A polynomial invariant of pseudo-Anosov maps

Joan Birman (with Peter Brinkmann and Keiko Kawamuro)

May 23, 2010
**Background:**

**Definition:** $S =$ **orientable hyperbolic surface**, closed or with finitely many punctures. Mapping class group of $S = \text{Mod} (S) = \pi_0(o.p.\text{Diff}(S))$.

A conjugacy class in $\text{Mod} (S)$ is

- **periodic** if it has a representative $F \in \text{Diff} S$ such that $[F]^k$ is the identity for some $k > 0$,
- **reducible** if it has a representative $F$ such that there is a family of simple loops $\mathcal{C} \subset S$ with $F(\mathcal{C}) = \mathcal{C}$ and each component of $S \setminus \mathcal{C}$ has negative Euler characteristic.
- **pseudo-Anosov** if neither periodic nor reducible. Theorem of Riven: this is the generic case.

**Theorem of Thurston (1970’s):** If $[F]$ is pseudo-Anosov, there exists a representative $F : S \to S$ and a pair of transverse measured foliations $\mathcal{F}^u, \mathcal{F}^s$ and a real number $\lambda$, the dilatation of $[F]$, such that $F$ multiplies the measure on $\mathcal{F}^u$ (resp. $\mathcal{F}^s$) by $\lambda$ (resp. $\frac{1}{\lambda}$).
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Assume \([F]\) pseudo-Anosov. Two interesting known invariants of \([F]\):

\begin{itemize}
  \item (1) \(\lambda\), an algebraic integer, or its \textit{minimum polynomial} \(m_{\lambda}(x)\).
  \item (2) If the mapping torus of \(F\) is complement of a knot \(K\) in a homology sphere, the Alexander polynomial \(\text{Alex}_K(x)\), is the characteristic polynomial of the action of the monodromy \(F\) on \(H_1(\text{fiber})\).
\end{itemize}

Our main new result: A new polynomial \(p(x)\) invariant of \([F]\), when \([F]\) is pseudo-Anosov.

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  \item Our \(p(x)\) is in general not irreducible, but if it is, it coincides with \(m_{\lambda}(x)\).
  \item In general, \(p(x)\) has a unique largest real root, and that root is \(\lambda\).
  \item If \(K\) a knot as above, our \(p(x)\) is in general not \(\text{Alex}_K(x)\), but for a special class of maps \([F]\) we have \(p(x) = \text{Alex}_K(x)\).
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In the course of the proof, we learned other new things. So I want to sketch how we approached the problem.
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Train tracks

Thurston introduced measured train tracks, as a way of recording properties of the measured foliations $\mathcal{F}^u, \mathcal{F}^s$ associated to a pseudo-Anosov mapping class $[F]$ on the surface $S$.

A train track $\tau$ is a branched 1-manifold embedded in the surface $S$.

Made up of smooth edges (called branches), disjointly embedded in $S$, and vertices (called switches).

Given a pseudo-Anosov mapping class $[F]$, there exists a train track $\tau \subset S$ that fills the surface, i.e., the complement of $\tau$ consists of possibly punctured discs, and $\tau$ is left invariant by $[F]$. Moreover, there exists a transverse (resp. tangential) measure on $\tau$ that encodes the structure of the stable (resp. unstable) foliation of $[F]$.

Measures on branches. Switch conditions satisfied at every switch.
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Bestvina and Handel’s proof of Thurston’s theorem

1995: Bestvina and Handel gave an algorithmic proof of Thurston’s theorem.

Proof shows: if \([F]\) is pseudo-Anosov, one may construct a graph \(G\) homotopic to \(S\) and an induced map \(f : G \rightarrow G\) that’s a homotopy-equivalence. \((f(\text{vertex})\ of\ G \rightarrow \text{vertex of}\ G,\ and\ f(\text{edge}) = \text{sequence of edges},\ an\ immersion\ on\ interior\ of\ each\ edge.\)

Proof is algorithmic. Construct candidates for \(G\) and for \(f : G \rightarrow G\).

Arrive at one which can be used to construct a measured train track \(\tau\) and recover the measured foliations \(\mathcal{F}^u, \mathcal{F}^s\). Candidates which cannot be so-used are discarded along the way.

Assuming that the algorithm has ended with a proof that \([F]\) is pseudo-Anosov, will have in hand the graph \(G\) and a map \(f : G \rightarrow G\), and (implicitly) a special measured train track constructed from it.
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Calculating $\lambda$, in the Bestvina-Handel setting

The graph $G$ is embedded in the surface $S$. If $S$ has one or more punctures, then $G$ is homotopy equivalent to $S$; in fact, each component of the complement of $G$ is a 2-cell, possibly with one puncture.

