists (for any $(x,\tau) \in \mathbb{R}^2$) and is unique. The function h may be thought of as follows : $h(x,\tau)$ is the position at time τ of a moving point starting at x at time 0 and following the vector field on the real line defined by σ. In other words, $h(x,\tau)$ is what we call the *flow* of the vector field σ.

By uniqueness, we have the *flow property*

$$h(x, \tau + \tau') = h(h(x,\tau), \tau') \tag{10}$$

Applying this to $\tau' = -\tau$, we get

$$x = h(h(x,\tau), -\tau) \tag{11}$$

Differentiating (10) with respect to τ, then taking $\tau = 0$, we get $\frac{\partial h}{\partial \tau}(x,\tau') = (\frac{\partial h(x,\tau)}{\partial \tau})_{\tau=0} \frac{\partial h}{\partial x}(h(x,0),\tau')$. Hence, by (15),

$$\frac{\partial h}{\partial \tau}(x,\tau) = \sigma(x)\frac{\partial h}{\partial x}(x,\tau) \tag{12}$$

Now adding $-\sigma(x)$ times the x-partial derivative of (12) with its τ-partial derivative, and regrouping terms, we get

$$\sigma^2 \frac{\partial^2 h}{\partial x^2} + \sigma\sigma' \frac{\partial h}{\partial x} - 2\sigma \frac{\partial^2 h}{\partial x \partial \tau} + \frac{\partial^2 f}{\partial \tau^2} = 0 \tag{13}$$

Third step Now we check that $X_t = h(x, B_t)$ almost surely. By (10) and (11), this is equivalent to

$$x = h(X_t, -B_t).$$

To prove this, we apply Ito's formula to $h(X_t, -B_t)$; since $h(x_0, -B_0) = h(x,0) = x$, we get

$$\begin{aligned} h(X_t,-B_t) - x &= \int_{0^t} \frac{\partial h}{\partial x}(X_s,-B_s)\ dX_s - \int_{0^t} \frac{\partial h}{\partial \tau}(X_s,-B_s)\ dB_s \\ &+ \frac{1}{2}\int_0^t \left(\frac{\partial^2 h}{\partial x^2}(X_s,-M_s)\sigma^2(X_s) - 2\frac{\partial^2 h}{\partial x \partial \tau}(X_s,-M_s)\sigma(X_s) + \frac{\partial^2 h}{\partial \tau^2}(X_s,-M_s) \right)\ ds \end{aligned}$$

So by (6) we have

$$h(X_t,-B_t) - x = \int_{0^t} \left(\sigma(X_s)\frac{\partial h}{\partial x}(X_s,-B_s) - \frac{\partial h}{\partial \tau}(X_s,-B_s) \right)\ dB_s$$

$$+\frac{1}{2}\int_0^t \left(\frac{\partial^2 h}{\partial x^2}(X_s,-M_s)\sigma^2(X_s) - 2\frac{\partial^2 h}{\partial x \partial \tau}(X_s,-M_s)\sigma(X_s) + \frac{\partial^2 h}{\partial \tau^2}(X_s,-M_s) + \frac{\partial h}{\partial x}(X_s,-M_s)\sigma(X_s)\sigma'(X_s) \right)\ ds$$

But the first integrand is 0 by (12) and the second is also 0 by (13). So the theorem is proved.

Remark 3. If we consider the deterministic smooth analog of (6) namely $dX_t = \sigma(X_{t)})db_{1(t))}$ where $b_1(t)$ is a deterministic smooth function, then the solution is $X_t = h(x, b_1(t))$

2. Reminder on vector fields and Lie Algebra

2.1. **Vector fields and Lie brackets.** A *vector field* in $\mathbb{R}^d$ is (simply) a $\mathbb{C}^\infty$-application form $\mathbb{R}^d$ to $\mathbb{R}^d$. If A is a vector field, we should think of $A(x)$, for x a *point* in $\mathbb{R}^d$, as a *vector* in $\mathbb{R}^d$, and if we have to draw this vector, we should choose x as its starting point. The coordinates of the vector $A(x)$ will be denoted by $A^j(X)$, $j = 1, \ldots, d$.

