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KNOTTED PERIODIC ORBITS IN DYNAMICAL SYSTEMS—I: LORENZ'S EQUATIONS

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§1. INTRODUCTION

THIS PAPER is the first in a series which will study the following problem. We investigate a system of ordinary differential equations which determines a flow on the 3-sphere S^3 (or $\mathbb{R}^3$ or ultimately on other 3-manifolds), and which has one or perhaps many periodic orbits. We ask: can these orbits be knotted? What types of knots can occur? What are the implications?

Knotted periodic orbits in dynamical systems do not appear to have been systematically studied, although there is one very well known example. Let (x_1, x_2, x_3, x_4) be rectangular coordinates in R^4 and let $S^3 \subset R^4$ be the subset of points satisfying $\Sigma_{i=1}^4 c_i^2 = 1$. Let (p, q) be a pair of coprime integers, and consider the system of ordinary differential equations:

$$\mathring{x}_1 = px_2 \quad \mathring{x}_3 = qz_4$$
$$\mathring{x}_2 = -px_1 \quad \mathring{x}_4 = -qx_3. \tag{1.1}$$

The flow determined by this system is given explicitly by the equations

$$x_1(t) = x_1 \cos pt + x_2 \sin pt$$
$$x_2(t) = -x_1 \sin pt + x_2 \cos pt$$
$$x_3(t) = x_3 \cos qt + x_4 \sin qt$$
$$x_4(t) = -x_3 \sin qt \times x_4 \cos qt, \tag{1.2}$$

which defines a 1-parameter family of transformations φ_t: $S^3 \to S^3$, $t \in [0, 2\pi]$. The non-wandering set is all of S^3, since every trace curve closes to a periodic orbit. With two exceptions, these orbits are torus knots of type (p, q) (or (p, q)) any two of which link non-trivially. If $p = q = 1$ the flow described by eqns (1.2) determines, of course, the well known Hopf fibration of S^3, given by π: $S^3 \to S^3/\sim$, where $\pi(x) \sim \pi(x')$ if x and x' lie on the same trace curve.

In this paper we study the periodic orbits which arise in the flow on S^3 determined by Lorenz's equations, a system of ordinary differential equations which were introduced by Lorenz in 1963[20]. The problem which Lorenz was attempting to deal with was this: in all known examples of differential equations the solutions appeared to fall into two categories—those which ultimately settled down to some sort of steady state behavior, and those which are periodic in time. On the other hand, in nature phenomena such as "cyclones and anticyclones, which continually arrange themselves into new patterns"[20] exhibit much more complicated behavior which has been described in various places as being random, chaotic or "turbulent". Lorenz opposed this viewpoint, adopting instead the point of view that what was needed was a study of more complicated systems. His paper[20] is entitled "Deterministic non-periodic flow".

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Starting with the Navier–Stokes equation, which governs the motion of a viscous, incompressible fluid, Lorenz introduced a truncation which enabled him to reduce the Navier–Stokes equation to a system of ordinary differential equations in 3 space variables x, y, z as a function of time:

$$\begin{aligned} \mathring{x} &= -10x + 10y \\ \mathring{y} &= rx - y - xz \\ \mathring{z} &= -8/3z + xy \end{aligned} \tag{1.3}$$

where r is a real parameter, the Rayleigh number, which will be taken to be about 24. In this paper we investigate the closed orbits in the solutions to eqns (1.3). We show that knots and links do occur, and indeed they are a most interesting class. An example is given in Fig. 1.1. It has 3 components, representing 3 distinct knot types.

Recall that a *link* L is a collection of pairwise disjoint oriented simple closed curves embedded in (oriented) S^3. The *link type* of L is its equivalence class under the relation $L \approx L'$ if there is an orientation preserving homeomorphism h: $(S^3, L) \to (S^3, L')$. A *knot* is a link consisting of a single component. The *trivial knot type* is the knot type of the unit circle $S^1 \subset S^2 \subset S^3$. The *genus* of L is the smallest integer g such that L is the boundary of an embedded orientable surface $M \subset S^3$ where M has genus g. The surface M is a (minimal) *Seifert surface* for L. A link is *fibered* if $S^3 - L$ is a smooth fiber bundle over S^1 with fiber a Seifert surface for L. A link is *amphicheiral* if there is an orientation reversing homeomorphism h: (S^3, L)–(S^3, L) possibly reversing the orientation of L.

Let T be a solid torus of revolution in $\mathbb{R}^3$, obtained by revolving a disc D^2 about the z-axis. An oriented link $L \subset \mathring{T}$ is said to be represented as a *closed n-string braid* with braid *axis* the z-axis if L meets each disc $D^2 \times \{\mathcal{O}\}$, $\mathcal{O} \in [0, 2\pi]$ transversely in

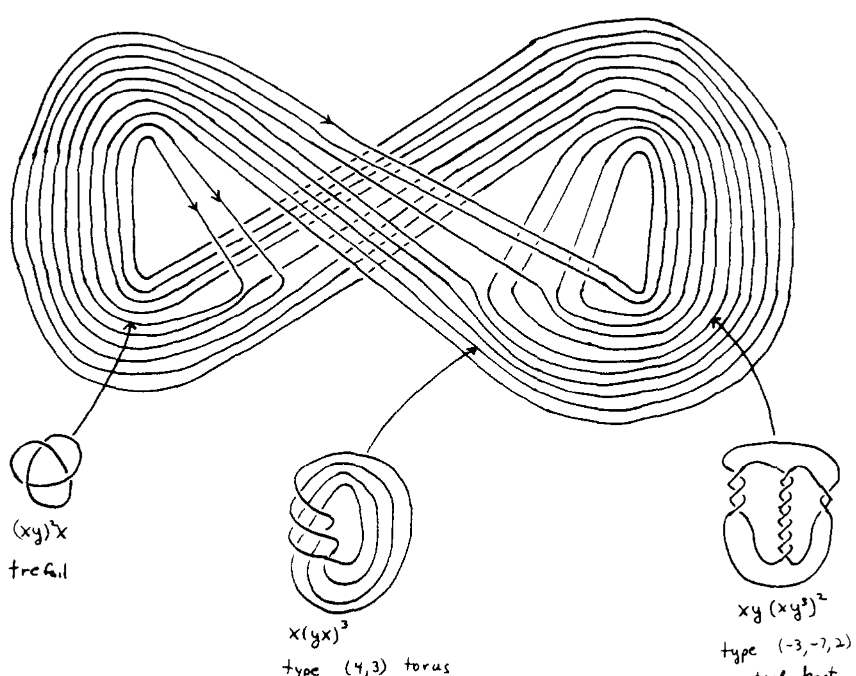


Fig. 1.1.