Let $e_1, \ldots, e_m$ be the unoriented edges of $G$. Then $f(e_j)$ is an edge path in $G$ for each $j$. The transition matrix of $f$ is the $m \times m$ matrix $T = (t_{i,j})$, where $t_{i,j}$ is the number of times $f(e_j)$ crosses $e_i$, counted without orientation, so that all entries of $T$ are non-negative integers. 

Pseudo-Anosov $\Longrightarrow$ $T$ irreducible.

It’s a Perron-Frobenius matrix. $T$ has a largest real eigenvalue $\lambda$. That’s the dilatation $\lambda$ of $[F]$. The eigenvalue $\lambda$ has algebraic multiplicity one, and the corresponding eigenspace is spanned by a positive vector. Left (resp. right) eigenvectors of $T$ are determined by the transversal (resp. tangential) measures on $\tau$.

(Eigenvectors also determine the measures on $\tau$. Nice. [BH] proof is algorithmic.)
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Study structure of $\det(T - xl) = \chi(T)$

Measures on $\tau$ are determined by $T$. [BH] prove can use $f : G \to G$ to construct $\tau$, and from it the foliations.

Interplay between $\tau$ and $G$. The map $f : G \to G$ can be used to construct $\tau$. But also $\tau$, with its measures, determines $G$. Recall the [BH] version of $\tau$, and the measures on $\tau$. Real edges and infinitesimal edges. 4 types of vertices.

Using [BH] version of $\tau$, there is a natural projection $\pi : \tau \to G$, defined by collapsing the infinitesimal edges to associated vertices. So assume [BH] algorithm has determined that $[F]$ is PA, and has produced a train track $\tau$, with associated collapsing map to $\pi : \tau \to G$. 

Odd vertex

Even vertex

Partial vertex

Evanescent vertex
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To study $\chi(T)$, introduce vector spaces $V(\tau)$ of weighted train tracks and $W(\tau)$ of ones whose weights satisfy branch conditions. Also $V(G)$ of ‘widths’ of edges of $G$. Then $\pi$ induces $\pi_\ast : V(\tau) \to V(G)$, space of ‘measures’ on the associated graph $G$. Let $f_\ast : V(G) \to V(G)$ denote the linear map induced by $f$. Let $W(G, f) = \pi_\ast(W(\tau)).$

It’s subspace of $V(G)$ whose elements admit an extension to a transverse measure on $\tau$. (Need it because ‘switch conditions’ natural for train tracks, but not for graphs.)

Using this result, we then prove that $W(G, f)$ is the kernel of a homomorphism $\delta$, and also that the resulting decomposition $V(G) \cong W(G, f) \oplus \operatorname{im}(\delta)$ is invariant under the action of $f_\ast$.

Since $f_\ast$ is represented by $T$, there is a corresponding product decomposition

$$\chi(T) = \chi(f_\ast) = \chi(f_\ast|_{W(G, f)})\chi(f_\ast|_{\operatorname{im}(\delta)})$$
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$$\chi(T) = \chi(f_*) = \chi(f_*|_{W(G,f)}) \chi(f_*|\text{im}(\delta))$$
A skew-symmetric form on the graph $G$

[F] define a skew-symmetric form on the space of transversal measures on a train track $\tau$. We extend their form to a skew-symmetric form on our space $W(G, f)$. We prove that this skew-symmetric form is invariant under the action of $f_*$.

Interesting aspect of our work: [PH] imply that the skew-symmetric form on $\tau$ non-degenerate, but we found examples where had to be degenerate. We give a complete description of space $Z$ of degeneracies.

Once again, the decomposition $W(G, f) \cong (W(G, f)/Z) \oplus Z$ turns to be invariant under $f_*$. So we have

$$\chi(T) = \chi(f_*) = (\chi(f_*|_{W(G,f)/Z})\chi(f_*|Z))\chi(f_*|\text{im}(\delta))$$
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$$\chi(T) = \chi(f_*) = (\chi(f_*|W(G,f)/Z))(\chi(f_*|Z))(\chi(f_*|\text{im}(\delta)))$$
Main Results

(1) *The polynomial* $\chi(T) = \chi(f_*)$ *factorizes as a product of three not necessarily irreducible factors* $\chi(f_*|_{W(G,f)/Z})$, $\chi(f_*|Z)$ *and* $\chi(f_*|\text{im}(\delta))$.

(2) The polynomial $\chi(f_*|_{W(G,f)/Z})$ *is an invariant of* $[F]$. It is palindromic. It contains the dilatation $\lambda$ as its largest real root. While it always contains the minimum polynomial of $\lambda$ as a factor, it does not, in general coincide with the minimum polynomial.

Jeffrey Carlson helped us to see the final step in the proof, in (2).