Exercise 1. *"Draw" the vector field $A(x^1, x^2) = (-x^2, x^1)$ in $\mathbb{R}^2$.*

Alternatively, we can see a vector field A as a (first-order) differential operator on $C^\infty(\mathbb{R}^d)$. Namely, if f is a function in $C^\infty(\mathbb{R}^d)$ we define Af as the function such that $Af(x) = \left(\frac{d}{dt}f(x + tA(x))\right)_{t=0}$.

Exercise 2. *Check that $Af(x) = \sum_{j=1}^d A^j(x)\frac{d}{dx^j}f(x)$.*

We thus write shortly $A = \sum_j A^j \frac{d}{dx^j}$.

Exercise 3. *Check that for two functions f, g, and a constant c, we have*

$$A(fg) = A(f)g + A(g)f,\ \ A(f+g) = A(f) + A(g), A(\lambda f) = \lambda A(f). \tag{14}$$

Prove that every application on $C^\infty(\mathbb{R}^d)$ that satisifies (14) comes from a vector field A as above

The *flow* of a vector field A is the function $h_A : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}^d$ defined by

$$\frac{\partial h}{\partial \tau}(x,\tau) = A(h(x,\tau)); h_A(x,0) = x \tag{15}$$

In other words, for a given x, $h(x,\tau)$ is the position at time τ of a moving object that starts at x and whose velocity vector at any time is the vector A at its position.

Example 4. The function h of the last section is the flow of σ seen as a vector field.

The Lie bracket of two vector fields A, B is denoted $[A,B]$ and is defined as follows : we see A, B as differntial operators that satisfy 14 and we define $[A,B]$ as the differential operator $AB - BA$.

Exercise 4. *Shows that $AB - BA$ also satisfy (14) (although AB does not)*

So we can see $[A,B]$ as a vector field.

Exercise 5. *Show that* $[A,B]^j = \sum_{k=1}^d (A^k \frac{dB^j}{dx^k} - B^k \frac{dA^j}{dx^k})$

The following property may help to understand intuitively the meaning of the Lie bracket : $h_A(h_B(x,\tau),\tau) - h_B(h_A(x,\tau),\tau) \simeq \tau^2[A,B]$. Here $\simeq$ means "is equivalent as τ goes to 0". In words : start at x follow A during a small time τ, then follows B during the same time τ; say you are at the point y. Now start at x, follow A during a small time τ, then follows B during the same time τ; say you are at z. Then the vector $\vec{yz}$ is close to $\tau^2[A,B](x)$.

2.2. **Lie algebra.**

Definition 5. For our purpose, we shall define a *Lie algebra* as a subset L of the set of all vector fields on $\mathbb{R}^d$ such that if $A, B \in L$, and $\lambda \in \mathbb{R}$ then λA, $A + B$, and $[A,B]$ are also in L.

In other words, a Lie algebra is a vectros space of vector fields that is stable by Lie brackets.

Definition 6. If $\{A^i\}$ is a set of vector fields, the *Lie algebra generated by the A_i's* is the smallest Lie algebra that contains the A_i's.

If L is a Lie algbera, we define $L^{(0)} = L$ and by induction $L^{(k+1)}$ as the Lie-algbera generated by the vector fields $[A,B]$ for all $A \in L$ and $B \in L^{(k)}$.

Definition 7. We say that a Lie algebra is *nilpotent* if $L^{(k)} = 0$ for some k. If p is the smallest such k we say that L is *nilpotent of step (or order) p*

If L is nilpotent of step 1 we say that L is *commutative*. This means that all lie brackets are zero.

Example 8. We work in $\mathbb{R}^2$ and call x and y the coordinates. Let $A_1 = \frac{\partial}{\partial x}$ and $A_2 = x\frac{\partial}{\partial y}$. Then $Z := [A_1, A_2] = \frac{\partial}{\partial y}$ and $[Z, A_1] = [Z, A_2] = 0$. Hence the Lie algebra L generated by A_1 and A_2 is the vector space generated by A_1, A_2 and Z, $L^{(1)} = \mathbb{R}Z$ and $L^{(2)} = 0$. The Lie algebra L is nilpotent of step 2.

