

# CONVERGING SEQUENCES OF $p$ -ADIC GALOIS REPRESENTATIONS AND DENSITY THEOREMS

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ABSTRACT. We compare different notions of convergence for sequences of finite-dimensional representations over a valued field. We then consider convergent sequences of Galois representations over  $C_p$ , prove a control result for the component groups of the algebraic envelopes of the representations, and deduce a uniform Chebotarev-type density theorem for the representations in the sequence.

In this paper we prove density theorems for *converging* sequences of continuous representations  $\rho_n: G_F \rightarrow \mathrm{GL}_d(\mathbb{C}_p)$ , with  $G_F$  the absolute Galois group of a number field  $F$  and  $\mathbb{C}_p$  the completion of an algebraic closure of  $\mathbb{Q}_p$ .

There is some play in what one means by *convergence* which is studied in the first part of the paper. We work in a more general context : let  $\rho_n: G \rightarrow \mathrm{GL}_d(K)$  be a sequence of representations of an arbitrary group  $G$  on a complete valued field  $K$  (of characteristic zero, or  $p > d$ ).

One might consider *trace-convergence*: the sequence of traces, and hence the sequence of characteristic polynomials, of  $\rho_n(g)$  converges for each  $g \in G$ . Then the limit of the trace is a well-defined  $K$ -valued pseudo-character of  $G$ , which by theory of pseudo-representations initiated by Wiles and developed by Taylor [Tay], is the trace of a representation of  $G$  defined over a finite extension  $K'$  of  $K$ , and unique up to semi-simplification. The stronger notion of *physical convergence* means that we can conjugate each  $\rho_n$  by an element of  $\mathrm{GL}_d(K)$  (depending on  $n$ ) so that the resulting homomorphisms converge entry by entry. Then there is at least one limit representation  $\rho$  defined over  $K$ , and well-defined up to semi-simplification. Of course physical convergence implies trace-convergence. We are interested in results in the other direction, as in applications (e.g. to Galois representations) the sequences which arise naturally are only trace-convergent (e.g. the sequence of Galois representations given by congruences between characters of a Hecke algebra), but sometimes the results we can prove about them (see the remarks at end of section 3) need physical convergence.

Before stating our results, we have to introduce a second distinction in the notions of convergence (both trace and physical). We may ask that the trace function (resp. the entries) converge simply or uniformly in  $G$ .

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So we consider in fact four notions of convergence, namely : *simple trace-convergence*, *uniform trace-convergence*, *simple physical convergence*, and *uniform physical convergence*.

Our first result (Theorem 1.2) states that if the  $\rho_n$  simply trace-converge, and if the limit pseudo-character is absolutely irreducible, then it is the trace of a representation defined over  $K$  which is the physical limit of the  $\rho_n$ . We show by an example that we cannot omit the hypothesis that the limit pseudo-character is absolutely irreducible, as long as we only assume simple trace-convergence. But our second result (Theorem 1.4) says that if the  $\rho_n$  uniformly trace-converges, and if the  $\rho_n$  are irreducible and the limit pseudo-character is a sum of *distinct* absolutely irreducible pseudo-characters, then it is the trace of a semisimple representation defined over  $K$  which is a uniform physical limit of the  $\rho_n$ .

Note that Theorem 1.2 was known in the case  $K = \mathbb{C}$ ,  $G$  finitely generated. Its proof used invariant theory. Our argument is completely different. Theorem 1.4 is new, as far as we know.

We also consider analogues of these questions for integral models of  $\rho_n$  which are no longer unique but of which there are only finitely many up to isomorphism. Here we assume that the field  $K$  is non-archimedean, with ring  $\mathcal{O}$ . We prove (Proposition 1.9) that in case of uniform convergence, when the limit representation has a stable lattice, then the  $\rho_n$  also have one for  $n$  big enough, and there is physical convergence in bases which are  $\mathcal{O}$ -bases of stable lattices.

In the second part of this paper, we specialize to the case of sequences of representations of a compact group taking values in  $\mathbb{C}_p$ . We consider the algebraic envelopes of the representations, i.e., the Zariski-closures of their images. Our goal is to control the component groups of the resulting algebraic groups. In particular, we prove that the order of these groups is bounded in a uniformly trace-convergent sequence of representations and that in the case when a limit representation is irreducible, all but finitely many of these component groups are quotients of the component group of the limit representation.

Control of component groups is needed in the last part of the paper, where we study density theorems for uniformly trace-convergent sequences of *Galois* representations. Here our point of view is that sequences of converging representations behave like one big representation with given specialisations, and one particularly interesting specialisation which corresponds to a limit representation which controls the behaviour of almost all elements of the sequence. The main theorems are Theorem 3.6, Theorem 3.7 and Theorem 3.8. Chebotarev density theorems for a *single*  $p$ -adic Galois representation were proved by Serre in [S2] and density 0 results about ramified primes in a single semisimple representation were proved in [Kh-Raj] and [KLR]. Thus the density theorems we prove in the last section may be regarded as generalisations of these results to the situation where we have at hand converging sequences of Galois representations rather than just a single representation.

Limits of Galois representations were previously studied in [Kh] in the residually irreducible case in which case the convergence results needed were available because of the results of Carayol ([Ca]).

## 1. LIMITS OF REPRESENTATIONS

**1.1. The limit representation.** Let  $G$  be a group,  $K$  be a complete, non-discrete, valued field and  $d \geq 1$  an integer. If  $K$  has finite characteristic  $p > 0$ , we assume that  $d < p$ . Let  $(\rho_n)_{n \in \mathbb{N}}$  be a sequence of  $d$ -dimensional representations of  $G$  over  $K$ .

**Definition 1.1.** *i) We say that  $(\rho_n)$  is trace-convergent if for all  $g \in G$ , the sequence  $(\text{tr}(\rho_n(g)))$  converges in  $K$ . Moreover, if the functions  $\text{tr}(\rho_n(\cdot))$  converge uniformly on  $G$ , we will say that  $(\rho_n)$  is uniformly trace-convergent.*

*ii) We say that  $(\rho_n)$  is physically convergent if for all  $n$ , there exists a  $K$ -basis of  $\rho_n$  such that the matrix coefficients  $c_{i,j}^n$  in this basis satisfy:*

$$\forall g \in G, (c_{i,j}^n(g))_{n \geq 0} \text{ converges in } K.$$

*The equivalence class of the representation  $G \rightarrow \text{GL}_n(K)$ ,  $g \mapsto \lim c_{i,j}(g)$  is called a physical limit of  $(\rho_n)$ . Moreover, if the functions  $c_{i,j}^n$  converge uniformly on  $G$ , we will say that  $(\rho_n)$  is uniformly physically convergent.*

Suppose  $(\rho_n)$  is trace-convergent, and let  $T$  be the  $K$ -valued function on  $G$  defined by

$$T(g) := \lim_{n \rightarrow \infty} \text{tr}(\rho_n(g)).$$

Then  $T$  is a  $K$ -valued pseudo-character on  $G$  in the sense of [Rou]. We claim it is  $d$ -dimensional. Indeed, if  $d' := \dim(T)$ , then  $d' \leq d$  and, by [Rou, Prop. 2.4] we have  $d' \equiv d \pmod{\text{char}(K)}$ , so that  $d' = d$ .

By loc. cit. Lemma 4.1,  $T$  is the trace of a semisimple representation  $\rho: G \rightarrow \text{GL}_d(\overline{K})$ , which is unique by the Brauer-Nesbitt theorem. Here,  $\overline{K}$  is an algebraic closure of  $K$ . We will call  $\rho$  the limit representation of  $(\rho_n)$ . It is a priori defined over a finite extension of  $K$ .

Note that if  $G$  is topological, and  $T$  is continuous (this happens e.g. if each  $\rho_n$  is continuous and  $(\rho_n)$  is uniformly trace-convergent), then  $\rho$  is continuous. (This is quite easy: for a proof see, e.g., [BC, lemma 7.1].)

## 1.2. Simple convergence and irreducible limit.

**Theorem 1.2.** *Assume that  $(\rho_n)$  is trace-convergent and that  $\rho$  is irreducible. Then the representations  $\rho_n$  are absolutely irreducible for  $n$  big enough,  $(\rho_n)$  is physically convergent, and  $\rho$  is defined over  $K$ .*

*If, moreover,  $(\rho_n)$  is uniformly trace-convergent, then it is uniformly physically convergent.*

Let  $A$  be the  $K$ -algebra of sequences  $(x_n)_{n \in \mathbb{N}}$  of elements of  $K$ , such that  $x_n$  converges in  $K$ . Let  $r \in \mathbb{N}$ , and  $f_r \in A$  be the sequence such that  $(f_r)_n = 0$  for  $n < r$  and 1 for  $n \geq r$ . Let  $A_r := A_{f_r}$ . The natural map  $A \rightarrow A_r$  is surjective, with kernel the ideal of sequences  $(x_n)$  with  $x_n = 0$  for  $n \geq r$ . Let  $\mathfrak{m} \subset A$  be the ideal of sequences converging to 0.

For the basic results and definitions concerning Azumaya algebras, we refer to [G, §5.1]. The reader should note that our rings  $A$ ,  $A_r$ ,  $A_m$  are by no means noetherian.

**Lemma 1.3.** (a) *The maximal ideals of  $A$  are exactly  $\mathfrak{m}$  and, for  $i \geq 0$ ,  $\mathfrak{m}_i := \{(x_n) \in A, x_i = 0\}$ .*

(b) *The canonical maps  $A_r \rightarrow A_m$  induce an isomorphism  $\varinjlim_r A_r \xrightarrow{\sim} A_m$ . Hence  $A_m$  is the ring of germs at  $\infty$  of converging sequences.*

(c)  *$A_m$  is a local Henselian ring.*

(d) *If  $B$  is an Azumaya algebra over  $A$ , then  $B \otimes_A A_r$  is isomorphic to  $D \otimes_K A_r$  for  $r$  big enough, where  $D := B/mB$ .*

*In particular, if  $B/m_i B$  is trivial for an infinite number of integers  $i$ , then  $D$  and  $B \otimes_A A_r$  are also trivial.*

*Proof.* (a) The ring  $A$  equipped with the sup. norm is a  $K$ -Banach algebra, so that each maximal ideal is closed. Let  $I$  be such an ideal. If  $I$  is not in  $\mathfrak{m}_i$ , it contains a sequence  $\delta_i$  such that  $(\delta_i)_n = 0$  if and only if  $n \neq i$ . So if  $I$  is none of the  $\mathfrak{m}_i$ ,  $I$  contains all the finite sequences, which are dense in  $\mathfrak{m}$ .

(b) Let  $f \in A \setminus \mathfrak{m}$ , then  $f_n \neq 0$  for all  $n \geq r$  for  $r$  big enough. Fix such an  $r$ , then the natural map  $A_f \rightarrow A_m$  does factor through  $A_r \rightarrow A_m$ .