precisely n points for some n. If L further has a regular projection onto the plane $z = 0$ such that all crossings are as in Fig. 1.2 then L is said to be a *positive braid.*

Algebraic† *knots* (and *links*) are a subclass of the class of fibered knots and links which arise in the following way:

Let $f(z_1, z_2)$ be a non-constant polynomial in 2 complex variables and let V be the zero-set of f, regarded as a subset of R^4. The set V describes a surface in R^4. Choose any $\mathbf{z}^0 = (z_1^0, z_2^0) \in V$, and let S_ϵ^3 be the 3-sphere boundary of a 4-ball of small radius ϵ centered at $\mathbf{z}^0$. Then $L = V \cap S_\epsilon^3$ is a 1-manifold which represents the trivial knot type in S_ϵ^3 if $\mathbf{z}^0$ is a regular point, but a non-trivial link in S_ϵ^3 if the Jacobian vanishes at $\mathbf{z}^0$ (see [21]). The class of all links which arise in this way are algebraic links. We note that every algebraic link is fibered, also every algebraic link is an iterated torus link (see §6), however, there are fibered links which are not iterated torus links and iterated torus links which are not fibered, also there are iterated knots and links which are not algebraic [9].

A link *splits* if there is an embedded 2-sphere $S^2 \subset S^3$ which is disjoint from L and separates the components of L into sub-links. Otherwise it is *unsplittable.*

We will call the totality of closed orbits in the solution to Lorenz's equations‡ the *master Lorenz link* L^*. Any finite subset of L^* is a *Lorenz link* L. Any component of L is a *Lorenz knot* K. Here are some of the properties of Lorenz knots and links which will be established in this paper.

(1) There are infinitely many inequivalent Lorenz knots. These include Lorenz knots of arbitrarily high genus, although for fixed genus g only finitely many distinct knot types occur.

(2) Every Lorenz knot and link is fibered. In particular, each finite subset of L^* is a fibered link.

(3) Every algebraic knot is a Lorenz knot, and some algebraic links are Lorenz links. In particular, all torus knots occur but some torus links do not.

(4) There are Lorenz knots which are not iterated torus knots; there are iterated torus knots which are Lorenz but not algebraic.

(5) Every Lorenz link is a closed positive braid, however there are closed positive braids which are not Lorenz.

(6) Every non-trivial Lorenz link of ≥ 2 components is unsplittable, also the algebraic and geometric linking numbers are positive and equal.

(7) Non-trivial Lorenz knots and links are non-amphicheiral.

(8) Non-trivial Lorenz links have positive signature.

Note. Both (7) and (8) are trivial consequences of a new theorem of Lee Rudolph [Ru].

In addition to the above, we give an algorithm for computing a presentation for

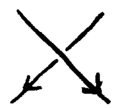


Fig. 1.2.

†The reader is warned that "algebraic knot" is used in the classical sense and is unrelated to Conway's rational tangles.

‡As will be explained later, we are making certain assumptions here about the Lorenz system which have not been proved with complete rigor. A more accurate description of L* would be that it is the collection of periodic orbits defined by the Lorenz "knot holder", H, however at this point in our discussion it is premature to introduce H.

$\pi_1(S^3 - L)$, and for computing an Alexander matrix and a Seifert matrix for any Lorenz link L. However, we were unable to characterize in a definitive way the groups or the Alexander polynomials or the Seifert forms which arise, leaving many unanswered questions.

As noted earlier, this paper is the first in a series. The second, "Knotted periodic orbits in dynamical systems—II: knot horders for fibered knots", will study knotted orbits which arise as the trace curves in the fibration of a fibered knot in S^3, when the gluing map is pseudo-Anosov. It will be seen that "twisted" Lorenz links (see Fig. 11.1) appear in every such flow.

§2. BRANCHED 2-MANIFOLDS AND THE LORENZ ATTRACTOR

In this section we review briefly the work of Williams[36, 37], which provided us with the basic tools needed for this paper. The reader who is interested in further details on the matters described here is referred to[36, 37], and thence to earlier references given there. Background material may be found in[4, 30].

For a system of ordinary differential equations such as (1, 3), as t changes the points of R^3 move simultaneously along trajectories, defining a flow φ_t: $R^3 \to R^3$ for each $t \in R$. We are interested in studying the closed orbits for this flow. Since (1.3) cannot be integrated in closed form, and since numerical integration is a very poor method for detecting closed orbits because the presence of small errors leads to incorrect conclusions, an indirect approach is necessary. The approach which Williams used is to find structures in R^3 relative to the flow φ_t which allow the periodic orbits to be collapsed onto a 2-dimensional subset of R^3, a "branched 2-manifold", where they may be described via the Poincaré map associated to an induced semi-flow, $\bar{\varphi}_t$, $t \geq 0$.

One proceeds from the eqns (1.3) to certain appropriate geometric properties and next to deduce from these geometric properties what the periodic orbits are.

Remark. This first step has not really been carried out but after Guckenheimer[15] we have a pretty good idea [16, 37] what the appropriate geometric properties *are*, at least. It remains a central problem to deduce these properties from (1.3) (or to show they fail). Meanwhile, these properties *do* describe a differential equation, so at the very least we are not working in a vacuum. One is also encouraged that many of the periodic orbits so deduced have indeed been found by computer to occur for (1.3). In particular Curry[8] has found many of these including x^2yx^2yxyxy (see Corollary 2.4.2) and six distinct knot types!

Thus "geometry" is described in terms of "branched manifolds," "hyperbolic structures," "symbolic dynamics," "strong stable manifolds," etc. and in as much as these are fairly specific to dynamical systems theory, we next present a symmary of these ideas. Suppose we are given a manifold M (for our purposes this can be S^3 or $\mathbb{R}^3$) and a C^r flow $\varphi_t (r \geq 1)$ on M. This is equivalent to a vector field or (autonomous) ordinary differential equation by existence, uniqueness and smooth dependence upon initial conditions–valid whenever M is complete, which is assumed.

2.1 *Definitions.* A point x of M is called *chain-recurrent* for φ_t provided that corresponding to any ϵ, $T > 0$ there exist points $x = x_0, x_1, \ldots, x_n = x$ and real numbers $t_o, t_1, \ldots, t_{n-1}$ all greater than T such that the distance $d(\varphi_{t_i}(x_i), x_{i+1}) < \in$ for all $0 \leq i \leq n-1$. The set of all such points, called the chain-recurrent set $\mathcal{R}$, is a compact set inveriant under the flow. This is, roughly speaking, what the computer sees when it thinks it has found a periodic orbit.