Exercise 6. *Same notations as in the example, but we change A_2 : $A_2 = x^n \frac{del}{\partial y}$. Show that the dimension of L as a vector space is $n+2$ and that it is nilpotent of step $n+1$.*

Exercise 7. *Same notations as in the example, but we change A_2 : $A_2 = e^x \frac{del}{\partial y}$. Show that the dimension of L as a vector space is 3 but that L is not nilpotent. (Remark : L is actually resoluble of step 2, in the sense that $L^{(1)}$ is abelian. But this is a weaker notion that nilpotent)*

Exercise 8. *We work in $\mathbb{R}^3$, and we note the coordinates x^i, $= 1, 2, 3$. Show that the Lie algebra generated in $\mathbb{R}^3$ by $\frac{d}{dx^1}$ and $\frac{d}{dx^2}$ is commutative. Show that the Lie algebra generated by $A_1 = \frac{d}{dx^1} + 2x^2 \frac{d}{dx^3}$ and $A_2 = \frac{d}{dx^2} - 2x^1 \frac{d}{dx^3}$ is nilpotent of step 2.*

2.3. **Iterated Lie brackets.** In this subsection we introduce notations that shall stay in use during all this talk.

Let $n \geq 1$ be an integer. We set $E = \{1, \dots, n\}$. For every integer p we define $E(p)$ has the set of all sequences of length at most p of elements of E. A typical element of E will be denote $I = (i_1, \dots, i_a)$ for some $1 \leq a \leq p$. We set

$$E(\infty) := \cup_{p=1}^{\infty} E(p).$$

Let $A_1, \dots, A_n$ be n vector fields on $\mathbb{R}^d$. For $I = (i_1, \dots, i_1) \in E(p)$. we define A_I by induction on a : for $a = 1$, we symply set

$$A_{(i)} := A_i.$$

For general I we set

$$A_{(i_1,\dots,i_a)} := [A_{(i_1,\dots,i_{a-1})}, A_{i_a}].$$

Exercise 9. *Show that*

$$A_{(i_1,i_2)} = -A_{(i_2,i_1)} \tag{16}$$

and that

$$A_{(i_1,i_2,i_3)} + A_{(i_2,i_3,i_1)} + A_{(i_3,i_1,i_2)} = 0 \tag{17}$$

It should be clear that the Lie algebra L generated by the A_i is the set of all finite sums of the forms $\sum_{I \in E(\infty)} f_I A_I$ where $f_I \in C^\infty(\mathbb{R}^d, \mathbb{R})$. In other words, as a module over the ring $C^\infty(\mathbb{R}^d, \mathbb{R})$, L is generated by the A_I, $I \in E(\infty)$.

If L is nilpotent of step p, then all the A_I for I of length greater than p will be zero. Hence L is generated (as a module) by the A_I's, $I \in E(p)$.

As a matter of fact, not all the $I \in E(p)$ are needed in order to generate L. For example, $A_{(i,i)}$ is always zero so is not needed ; if we have $A_{(i_1,i_2)}$ we don't need $A_{(i_2,i_1)}]$ which is just the opposite by (16) ; if we have $A_{(i_1,i_2,i_3)}$ and $A_{(i_2,i_3,i_2)}$ we don't need $A_{(i_3,i_1,i_2)}$ which is the opposite of the sum of the first two by (17).

It is easy to see, less to write, that for each p we can choose a minimal subset $F(p)$ of $E(p)$ such that we can retrieve the A_I for $I \in E(p)$ as linear combinations of the A_I for $I \in F(p)$ using the rules (16) and (17). Hence if L is nilpotent of step p the A_I's, $I \in F(p)$ generate L as a module.

Example 9. If $n = 2$, we can choose $F(2) = \{(1), (2), (1, 2)\}$. Note that $\{(1), (2), (2, 1)\}$ is an equally correct choice. We can take $F(3) = F(2)$.

If $n = 3$, we can choose $F(2) = \{(1), (2), (3), (1, 2), (1, 3), (2, 3)\}$.

Exercise 10. *Find a correct choice for $F(3)$ when $n = 3$.*

2.4. **Frobenius theorem.** Let L be n-dimensional vector space of vector fields in $\mathbb{R}^d$. Foe each $x \in \mathbb{R}^d$, we denote by L_x the set of vectors $A(x)$ for $A \in L$. This is a subspace of $\mathbb{R}^d$, of dimension at most n. It is of course possible that $\dim L_x < n$ and this number may even depend of x.

