A polynomial invariant of pseudo-Anosov maps

Joan Birman (with Peter Brinkmann and Keiko Kawamuro)

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Background:

Defintion: $S=$ orientable hyperbolic surface, closed or with finitely many punctures. Mapping class group of S = Mod (S) $= \pi_0(\mathrm{o.p.Diff}(S))$.

A conjugacy class in Mod (S) is

- periodic if it has a representative $F \in \mathrm{Diff} S$ such that $[F]^k$ is the identity for some $k > 0$,
- reducible if it has a representative F such that there is a family of simple loops $\mathcal{C} \subset S$ with $F(\mathcal{C}) = \mathcal{C}$ and each component of $S \setminus \mathcal{C}$ has negative Euler characteristic.
- pseudo-Anosov if neither periodic nor reducible. Theorem of Riven: this is the generic case.

Theorem of Thurston (1970's): If $[F]$ is pseudo-Anosov, there exists a representative $F : S \to S$ and a pair of transverse measured foliations $\mathcal{F}^u, \mathcal{F}^s$ and a real number λ, the dilatation of $[F]$, such that F multiplies the measure on $\mathcal{F}^u$ (resp. $\mathcal{F}^s$) by λ (resp. $\frac{1}{\lambda}$).

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Assume $[F]$ pseudo-Anosov. Two interesting known invariants of $[F]$:

(1) λ, an algebraic integer, or its minimum polynomial $m_\lambda(x)$.

(2) If the mapping torus of F is complement of a knot K in a homology sphere, the Alexander polynomial $Alex_K(x)$, is the characteristic polynomial of the action of the monodromy F on $H_1(fiber)$.

Our main new result: A new polynomial $p(x)$ invariant of $[F]$, when $[F]$ is pseudo-Anosov.

- Our $p(x)$ is in general not irreducible, but if it is, it coincides with $m_\lambda(x)$.
- In general, $p(x)$ has a unique largest real root, and that root is λ.
- If K a knot as above, our $p(x)$ is in general not $Alex_K(x)$, but for a special class of maps $[F]$ we have $p(x) = Alex_K(x)$.

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Train tracks

Thurston introduced measured train tracks, as a way of recording properties of the measured foliations $\mathcal{F}^u, \mathcal{F}^s$ associated to a pseudo-Anosov mapping class $[F]$ on the surface S.

A train track τ is a branched 1-manifold embedded in the surface S.

Made up of smooth edges (called branches), disjointly embedded in S, and vertices (called switches).

Given a pseudo-Anosov mapping class $[F]$, there exists a train track $\tau \subset S$ that fills the surface, i.e., the complement of τ consists of possibly punctured discs, and τ is left invariant by $[F]$. Moreover, there exists a transverse (resp. tangential) measure on τ that encodes the structure of the stable (resp. unstable) foliation of $[F]$.

Measures on branches. Switch conditions satisfied at every switch.

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Bestvina and Handel's proof of Thurston's theorem

1995: Bestvina and Handel gave an algorithmic proof of Thurston's theorem.

Proof shows: if $[F]$ is pseudo-Anosov, one may construct a graph G homotopic to S and an induced map $f : G \to G$ that's a homotopy-equivalence. (f(vertex) of $G \to$ vertex of G, and f(edge) $=$ sequence of edges, an immersion on interior of each edge.

Proof is algorithmic. Construct candidates for G and for $f : G \to G$.

Arrive at one which can be used to construct a measured train track τ and recover the measured foliations $\mathcal{F}^u, \mathcal{F}^s$. Candidates which cannot be so-used are discarded along the way.

Assuming that the algorithm has ended with a proof that $[F]$ is pseudo-Anosov, will have in hand the graph G and a map $f : G \to G$, and (implicitly) a special measured train track constructed from it.

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Calculating λ, in the Bestvina-Handel setting

The graph G is embedded in the surface S. If S has one or more punctures, then G is homotopy equivalent to S; in fact, each component of the complement of G is a 2-cell, possibly with one puncture.

Let $e_1, \ldots, e_m$ be the unoriented edges of G. Then $f(e_j)$ is an edge path in G for each j. The *transition matrix* of f is the $m \times m$ matrix $T = (t_{i,j})$, where $t_{i,j}$ is the number of times $f(e_j)$ crosses e_i, counted without orientation, so that all entries of T are non-negative integers. $[F]$ pseudo-Anosov $\implies T$ irreducible.

It's a Perron-Frobenius matrix. T has a largest real eigenvalue λ. That's the dilatation λ of $[F]$. The eigenvalue λ has algebraic multiplicity one, and the corresponding eigenspace is spanned by a positive vector. Left (resp. right) eigenvectors of T are determined by the transversal (resp. tangential) measures on τ.