A compact invariant set K for a flow φ_t is said to have a *hyperbolic structure* provided that the tangent bundle of M restricted to K is the Whitney sum of three bundles $E^s \oplus E^u \oplus E^c$ each invariant under $D\varphi_t$ for all t and also

(a) The vector field tangent to φ_t spans E^c.
(b) There are $C, \lambda > 0$, such that

$$\|D\varphi_t(v)\| \leq C\,e^{-\lambda t}\|v\| \text{ for } t \geq 0 \text{ and } v \in E^s$$

$$\|D\varphi_t(v)\| \geq C\,e^{\lambda t}\|v\| \text{ for } t \geq 0 \text{ and } v \in E^u.$$

What this says, intuitively, is that the flow exponentially expands some directions, contracts others and leaves the flow direction roughly constant. A good example is the geodesic flow on a surface of negative curvature. Simpler examples (Fig. 2.1): Let there be a periodic orbit Λ through x_0 and let y be a small disk transverse to Λ at x_o. Then the first return map f: $y \to y$ is defined if y is small enough. Then df at x_0 is an automorphism of E_x, the tangent bundle to y at x. Then Λ has a hyperbolic structure iff df_{x0} has no eigenvalue of absolute value 1. [In this case, the eigenspaces $E^u_x, E^s_x \subset E_x$ corresponding to eigenvalues which are outside (respectively inside) the unit circle will be the fibers of E^u and E^s at x, if we happened to choose y just right.]

It is shown in [13] that the condition that a flow have hyperbolic chain-recurrent set is equivalent to Axiom A of Smale [30] and the no-cycle property. Results of Smale [*Sm*] then show that the chain-recurrent set $\mathcal{R}$ is the union of a finite number of disjoint, compact, invariant pieces called *basic sets*, each of which contains a dense orbit.

The bundles E^u and E^s are the infinitesimal versions of unstable and stable manifolds, which we define next.

If X is a subset of a hyperbolic set of a flow we define the stable and unstable manifolds $W^s(X)$ and $W^u(X)$ as follows

$$W^s(X) = \{Y | d(\varphi_t y, \varphi_t(X)) \to 0 \text{ as } t \to \infty\}$$

$$W^u(X) = \{y | d(\varphi_t y, \varphi_t(X)) \to 0 \text{ as } t \to -\infty\}.$$

If X is a point x then $W^s(X)$ is called the *strong stable manifold* of x, denoted $W^{ss}(w)$. If X is the orbit containing x then $W^s(X)$ is called the *weak stable manifold* of x. In both these cases $W^s(X)$ is in fact a manifold [17].

We next define suspension and subshift of finite type (see [2, 27]).

Let Σ_n be the space of all doubly infinite sequences

$$\underline{n} = \ldots n_{-1}n_0n_1n_2 \ldots, \quad n_i = 1, \ldots, n.$$

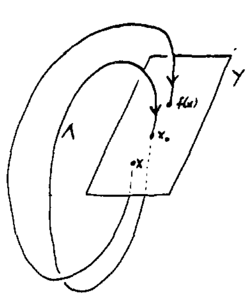


Fig. 2.1.

under the product topology. The shift map $s: \Sigma_n \to \Sigma_n$ is defined by $s(\underset{\sim}{n})_i = n_{i+1}$, $i \in z$. Given an $n \times n$ matrix A of 0's and 1's, let $\Sigma(A)$ denote the subset of Σ_n consisting of all sequences $\underset{\sim}{n}$ such that for each i, the (n_i, n_{i+1}) term of the matrix A is 1. Then the shift map s leaves $\Sigma(A)$ invariant. Then $s: \Sigma_n \to \Sigma_n$ is the *full n shift* (Bernoulli n-shift) and $s: \Sigma(A)$ is the *subshift of finite type, corresponding to A.* Given a map $f: X \to X$ by the *suspension of f* is meant

(a) the mapping torus of $f = T_f = X \times E/Z$ where a generator of Z is $(x, s) \to (fx, s+1)$.

(*b*) *The flow* φ_t induced on T_f by the trivial flow $\psi_t(x, s) = (x, s+t)$.

It follows from a theorem of Rufus Bowen that any 1-dimensional basic set of a flow is a suspension of a subshift of finite type, herein called a *Bowen–Parry flow.*

2.2 *The knot-holder H.* In Fig. 2.2 we have indicated the version of the (geometric) Lorenz flow we will use throughout this paper. To begin to understand this picture, at least near the origin, note that the Lorenz eqns (1.3) have $\theta = (0,0,0)$ as a rest point, and have the matrix

$$\begin{bmatrix} -10 & 10 & 0 \\ 24 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix}$$

as linear part. The eigen direction and corresponding eigenvalues are (roughly)

$$(0,0,1),\ -8/3;$$

$$(1, 2.06, 0), 10.6;$$

and

$$(1, -1.16, 0), -21.6.$$

Then the stable manifold at $(0,0,0)$ is spanned by $(0,0,1)$ and $(1,-1.16,0)$. However, it seems to be the case that there is a whole "field" of stable directions corresponding to $(1,-1.16,0)$ near $(0,0,0)$. This is what was hypothesized in [36], and from this assumption, one deduces (or presumably could deduce) a strong stable foliation roughly parallel to $(1,-1.16,0)$ near the origin†. Then one collapses out this foliation

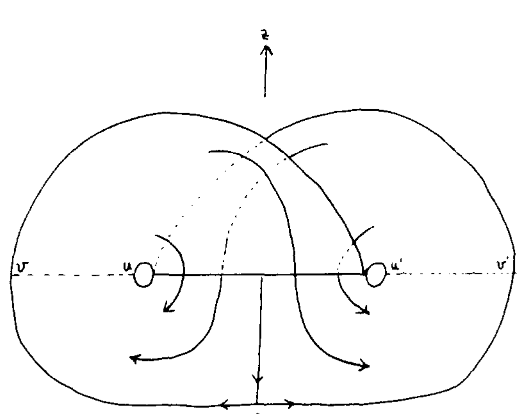


Fig. 2.2.

†At this writing Sinai and Vul[33] seem to have proved something like this.

and is left with equations like

$$\dot{z} = -8/3\ z$$
$$\dot{w} = 10.6\ w \qquad (2.1)$$

(holding approximately, near $\mathcal{O}$) where w corresponds to the vector $(1, 2.06, 0)$. Then one has a hyperbolic point at $\mathcal{O}$ with vertical stable direction, and unstable direction w which forms the boundary of H near $\mathcal{O}$. The (positive) orbit through m, $\varphi_t(m)$, $t \geq 0$ approaches $\mathcal{O}$ as a limit as $t \to \infty$. For $x \in I - \{m\}$, where I is the branch line in Fig. 2.2, there is a first return (or Poincaré) image, $f(x)$. The orbit from x to $f(x)$ goes around the left hole and in front for $x < m$, around the right hole in back for $x > m$. As $8/3 < 10$ it follows that $f' \to \infty$ as we approach m. Thus it is realistic to assume that $f' > 1$ always as we do. Finally, we point out that there appears to be a region ($r \sim 15$) where the Lorenz equations have a suspension of the full two shift as chain recurrent set. This can be treated by a semi-flow that differs little from φ_t. Instead of the singular point, one allows orbits at the bottom center to over flow and "wander" to sinks. The first return map would look like that of Fig. 2.3. One sees fairly easily that this alteration does not change the topological nature of the periodic orbits.

