My favorite paper

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1 Introduction

My mathematical colleague and friend Robion Kirby asked me to write a few words, for the volume he is preparing for the website *Celebratio Mathematica*. The topic he suggested was to choose my favorite in the list of my research papers that he is assembling. He asked a good question, and the question lead me to ruminate a bit about the events that had surrounded the paper that I ultimately decided had been especially rewarding: my work with Mike (Hugh Michael) Hilden, *Isotopies of homeomorphisms of Riemann surfaces* [2]. In particular, I want to say a few words about three aspects of creative research in mathematics that were present in that work, all of which have been very important to me:

(1) The pleasure in truly understanding new ideas in mathematics;

(2) The rare but precious creative insight, the 'aha' moment; and

(3) The experience of understanding the ability and creativity of another human being. My deepest personal and mathematical friendships have come about through collaborative work, and the joy in discovery is truly wonderful when it's a shared joy.

2 Background

Thirteen years after graduating from college, and 2 weeks after the birth of my third and youngest child, I made my first tentative move toward a career in math by enrolling in an evening grad course in Linear Algebra at NYU's Courant Institute. The total break from home duties to study math was a very

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welcome breath of fresh air and the pleasure I experienced when I understood something new was exhilarating.

One course soon lead to two, but for several years I was working alone. I eventually got to know one of my fellow students, Orin Chein, well and we would meet after lectures to go over our notes from earlier lectures, adding a new element to my studies: the pleasure in exchanging ideas with others. One of the courses on which Orin and I worked especially hard was the course offering in topology, taught by Professor Jacob Schwartz, who had told his students that he made it a practice to learn new mathematics by teaching every course that Courant offered at least once. He added that he would be learning the tools of topology with us! His presentation was brilliant and inspiring. Surprisingly, it seemed sufficiently different from the presentations in the books we consulted that it made it difficult to fill in gaps in our understanding, and harder still to find examples.

My thesis advisor, Wilhelm Magnus, was a fine mentor, and he was very sensitive to my interests, which by then had evolved in directions far from the core research topics at Courant. He told me about his own work from the 1930's on the mapping class group of a torus with 2 points removed, and suggested that I think about removing 3 points. He also mentioned, in passing, recent work [3] that he had noticed by Fadell, Neuwirth and others about a new way to think about Artin's braid group that lead in a natural way to the then-new concept of braid groups of surfaces. With the wisdom of hindsight I now realize that his instincts in singling out [3] were prescient. The task at hand was to understand what happened to the mapping class group when one passed from a surface $S_{g,n}$ of genus g with n marked points to $S_{g,0}$? Homotopy groups were natural in this setting, because the mapping class group of $S_{g,n}$ is the group $\pi_0(\mathrm{Diff} S_{g,n})$. The key insight in [3] was that the n-strand braid group of a surface Σ can be defined to be the fundamental group of the space of n distinct points on Σ, the key special case being Artin's braid group, which occurs when one chooses $\Sigma = S_{0,1}$. Still, I do not think that Magnus anticipated the work that ultimately became my thesis, because of his very genuine surprise when I told him about the long exact sequences of (sometimes non-abelian) homotopy groups that Orin and I had learned about from Jacob Schwartz.

The theorem that I eventually proved is now known as the *Birman exact sequence.* It identifies the kernel of the homomorphism $\pi_0(\mathrm{Diff} S_{g,n}) \to \pi_0(\mathrm{Diff} S_{g,0})$ as the n-strand braid group of S_g modulo its center. In the case $n = 1$ the kernel reduces to $\pi_1(S_g, \star)$. That was very nice, but there was still something missing: how to construct a group of diffeomorphisms of S_g that would be isomorphic to $\pi_1(S_g, \star)$. My pleasure in discovering the map that is now known as the *point pushing map* was one of those 'aha' moments in mathematics. I remember to this day where I was standing in our home when I suddenly understood how to construct, in the mapping class group of an orientable surface S with a marked point $*$, an arbitrary element in a subgroup of Diff $(S_g, \star)$ that would be isomorphic to $\pi_1(S_g, \star)$.

By the time that I received my PhD in Mathematics, it was clear to me that (i) I really liked the challenges of research, that (ii) working alone was possible and had its rewards, but that it might not be not ideal for me, and that (iii) my thesis was at best a small step toward understanding the *real* problem, namely the mapping class group of $S_{g,0}$, that is the group $\pi_0 \mathrm{Diff} S_{g,0}$.