Theorem 10 (Frobenius)**.** *Assume that for all x, L_x has dimension n. Assume moreover that L is a Lie algebra. Choose a point $x_0 \in \mathbb{R}^d$. Then there is an application $u : R^n \to \mathbb{R}^d$ injective, such that for all $\tau \in \mathbb{R}^n$, the image of the differential of u at τ is $L_{u(\tau)}$.*

Note that in particuar, the differential of u at τ, that we shall note du_τ is also injective by the nullrank result. We say that u is immersive. This means that the image of any sufficiently small open set $\in \mathbb{R}^n$ is a nice *subvariety* of $\mathbb{R}^d$, whose tangent space at any point x is L_x. However, this may not hold globally.

3. Generalization of Doss' theorem in dimension d : statements

We consider now a n-dimensional Brownian motion B_t that we see as a column vector whose components are denoted B_t^i for $i = 1, \dots, n$. We consider a d-dimensional differential equation :

$$dX_t = A(X_t) \circ dB_t + A_0(X_t)\, dt \tag{18}$$

$$X_0 = x \in \mathbb{R}^d, \tag{19}$$

where A is a $d \times n$ matrix whose coefficients are in $C^\infty(\mathbb{R}^d)$ and A_0 is a vector field on $\mathbb{R}^d$. The columns of A are denoted $A_1, \dots, A_n$ and are vector fields on $\mathbb{R}^d$. We can rewrite the equation as

$$dX_t = \sum_{i=1}^{n} A_i(X_t) \circ dB_t^i + A_0(X_t)dt$$

This equation describes a diffusion of particules moving along a drift A_0 plus n vector fields, each of them beeing driven by a Brownian motion.

In the spirit of our proof of the one-dimensional case above, we will assume, although it is not at all necessary, that there is no drift :

$$A_0 = 0 \tag{20}$$

Our first result is a direct generalization of the first theorem :

Theorem 11 (Doss)**.** *Assume that there is a function $h : \mathbb{R}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}^d$ such that for all $x \in \mathbb{R}^d$ and for all $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we have*

$$\frac{\partial h}{\partial y_i}(x, y) = A_i(h(x, y)) \text{ for } i = 1, \dots, n \tag{21}$$

$$h(x, 0) = x \tag{22}$$

Then almost surely we have for all t

$$X_t = h(x, B_t)$$

The proof is exactly the same as the one of Theorem 1.

We note that the condition (21) is an extremely strong condition on the vector fields A_i's. It is called *total integrability.*

Exercise 11. *Show that this condition implies that* $[A_i, A_j] = 0$ *for all* i, j. *In particular, the Lie algebra* L *generated by the* A_i*'s is generated by the* A_i *as a module and is commutative*

As we shall see below, this condition is really necessary in order to be able to represent the solution X_t as a determinist function of the B_t^i (and the initial condition x) only. To state a result that is true in more general context, we need to introduce more processes derived from the B_t^i, $i = 1, \dots, n$.

Definition 12. For $I \in E(\infty)$, we define B_t^I by induction on the length a of I. If $a = 1$ and $I = (i)$ we set $B_t^{(i)} := B_t^i$. For $a > 1$ and $I = (i_1, \dots, i_a)$ we set

$$B_t^I := \int_0^t B_s^{(i_1,\dots,i_{a-1})} \circ dB_{i_a}.$$

Exercise 12. *Compute* $B_t^{(i,j)}$ *in terms of Ito's integral instead of Stratonovoch'*

For $p > 1$ an integer, we set $B_t^{F(p)} := (B_t^I)_{I \in F(p)}$. This is a process valued in the space $\mathbb{R}^{F(p)}$ which has dimension $F(p)$

Theorem 13 (Yamato 1979)**.** *If the Lie algebra* L *generated by the* A_i*'s is nilpotent of step* p*, then there is a deterministic* C^∞ *function* $h : \mathbb{R}^d \times \mathbb{R}^{F(p)} \to \mathbb{R}^d$ *such that almost surely, for all* t*,*

$$X_t = h(x, B_t^{F(p)}) \tag{23}$$

Conversely, there is a p *and a function* h *as above such that (23) holds for all* $x \in \mathbb{R}^d$ *almost surely for all* t*, then the Lie algebra* L *is nilpotent of step at most* p*,*

So in particular, if the conclusion of Theorem 11 holds, we see that L is nilpotent of step 1 (or zero, if all the X_i are zero), that is commuative.