The reader familiar with earlier papers [16, 20, 37] will note a slight change in the branched manifold of Figure 2.2. The two holes are bounded by "trivial" periodic orbits, so that the first return map f (Fig. 2.4) will be a map from $I - m$ onto I which covers I two complete times with $f' > \lambda > 1$. This *seems* to occur on the boundary between the "pre-turbulent" and "turbulent" regions (see [18]) of the equations. The point is that H, φ_t has *all* the periodic orbits that occur in any Lorenz attractor. This makes for a simpler description though we pay the price of having two orbits we don't really want, namely two unknotted orbits, one encircling each "hole". The two trivial orbits constitute exceptions to most of our results below–and we indicate this by such phrases "except for the trivial orbits...", etc.

2.3 *Why Branches*? For the reader not familiar with earlier use of branch manifolds, we indicate how they arise, schematically, in Figure 2.5. One has a closed neighborhood N, a flow φ_t with $\varphi_t(N) \subset N$, $t > 0$, and a foliation by "strong stable manifolds" W^s through each point $x \in N$; these satisfy $\varphi_t(W^s_x) = W^s(\varphi_t(x))$. Then we form a quotient space q: $N \to B$ by $x' \sim x''$ iff x' and x'' lie in a connected component of some $W^s(x) \cap N$.

The example given in Fig. 2.5 is too rudimentary to have periodic orbits. But when periodic orbits do occur this process of collapsing the strong stable manifolds does not change their nature.

Proposition 2.3.1. *The collapsing map q is $1-to-1$ on the union of all the periodic orbits.*

Remark. In §1, when we stated our main results, we noted that there are infinitely

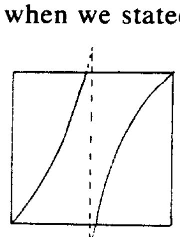


Fig. 2.3.

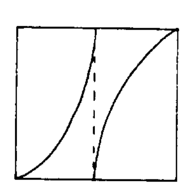


Fig. 2.4.

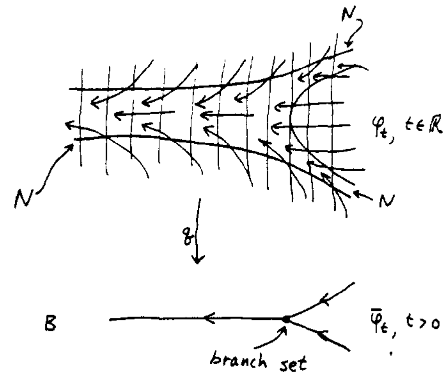


Fig. 2.5.

many distinct Lorenz knots. One of the remarkable consequences of the proposition just stated is that all of these knots exist *simultaneously* as a dense embedded subset of the Lorenz knot-holder. Concrete examples of sets of distinct Lorenz knots which "fit together" on the knot holder H are extremely interesting (e.g., see fig. 1.1).

Proof of Proposition 2.3.1. Let x be a periodic point and recall $W^{ss}(x) = \{y | d(\varphi_t y, \varphi_t x) \to 0 \text{ as } t > 0\}$. Then $W^{ss}(x)$ does not intersect any other periodic orbit, as each 2 periodic orbits are a positive distance apart. Nor could $W^{ss}(x)$ intersect the orbit through x again, as such a point is "out of phase" with x. Finally, this collapsing of the neighborhood can be carried out in $\mathbb{R}^3$ in the case of the (geometric) Lorenz attractor by, say, a deformation H_s, $0 \leq s \leq 1$. Here H_0 is the identity, and H_s is a homeomorphism for $s < 1$. It follows that the union of any finite set of periodic orbits under ϕ_t, is isotopic to their image under $q = H_1$. ||

2.4 *Symbolic dynamics of the semi-flow.* We next outline some of the "symbolic dynamics" of the semi-flow $\bar{\phi}_t$ and its first return map f. Though this material is well known[16, 36], we include it for completeness. Let $a \in I$ and define the finite or infinite sequence

$$k(a) = k_0(a), k_1(a), k_2(a), \ldots$$

by

$$k_0(a) = \begin{cases} x \text{ if } a \text{ is to the left of } m \\ 0 \text{ if } a = m \\ y \text{ if } a \text{ is to the right of } m; \text{ and} \end{cases} \tag{2,2}$$

$k_i(a)$ is defined iff $f^i(a)$ is defined and

$$k_i(a) = \begin{cases} x \text{ if } f^i(a) < m, \\ 0 \text{ if } f^i(a) = m, \text{ and} \\ y \text{ if } f^i(a) > m. \end{cases} \tag{2.3}$$

Then sequences k are lexicographically ordered by setting $x < 0 < y$. In [37] we were concerned with actual Lorenz attractors and hence the $k(a)$ were subject to restrictions; here, however, we are using the limiting case, where *all* sequences occur.

PROPOSITION 2.4.1. *The map $a \to k(a)$ is a 1-to-1 order preserving correspondence between the points of the branch set and the lexicographical ordering of the set of all sequences $k_0, k_1, \ldots$ such that*

$$(a)\quad \textit{each } k_i = x, y, \textit{or } 0$$

$$(b)\quad \textit{the sequence terminates with } k_i \text{ iff } k_i = 0.$$

COROLLARY 2.4. *The periodic orbits of φ_t correspond 1-to-1 with the cyclic permutation classes of finite aperiodic words w in the free monoid generated by x and y.*

Notation. We let $\Lambda(w)$ be the periodic orbit corresponding to w. We will sometimes refer to the cyclic permutation class of a word w as a *cyclic word.*

Proof of Proposition 2.4.1. Let Λ be a periodic orbit under φ_t and let a be the left-most point of Λ on the branch line. Thus, for some $n, f^n(a) = a$, and we choose n to be the (minimum) period of a. Then $k(a)$ is a periodic sequence

$$k_0(a), k_1(a), \ldots, k_{n-1}(a), k_1(a), \ldots.$$

Let $w = k_0(a), k_1(a), \ldots, k_{n-1}(a)$. Were w periodic, say $w = u^r$ where $n = rq$, then the orbits of a and $f^q(a)$ would lie together on the same side of the midpoint m for n iterations of f--and hence for all iterations of f. This is impossible as the derivative $f' > \lambda > 1$. It follows that the first time, say j, where the orbit of a and that of $f^i(a)$, $i = 1, \ldots, n-1$, are on a different side of m, we have $f^j a < m < f^{i+j}(a)$. In other words, $w(a) < w(f^i(a)), i = 1, \ldots, n-1.$ ‖