(c) We must show that if a sequence of monic polynomials  $P_n \in K[T]$  of a fixed degree converges to  $P$ , and  $P$  has a simple root  $x$ , then for all  $n$  big enough, there exists a root  $x_n \in K$  of  $P_n$ , such that  $x_n \rightarrow x$ . Suppose first that  $K$  is non-archimedean, then for  $n$  big enough,  $|P_n(x)| < |P'_n(x)|^2$  and Newton's method gives a root  $x_n \in K$  of  $P_n$  such that  $|x - x_n| \leq |P_n(x)|/|P'_n(x)|$ , and we are done. If  $K = \mathbb{R}$ , then for each  $\varepsilon > 0$  small enough and  $n$  big enough,  $P_n(x + \varepsilon)P_n(x - \varepsilon) < 0$ . In particular,  $P_n$  has a real root for  $n$  big enough, and we can choose  $x_n$  to be one of the closest to  $x$ . If  $K = \mathbb{C}$  this is simply the continuity of roots of polynomials.

(d) Let  $B$  be an Azumaya algebra over  $A$ . We call  $B_r := B \otimes_A A_r$ ,  $B_\infty := B \otimes_A A_m$ . By (b),  $B_\infty$  is the inductive limit of  $B_r$  when  $r$  grows. As  $A_m$  is henselian by (c) and Azumaya theorem [G, thm. 6.1],  $B_\infty$  is isomorphic to  $D \otimes_K A_m$  where  $D := B/mB$ . For  $r \in \mathbb{N} \cup \{\infty\}$ , let  $C_r := D \otimes_K A_r$ . As  $B$  and  $C$  are finitely presented as  $A$ -modules and by (b), any  $A_m$ -module isomorphism  $\varphi_\infty : B_\infty \rightarrow C_\infty$  comes from an  $A_r$ -module isomorphism  $\varphi_r : B_r \rightarrow C_r$  for  $r$  big enough. If we assume moreover that  $\varphi_\infty$  is a ring homomorphism then for  $r' > r$  big enough,  $\varphi_r \otimes_{A_r} A_{r'} : B_{r'} \rightarrow C_{r'}$  is also a ring homomorphism. Indeed, there are only a finite number of products to check by linearity, and we are done by (b).  $\square$

We now prove Theorem 1.2. Let  $\text{Tr}: G \rightarrow A$  be the function defined by  $\text{Tr}(g)_n := \text{tr}(\rho_n(g))$ , which is an element of  $A$  by assumption. We note first that  $\rho_n$  is absolutely irreducible for all  $n$  big enough.

By assumption,  $\rho$  is absolutely irreducible. By the non-degeneracy of the trace in  $M_d(\overline{K})$  we can find  $d^2$  elements  $g_s \in G$  such that  $\det(\text{tr}(\rho(g_s g_t))) \in K^*$ . By continuity,  $\det(\text{tr}(\rho_n(g_s g_t)))$  is also non-zero for  $n$  big enough, i.e. the  $(\rho_n(g_s))_s$  form a  $K$ -basis of  $M_d(K)$ , as we wanted.

So we can assume that all the  $\rho_n$  are absolutely irreducible. By hypothesis,  $\rho$  also is absolutely irreducible, so that by the lemma (a),  $\text{Tr} \bmod I$  is absolutely irreducible for each maximal ideal  $I$  of  $A$ . We can then apply Rouquier's theorem ([Rou, Theorem 5.1]) that there exists an Azumaya algebra  $B$  over  $A$  and a surjective  $A$ -algebra homomorphism  $A[G] \rightarrow B$  whose reduced trace coincide with  $\text{Tr}$  on  $G$ . As  $\rho_n$  is absolutely irreducible (and defined over  $K$  by hypothesis), the  $K$ -algebra  $B/m_n B$  is then isomorphic to  $M_d(K)$  for all  $n$ . By lemma (d), it follows that  $B_r$  is isomorphic with  $M_d(A_r)$  for some  $r$ . This concludes the first point of the proof.

Consider the representation  $\rho': G \rightarrow \text{GL}_d(A_r)$ , whose trace is  $\text{Tr}$ , constructed in the previous paragraph. We know that the induced morphism  $A_r[G] \rightarrow M_d(A_r)$  is surjective. It implies that the  $A_r$ -dual of  $M_d(A_r)$  is generated as  $A_r$ -module by linear forms of the form:  $x \mapsto \text{Tr}(xh)$ , for some  $h$  in  $G$ . Applying this to the  $(i, j)$ -matrix coefficient  $c_{i,j}$ , we get that there exists a finite number of  $a_k \in A$  and  $g_k \in G$  such that

$$\forall g \in G, c_{i,j}(g) = \sum_k a_{k,i,j} \text{Tr}(gg_k).$$

As sequence of functions on  $G$ , this implies that  $c_{i,j}$  converges uniformly.  $\square$

**Remark:** As the above proof shows, the result holds in the context of representations of  $A$ -algebras: if  $R$  is any  $A$ -algebra equipped with a faithful  $d$ -dimensional pseudo-character  $T = (T_n) : R \rightarrow A$  such that  $\lim T$  is absolutely irreducible, then for  $r$  big enough,  $R \otimes_A A_r$  is isomorphic to  $M_d(A_r)$  as  $A_r$ -algebra.

**Remark:** (i) Assume that  $G$  is a topological group, that the  $\rho_n$  are continuous, uniformly trace-convergent, and that  $\rho$  is irreducible, then  $\rho$  is also continuous by the theorem.

(ii) When  $(\rho_n)$  is trace-convergent but  $\rho$  is reducible,  $(\rho_n)$  need not converge physically in general, as the following example shows.

Let  $A$  be the ring introduced in section 1.2,  $\mathfrak{m}$  its maximal ideal of sequences converging to zero, and  $A' \supset A$  the ring of bounded,  $K$ -valued, sequences. We have  $\mathfrak{m}A' \subset \mathfrak{m}$  and  $(A \setminus \mathfrak{m}) + \mathfrak{m} \subset (A \setminus \mathfrak{m})$ . We can thus consider the following group  $G \subset \text{GL}_2(A')$  of matrices:

$$\left( \begin{array}{cc} A \setminus \mathfrak{m} & A' \\ \mathfrak{m} & A \setminus \mathfrak{m} \end{array} \right) \cap \text{GL}_2(A')$$

Let  $\rho': G \rightarrow \mathrm{GL}_2(A')$  be the canonical representation, and  $\rho_n$  its  $n$ th-coordinate projection,  $\rho_n: G \rightarrow \mathrm{GL}_2(K)$ . Then  $\rho_n$  is trace-convergent by construction, but not physically.

Here is a proof of this last fact. If a  $\rho': G \rightarrow \mathrm{GL}_2(A)$  commutes with trace, we can conjugate it such that the constant element  $(-1, 1)$  acts diagonally by  $(-1, 1)$ . In that base, because of the trace identity, every diagonal matrix maps to itself. If  $\rho'_n: G \rightarrow \mathrm{GL}_2(K)$  denotes the projection of  $\rho'$  on the  $n$ th coordinate,  $\rho'_n$  has the same trace as the irreducible representation  $\rho_n$ , so that it factors through the  $n$ th coordinate  $G \rightarrow \mathrm{GL}_2(K)$ , which is surjective. Call the induced map  $\mathrm{GL}_2(K) \rightarrow \mathrm{GL}_2(K)$  the  $n$ th-component of  $\rho'$ . As  $\rho_n$  is absolutely irreducible, the  $n$ th component of  $\rho'$  is an inner embedding  $\mathrm{GL}_2(K) \rightarrow \mathrm{GL}_2(K)$ , which is the identity on all diagonal matrices. It is therefore a diagonal conjugation and preserves the standard upper and lower Borels by multiplying the coordinate by a non-zero element, say  $x_n \in K^*$  for the upper, and so by  $x_n^{-1}$  for the lower. We get then a map  $A' \rightarrow A$  given by  $(b_n) \mapsto (x_n b_n)$ . Taking  $(b_n) = (1)$  and  $(b'_n)$  with  $b'_{2n} = 1$ ,  $b'_{2n+1} = 0$  implies that  $x_n$  converges to 0. But we get also a map  $m \rightarrow A$ ,  $(c_n) \mapsto (x_n^{-1} c_n)$ . Let  $(c_n)$  be given by  $c_{2n} = 0$  and  $c_{2n+1} = x_{2n+1}$ , we get a contradiction.

### 1.3. Uniform convergence and multiplicity-free limit.

**Theorem 1.4.** *Assume that the representations  $\rho_n$  are absolutely irreducible for  $n$  big enough, that  $(\rho_n)$  is uniformly trace-convergent and that  $\mathrm{tr}(\rho)$  is a sum of pairwise distinct,  $K$ -valued, absolutely irreducible pseudo-characters. Then  $\rho$  is defined over  $K$  and  $(\rho_n)$  is uniformly physically convergent to  $\rho$ .*

**Remark:** It is easy to give examples where  $(\rho_n)$  is physically convergent to several non-isomorphic representations (which have of course isomorphic semi-simplifications). The above theorem asserts that we can make the  $(\rho_n)$  physically converge, and what is more, to converge to the semisimple limit. The methods of the proof below are close in spirit to those of [BG].

We now begin the proof of Theorem 1.4. Let  $A$ ,  $A_r$  and  $\mathfrak{m}$  be as in section 1.2. Let  $S := K^{\mathbb{N}} \supset A$  be the  $K$ -algebra of all  $K$ -valued sequences,  $S_r$  be the quotient of  $S$  by the ideal of sequences which are zero after  $r$ , and let  $B \supset A_{\mathfrak{m}}$  be the  $K$ -algebra of germs at  $\infty$  of elements of  $S$ , that is  $B = \varinjlim_r S_r$ . The representations  $\rho_n$  of the assumption altogether give rise to a representation  $\rho': G \rightarrow \mathrm{GL}_d(S)$  whose trace is  $A$ -valued. Let  $(T_i)_{i=1, \dots, s}$  be the pairwise distinct, absolutely irreducible,  $K$ -valued, pseudo-characters of the assumption, and  $d_i := \dim(T_i)$ .

**Lemma 1.5.** *For  $r$  big enough, there are  $s$  orthogonal idempotents  $e_1, \dots, e_s$  in  $\rho'(A_r[G]) \subset M_d(S_r)$  satisfying:*

- i)  $e_1 + \dots + e_s = 1$ ,
- ii) for each  $i$ ,  $\mathrm{tr}(e_i) = d_i$ ,
- iii) for each  $i \neq j$ , and  $x, y \in \rho'(A_r[G])$ ,  $\mathrm{tr}(e_i x e_j y e_i) \in \mathfrak{m}$ ,
- iv) for each  $i$ ,  $e_i \rho'(A_r[G]) e_i$  is isomorphic as  $A_r$ -algebra to  $M_{d_i}(A_r)$ .