3 My favorite paper

My first job was as an Assistant Professor of Mathematics at Stevens Institute of Technology. Its pleasant green campus was on the banks of the Hudson River, with incredible views of New York City across the river. Math was in a low white clapboard building, and the day I arrived to begin teaching I was greeted by a graduate student, Hugh Michael Hilden, who introduced himself as 'Mike'. His office was near mine. We had lunch together, and he told me that he would be getting his degree the following May, having solved his thesis problem, but that he didn't really like his thesis area. He wanted to know, what was I working on? I told him about my obsessive wish to uncover the structure of the group $\pi_0 \mathrm{Diff} S_{g,0}$, the first interesting case (here I was remembering that Magnus had asked me about the mapping class group of the 3-times punctured torus) being $g = 2$. I told him about its potential central role of mapping class groups in 3-manifold topology, a topic that interested him very much, even though it was far from his thesis. I was delighted by this unexpected turn of events.

That was the first of many lunchtime discussions, and it made the year exceptionally interesting. Mike was a fast learner, and I was very happy to be both teaching classes with some talented students, and having stimulating lunchtime discussions about research. I told Mike many things about Artin's braid group B_n, the mapping class group of the disc with n marked points, and its close relative, the mapping class group of the sphere.

Focussing on $\pi_0(\mathrm{Diff} S_{2,0})$ we began to understand its special nature: unlike the groups $\pi_0(\mathrm{Diff} S_{g,0}), g > 2$, the group $\pi_0(\mathrm{Diff} S_{2,0})$ had a center, the mapping class of a diffeomorphism $\mathfrak{h} : S_{2,0} \to S_{2,0}$ of order 2 with 6 fixed points. The map $\mathfrak{h}$ was known as the *hyperelliptic involution*. It had played an important role in algebraic geometry. We soon realized that $S_{2,0}/\mathfrak{h}$ was isomorphic to the sphere $S_{0,6}$ with 6 marked points. Mike told me about branched covering spaces, which suggested to us that we study the *branched covering space projection* $\mathfrak{h}^* : S_{2,0} \to S_{0,6}$. We asked: when does $f \in \mathrm{Diff}(S_{2,0})$ project to $S_{0,6}$? It was easy to see that, up to isotopy, *every* f projected, that's what it meant for the mapping class group of $S_{2,0}$ to have a center. But that was not the issue. There were isotopies of $S_{2,0}$ that did not project to isotopies of $S_{0,6}$.

We pondered that matter for a long time, and then the 'aha' moment came, but this time it was with a bonus: I honestly cannot remember whether it was Mike or me or both of us who had the key idea to focus on one of the 6 points $q_i, i = 1, 2, \ldots, 6$ that were fixed by the hyperelliptic involution $\mathfrak{h}$, say q_i, and consider its orbit $f_t(q_i)$ under an isotopy f_t of the surface from f_0 to the identity map f_1. Could the loop $f_t(q_i)$ represent a non-trivial element of $\pi_1(S, q_i)$? We proved it could not, and then went on to the more tedious work of modifying the given isotopy f_t to a new one f'_t that kept q_i fixed for every $t \in [0, 1]$. Ultimately, with more tedious work, we constructed an isotopy f''_t that commuted with $\mathfrak{h}$ at every t and at every point q on the surface[1]. We were on our way to writing the paper that was ultimately published as [2], my favorite paper.

References

[1] Joan S.Birman and Hugh M. Hilden, *Mapping class groups of closed surfaces as covering spaces*, Advances in the theory of Riemann surfaces, Editors Ahlfors, Bers et al, Annals of Math. Studies **66** (1971), 81-115.

[2] Joan S.Birman and Hugh M. Hilden, *On isotopies of homeomorphisms of Riemann surfaces*, Annals of Mathematics, **97** (No. 3), May 1973, 424-439.

[3] E. Fadell and L. Neuwirth, *Configuration spaces*, Math Scand. **10** 1962, 111-118.

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[1]See [1] for more on this viewpoint. It *motivated* the work in [2], which used slicker and less transparent arguments, but also proved much more. It's sad that motivation is sometimes concealed in this way, but that's part of mathematics too, and perhaps should be the topic for a different discussion at another time.