We indicate why $a \mapsto w(a)$ is onto. For this purpose we assume f has the form

$$f(x) = \begin{cases} 2x, & x < m \\ 2x - 1, & x > m \end{cases}.$$

Set I_x = left half of the branch set, I_y the right half, I_{xx} the left half of I_x, I_{xy} the right half etc. Note that $f^{-1}|I_x$ is a well defined contraction of I_x onto itself and thus has a unique fixed point (the left end-point of I). Similarly $f^{-2}|I_{xy}$ is a well defined contraction of I_{xy} into itself and hence has a fixed point, say a. Then $f^2(a) = a$, so that $k(a) = x, y, x, y, \ldots,$ and $w(a) = xy$. Similarly, we define I_w for $w = w_0, \ldots, w_{n-1}$ and see that $f^{-n}|I_w$ is a well defined contraction, that its unique fixed point, a, satisfies $k(a) = w_0, w_1, \ldots, w_{n-1}, w_0, w_1, \ldots$, so that $w(a) = w$.

By following a periodic orbit around one finds its "word" w, which is (essentially) an element of $\pi_1(H)$. This process is reversible and we next give an algorithm. (The reader may prefer to read the examples instead of the general description.)

ALGORITHM 2.4.3. *Given a finite set of distinct aperiodic cyclic words $w_1, \ldots, w_n$,* i.e. such that no w_i is a cyclic permutation of w_j for $j \neq i$, *the corresponding link can be drawn on H as follows. Let $\bar{w}_i$ be the infinite periodic word $w_i w_i w_i \ldots$ and let s be the operation of removing the first letter from a word. Let $s^1 = s$ and define s^j inductively by $s^j = s(s^{j-1})$. Then the set $\mathcal{S} = [\{s^j(\bar{w}_i): j = 0, 1, \ldots\}, i = 1, 2, \ldots, n.]$ has m (distinct) words in it, where m is the sum of the lengths of the words $w_1, \ldots, w_n$. This uses aperiodicity and the fact that no w_j is a cyclic permutation of $w_i, i \neq j$. Alphabetize them and note that the points $k^{-1}(s^j(\bar{w}_i))$ occur on I in just this order, where k is the 1-to-1 map of 2.4.1. Thus, up to topological equivalence, we can use any points $P(s^j(\bar{w}_i))$ which occur in this order. Then one can trace out the orbits corresponding to*

w_i by starting at $P(\bar{w}_i)$, connecting it to $P(s^1(\bar{w}_i))$, proceeding to the right or left on H depending upon whether $\bar{w}_i$ begins with x, or y. Continue by connecting $P(s^1(\bar{w}_i)$, to $P(s^2\bar{w}_i)$, in the same manner, etc.

Example (*See Fig.* 2.6). $w_1 = xy$, $w_2 = wy^2$. Then $\bar{w}_1 = xyxy\ldots\ldots$, $\bar{w}_2 = xyyxyy\ldots$. The set $\mathscr{S}$ is $[\{xyxy\ldots\ldots, yxyx\ldots\ldots\}, \{xyyxyy\ldots, yyxyyx\ldots, yxyyxy\ldots\}]$. Alphabetizing, according to the rule $x < y$, we obtain $1 \leftrightarrow xyxy\ldots$, $2 \leftrightarrow xyyxyy\ldots$, $3 \leftrightarrow xyxyx\ldots$, $4 \leftrightarrow yxyyxy\ldots$, $5 \leftrightarrow yyxyyx\ldots$. Thus the cyclic permutation associated to w_1 is $(1, 3)$, while that for w_2 is $(2, 5, 4)$.

A harder example is given in Fig. 1.1. There are 3 words $w_1 = (xy^2)x$, $w_2 = x(yx)^3$, $w_3 = xy(xy^3)^2$.

§3. NOTATIONS AND TERMINOLOGY

We restate Corollary 2.4.2. *and Algorithm* 2.4.3 *as*:

PROPOSITION 3.1. (a) *The periodic orbits of φ_t are in* 1-*to*-1 *correspondence with the cyclic permutation classes of positive aperiodic words in x and y, referred to from now on as Lorenz words.*

(b) *The set of all Lorenz links is in* 1-*to*-1 *correspondence with the collection of all finite sets $w_1, \ldots, w_k$, where each w_j* is as in (a), *and no w_j is a cyclic permutation of any w_s, $s \neq j$, $s, j = 1, \ldots, k$.*

Algorithm 2.4.3 allows one to construct the Lorenz link associated to a collection of Lorenz words. It well be convenient to have alternative descriptions. Accordingly, we will introduce in this section the concepts of Lorenz braids, Lorenz permutations, string index, braid index, crossing number, trip number, rank, genus and braid word.

Let L be a Lorenz link. Then L is a subset of the knot holder H pictured in Fig. 2.2. Cut H open along lines uv and $u'v'$ in Fig. 2.2. Denote the cut-open holder by $\tau(H)$. Then $\tau(L) \subset \tau(H)$ may be unfolded to an open braid $\beta(L) = \beta$ on n strings for some integer n. Clearly the braid β determines L and conversely. We call it the *Lorenz braid* associated to L. An example is given in Fig. 3.1.

To describe the braid, note that the strings have a natural ordering, from left to right. Number them $1, \ldots, n$ on the bottom and on the top. These strings fall into 2 groups of parallel strands, a left group of p strands and a right group of q strands, $p + q = n$, where the strands in the left group always pass over those in the right group, but strands in the same group never cross one-another. Clearly the braid is uniquely determined by a pair of integers (p, q) and a permutation π_* which is induced by the first return map f. Even more, p, q and the permutation π_* are

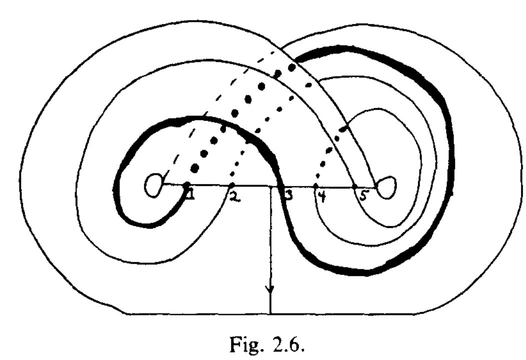


Fig. 2.6.