*Proof.* Let  $R := \rho'(A_{\mathfrak{m}}[G]) \subset M_d(B)$ ,  $T = \text{tr}(\rho')$ ,  $\overline{R} := R/\mathfrak{m}R$ ,  $\overline{T} := T \bmod \mathfrak{m}$  and

$$\text{Ker}(\overline{T}) := \{x \in \overline{R}, \forall y \in G, \overline{T}(xy) = 0\}.$$

Let  $\pi : K[G] \rightarrow \overline{R}$  be the surjective  $K$ -algebra morphism which sends  $g$  to the reduction of  $\rho'(g)$ . By hypothesis,  $\overline{T} \circ \pi = \sum_{i=1}^s T_i$ . Now choose (cf. [Rou, thm 4.2]) an irreducible representation  $\overline{\rho}_i : K[G] \rightarrow M_{d_i}(\overline{K})$  whose trace is  $T_i$ , and let  $\overline{\rho} = \oplus_{i=1}^s \overline{\rho}_i$ . Because the  $\overline{\rho}_i$  are pairwise non-isomorphic, the image  $\overline{\rho}(K[G])$  is  $\oplus_{i=1}^s R_i$ , with  $R_i = K[G]/(\text{Ker}(T_i))$  a central simple algebra over  $K$ . Because  $\overline{\rho}$  is semisimple we have  $\text{Ker}(\overline{T} \circ \pi) = \text{Ker}(\overline{\rho})$  by [Tay, Th. 1.1.]. Then  $\overline{\rho}$  induces an isomorphism  $\overline{R}/(\text{Ker}(\overline{T})) \simeq \oplus_{i=1}^s R_i$  such that the reduced trace of  $R_i$  is  $T_i$ .

We call  $\epsilon_i$ , for  $i = 1, \dots, s$ , the unit of  $R_i$ , seen as a central idempotent of  $\overline{R}/(\text{Ker}(\overline{T}))$ . As  $A_{\mathfrak{m}}$  is local henselian and  $R$  is integral over  $A_{\mathfrak{m}}$  by Cayley-Hamilton theorem, [Bki, III, §4, exercice 5(b)] implies that there exist orthogonal idempotents  $f_i \in R$ ,  $i = 1, \dots, d$  lifting  $\epsilon_i$ . By construction, we have  $\text{tr}(f_i) \equiv d_i \pmod{\mathfrak{m}}$ . Note that if  $f \in M_d(B)$  is an idempotent, then its trace is the germ of a sequence of integers between 0 and  $d$ . If moreover  $\text{tr}(f) \in A_{\mathfrak{m}}$ , then this sequence is eventually constant. In particular,  $\text{tr}(f_i) = d_i$ , and so  $f_1 + \dots + f_s = 1$ .

Now, fix  $e_i \in A[\rho'(G)]$  be a lift of  $f_i$ . For  $r$  big enough, we have  $e_i e_j = \delta_{i,j} e_i$  and  $\sum_i e_i = 1$  in  $M_d(B_r)$ , and also  $\text{tr}(e_i) = d_i \in A_r$ . This proves i) and ii). For iii), it suffices to prove that the image of  $(e_i x e_j y e_i)$  is zero in  $\overline{R}/\text{ker}(\overline{T}) \simeq \oplus_{i=1}^s R_i$ . But this image is  $(\epsilon_i \overline{x} \epsilon_j \overline{y} \epsilon_i)$  which is obviously zero. It remains to prove iv). Let  $R' := e_i R e_i \subset M_{d_i}(S)$ ,  $T'$  the restriction of  $T$  to  $R'$ , then  $T' \bmod \mathfrak{m} = T_i$  is absolutely irreducible, and  $T'$  is faithful if  $r$  is big enough so that all the representations  $\rho_n$ ,  $n \geq r$ , are absolutely irreducible. By Theorem 1.2 (see the remark immediately following the proof of the theorem),  $R'$  is isomorphic as  $A_r$ -algebra to  $M_{d_i}(A_r)$  for  $r$  big enough.  $\square$

In particular, assertion iv) implies that each irreducible factor of  $\rho$  is defined over  $K$ . Forgetting the first  $r$  terms of our sequence, we may assume  $r = 0$  in the preceding lemma, so we drop the  $r$  in  $S_r$  and  $A_r$ .

For the convenience of the reader, we first prove the theorem in the case where  $\rho$  is a sum of pairwise distinct *one-dimensional characters*, i.e  $s = d$ ,  $d_i = 1, \forall i$ . We will return to the general case, which requires no new idea but a great deal of additional notation, at the end of the proof.

$\hookrightarrow$  From i) and ii) of the previous lemma, we can choose an  $S$ -basis  $(E_i)$  of  $S^d$  such that for each  $n$ ,  $K.E_i^n = e_i^n(K^d)$ . For  $y \in M_d(S)$ , we note  $y_{i,j}$  the  $(i, j)$ -component of  $y$ . We note  $E_{i,j}$  the matrix whose  $(i, j)$ -coefficient is one and others are zero. Note that  $E_{i,i} = e_i$  and that for  $y \in M_d(S)$ ,  $e_i y e_j = y_{i,j} E_{i,j}$ . Now for each  $i, j \in \{1, \dots, d\}$  and  $n \in \mathbb{N}$  we define

$$x_{i,j}^n := \inf_{g \in G} v(\rho'(g)_{i,j}^n).$$

Here  $v$  is a fixed  $\mathbb{R}$ -valued valuation of  $K$ . Note that for each  $n \in \mathbb{N}$ , and each  $i, j$  we have  $x_{i,j}^n \in \mathbb{R} \cup \{-\infty\}$  because  $\rho_n$  is absolutely irreducible. As a consequence, it makes sense to add and compare those numbers.

**Lemma 1.6.** *There exists a real number  $N$  such that for each  $i, j, k$  pairwise distinct in  $\{1, \dots, d\}$ , and each  $n \in \mathbb{N}$ , we have*

$$x_{i,j}^n \leq x_{i,k}^n + x_{k,j}^n + N$$

*Proof.* We can write all idempotents  $e_i$  as finite sums of elements of  $\rho'(G)$  with coefficients in  $A$  : there is an  $l$  such that for each  $i$

$$(1) \quad e_i = \sum_{s=1}^l a_{i,s} \rho'(h_{i,s})$$

and all the coefficients  $a_{i,s}$  are convergent (hence bounded) sequences in  $K$ . We define  $-N$  as

$$-N := v(l) + \inf_{i,s,n \in \mathbb{N}} v(a_{i,s}^n).$$

Now we fix an  $n \in \mathbb{N}$ , and  $i, j, k \in \{1, \dots, d\}$  and choose a real  $\varepsilon > 0$ . We choose  $g$  and  $g'$  such that  $v(g_{i,k}^n) \leq x_{i,k}^n + \varepsilon$  and  $v(g'_{k,j}) \leq x_{k,j}^n + \varepsilon$ . We have

$$g_{i,k} g_{k,j} E_{i,j} = e_i g e_k g' e_j = \sum_{k=1}^l a_{k,s} (gh_{i,s} g')_{i,j} E_{i,j},$$

so

$$x_{i,k}^n + x_{k,j}^n + 2\varepsilon \geq v(g_{i,k}^n g_{k,j}^n) = v\left(\sum_{k=1}^l a_{k,s}^n (gh_{i,s} g')_{i,j}^n\right).$$

Now

$$\left| \sum_{k=1}^l a_{k,s}^n (gh_{i,s} g')_{i,j}^n \right| \leq l \sup_{k,s,n} |a_{k,s}^n| \sup_{g \in G} |g_{i,j}^n|,$$

and so  $v(\sum_{k=1}^l a_{k,s}^n (gh_{i,s} g')_{i,j}^n) \geq -N + x_{i,j}^n$ . This concludes the proof.  $\square$

We now use uniform trace-convergence to prove the following lemma

**Lemma 1.7.** *For each  $i \neq j$ ,  $x_{i,j}^n + x_{j,i}^n$  goes to infinity with  $n$ .*

*Proof.* By hypothesis there is a sequence  $\delta_n \in \mathbb{R} \cup \{-\infty\}$  which goes to infinity, such that for each  $g \in G$ , we have  $v(\text{tr}(\rho_n(g)) - \lim \text{tr}(\rho(g))) > \delta_n$ .

We have by (1) (applied twice to  $e_i$  and once to  $e_j$ )

$$e_i \rho'(g) e_j \rho'(g') e_i = \sum_{s,s',s''} a_{i,s} a_{j,s'} a_{i,s''} \rho'(h_{i,s} g h_{j,s'} g' h_{i,s''})$$

which proves that there exists a sequence  $\delta'_n$  which goes to infinity such that for all  $g, g' \in G$ ,  $i, j \in \{1, \dots, d\}$

$$v(\text{tr}(e_i \rho_n(g) e_j \rho_n(g') e_i)) - \lim(\text{tr}(e_i \rho'(g) e_j \rho'(g') e_i)) > \delta'_n.$$

But by Lemma 1.5, (iii), we have  $\lim(\text{tr}(e_i \rho'(g) e_j \rho'(g') e_i)) = 0$ . Hence

$$v(\text{tr}(e_i \rho_n(g) e_j \rho_n(g') e_i)) > \delta'_n,$$

that is

$$v(g_{i,j}^n g'_{j,i}^n) > \delta'_n.$$

Taking inf on  $g$  and  $g'$ , we get

$$x_{i,j}^n + x_{j,i}^n > \delta'_n.$$

□

In particular, for  $n$  big enough, all the  $x_{i,j}^n$  are true real numbers, so that we can assume that  $x_{i,j}^n \in \mathbb{R}$  for all  $n, i, j$ . The following lemma is a simple matter of real inequalities.

**Lemma 1.8.** *Let  $x_{i,j}$ ,  $i \neq j \in \{1, \dots, d\}$ , be a family of sequences of real numbers, and a real  $N$ , such that  $x_{i,j} + x_{j,i}$  goes to infinity and  $x_{i,j} \leq x_{i,k} + x_{k,j} + N$ . Then there exist sequences of integers  $u_i$ ,  $i = 1, \dots, d$ , such that for each  $i \neq j$ ,  $x_{i,j} - (u_i - u_j)$  goes to infinity.*

*Proof.* First we may assume that  $N = 0$ . Indeed let  $x'_{i,j} = x_{i,j} + N$ , we have  $x'_{i,j} \leq x'_{i,k} + x'_{k,j}$ , and the other hypothesis as well as the conclusion remain unchanged. Note that in the conclusion we may also choose the numbers  $u_i$  to be real instead of integer, for the integer parts of the  $u_i$  will also work. We may also assume that  $x_{i,j}^n + x_{j,i}^n \geq 0$  for all  $i, j, n$ .