(Eigenvectors also determine the measures on τ.
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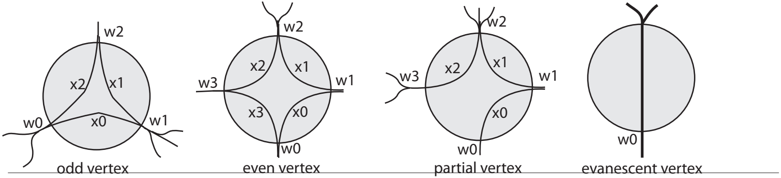
Study structure of $\det(T - xI) = \chi(T)$

Measures on τ are determined by T. [BH] prove can use $f : G \to G$ to construct τ, and from it the foliations.

Interplay between τ and G. The map $f : G \to G$ can be used to construct τ. But also τ, with its measures, determines G. Recall the [BH] version of τ, and the measures on τ. Real edges and infinitesimal edges. 4 types of vertices.



Using [BH] version of τ, there is a natural projection $\pi : \tau \to G$, defined by collapsing the infinitesimal edges to associated vertices. So assume [BH] algorithm has determined that $[F]$ is PA, and has produced a train track τ, with associated collapsing map to $\pi : \tau \to G$.

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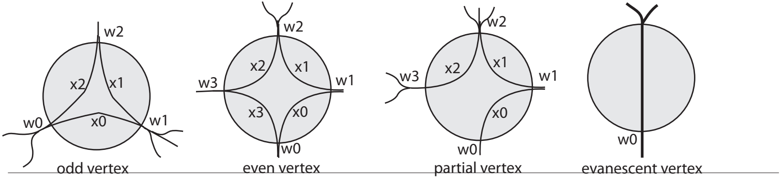

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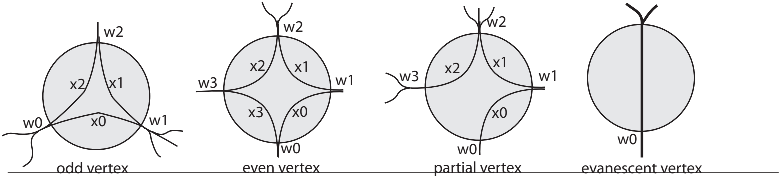

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To study $\chi(T)$, introduce vector spaces $V(\tau)$ of weighted train tracks and $W(\tau)$ of ones whose weights satisfy branch conditions. Also $V(G)$ of 'widths' of edges of G. Then π induces $\pi_* : V(\tau) \to V(G)$, space of 'measures' on the associated graph G. Let $f_* : V(G) \to V(G)$ denote the linear map induced by f. Let $W(G, f) = \pi_*(W(\tau))$.

It's subspace of $V(G)$ whose elements admit an extension to a transverse measure on τ. (Need it because 'switch conditions' natural for train tracks, but not for graphs.)

Using this result, we then prove that $W(G, f)$ is the kernel of a homomorphism δ , and also that the resulting decomposition $V(G) \cong W(G, f) \oplus \mathrm{im}(\delta)$ is invariant under the action of f_*.

Since $f_\star$ is represented by T, there is a corresponding product decomposition

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A skew-symmetric form on the graph G

[F] define a skew-symmetric form on the space of transversal measures on a train track τ. We extend their form to a skew-symmetric form on our space $W(G, f)$. We prove that this skew-symmetric form is invariant under the action of f_*.

Interesting aspect of our work: [PH] imply that the skew-symmetric form on τ non-degenerate, but we found examples where had to be degenerate. We give a complete description of space Z of degeneracies.

Once again, the decomposition $W(G, f) \cong (W(G, f)/Z) \oplus Z$ turns to be invariant under f_*. So we have

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Main Results

(1) *The polynomial* $\chi(T) = \chi(f_*)$ *factorizes as a product of three not necessarily irreducible factors* $\chi(f_*|_{W(G,f)/Z})$, $\chi(f_*|_Z)$ *and* $\chi(f_*|_{\mathrm{im}(\delta)})$.

(2) *The polynomial* $\chi(f_*|_{W(G,f)/Z})$ *is an invariant of* $[F]$. *It is palindromic. It contains the dilatation* λ *as its largest real root. While it always contains the minimum polynomial of* λ *as a factor, it does not, in general coincide with the minimum polynomial.*

Jeffrey Carlson helped us to see the final step in the proof, in (2).

(3) *The polynomial* $\chi(f_*|_Z)$ *is an invariant of* $[F]$. *It encodes information about how* f *permutes the punctures of* S. *It is palindromic or anti-palindromic, and all of its roots are roots of unity.*

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(4) *The polynomial* $\chi(f_*|_{\mathrm{im}(\delta)})$ *records the way that* f_* *permutes the vertices of* τ*. Palendromic or anti-palendromic times a power of* x*, and all of its roots are zero or roots of unity. Not an invariant of* $[F]$*.*

(5) $\chi(f_*) = \chi(f_*|_{W(G,f)/Z})\chi(f_*|Z)\chi(f_*|_{\mathrm{im}\delta})$ *is a palindromic or antipalindromic polynomial, possibly up to multiplication by a power of* x*.*

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