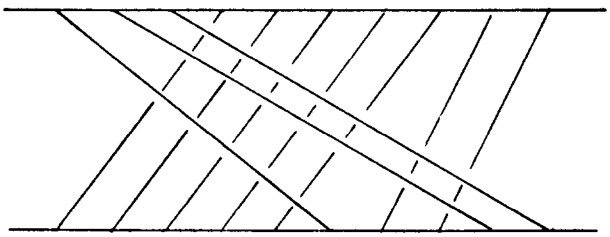


Fig. 3.1.

uniquely defined by the array $\pi_i, \pi_2, \ldots, \pi_p$, where $\pi_i = \pi_*(i)$, since the remaining q $\ldots < \pi_p = n$. We call any such finite sequence $\pi_1, \ldots, \pi_p$ which arises from a Lorenz braid, a *Lorenz permutation*, and often identify it with the permutation π_* which it determines. The symbol π will sometimes be used for the sequence $\pi_1, \ldots, \pi_p$. In the example of Fig. 3.1 we have $\pi = (6, 9, 10)$. The example in Fig. 1.1 had the Lorenz permutation $\pi = (5, 7, 11, 12, 13, 14, 15, 18, 21, 22)$.

PROPOSITION 3.2. *A sequence* $\pi_1, \ldots, \pi_p$ is a Lorenz permutation if and only if:

(a) $1 < \pi_1 < \ldots < \pi_p$; *and* (b) *if the resulting permutation* π_* *splits as a product of* $k \geq 2$ *disjoint cycles, then no two cycles are parallel, i.e. no two cyclic factors* μ, η *of* π_* *satisfy* (i) *length of* μ = *length of* $\eta = r$; (ii) $\mu_i = \eta_i + j$, $i = 1, \ldots, r$ for some j.

Proof. The condition $1 < \pi_1$ assures that no "trivial orbits" (see §2.2) occur. Assume $1 < \pi_1 < \ldots < \pi_p = n$ and that the resulting permutation π_* satisfies (b). We may think of π_* as a 1-to-1 map on the finite set $\{1, \ldots, n\}$. In fact π_* is essentially $f|I_0$ where $f\colon I \to I$ is the Poincaré map and I_0 is a union of periodic orbits of f. We must show that π_* defines a set $w_1, \ldots, w_r$ of Lorenz words which determine a Lorenz link.

We consider 2 cases.

Case 1. π_* is a cyclic permutation. Then the orbit of 1 under the powers of π_* contains all of the symbols $1, 2, \ldots, \pi_p = n$, also n is the smallest integer such that $\pi_*^n(1) = 1$. Define integers $a_1, b_1, \ldots, a_t, b_t$, where $a_1 + b_1 + \ldots + a_t + b_t = n$ by the inequalities:

$$\pi_*(1) < \pi_*^2(1) < \ldots < \pi_*^{a_1}(1) > \pi_*^{a_1+1}(1) > \ldots > \pi_*^{a_1+b_1}$$
$$< \pi_*^{a_1+b_1+1}(1) < \ldots < \pi_*^{a_1+b_1+a_2}(1) > \ldots > \pi_*^{a_1+b_1+\ldots+a_t+b_t} \qquad (1).$$

Let $w = w(\pi) = x^{a_1}y^{b_1}x^{a_2}\ldots x^{a_t}y^{b_t}$ and let $\Lambda(w)$ be the unique periodic orbit of φ_t associated to $w(\pi)$. To recover π from $w(\pi)$, let z to the left-most point of $\Lambda(w)$ on I. Then z is a point of period n under f, and we have

$$f(z) < f^2(z) < \ldots < f^{a_1}(z) > f^{a_1+1}(z) > \ldots \qquad (**)$$

reproducing the previous inequalities exactly. Hence $\pi = (\pi_1, \ldots, \pi_p)$ is a Lorenz permutation.

General case. We associate to π_* its cyclic factors and to each of these the corresponding periodic orbit via case 1. Thus to complete the proof that π is a Lorenz permutation it well suffice to show that distinct cyclic factors of π_* yield distinct periodic orbits.

To this end, let μ, η be two cyclic factor of π_*, let $\Lambda(\mu)$, $\Lambda(\eta)$ be the periodic orbits attached to them by case 1, and assume on the contrary that these are the same, 1,e. $\Lambda(\mu) = \Lambda(\eta)$. Then, μ and η are the same length, say r; let α be the least integer in the orbit of μ, β the least integer in the orbit of η. We assume $\alpha < \beta$ as the other case is similar.

Claim. $\eta(\beta) - \mu(\alpha) > \beta - \alpha$. For η and μ are just restrictions of π_* and in the range α to β, π_* is increasing. That is to say $\pi_*(1) < \pi_*(2) < \ldots < \pi_*(p) > \pi_*(p+1)$ and

$$\pi_*(p+1) < \pi_*(p+2) < \ldots < \pi_*(n).$$

Now under the assumption that $\Lambda(\mu) = \Lambda(\eta)$, it follows that $\eta(\beta)$ and $\mu(\alpha)$ are either both $\leq p$ or both $\geq p+1$. Thus $\eta^2(\beta) - \mu^2(\alpha) \geq \eta(\beta) - \mu(\alpha)$. This property continues, as the orbits $\Lambda(\eta)$ are the same. Hence

$$\beta - \alpha = \eta^r(\beta) - \mu^r(\alpha) > \ldots > \eta(\beta) - \mu(\alpha) \geq \beta - \alpha$$

so that equality holds in every case. Let $j = \beta - \alpha$. Then

$$\mu = (\alpha, \mu(\alpha), \ldots, \mu^{r-1}(\alpha))$$

and

$$\eta = (\beta, \eta(\beta), \ldots, \eta^{r-1}(\beta)).$$

Then μ and η are parallel cycles, contradicting part (b). Thus π is a Lorenz permutation which proves the if part of 3.2.

The only if part is essentially trivial, because any Lorenz link corresponds to a set $\{w_1, \ldots, w_k\}$ of words satisfying condition (b) of Proposition 3.1. Furthermore, the w_i correspond to cyclic factors of the permutation π_*. As no w_i is a cyclic permutation of any w_s, it follows that no two cyclic factors of π_* are parallel. This completes the proof of Proposition 3.2. ‖

Let β be a Lorenz braid and let $\pi = (\pi_1, \ldots, \pi_p)$ be the associated Lorenz permutation. Let L be the Lorenz link defined by β or π. Let $w_1, \ldots, w_r$ be the Lorenz words which define L, where $w_i = x^{a_{i1}} y^{b_{i1}} x^{a_{i2}} \ldots y^{b_{it_i}}$, $1 \leq i \leq r$. The following terminology will be used in the paper:

(1) The *string index* is the number n of strings in β. It is the sum $n_1 + n_2 + \ldots + n_r$ of the *letter lengths* $n_i = a_{i1} + b_{i1} + a_{i2} + b_{it_i}$ of w_i, $1 \leq i \leq r$; it is also the last entry $n = \pi_p$ in the Lorenz permutation π.