Choose  $n \in \mathbb{N}$ . For each  $l \in \{1, \dots, d\}$ , we define  $u^n(l) \in \mathbb{R}^d$  by  $u^n(l)_i = -x_{l,i}^n$  if  $i \neq l$ , and  $u^n(l)_l = 0$ . For each  $i \neq j \in \{1, \dots, d\}$ , we check easily that

$$u^n(l)_i - u^n(l)_j \leq x_{i,j}^n.$$

Now we consider  $u$  the barycenter of the  $u(l)$ 's, that is

$$u^n = \frac{1}{d} \sum_{l=1}^d u^n(l)$$

We have

$$\begin{aligned} x_{i,j}^n - (u_i^n - u_j^n) &= \frac{1}{d} \sum_{l=1}^d (x_{i,j}^n - (u^n(l)_i - u^n(l)_j)) \\ &= \frac{1}{d} \left( \sum_{l=1, l \neq j}^d (x_{i,j}^n - (u^n(l)_i - u^n(l)_j)) \right) + \frac{1}{d} (x_{i,j}^n - u^n(j)_i) \\ &\geq \frac{1}{d} (x_{i,j}^n + x_{j,i}^n) \end{aligned}$$

This last inequality makes clear that  $x_{i,j}^n - (u_i^n - u_j^n) \xrightarrow[n \rightarrow \infty]{} \infty$ . □

Now we can finish the proof of the theorem. We may assume that  $v$  takes the value 1, and choose an element  $\varpi \in K^*$  such that  $v(\varpi) = 1$ . We choose now  $u_i$  as in the preceding lemma, and consider the following new basis of  $S^d$  :  $F_i := (\varpi^{u_i^n})_n E_i$ . In this basis, the  $(i, j)$  coefficient of  $\rho'(g)$ , is equal to  $\varpi^{u_i^n - u_j^n} \rho'(g)_{i,j}$ , whose  $n$ th term has valuation greater or equal than  $x_{i,j}^n - (u_i^n - u_j^n)$ . If  $i \neq j$ , this shows that the  $(i, j)$ -coefficient of  $\rho'(g)$  converges to zero uniformly in  $g$ . Moreover, the diagonal coefficients, which are still  $\rho'(g)_{i,i}$ , are uniformly convergent, as they are equal to  $\text{tr}(e_i \rho(g) e_i)$ . We thus have shown that the sequences of matrices of  $\rho'(g)$ , in the basis  $(F_i)$ , converge uniformly to the diagonal matrix  $(\chi_i(g))$ , which is what we wanted.

We now return to the general case with  $d_i \geq 1$ , by indicating only the modifications of the proof. We choose first an  $S$ -basis  $(F_\alpha)_{1 \leq \alpha \leq d}$  of  $S^d$  adapted to the idempotents  $e_i$ . That means that  $\forall \alpha \in \{1, \dots, d\}$  there exists  $i$  (necessarily unique) such that  $e_i F_\alpha = F_\alpha$ , we then say that  $\alpha$  belongs to  $i$ . Define the

$$x_{\alpha,\beta}^n := \inf_{g \in G} v(\rho'(g)_{\alpha,\beta}^n),$$

and  $x_{i,j}^n = \inf_{\alpha,\beta} x_{\alpha,\beta}^n$  where  $\alpha$  (resp.  $\beta$ ) belongs to  $i$  (resp.  $j$ ). As a consequence of lemma 1.5 iv), there exists a constant  $N' \in \mathbb{R}$  such that for any  $n \in \mathbb{N}$ , any  $i, j$ , and  $\alpha$  belonging to  $i$ ,  $\beta$  belonging to  $j$ , then  $x_{\alpha,\beta}^n \leq x_{i,j}^n + N'$ . Then, it is easy to see that lemmas 1.6 and 1.7 hold for the  $x_{i,j}^n$  with the same proof (note however that 1.7 does not hold for the  $x_{\alpha,\beta}^n$ 's.) . We conclude as above.  $\square$ .

**Remarks:** As we have already seen, the uniform convergence hypothesis cannot be omitted. Moreover, we can not omit the hypothesis that the  $\rho_n$  are absolutely irreducible, as shown by this counterexample :

Let  $K$  be non-archimedean, let  $A$  be as before, and let  $A_u$  be the subring of  $A$  of sequence  $x_i$  such that  $v(x_i) \geq 0$  and  $v(x_i - \lim x_i) \geq i$  for all  $i \in \mathbb{N}$ . Let  $G$  be the group

$$\begin{pmatrix} A_u^* & A \\ 0 & A_u^* \end{pmatrix}$$

Let  $\rho' : G \rightarrow \text{GL}_2(A)$  be the canonical representation, and  $\rho_n$  its  $n$ th-coordinate projection,  $\rho_n : G \rightarrow \text{GL}_2(K)$ . Then  $\rho_n$  is uniformly trace-convergent by construction, but not uniformly physically. Moreover the  $\rho_n$  are simply physically convergent, but have no semisimple physical limit. (We leave the proofs to the reader.)

We do not know whether the hypothesis that the limit pseudo-character is multiplicity free is really necessary. We believe it is not, but that a new idea would be needed to remove it.

**1.4. Lattices.** Assume that  $K$  is non-archimedean, and denote by  $\mathcal{O}$  its valuation ring and  $m$  its maximal ideal. Let  $\tau$  be a representation of  $G$  on a finite dimensional  $K$ -vector space  $V$ . A *stable lattice* of  $\tau$  is a finitely generated sub- $\mathcal{O}$ -module of  $V$  which is stable by  $\tau(G)$  and generates  $V$  as

$K$ -vector space. As  $\mathcal{O}$  is a valuation ring, such a lattice is automatically free as  $\mathcal{O}$ -module, of rank  $\dim_K(V)$ .

If  $\tau$  has a stable lattice, then  $\tau(G)$  is bounded and  $\text{tr}(\tau(G)) \subset \mathcal{O}$ . Conversely, we can ensure that  $\tau$  admits at least a stable lattice in the following three cases:

- i)  $\tau(G)$  has compact closure in  $\text{GL}(V)$ ,
- ii)  $K$  is discretely valued,  $\tau$  is absolutely semisimple and  $\text{tr}(\tau(G)) \subset \mathcal{O}$ .
- iii)  $\text{tr}(\tau(G)) \subset \mathcal{O}$  and  $\text{tr}(\tau(\cdot)) \bmod m$  is an absolutely irreducible pseudo-character.

**Remark:** Let  $\mathcal{O}_p$  denote the ring of integers in  $\mathbb{C}_p$  and  $G$  the subgroup of  $\text{GL}_2(\mathbb{C}_p)$  of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, d \in \mathcal{O}_p^*, v(b) \geq -\sqrt{2}, v(c) \geq \sqrt{2}.$$

Then  $G$  is bounded,  $\text{tr}(G) \subset \mathcal{O}_p$ ,  $G$  acts irreducibly on  $\mathbb{C}_p^2$  but there is no stable lattice.

**Proposition 1.9.** *Suppose  $(\rho_n)$  is uniformly physically convergent to  $\rho$ , and assume that  $\rho$  has a stable lattice. Then, for  $n$  big enough, there exists a  $K$ -basis of  $\rho_n$  which generates over  $\mathcal{O}$  a stable lattice and such that the matrix coefficients  $c_{i,j}^n$  in this basis converges uniformly. In particular,  $\rho_n$  has a stable lattice for  $n$  big enough.*

*Proof.* By assumption, we can assume that there is a representation  $\rho': G \rightarrow \text{GL}_d(A)$  whose  $n$ th-coordinate is  $\rho_n$ , and whose  $(i, j)$ -coefficients are uniformly converging as sequence of functions on  $G$ . We choose a basis of a stable lattice of the limit representation  $\rho = \lim \rho'$ , and fix  $P \in \text{GL}_d(K)$  so that  $P\rho(G)P^{-1} \subset \text{GL}_d(\mathcal{O})$ . If  $P'$  in  $\text{GL}_d(A_r)$  is the constant sequence of matrices  $(P, P, P, \dots)$ , then conjugating  $\rho'$  by  $P'$  allows us to assume that  $\rho(G) \subset \text{GL}_d(\mathcal{O})$ .

Now, by the first sentence of the proof, we can choose an integer  $N$  such that

$$\forall n \geq N, \forall i, j, \sup_{g \in G} |c_{i,j}^n(g) - c_{i,j}^\infty(g)| < 1.$$

As  $K$  is non-archimedean, we have  $c_{i,j}^n(g) \in \mathcal{O}$  for all  $n \geq N, g \in G$ . □

**Remark:** Assume that  $(\rho_n)$  is uniformly physically convergent to  $\rho$  and that  $(\rho_n)$  has a stable lattice for all  $n$ . Then it does not follow that  $\rho$  has a stable lattice, as the following counter-example shows.

Let  $(\beta_n)_{n \geq 0}$  and  $(\gamma_n)_{n \geq 0}$  be strictly decreasing sequences of real numbers converging to  $-\sqrt{2}$  and  $\sqrt{2}$  respectively. Fix  $K = \mathbb{C}_p$  and  $A$  as in the first section. Consider the subset  $G \subset \text{GL}_2(A)$  of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that:  $\lim a \in \mathcal{O}^*$  and  $v(a_n - \lim a) > n$ ,  $\lim d \in \mathcal{O}^*$  and  $v(d_n - \lim d) > n$ ,  $v(b_n) > \beta_n$  and  $v(b_n - \lim b) > n - \sqrt{2}$ ,  $v(c_n) > \gamma_n$  and  $v(c_n - \lim c) > n + \sqrt{2}$ .

It is easy to check this is indeed a subgroup. Let  $\rho': G \rightarrow \mathrm{GL}_2(A)$  be the canonical representation, and  $\rho_n$  its  $n$ th-coordinate projection,  $\rho_n: G \rightarrow \mathrm{GL}_2(\mathbb{C}_p)$ . Then  $\rho_n$  is uniformly physically convergent by construction. We see easily that the image of  $\rho_n$  is the subgroup of  $\mathrm{GL}_2(\mathbb{C}_p)$  of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with  $a, d \in \mathcal{O}^*$  and  $v(b) > \beta_n$ ,  $v(c) > \gamma_n$ . As  $\beta_n + \gamma_n > 0$ ,  $\rho_n$  has a stable lattice. Moreover, the image of the limit representation  $\rho$  is the group of the above remark, which has no stable lattice.

### 1.5. Simple versus uniform physical convergence.

**Proposition 1.10.** *Assume that  $K$  is non-archimedean,  $G$  is a compact group, and the representations  $\rho_n$  are continuous, with a simple physical limit  $\rho$ . Then  $\rho_n$  converges uniformly to  $\rho$  in each of the following two cases:*

- (i) *the group  $G$  is topologically finitely generated ;*
- (ii) *the limit representation  $\rho$  is continuous.*

*Proof.* First suppose we are in case (i). Up to conjugation, we may assume  $\rho(G) \subset \mathrm{GL}_d(\mathcal{O})$ . Let  $g_1, \dots, g_k$  be a family of topological generators of  $G$ . For all  $n$  big enough, and all  $i \in \{1, \dots, k\}$ , each  $\rho_n(g_i)$  is in  $\mathrm{GL}_d(\mathcal{O})$  and the sequence. Moreover for all integer  $N$ , and for all  $i$ , the sequence  $\rho_n(g_i) \pmod{\varpi^N}$  is eventually constant. Choose an  $n_0$  such that for all  $i$ , the  $\rho_n(g_i) \pmod{\varpi^N}$ 's are constant for  $n \geq n_0$ . Then it is clear that the sequences  $\rho_n(g) \pmod{\varpi^N}$  are constant for all  $n \geq n_0$ ,  $g \in G$ . That is,  $\rho_n$  is uniformly convergent.