Exercise 13. *In the situation of example 8, solve equation (18) and prove that* $X_t = (x + B_t^1, y + \int_0^t B_t^1 \ dB_2^t)$.

Exercise 14. *In the situation of example 8, solve equation (18) and prove that* $X_t = (x^1 + B_t^1, x^2 + B_t^2, x^3 + 2(x^2 B_t^1 - x^1 B_t^2) + 2(B_t^1 B_t^2 - 2B_t^{1,2}$.

4. Proof of the converse assertion of Theorem 13

We start with the proof of the converse, which will also provide us some useful lemma for the "analysis" part in the proof of the direct part.

4.1. **Computing the differential of a function of the B_I's.** We shall work with the space $\mathbb{R}^{E(p)}$. We shall denote by y^I, $I \in E(p)$ a coordinate on this space. We shall denote by $B_t^{E(p)}$ the process $(B_t^I)_{I\in E(p)}$ valued in this space.

Let $u : \mathbb{R}^{E(p)} \to \mathbb{R}$ be a smooth function. Then $u(B_t^{E(p)})$ is an $\mathbb{R}$-valued process. Ito's formula (in the simplified form it takes for the Stratonovich integral) for that process gives us

$$du(B_t^{E(p)}) = \sum_{I\in E(p)} \frac{\partial u}{\partial y^I}(B_t^{E(p)}) \circ dB_t^I$$

By definition

$$dB_t^{(i_1,\dots,i_a)} = B_t^{(i_1,\dots,i_{a-1})} \circ dB_t^{i_a}.$$

Hence

$$du(B_t^{E(p)}) = \sum_{i\in E}(Q_i u()B_t^{E(p)}) \circ dB_t^i \tag{24}$$

where we have defined vector fields Q_i in $\mathbb{R}^{E(p)}$ by

$$Q_i = \frac{\partial}{\partial y^i} + \sum_{\substack{a+1\le p\\ j_1,\dots,j_a\in E}} y^{(j_1,\dots,j_a)} \frac{\partial}{\partial y^{(j_1,j_2,\dots,j_a,i)}} \tag{25}$$

4.2. **Study of the vector fields Q_i.** The Q_i's we have just defined are perfectly explicit vector fields. We shall try to understand how they behave and how is made up the Lie algebra L they generate.

We define iterated brackets Q_I for $I \in E(\infty)$ in the same way as the A_I ; we may compute explictly the Q_I.

Example 14. Take $E = \{1, 2\}$ and $p = 2$. Then

$$\begin{aligned} Q_1 &= \frac{\partial}{\partial y^1} + y^1 \frac{\partial}{\partial y^{(1,1)}} + y^2 \frac{\partial}{\partial y^{(2,1)}} \\ Q_1 &= \frac{\partial}{\partial y^2} + y^1 \frac{\partial}{\partial y^{(1,2)}} + y^2 \frac{\partial}{\partial y^{(2,2)}} \end{aligned}$$

so

$$[Q_1, Q_2] = Q_{(1,2)} = -Q_{(2,1)} = \frac{\partial}{\partial y^{(1,2)}} - \frac{\partial}{\partial y^{(2,1)}}$$

and $Q_{(1,1)} = Q_{(2,2)} = 0$.

More generally :

Lemma 15. *The Lie alegbra L generated by the Q_i is nilpotent of set p. Its dimension as a vector space is exactly $|F(p)|$ and the Q_I's, $I \in F(p)$ define an $\mathbb{R}$-basis of L.*

Exercise 15. *Prove this lemma*

4.3. **End of the proof.** Now we begin to guess what will be the strategy : relating as closely as possible the (unknown) A_i and the (explicit) Q_i (even if they do not live in the same space) We now prove a simple lemma in this direction

Lemma 16. *If u is as above, and $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ is $\mathbb{C}^\infty$ and if they satisfy almost surely for all t*

$$u(B_t^{E(p)}) = g(X_t),$$

where X_t is a solution of the differential equation above, then we have almost surely for all t $Q_i u(B_t^{E(p)}) = A_i g(X_t)$

Proof — By (24) and (18) we have

$$\sum_{i\in E} Q_i(u)(B_t^{E(p)}) \circ dB_t^i = \sum_{i\in E} A_i(g)(X_t) \circ dB_t^i$$

which implies easily the result. □

Remark 17. This lemma is the only point in this proof where we use the stochastic context. Everywhere else we could replace each B_t^i by a simple deterministic differentiable function $b^i(t)$ and Stratonovich integrals by ordinary ones, and the same formulas would hold. But this lemma becomes obviously false if we replace B_t^i by $b^i(t)$.