Note that the symbol n used here has exactly the same meaning as it did in §2, since the string index is precisely the number of points in the intersection of a Lorenz link L with the branch set I.

(2) The *braid index* of a knot is the minimum string index among all closed braid representatives of that knot. It is a knot invariant.

(3) The *crossing number c* is the number of double points in the projected image of the Lorenz braid β. If β is written as a product of elementary braids σ_i, in which the ith strand crosses over the the $i+1$st strand once, from left to right, with no other double points, then $\beta = \sigma_{\mu_1}\sigma_{\mu_2}\ldots\sigma_{\mu_c}$ $(1 \leq \mu_i \leq n-1)$ and c is the letter length of β.

The integer c may also be computed from π by the formula

$$c = \sum_{i=1}^{p} (\pi_i - i). \tag{3.1}$$

(We omit a proof because this formula is obvious if one draws a picture.) A somewhat more complicated formula for computing c directly from $w_1, \ldots, w_r$ is given in §8.

(4) The *trip number* t is the sum $\Sigma_{i=1}^{r} t_i$, where t_i is the number of syllables in w_i, a syllable being a maximal subword of w_i of the form $x^{a_{ij}}y^{b_{ij}}$. It may be computed from π by the formula

$$t = \text{cardinality}\,\{\pi_j \in \pi \,|\, \pi_j > p\} \tag{3.2}$$

(To see this, note that t_i is the number of x-symbols in w_i which are followed by y-symbols).

(4) The *rank* r of L is the rank of the group $H_1(M)$, where M is an orientable surface of minimal genus spanned by L.

(5) The *genus* g of L is the genus of M, where as above M is an orientable surface of minimal genus spanned by L.

(6) The *braid word* associated to a braid β on n strings is a representation of β by a word in the generators $\sigma_1, \ldots, \sigma_{n-1}$ of the braid group B_n, where σ_i is an elementary braid in which the ith string crosses over the $i+1$st.

§4. LORENZ LINKS ARE UNSPLITTABLE AND WELL-TANGLED.

THEOREM 4.1. *Let K_1, K_2 be any two Lorenz links, neither of which is a trivial orbit. Then the algebraic and geometric linking numbers are positive and equal. Thus in particular Lorenz links are unsplittable.*

Proof of 4.1. Each Lorenz knot is naturally oriented by the sense of the semi-flow on the knot holder H. Thus if K_1 crosses over K_2 at all in the projected image it crosses from left to right and $lk(K_l, K_2)$ is the number of such crossings. Hence it is only necessary to show that K_1 crosses K_2. This becomes obvious as soon as one draws a few pictures. (For example, see Fig. 2.6, where the linking number of the two orbits is 1.)

Remark. Lorenz links are "well-tangled", i.e. any two components of any finite sublink of L^* have positive linking number. We can think of L^* as a vast rats nest of knots. It will turn out that "almost all" of them have non-trivial knot types.

§5. POSITIVE BRAIDS AND THE FIBRATION THEOREM

Cutting open the knot-holder of Fig. 2.2 along the lines uv and $u'v'$ one sees immediately that every Lorenz link has a representation as a closed positive braid. We will sharpen this result to:

THEOREM 5.1. *If K is a Lorenz link of trip number t then K may be represented by a positive braid on t strands. The goal of this section will be to study the consequences of Theorem* 5.1.

The class of all links which admit representations as positive braids was first studied by Burau[5], later generalized by Murasugi[24] and still later rediscovered and reinterpreted by Stallings[32]. They are of particular interest in our work because (Theorem 5.2) closed positive braids are fibered links, and the genus can be computed from the braid representation by an easy combinatorial formula. In particular (Corollary 5.3) Lorenz links are fibered, and the genus of a Lorenz knot is bounded below by $(t)(t-1)/2$ where t is the trip number. Moreover, a Lorenz knot is unknotted if and only if it has trip number 1. Related results hold for Lorenz links, with rank r replacing genus g. Later, in Section 8, we will develop further consequences: a zeta function for the flow which records information about the knot types of the periodic orbits, instead of just the number of periodic orbits of each period. Finally, Corollaries 5.4 and 5.5 are other interesting consequences of the positive braid representation: Lorenz knots have positive signature and are non-amphicheiral.

We conjecture that the trip number t is actually the braid index for a Lorenz link, i.e. that a Lorenz link cannot be represented by a braid on fewer than t strings. A partial result which is a step toward a proof is given in Theorem 5.7. Such a result would be extremely interesting, because it would imply that trip number is a knot invariant.