Now suppose we are in case (ii). Failure of uniform convergence means there exists an open neighborhood of the identity  $U$  in  $\mathrm{GL}_n(K)$  such that for all integers  $N$  there exists  $x_N$  in  $G$  and  $i, j > N$  such that  $\rho_i(x_N)\rho_j(x_N)^{-1} \notin U$ . Fix an open neighborhood  $V$  of the identity so that  $V^3 \subset U$ . As the topology at 1 is generated by open subgroups, we may assume  $V$  a subgroup. Let  $x$  be the limit of a convergent subsequence of  $x_N$  in  $G$ . Pointwise convergence at  $x$  means that there exists  $M$  such that for all  $i, j > M$ ,  $\rho_i(x)\rho_j(x)^{-1} \in V$ . Choose  $N > M$  for which  $x_N$  belongs to our subsequence converging to  $x$ . Then there exist  $i, j > N > M$  such that either  $\rho_i(x_N)\rho_i(x)^{-1}$  or  $\rho_j(x)\rho_j(x_N)^{-1} \notin V$ . Either way, there exist  $k > N$  such that  $\rho_k(x_N)\rho_k(x)^{-1} \notin V$ . We can therefore extract a subsequence of the representations  $\rho$  and a subsequence of terms of the form  $y_N x_N^{-1}$  which violates the following lemma.  $\square$

**Lemma 1.11.** *Let  $G$  be a compact topological group,  $H$  any topological group,  $V$  an open subgroup of  $H$ ,  $\rho_i : G \rightarrow H$  a sequence of continuous*

homomorphisms converging pointwise to the continuous homomorphism  $\rho$  and  $y_i \in G$  converging pointwise to the identity, then  $\rho_i(y_i) \in V$  for some  $i$ .

*Proof.* We iteratively construct a strictly monotone sequence  $a_1, a_2, \dots$  of positive integers such that

- 1)  $\rho(y_{a_i}) \in V$  for all  $i$ .
- 2)  $\rho_{a_n}(y_{a_i}) \in V$  for all  $i < n$ .
- 3)  $\rho_{a_i}(y_{a_n}) \in V$  for all  $i < n$ .

Note that as long as (1) holds for  $i < n$ , (2) holds for all sufficiently large  $a_n$  by pointwise convergence of the representation sequence; and (3) holds for all sufficiently large  $a_n$  by continuity of each  $\rho_i$ . Replacing  $\rho_i$  and  $y_i$  by subsequences, therefore, we can arrange that  $\rho_i(y_j) \in V$  if and only if  $i \neq j$ . Now let  $z_n = y_1 \cdots y_n$ . This gives a sequence of points such that  $\rho(z_n) \in V$  and  $\rho_i(z_n) \notin V$  for all  $i$ . Let  $z$  be a limit point of this sequence. Then  $\rho_i(z) \notin V$  for all  $i$ , and  $\rho(z) \in V$ , contrary to pointwise convergence.  $\square$

## 2. COMPONENT GROUPS FOR ALGEBRAIC ENVELOPES

Throughout this section,  $\pi_0(G)$  denotes the group  $G/G^\circ$  of connected components of a linear algebraic group  $G$ .

### 2.1. Preliminary lemmas.

**Lemma 2.1.** *Let  $G \subset \mathrm{GL}_d$  be a linear algebraic group defined over an algebraically closed field  $K$  of characteristic zero. Let  $g \in G(K)$  be an element of  $G$  whose image in  $\pi_0(G)$  has order  $m$ . Then the subgroup of  $K^\times$  generated by the eigenvalues of  $g$  contains a primitive  $m$ th root of unity.*

*Proof.* Let  $g = g_s g_u$  denote the Jordan decomposition of  $g$ , and let  $C$  (resp.  $C_s, C_u$ ) denote the Zariski-closure of the cyclic group  $\langle g \rangle$  (resp.  $\langle g_s \rangle, \langle g_u \rangle$ ). By [Bor, 4.7],  $C_s, C_u \subset C$ , and the product map gives an isomorphism  $C_s \times C_u \cong C$ . The map  $t \mapsto \exp(t \log(g_u))$  gives an isomorphism from the additive group  $\mathbb{G}_a$  to  $C_u$ , so  $C_u \subset C \subset G$  implies  $C_u \subset G^\circ$ . The same observations apply to powers of  $g$ . If  $C^k$  (resp.  $C_s^k$ ) denotes the Zariski-closure of  $\langle g^k \rangle$  (resp.  $\langle g_s^k \rangle$ ), then we have equivalences

$$g_s^k \in G^\circ \Leftrightarrow C_s^k \subset G^\circ \Leftrightarrow C^k \subset G^\circ \Leftrightarrow g^k \in G^\circ \Leftrightarrow k \in m\mathbb{Z}.$$

There is a natural surjection from  $\mathbb{Z}/m\mathbb{Z}$  to  $C_s/C_s^m$  sending 1 to the class represented by  $g_s$ . It is an isomorphism because  $g_s^k \in C_s^m$  implies  $k \in m\mathbb{Z}$ . By [Bor, 8.4],  $C_s^m$  and  $C_s$  are diagonalizable groups. As  $K$  is of characteristic zero, the functor  $C \mapsto X^*(C)$  is an equivalence of categories [Bor, 8.3], so the inclusion  $C_s^m \rightarrow C_s$  corresponds to a surjection  $X^*(C_s) \rightarrow X^*(C_s^m)$  with kernel cyclic of order  $m$ ; let  $\chi$  be an element of  $X^*(C_s)$  lying in this kernel. Then  $\chi^k(g_s) = 1$  if and only if  $\chi^k(C_s) = 1$ , and the latter condition is equivalent to  $k \in m\mathbb{Z}$ . Finally, the inclusion of  $C_s$  in  $\mathbb{G}_m^d \subset \mathrm{GL}_d$  gives a surjective homomorphism  $\mathbb{Z}^d \rightarrow X^*(C_s)$ , so  $\chi$  corresponds to a  $d$ -tuple of integers  $(a_1, \dots, a_d)$ . If  $g_s$  maps to the diagonal matrix with entries

$(\lambda_1, \dots, \lambda_d)$ , then  $(\lambda_1^{a_1} \cdots \lambda_d^{a_d})^k = 1$  if and only if  $k \in m\mathbb{Z}$ , so  $\lambda_1^{a_1} \cdots \lambda_d^{a_d}$  is a primitive  $m$ th root of unity.  $\square$

**Lemma 2.2.** *There exists a function  $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  such that if  $G$  is a closed subgroup of  $\mathrm{GL}_d$  defined over an algebraically closed field  $K$  of characteristic 0, and every element of  $\pi_0(G)$  has order  $\leq m$ , then  $|\pi_0(G)| \leq f(m, d)$ .*

*Proof.* By [Mos], there exists a Levi decomposition  $G = MN$ , where  $N$  is the unipotent radical of  $G$  and therefore  $\pi_0(G) \cong \pi_0(M)$ . Without loss of generality, therefore, we may assume  $G$  is reductive. Up to  $K$ -isomorphism there are only finitely many possibilities for  $G^\circ$  given  $d$ , and each such  $G^\circ$  admits only finitely many equivalence classes of  $d$ -dimensional representation. Conjugation by any element  $g \in N_{\mathrm{GL}_d}(G^\circ)$  induces an automorphism of  $G^\circ$  which is inner if and only if  $g \in Z_{\mathrm{GL}_d}(G^\circ)G^\circ$ . It follows that the quotient  $N_{\mathrm{GL}_d}(G^\circ)/Z_{\mathrm{GL}_d}(G^\circ)G^\circ$  is contained in the outer automorphism group of the reductive Lie group  $G^\circ$  and is therefore discrete. As it is a linear algebraic group, it is finite. We have a homomorphism from  $\pi_0(G)$  to this quotient, so to bound the order of the former it suffices to bound the order of the kernel of the homomorphism, i.e.,

$$\begin{aligned} (G \cap Z_{\mathrm{GL}_d}(G^\circ)G^\circ)/Z(G^\circ)G^\circ &\subset Z_{\mathrm{GL}_d}(G^\circ)G^\circ/G^\circ \\ &= Z_{\mathrm{GL}_d}(G^\circ)/Z_{\mathrm{GL}_d}(G^\circ) \cap G^\circ \\ &= Z_{\mathrm{GL}_d}(G^\circ)/Z(G^\circ). \end{aligned}$$

This latter group is determined by  $G^\circ$  together with its ambient representation, for which there are only finitely many possibilities. For any fixed linear group  $Z_{\mathrm{GL}_d}(G^\circ)/Z(G^\circ)$  in characteristic 0, Jordan's theorem gives an upper bound to the order of a finite subgroup whose elements all have bounded order.  $\square$

**Lemma 2.3.** *Let  $\lambda_1, \dots, \lambda_d \in \mathbb{C}_p^\times$  be units and  $F \subset \mathbb{C}_p$  be a finite extension of  $\mathbb{Q}_p$  with ring of integers  $\mathcal{O}_F$ . Suppose that the elementary symmetric polynomials of  $\lambda_1, \dots, \lambda_d$  have values in  $\mathcal{O}_F + p\mathcal{O}_p$ . Then for every element  $\lambda$  of the multiplicative group  $\langle \lambda_1, \dots, \lambda_d \rangle$ , there exists a monic polynomial  $P_\lambda(x) \in \mathcal{O}_F[x]$  such that  $P_\lambda(\lambda)$  is divisible by  $p$  and  $\deg(P_\lambda) = d!$ .*

*Proof.* We write  $\lambda$  as  $\lambda_1^{a_1} \cdots \lambda_d^{a_d}$  and define  $Q_\lambda(x)$  to be the monic polynomial whose roots are  $\lambda_{\sigma(1)}^{a_1} \cdots \lambda_{\sigma(d)}^{a_d}$ , as  $\sigma$  ranges over  $S_d$ . The elementary symmetric polynomials in these roots lie in the ring generated by the elementary symmetric polynomials in  $\lambda_1, \dots, \lambda_d$  together with  $(\lambda_1 \cdots \lambda_d)^{-1}$ . By hypothesis, the elementary symmetric polynomials in  $\lambda_1, \dots, \lambda_d$  lie in the ring  $\mathcal{O}_F + p\mathcal{O}_p$ . The same is true of  $(\lambda_1 \cdots \lambda_d)^{-1}$  since  $\lambda_1 \cdots \lambda_d$  can be written  $u_F + pe$ , where  $u_F$  is a unit in  $\mathcal{O}_F$ . Thus  $Q_\lambda(x)$  is monic with coefficients in  $\mathcal{O}_F + p\mathcal{O}_p$ . It follows that there exists a monic polynomial  $P_\lambda(x)$  of the same degree with coefficients in  $\mathcal{O}_F$  which is congruent to  $P_\lambda(x) \pmod{p}$ . Thus  $p$  divides  $P_\lambda(\lambda)$ .  $\square$