(3) The polynomial $\chi(f_*|Z)$ *is an invariant of* $[F]$. It encodes information about how $f$ permutes the punctures of $S$. It is palindromic or anti-palindromic, and all of its roots are roots of unity.
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(1) The polynomial $\chi(T) = \chi(f_*)$ factorizes as a product of three not necessarily irreducible factors $\chi(f_*|_{W(G,f)/\mathbb{Z}})$, $\chi(f_*|\mathbb{Z})$ and $\chi(f_*|\text{im}(\delta))$.

(2) The polynomial $\chi(f_*|_{W(G,f)/\mathbb{Z}})$ is an invariant of $[F]$. It is palindromic. It contains the dilatation $\lambda$ as its largest real root. While it always contains the minimum polynomial of $\lambda$ as a factor, it does not, in general coincide with the minimum polynomial.

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(3) The polynomial $\chi(f_*|\mathbb{Z})$ is an invariant of $[F]$. It encodes information about how $f$ permutes the punctures of $S$. It is palindromic or anti-palindromic, and all of its roots are roots of unity.
Main Results

(1) The polynomial $\chi(T) = \chi(f_*)$ factorizes as a product of three not necessarily irreducible factors $\chi(f_*|W(G,f)/Z)$, $\chi(f_*|Z)$ and $\chi(f_*|\text{im}(\delta))$.

(2) The polynomial $\chi(f_*|W(G,f)/Z)$ is an invariant of $[F]$. It is palindromic. It contains the dilatation $\lambda$ as its largest real root. While it always contains the minimum polynomial of $\lambda$ as a factor, it does not, in general coincide with the minimum polynomial. Jeffrey Carlson helped us to see the final step in the proof, in (2).

(3) The polynomial $\chi(f_*|Z)$ is an invariant of $[F]$. It encodes information about how $f$ permutes the punctures of $S$. It is palindromic or anti-palindromic, and all of its roots are roots of unity.
The polynomial $\chi(f_*|\text{im}(\delta))$ records the way that $f_*$ permutes the vertices of $\tau$. Palendromic or anti-palendromic times a power of $x$, and all of its roots are zero or roots of unity. Not an invariant of $[F]$.

$\chi(f_*) = \chi(f_*|W(G,f)/Z)\chi(f_*|Z)\chi(f_*|\text{im}\delta)$ is a palindromic or antipalindromic polynomial, possibly up to multiplication by a power of $x$.

Remark: Thurston had established a result that sounds the same as (5) above. In fact the two results are different. He introduced the orientation cover $\tilde{S}$ of $S$ and proved that the dilatation of $[F]$ is an eigenvalue of the covering map, i.e., the dilatation is a root of a symplectic polynomial that has degree $2\tilde{g}$, where $\tilde{g}$ is the genus of $\tilde{S}$. We note that our polynomial $\chi(f_*|W(G,f)/Z)$ has a degree $\neq 2\tilde{g}$ in general, and also that the symmetries of the three factors whose product is $\chi(f_*)$ arise for three distinct reasons.
The polynomial $\chi(f_\ast|\text{im}(\delta))$ records the way that $f_\ast$ permutes the vertices of $\tau$. Palendromic or anti-palendromic times a power of $x$, and all of its roots are zero or roots of unity. Not an invariant of $[F]$.

\begin{equation}
\chi(f_\ast) = \chi(f_\ast|\mathcal{W}(G,f)/\mathbb{Z})\chi(f_\ast|\mathbb{Z})\chi(f_\ast|\text{im}\delta)
\end{equation}
is a palindromic or antipalindromic polynomial, possibly up to multiplication by a power of $x$.

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(4) The polynomial $\chi(f_*|_{\text{im}(\delta)})$ records the way that $f_*$ permutes the vertices of $\tau$. Palendromic or anti-palendromic times a power of $x$, and all of its roots are zero or roots of unity. Not an invariant of $[F]$.

(5) $\chi(f_*) = \chi(f_*|_{W(G,f)/Z})\chi(f_*|Z)\chi(f_*|_{\text{im}\delta})$ is a palindromic or antipalindromic polynomial, possibly up to multiplication by a power of $x$.

Remark: Thurston had established a result that sounds the same as (5) above. In fact the two results are different. He introduced the orientation cover $\tilde{S}$ of $S$ and proved that the dilatation of $[F]$ is an eigenvalue of the covering map, i.e., the dilatation is a root of a symplectic polynomial that has degree $2\tilde{g}$, where $\tilde{g}$ is the genus of $\tilde{S}$. We note that our polynomial $\chi(f_*|_{W(G,f)/Z})$ has a degree $\neq 2\tilde{g}$ in general, and also that the symmetries of the three factors whose product is $\chi(f_*)$ arise for three distinct reasons.