We can now finish the proof of the converse part of the theorem. our hypotheis is that $X_t = h(x, B_t^{F(p)})$ for some deterministic function $h : \mathbb{R}^d \times \mathbb{R}^{F(p)} \longrightarrow \mathbb{R}^d$. we shall assume something even weaker : that h goes from $\mathbb{R}^d \times \mathbb{R}^{E(p)} \rightarrow \mathbb{R}^d$ an that $X_t = h(x, B_t^{E(p)})$, This is an almost sure equality in $\mathbb{R}^d$, true for all t.

Applying Lemma 16 to the jth coordinate of this equality, we get

$$A_i^j(X_t) = Q_i h^j(B_t^{E(p)})$$

Applying Lemma 16 $k-1$ times again, we get

$$A_{i_1} \dots A_{i_k} A_{i_k}^j(X_t) = Q_{i_1} \dots Q_{i_k} h^j(B_t^{E(p)})$$

from which we get easily, for $I \in E(\infty)$

$$A_I^j(X_t) = Q_I h^j(B_t^{E(p)})$$

Now if the length of I is greater than p the RHS is zero by Lemma 15. Hence so is the LHS, and finally we get $A_I = 0$. This proves that L is nilpotent of step at most p.

5. Proof of the direct part of Theorem 13

5.1. **Analysis.** We want afunction h such that if X_t is defined by (23), then the stochastic differential equation 21 holds.

The formula (21) implies

$$X_t^j = h^j(x, B_t^{F(p)})$$

Let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ be simply the coordinate x^j. Then the LHS is just $g(X_t)$ and we can apply (for x fixed) Lemma 16 that gives us

$$A_i X_t^j = (Q_i h^j)(x, B_t^{F(p)}),\ \forall j = 1, \dots, d$$

or

$$A_i X^t = (Q_i h)(x, B_t^{F(p)} \tag{26}$$

if we understand the action of the derivation Q_i to be componentwise.

So the question is : for z fixed, can we find an h that satisfies the initial condition and the condition (26). A necessary condition for that is clearly that all linear relation between the Q_I is also satisfied by the A_I. But this holds, as it follows from lemma **??** that the only linear relation satisfied by the Q_I are the ones that hold in any Lie algebra nilpotent of step p. Is this necessary condition also a sufficienmt one : the answer is yes, and the proof is just a technical matter based on the Frobenius theorem.

5.2. **Synthesis or The proof.** We want to apply Frobenius' theorem. Basically, we would like to apply it to the Lie algebra generated by the A_i. But this leads to several problems. For one thing, there is no reason that the dimension of the space generated by the $A_I(x)$ for $I \in F(p)$ be independant of x.

Wo will consider an auxiliary Lie Algebra of vector field on the space $\mathbb{R}^d \times \mathbb{R}^{F(p)}$. On $\mathbb{R}^d$, we have the n vector fields A_i's, and more generally all the A_I's for $I \in F(p)$. We can consider them as vector fields on $\mathbb{R}^d \times \mathbb{R}^{F(p)}$, constant on the second factor. On $\mathbb{R}^{F(p)}$, we have the n vector fields Q_i and more geneerally the Q_I's for $I \in F(p)$. We can consider them as vector fields on $\mathbb{R}^d \times \mathbb{R}^{F(p)}$, constant on the first factor. We remark that $[A_i, Q_j] = 0$ for all i, j.

Now we consider the n vector fields $A_i + Q_i$ on this space, and we call L the Lie algebra they generates. In the computation of the iterated brackets of those operators, A_i and Q_j will never interact, so they will be the $A_I + Q_I$. Hence, since A_I (by hypothesis) and Q_I (by lemma **??**) are zero if I has length greter than p, then L is nilpotent of step p and generated as a vector space by the $A_I + Q_I$, $I \in F(p)$. Moreover, by lemma **??**, L has dimension $|F(p)|$ and so have all the $L_{(x,y)}$ for all $x \in \mathbb{R}^d, y \in \mathbb{R}^{F(p)}$.

To be continued.

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