## 2.2. Variation in $\pi_0(G_n)$ for a convergent sequence of representations.

**Theorem 2.4.** *Let  $\Gamma$  be a compact group, and let  $\rho_n: \Gamma \rightarrow \mathrm{GL}_d(\mathbb{C}_p)$  denote a uniformly trace-convergent sequence of continuous representations. Let  $G_n$  denote the Zariski closure of  $\rho_n(\Gamma)$ . Then  $|\pi_0(G_n)|$  is bounded.*

*Proof.* We know that the representations  $\rho_n$  uniformly trace converge to a continuous representation  $\rho: \Gamma \rightarrow \mathrm{GL}_d(\mathbb{C}_p)$  by using the theory of pseudo-representations as in Section 1. Further by [KLR, Lemma 2.2] all the representations  $\rho_n$  and  $\rho$  may be assumed to be valued in  $\mathrm{GL}_d(\mathcal{O}_p)$ . In its reduction (mod  $p$ ) under  $\rho$ ,  $\Gamma$  has finite image, so that all its entries lie in  $\mathcal{O}_F/p$  for some finite extension  $F/\mathbb{Q}_p$ . For large enough  $n$ , the mod  $p$  characteristic polynomials of  $\rho_n$  and  $\rho$  agree. Thus to prove the theorem we may assume without any loss of generality that they do so for all  $n$ . If  $\rho_n(g)$  lies in  $G_n(\mathbb{C}_p) \setminus G_n^\circ(\mathbb{C}_p)$ , by Lemma 2.1, some non-trivial root of unity  $\zeta$  lies in the group generated by the eigenvalues of  $\rho_n(g)$ . We claim that there exists an upper bound on the order of  $\zeta$  depending only on  $n$  and  $F$ . If  $\zeta$  has order  $p^k m$ ,  $\zeta^{p^k}$  and  $\zeta^m$  both lie in the group generated by eigenvalues of  $\rho_n(g)$ . It suffices, therefore, to prove that the order of  $\zeta$  is bounded in the case that this order is prime to  $p$  and in the case that it is a power of  $p$ .

By Lemma 2.3, if the order of  $\zeta$  is prime to  $p$ , the reduction of  $\zeta$  modulo the maximal ideal of  $\mathcal{O}_p$  must satisfy a polynomial equation of degree less than or equal to  $d!$  over the residue field of  $\mathcal{O}_F$ . This gives a bound on the order. If  $\zeta$  is of prime power order,  $p^k$ , then the valuation of  $\lambda := 1 - \zeta$  is  $\frac{1}{p^k - p^{k-1}}$  times the valuation of  $p$ . If the ramification degree of  $F$  over  $\mathbb{Q}_p$  is  $e$ , then  $1, \lambda, \lambda^2, \dots, \lambda^m$  are linearly independent over  $\mathcal{O}_F/p\mathcal{O}_F$  as long as  $em < p^k - p^{k-1}$ . Thus, if  $p^k - p^{k-1} > ed!$ ,  $\zeta$  cannot satisfy a monic degree  $d!$  polynomial equation (mod  $p$ ) with coefficients in  $\mathcal{O}_F$ .

By Lemma 2.1, there exists  $m$  such that for all  $n \gg 0$ , every element of  $\pi_0(G_n)$  has order less than  $m$ . Thus Lemma 2.2 gives an upper bound of  $f(m, d)$  on  $|\pi_0(G_n)|$  for  $n \gg 0$  which proves the theorem.  $\square$

In general, as the examples in section 2.3 illustrate, there is little that can be said about the relation between the  $\pi_0(G_n)$  and  $\pi_0(G)$ . In the irreducible case, however, we have the following theorem which makes crucial use of Theorem 2.4:

**Theorem 2.5.** *Let  $\Gamma$  be a compact group, and let  $\rho_n: \Gamma \rightarrow \mathrm{GL}_d(\mathbb{C}_p)$  denote a uniformly trace-convergent sequence of continuous representations. Then we know that there is a continuous semisimple representation  $\rho: \Gamma \rightarrow \mathrm{GL}_d(\mathbb{C}_p)$  such that the representations  $\rho_n$  uniformly trace converge to  $\rho$ . Let  $G_n$  (resp.  $G$ ) denote the Zariski closure of  $\rho_n(\Gamma)$  (resp.  $\rho(\Gamma)$ ), regarded as a subgroup of  $\mathrm{GL}_d$ . Suppose that  $\rho$  is irreducible. Then for all  $n \gg 0$  there exists a surjective homomorphism  $\pi_0(G) \rightarrow \pi_0(G_n)$ .*

*Proof.* By Theorem 1.2 we know that  $\rho_n$  is irreducible for large enough  $n$ , and thus in proving the theorem we may assume without any loss of

generality that  $\rho_n$  are irreducible for all  $n$ . As  $\rho_n$  and  $\rho$  are irreducible, the identity components  $G_n^\circ$  and  $G^\circ$  are reductive. (By results of Section 1 we also know that  $(\rho_n)$  uniformly physically converges to  $\rho$ , although we will not need this in the proof.) Let  $V = \mathbb{C}_p^d$ , regarded as a representation space of  $G_n$ . (In what follows we several times use the hypothesis of irreducibility of the limit  $\rho$  without explicit mention, and examples of section 2.3 show why this hypothesis is necessary.)

We would like to prove that there exists a finite set  $S \subset \mathbb{Z}^d$  and an integer  $N$  (independent of  $n$ ) such that for all  $g \in G_n(\mathbb{C}_p) \setminus G_n^\circ(\mathbb{C}_p)$  with eigenvalues  $\lambda_1, \dots, \lambda_d$ , there exists  $(a_1, \dots, a_d) \in S$  such that  $\lambda_1^{a_1} \dots \lambda_d^{a_d}$  is a non-trivial root of unity of order less than or equal to  $N$ . The dimension data, consisting of the number of distinct irreducible representations, their dimension, and their multiplicity, arising from the decomposition of  $V$  as a  $G_n^\circ$  representation (Clifford theory) admits only finitely many possibilities as  $n$  varies. Thus by partitioning the given sequence into finitely many subsequences we may assume that the dimension data is independent of  $n$ .

We consider two cases according to whether or not  $g$  preserves every summand in the decomposition of  $V$  as  $G_n^\circ$  representation. If not,  $g$  induces a non-trivial permutation action on the isotypic components  $V_1^e, \dots, V_k^e$  of  $V|_{G_n^\circ}$ . If a linear transformation  $T$  cyclically permutes  $r$  independent subspaces of order  $n$ , then  $T$  and  $\zeta_r T$  are conjugate, for  $\zeta_r$  a primitive  $r$ th root of unity, and therefore the eigenvalues of  $T$  form a homogeneous space under the action of the group of  $r$ th roots of unity. Thus we can take  $S$  to consist of all vectors in  $\mathbb{Z}^d$  obtained by permuting the coordinates of  $(1, -1, 0, \dots, 0)$  and  $N = k$ .

If  $g$  preserves each  $V_i^e$ , then without loss of generality, we may assume its image in  $\mathrm{GL}(V_1^e)$  does not lie in the image  $H_n$  of  $G_n^\circ(\mathbb{C}_p) \rightarrow \mathrm{GL}(V_1^e)$ . Let  $D_n$  denote the derived group of  $H_n$ , so  $D_n$  is connected and semisimple, and  $H_n = D_n Z_n$ , where  $Z_n$  is either  $\{1\}$  or the group  $\mathbb{G}_m$  of scalar matrices in  $\mathrm{GL}(V_1^e)$ . By classification, there are only finitely many isomorphism classes of semisimple groups of dimension less than  $d^2$  over  $\mathbb{C}_p$  and finitely many equivalence classes of representations of dimension less than or equal to  $d$  for each; so up to conjugation in  $\mathrm{GL}(V_1^e)$  there are finitely many possibilities for  $D_n$  and therefore for  $H_n$ . Without loss of generality, therefore, we may pass to an infinite subsequence of  $(\rho_n)$  such that the vector spaces  $V_1^e$  are all isomorphic, the  $H_n$  mutually isomorphic, and the representations of  $H_n$  on  $V_1^e$  equivalent. By a well-known theorem [DMOS], there exist non-negative integers  $m_1$  and  $m_2$  (independent of  $n$ ) such that  $H_n$  is the pointwise stabilizer of

$$W_n := ((V_1^e)^{\otimes m_1} \otimes (V_1^e)^{* \otimes m_2})^{G_n^\circ}$$

in  $\mathrm{GL}_d$ . As the image  $\bar{g}$  of  $g$  in  $\mathrm{GL}(V_1^e)$  normalizes  $H_n$ , it stabilizes  $W_n$ , and acts non-trivially on it, but its  $|\pi_0(G_n)|$ th power must act trivially. But now as  $|\pi_0(G_n)|$  is bounded independently of  $n$  by Theorem 2.4, it follows that  $\bar{g}$  has an eigenvector with eigenvalue which is a non-trivial root of unity

of some bounded order  $N$  (with  $N$  independent of  $g$ ). Furthermore this eigenvalue can be written  $\lambda_1^{a_1} \cdots \lambda_d^{a_d}$ , with  $-m_2 \leq a_1, \dots, a_d \leq m_1$  and  $\lambda_i$  the eigenvalues of  $\bar{g}$ .

Let  $\Gamma^\circ = \rho^{-1}G^\circ(\mathbb{C}_p)$ , and let  $X \subset G^\circ$  denote the closed subset consisting of elements with eigenvalues  $\lambda_1, \dots, \lambda_d$  such that  $\lambda_1^{a_1} \cdots \lambda_d^{a_d}$  is a non-trivial root of unity of order  $\leq N$  for some  $(a_1, \dots, a_d) \in S$ . By Proposition 3.5 below, which again uses crucially Theorem 2.4, the Zariski closure of  $\rho_n(\Gamma^\circ)$  is connected. We can therefore define a surjective homomorphism from  $\pi_0(G)$  to  $\pi_0(G_n)$  by lifting to  $\Gamma$ , mapping by  $\rho_n$  to  $G_n(\mathbb{C}_p)$ , and projecting onto  $\pi_0(G_n)$ .  $\square$

**2.3. Examples.** The examples in this section are intended to give some perspective on Theorems 2.4 and 2.5.

Let  $p$  be an odd prime. Let  $e: \mathbb{Z}_p \rightarrow \mathbb{C}_p$  denote the exponential map  $\exp(pz)$  given by the (convergent) power series

$$e(z) := \sum_{i=0}^{\infty} \frac{p^i}{i!} z^i.$$

Let  $a_1, a_2, \dots$  denote a sequence in  $\mathbb{Z}$  which converges  $p$ -adically to an irrational element  $a \in \mathbb{Z}_p \setminus \mathbb{Q}$ . Let  $\Gamma = \mathbb{Z}_p$  and

$$\rho_n(z) = \begin{pmatrix} e(z) & 0 \\ 0 & e(a_n z) \end{pmatrix}$$

The Zariski closure of  $\rho_n(z)$  is  $\mathbb{G}_m$  embedded in  $\mathrm{GL}_2$  as

$$\begin{pmatrix} t & 0 \\ 0 & t^{a_n} \end{pmatrix}.$$

The limit representation  $\rho$  is given by

$$\rho(z) = \begin{pmatrix} e(z) & 0 \\ 0 & e(az) \end{pmatrix}$$

whose envelope is  $\mathbb{G}_m^2$ , the group of all invertible diagonal matrices, because  $a \notin \mathbb{Q}$ .

In this case, the envelopes  $G_n$  and  $G$  are all connected, but because the dimension jumps for the limit representation, we can easily modify the example either to prevent  $|\pi_0(G_n)|$  from converging at all as  $n \rightarrow \infty$  or to allow convergence to a value different from  $|\pi_0(G)|$ . For instance, we may set  $\Gamma = \mathbb{Z}_p \times \mathbb{Z}/2\mathbb{Z}$  and define

$$\rho_n(z, k) = (-1)^k \begin{pmatrix} e(z) & 0 \\ 0 & e(a_n z) \end{pmatrix}, \quad \rho(z, k) = (-1)^k \begin{pmatrix} e(z) & 0 \\ 0 & e(az) \end{pmatrix}.$$

Then  $G_n$  has 1 or 2 components depending on whether  $a_n$  is odd or even. Since  $p > 2$ , the parity of a  $p$ -adically convergent sequence of integers need not stabilize. If all the  $a_n$  are even, then  $|\pi_0(G_n)| = 2$  for all  $n$ , but  $|\pi_0(G)| = |\pi_0(\mathbb{G}_m^2)| = 1$ .

We also remark that the isomorphism class of  $G_n^\circ$  need not stabilize as  $n \rightarrow \infty$ , and even if it does stabilize, it need not coincide with that of  $G^\circ$ . For example, if  $\Gamma = \mathbb{Z}_p^2$ , and  $a_n$  is a sequence of  $p$ -adic integers converging to 0, we can set

$$\rho_n(z_1, z_2) = \begin{pmatrix} e(z_1) & 0 \\ 0 & e(a_n z_2) \end{pmatrix}, \quad \rho_n(z_1, z_2) = \begin{pmatrix} e(z_1) & 0 \\ 0 & 1 \end{pmatrix}.$$

In this example,  $G_n$  is isomorphic to  $\mathbb{G}_m$  whenever  $a_n \neq 0$  and otherwise to  $\mathbb{G}_m^2$ , and of course  $G$  is isomorphic to  $\mathbb{G}_n$ .

Finally, it may even happen that  $G_n$  is reductive for infinitely many values of  $n$  and unipotent for infinitely many values. For example, let  $\Gamma = \mathbb{Z}_p$  and  $a_n$  be a sequence of  $p$ -adic integers converging to 0. Let

$$\rho(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

and let

$$\rho_n(z) = \begin{cases} \begin{pmatrix} 1 & \frac{e(a_n z) - 1}{e(a_n) - 1} \\ 0 & e(a_n z) \end{pmatrix} & \text{if } a_n \neq 0, \\ \rho(z) & \text{if } a_n = 0. \end{cases}$$

Thus  $G_n$  is isomorphic to  $\mathbb{G}_a$  or  $\mathbb{G}_m$  depending on whether  $a_n$  is or is not equal to zero, and  $G$  is isomorphic to  $\mathbb{G}_a$ .

### 3. DENSITY THEOREMS FOR CONVERGING SEQUENCES

In this section we consider only continuous Galois representations to  $\mathrm{GL}_n(\mathbb{C}_p)$ . We fix the following situation and notation for all of this section.

Let  $X$  be a subvariety of  $\mathrm{GL}_d$  defined by a finite set  $\{f_1, \dots, f_t\}$  of the coordinate ring  $A$  of  $\mathrm{GL}_d$  that we assume can be chosen so that each  $f_i$  is conjugation invariant. We call such a  $X$  a characteristic subvariety. Consider a compact subgroup  $\Gamma$  of  $\mathrm{GL}_d(\mathbb{C}_p)$ , that by [KLR, Lemma 2.2] we can assume to be in  $\mathrm{GL}_d(\mathcal{O}_p)$  (by conjugating), and consider a Haar measure  $\mu$  on  $\Gamma$ . By saying that a point  $\gamma$  of  $\Gamma$  *lands inside  $X$  mod  $p^m$*  (or  $\Gamma$  is in  $X$  mod  $p^m$ , or is in a *tubular neighborhood of  $X$  of radius  $p^{-m}$* ), we will mean that  $v(f_i(\gamma)) > m$  for  $1 \leq i \leq t$  ( $\gamma$  lands inside  $X$  means this should hold for all  $m$ !). Here  $v$  is the valuation of  $\mathbb{C}_p$  normalised so that  $v(p) = 1$ . We say that  $X$  is *thin* with respect to  $\Gamma$  if for every finite subset  $\{g_1, \dots, g_m\}$  of the coordinate ring  $A$  of  $\mathrm{GL}_d$  such that  $V(g_1, \dots, g_m) \cap G = X$ , we have

$$\lim_{\alpha \rightarrow \infty} \mu(\{\gamma \in \Gamma \mid \forall i \ v(g_i(\gamma)) > \alpha\}) = 0.$$

We recall [KLR, Proposition 2.3], slightly reformulated, as we will repeatedly use it below:

**Proposition 3.1.** *Let  $\Gamma$  denote a compact subgroup of  $\mathrm{GL}_d(\mathbb{C}_p)$ ,  $\mu$  Haar measure on  $\Gamma$ ,  $G$  the Zariski closure of  $\Gamma$  in  $\mathrm{GL}_d$ , and  $X$  a subvariety of*

$\mathrm{GL}_d$  that intersects all components of  $G$  with positive codimension. Then  $X$  is thin with respect to  $\Gamma$ .

Let  $F$  be a number field and write  $G_F$  for its absolute Galois group. We consider uniformly trace-convergent sequence of continuous representations  $\rho_n: G_F \rightarrow \mathrm{GL}_d(\mathbb{C}_p)$  which by the theory of pseudo-representations (see section 1.1) is uniformly trace-convergent to a continuous, semisimple representation  $\rho: G_F \rightarrow \mathrm{GL}_d(\mathbb{C}_p)$ . (This also implies that in fact the characteristic polynomials of the  $\rho_n(g)$  converge to those of  $\rho(g)$  uniformly in  $g$ .)

**3.1. Frobenius polynomials at almost all ramified primes.** By [KLR, Lemma 2.2] we may assume that all the representations  $\rho_n$ , and  $\rho$  itself, are valued in  $\mathrm{GL}_d(\mathcal{O}_p)$ .

**Proposition 3.2.** *There exists a finite set  $S$  of places of  $F$  such that for all primes outside  $S$ , the ramification of all  $\rho_n$  is tame and unipotent (or trivial).*

*Proof.* We first exclude from discussion the finite set of places of  $F$  of residue characteristic  $p$ .

From the assumptions on  $\rho_n$  it follows that the number of residual mod  $p$  representations that arise from reducing any integral model of  $\rho_n$  modulo the maximal ideal of  $\mathcal{O}_p$  and semisimplifying is finite (for all  $i$  at once). If  $\rho_n$  is wildly ramified at any remaining place  $q$  of  $F$ , then  $\bar{\rho}_n$  and indeed  $\bar{\rho}_n^{ss}$  is already wildly ramified at  $q$ . Thus we see that the number of places of  $F$  whose ramification in any of the  $\rho_n$  is wild is finite. We exclude this set from discussion.

Let  $q$  be a place of  $F$  that has not been excluded. Then the image of a decomposition group  $D_q$  at  $q$  under any  $\rho_n$  factors through its tame quotient, which is topologically generated by  $\sigma_q$  and  $\tau_q$  with the relation

$$(2) \quad \sigma_q \tau_q \sigma_q^{-1} = \tau_q^{\|q\|}.$$

Here  $\sigma_q$  induces the  $q$ th power map on residue fields and  $\tau_q$  is a (non-canonical) generator of tame inertia. From this it follows that the eigenvalues of  $\rho_n(\tau_q)$  for any  $n$  are roots of unity. It is easy to see that for some  $m \gg 0$  that depends only on  $d$ , if  $\zeta$  is a root of unity such that  $(\zeta - 1)^d$  is 0 mod  $p^m$ , then  $\zeta = 1$ . Thus we see that if  $\rho_n(\tau_q)$  has the same characteristic polynomial mod  $p^m$  as that of the identity, then  $\rho_n(\tau_q)$  is unipotent. We fix  $N$  such that the characteristic polynomials of  $\rho_i(g)$  and  $\rho_j(g)$  are congruent to one another mod  $p^m$  for all  $i, j \geq N$  and  $g \in G_F$ . Taking  $g = \tau_q$ , we see that if for some  $i$   $\rho_i$  fails to have unipotent ramification at  $q$ , then for some  $j \leq N$ ,  $\rho_j(\tau_q)$  is not congruent to the identity (mod  $p^m$ ). As the mod  $p^m$  reductions of the representations  $\rho_i$  have finite images, the set of possible places  $q$  is finite.

□

The utility of Proposition 3.2 is that given uniformly trace-converging  $\rho_n$ , for a place  $q$  outside the finite set of places excluded in its statement, one can define the *characteristic polynomial of  $\rho_n(\text{Frob}_q)$*  as that of  $\rho_n(\sigma_q)$  for any  $\sigma_q$  that lifts the  $q$ th power map on residue fields. Using unipotence of  $\rho_n(\tau_q)$  and the tame inertia relation above we see that this is independent of choice of  $\sigma_q$  (proof: from this relation we see that  $\sigma_q$  preserves the kernel of  $(\tau_q - 1)^i$  for any  $i$ , and thus as  $\rho(\tau_q)$  is unipotent it follows easily that  $\rho(D_q)$  can be conjugated into upper triangular matrices over an algebraic closure with  $\tau_q$  mapped to strictly upper triangular element.) Thus given a characteristic subvariety  $X$  of  $\text{GL}_d$ , we can with some abuse talk of  $\rho_n(\text{Frob}_q)$  landing in  $X$ , as this condition will depend only on the characteristic polynomial of  $\rho_n(\text{Frob}_q)$ . In fact, one can prove slightly more: the conjugacy class of every element in the Frobenius coset is the same:

**Corollary 3.3.** *Let  $I_q \subset D_q$  denotes the inertia group at  $q$ . For  $q \notin S$ ,  $\rho_n(\sigma_q I_q)$  lies in a single  $\text{GL}_d(\mathbb{C}_p)$ -conjugacy class for all  $n$ .*

*Proof.* As we are working in characteristic zero, the log and exp maps give mutually inverse bijections between the variety of unipotent elements in  $\text{GL}_d$  and the variety of nilpotent elements in  $M_d$ . For fixed  $n$  and a fixed choice of  $\tau_q$ , let  $N_\tau = \log \rho_n(\tau_q)$ . Then (2) implies that

$$\rho_n(\tau_q)^{\|q\|} \rho_n(\sigma_q) \rho_n(\tau_q)^{-\|q\|} = \rho_n(\sigma_q) \rho_n(\tau_q)^{\|q\|-1},$$

and therefore, for all  $t \in \mathbb{C}_p$ ,

$$\rho_n(\tau_q)^{\|q\|} \rho_n(\sigma_q) \exp(tN_\tau) \rho_n(\tau_q)^{-\|q\|} = \rho_n(\sigma_q) \exp((t + \|q\| - 1)N_\tau),$$

If  $O \cong \mathbb{G}_a$  denotes the Zariski-closure of

$$\{\rho_n(\sigma_q) \exp(tN_\tau) \mid t \in \mathbb{C}_p\},$$

this implies that conjugation by  $\rho_n(\tau_q)^{\|q\|}$  acts on  $O$  without points of finite order. Any orbit of this action is Zariski-dense in  $O$ , and it follows that every  $\text{GL}_d$  conjugacy class in  $O$  is Zariski-dense. As  $O$  is connected, this implies that there is a single orbit. As  $\rho_n$  is tamely ramified at  $q$  and  $\tau_q$  is a topological generator of the tame inertia group,  $\rho_n(\sigma_q I_q) \subset O(\mathbb{C}_p)$ .  $\square$

**Corollary 3.4.** *With notations as above:*

- $d > 1$ : For any  $n$ , and  $q \notin S$ , if  $\rho_n$  is ramified at  $q$ , then  $\rho_n(\sigma_q)$  has two eigenvalues with ratio  $\|q\|$ .
- $d = 1$ : The union of the ramifying sets for all  $\rho_n$  is finite.

*Proof.* See [KLR, Lemma 2.6].  $\square$

In the rest of the paper we will implicitly exclude from the discussion the finite set  $S$  in Proposition 3.2.

**3.2. Density theorems.** As in section 2.2,  $G_n$  and  $G$  denote the Zariski-closures of  $\rho_n$  and  $\rho$  respectively. The following proposition is key to proving the density theorems below.

**Proposition 3.5.** *Let  $X$ , a characteristic subvariety of  $\mathrm{GL}_d$ , intersect all the components of  $G$  with positive codimension. Then for  $n \gg 0$ ,  $X$  intersects all the components of  $G_n$  with positive codimension.*

*Proof.* Assume the contrary. Let  $\Gamma$  be the image of the limiting representation  $\rho$ . We get a contradiction by proving the following claim: if  $\Gamma_m$  is the (finite) reduction of  $\Gamma \bmod p^m$ , then there exists  $\alpha > 0$  independent of  $m$  such that at least  $\alpha|\Gamma_m|$  elements of  $\Gamma_m$  land inside  $X \bmod p^m$ .

This will contradict the hypothesis that  $X$  has proper intersection with all components of  $G$  as, under the assumptions Proposition 3.1 proves that  $X$  is *thin* with respect to  $\Gamma$ .

We prove the claim by observing that: (i) by assumption for infinitely many  $n$ ,  $X$  contains a connected component of  $G_n$ , (ii) for a given tubular neighborhood  $U$  of the *characteristic* subvariety  $X$  the image of  $g \in G_F$  under  $\rho$  is in  $U$  if and only if  $\rho_n(g) \in U$  for  $n \gg 0$ , (iii) the number of connected components of  $G_n$  is bounded by some number  $r$  by Theorem 2.4.

From these three facts we see that we can in fact take  $\alpha$  to be  $\frac{1}{r}$ .  $\square$

**Theorem 3.6.** *Let  $X$  be a characteristic subvariety of  $\mathrm{GL}_d$  such that  $X$  intersects all the components of  $G$  with positive codimension. Then there is a  $N \gg 0$ , such that the set of places  $q$  of  $F$  such that  $\rho_n(\mathrm{Frob}_q) \in X$  for even one  $n > N$  is of Dirichlet density zero.*

*Proof.* Choose  $N$  such that  $X$  does not contain any component of  $G_n$  for  $n > N$  using Proposition 3.5 above. Let  $\Gamma$  be the image of  $\rho$  and  $\mu$  a Haar measure on it. We want to show that the upper density of primes  $q$  such that  $\rho_n(\mathrm{Frob}_q)$ , for even one  $n > N$ , lands in  $X$  can be made  $< \epsilon$  for any given  $\epsilon > 0$ . As before we claim this follows easily using Proposition 3.1, which proves that  $X$  is thin with respect to  $\Gamma$ . Namely, using loc. cit. and the classical Chebotarev density theorem for finite Galois extensions of number fields (the reader may also look at [KLR, Th. 2.4] for similar conclusions, and proof of [KLR, Th. 2.5] for a similar argument), we get that for  $m \gg 0$ , the upper density of  $q$  such that  $\rho(\mathrm{Frob}_q)$  lands in a tubular neighborhood  $U_m$  of  $X$  of radius  $p^{-m}$  is  $< \epsilon$ . Now as  $(\rho_n)$  uniformly trace-converges to  $\rho$ , and as  $X$  is characteristic, we see that there is a  $N' \gg 0$  such that  $\rho_n(g) \in U_m$  for  $n > N'$  if and only if  $\rho(g) \in U_m$ . Further the density of  $q$  such that  $\rho_n(\mathrm{Frob}_q)$  lands in  $X$  for any  $N < n < N'$  is of density 0 as  $X$  intersects all components of  $G_n$  with positive codimension and then we can again use Proposition 3.1 and the classical Chebotarev density theorem. Thus treating the cases  $N < n < N'$  and  $n > N'$  separately we see that the upper density of primes  $q$  such that  $\rho_n(\mathrm{Frob}_q)$ , for even one  $i > N$ , lands in  $X$  can be made  $< \epsilon$ .  $\square$

**Remark:** In the case when all the representations  $\rho_n$  are unramified outside a fixed finite set of places, we do not know (even assuming that the representations are  $\mathrm{GL}_d(\mathbb{Q}_p)$  valued) if there are quantitative refinements of the theorem above like the ones for a single representation proved in Théorème 10 of [S2].

We now prove a result about density of primes that ramify in uniformly trace-converging sequences, which has a precursor in [Kh], and is close to the proof of [KLR, Theorem 2.5] which proves the statement for a single representation (when the representation is valued in a  $\mathrm{GL}_d(K)$ , with  $K$  a finite extension of  $\mathbb{Q}_p$ , the result for a single representation goes back to [Kh-Raj]).

**Theorem 3.7.** *If  $\rho_n: G_F \rightarrow \mathrm{GL}_d(\mathbb{C}_p)$  is a sequence of irreducible representation which trace-converges uniformly to an irreducible  $\rho: G_F \rightarrow \mathrm{GL}_d(\mathbb{C}_p)$ , then the union over  $n$  of the sets of primes ramified in  $\rho_n$  has Dirichlet density zero.*

*Proof.* We can exclude the case of  $d = 1$  by Corollary 3.4. Let  $\varepsilon$  denote the  $p$ -adic cyclotomic character. Consider the direct sum representations  $\rho_n \oplus \varepsilon: G_F \rightarrow \mathrm{GL}_d \times \mathrm{GL}_1$  and  $\rho \oplus \varepsilon: G_F \rightarrow \mathrm{GL}_d \times \mathrm{GL}_1$ . These again trace-converge uniformly, and it suffices to prove the theorem for them instead. The proof would follow from Theorem 3.6, Proposition 3.2 and Corollary 3.4, if we knew the following:

*Claim:* Let  $H$  denote the Zariski closure of  $\rho \oplus \varepsilon(G_F)$ . Thus  $H \subset G \times \mathrm{GL}_1$ , with  $G$  the Zariski closure of  $\rho$ , and  $H$  projects onto each factor. Let  $X \subset H$  denote the subvariety of pairs  $(g, c) \in H$  such that  $g$  and  $gc$  have at least one eigenvalue in common. The claim is that  $X$  is of codimension greater than one in each component of  $H$ .

The  $X$  of the claim is characteristic in  $\mathrm{GL}_d \times \mathrm{GL}_1$  and hence Theorem 3.6 applies to it. Thus we are done once the claim is proved.

To check the claim as  $\rho$  is irreducible and thus centralised only by scalars, we can go modulo the centre of  $G$ , as this does not change anything and thus assume that  $G$  is semisimple. By Goursat's lemma,  $H$  is the pullback of the graph of an isomorphism between a quotient of  $G$  and a quotient of  $\mathrm{GL}_1$ . Every quotient of  $\mathrm{GL}_1$  is a torus and  $G$  admits no non-trivial toric quotient, so  $H = G \times \mathrm{GL}_1$ . For each  $g$  there are only finitely many possible values of  $c$  such that  $g$  and  $gc$  have an eigenvalue in common, so  $X$  is of codimension  $\geq 1$  in each component of  $H$ . □

We also have the following result about equidistribution of Frobenius elements in groups of connected components for converging sequences that is a simple consequence of Theorem 2.5 and the classical Chebotarev density theorem.

**Proposition 3.8.** *Assume that  $\rho$  is irreducible. Let  $\rho_n^\circ: G_F \rightarrow \pi_0(G_n)$  (resp.  $\rho^\circ: G_F \rightarrow \pi_1(G)$ ) denote the homomorphisms obtained by composing*

$\rho_n: G_F \rightarrow G_n(\mathbb{C}_p)$  (resp.  $\rho: G_F \rightarrow G(\mathbb{C}_p)$ ) with the quotient map by  $G_n^\circ(\mathbb{C}_p)$  (resp.  $G^\circ(\mathbb{C}_p)$ ), and let  $K_n$  (resp.  $K$ ) be the fixed fields of their kernels. For  $n \gg 0$ , each  $K_n$  is contained in  $K$ . For a conjugacy class  $C$  of  $\text{Gal}(K/F)$  we denote by  $C_n$  its image in  $\text{Gal}(K_n/F)$  ( $n \gg 0$ ). The density of places whose Frobenius under  $\rho_n^\circ$  lie in  $C_n$  for all  $n \gg 0$  and in  $C$  under  $\rho^\circ$ , is  $|C|/|\text{Gal}(K/F)|$ .

*Proof.* By considering the restriction  $\rho|_{G_K}$  and using Theorem 2.5 we see that all the Zariski closures of the images of  $\rho_n|_{G_K}$  for  $n \gg 0$  are connected, and thus for  $n \gg 0$ , the fields  $K_n$  are contained in  $K$ . The rest follows from a direct application of the classical Chebotarev density theorem.  $\square$

### Remarks:

1. Because of results of section 1, Proposition 3.5 and Theorem 3.6 also work when one simply assumes that  $X$  is a conjugation invariant subvariety and the limit  $\rho$  is multiplicity-free as then by Theorem 1.4 the sequence  $\rho_n$  uniformly physically converges.

2. Even if a characteristic  $X$  intersect all components of  $G_n$  with positive codimension, it might still happen that the density of Frobenius elements that land in  $X$  under  $\rho_n$  for any  $n \gg 0$  is 1. Of course such a putative  $X$  will contain some component of  $G$  because of the results of this section.

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