Proof of Theorem 5.1. In Fig. 5.1 we show how the knot holder of Fig. 2.2 can be

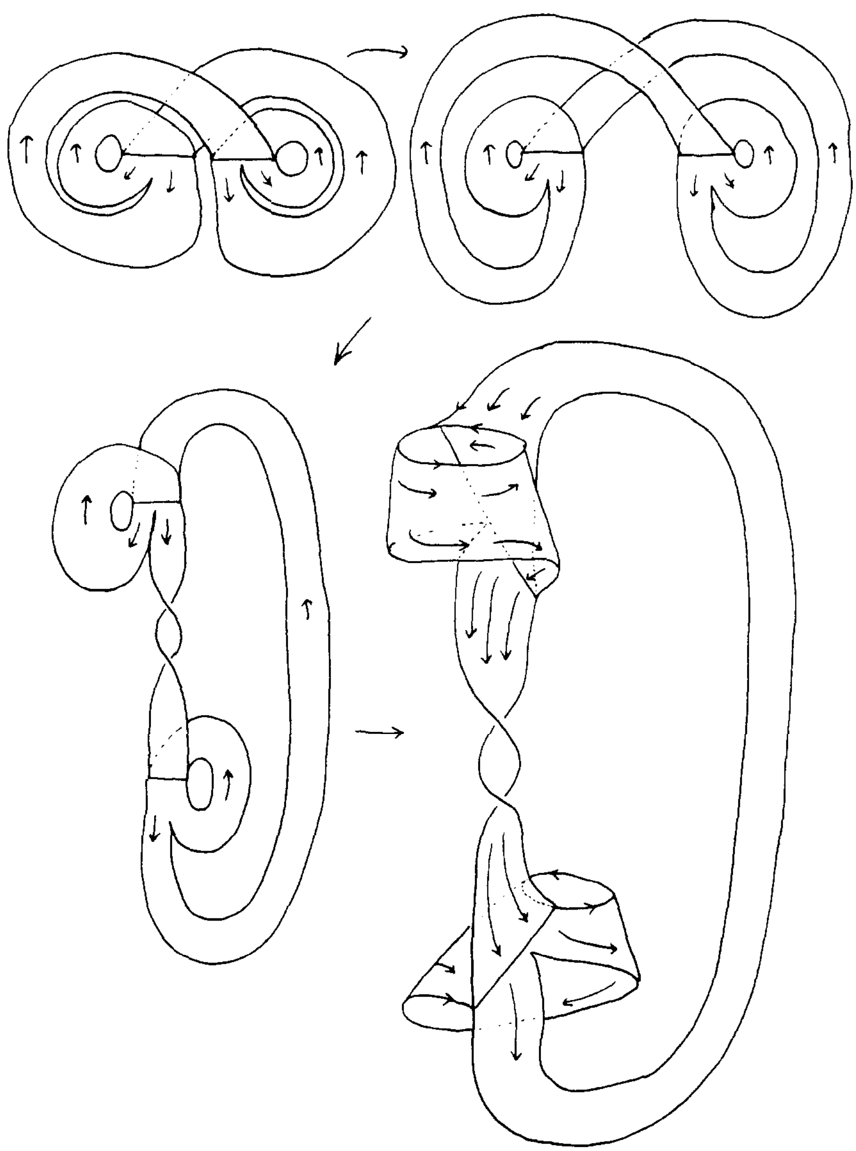


Fig. 5.1.

cut open and stretched out. The last version can be described as an annulus with a full twist with two truncated cones attached along lines of tangency. The top cone is in front of the annulus and slants down to the right; the bottom cone is in back of the annulus and slants to the left. The number of strands on the annulus is the number of arcs of K passing from the left side of the branch line to the right side (equal to the number passing from the right side to the left), i.e. the trip number t. That K is a braid in this presentation is just the fact that the vector field can be chosen on either cone to have a downward component everywhere except at the top, where it is horizontal. (The tops of the two cones are the two "trivial" Lorenz knots. See §2.) That this braid is positive (Fig. 1.2) is checked at the various double points which occur in the projection shown in the last picture of Fig. 5.1. These double points occur on the two cones and on the twisted band that joins them. On the twisted band all crossings are clearly positive. On the top cone there are three types of crossings where, say, α crosses in front of β: *type* 1: α on the front of the cone, β on the back of the cone; *type* 2: α on the front of the cone, β on the back-most strip; *type* 3: α on the back of the cone, β on the back-most strip. Types 1 and 2 are positive. Type 3 is negative, however each crossing of type 3 is preceded immediately by a positive crossing of type 1, so the net crossing is 0 (after a small deformation). The bottom cone is similar.

Recall that a link $L \subset S^3$ is said to be *fibered* if there is a locally trivial fibration π: $(S^3 - L) \to S^1$ with fiber an orientable surface M, $\partial M = L$. Let L be any oriented link in S^3, and let M be a spanning surface of minimum genus for L. Let η: $M \to (S^3 - M)$ be the map which pushes M off itself along the outward drawn normal, inducing a homomorphism η_*: $\pi_1 M \to \pi_1(S^3 - M)$. The fact that M has minimal genus implies (via the loop theorem [P]) that η_* is one-to-one. It is a consequence of the fibration theorem of Stallings [St 1] that L is fibered if and only if η_* is onto.

THEOREM 5.2. *Let L be an indecomposable link of multiplicity μ. Suppose that L has a regular projection which exhibits it as a positive n-string braid, and that in this projection there are c crossings. Then L is fibered, and its genus g is given by the formula*

$$2g = c - n + 2 - \mu \tag{5.1}$$

$$r = 2g + \mu - 1 = c - n + 1. \tag{5.2}$$

In particular, each finite sublink of the master Lorenz link L^* is a fibered link.

Remark. The fact that positive braids define fibered links was stated more-or-less without proof in [32], and while the proof follows from observations in that paper it did not seem totally obvious to us. A proof is also implicit in the work of Murasugi[24], who established that η_* is surjective. This implies, by[31], which was evidently not well-known at the time that[24] was published, that L is fibered. (We say this because[24] does not mention this consequence of the surjectivity of η_*.) In view of all of these, we give a complete proof here.

Proof of Theorem 5.2. The positive braid representation of L yields in a natural way a Seifert surface M_n for L consisting of n discs joined by c half-twisted bands, one for each braid crossing (see Fig. 5.2) with, say k_i strips joining the ith disc to the $(i+1)$st disc for each $i = 1, \ldots, n-1$. We prove that η_* is an isomorphism by induction on n. If $n = 1$, then $\pi_1 M_1 \cong \pi_1(S^3 - M_1) \cong \{1\}$. Note that if we delete the nth disc by cutting open the bands joining the $(n-1)$st and nth discs we will get a new

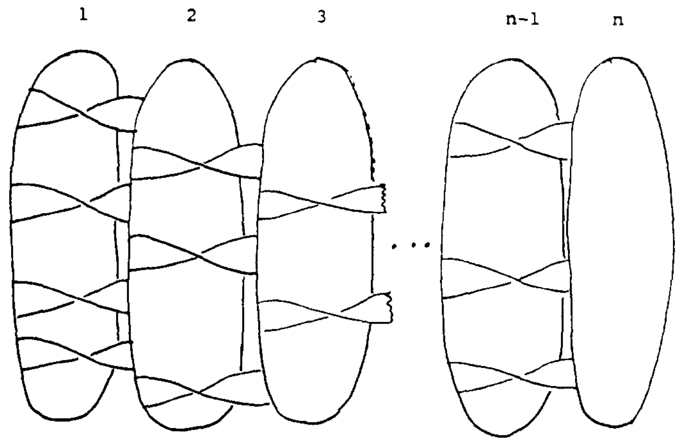


Fig. 5.2.

positive braid which is the boundary of a modified surface M_{n-1} and a new push-off map η_*' by restriction of the old. Since there are k_{n-1} bands joining the $(n-1)$st and nth discs, we have:

$$\pi_1(S^3 - M_n) = \pi_1(S^3 - M_{n-1})^* F_{k_{n-1}} \tag{5.3}$$

where $F_{k_{n-1}}$ is the free group of rank k_{n-1}. It is visibly clear (see Fig. 5.3) that

$$\eta_*(\pi_i M_n) = \eta_*(\pi_1 M_{n-1})^* F_{k_{n-1}}. \tag{5.4}$$

Induction on n implies that η_* is an isomorphism.

Now, $\pi_1 M_n = \pi_1 M$ is constructed from n discs (one for each braid string) which are joined in pairs by c bands (one for each elementary braid). An Euler characteristic computation gives eqns (5.1) and (5.2). ||

COROLLARY 5.3. *A Lorenz knot is unknotted if and only if it has trip number* 1. *A* 2-*trip Lorenz knot is a torus knot of type* $(2, 2m+1)$, $m \geq 1$. *For trip number t the genus is bounded below by*:

$$2g \geq (t-1)(t). \tag{5.5}$$

Moreover, this bound is sharp.

Proof of Corollary 5.3. The fact that each 1-trip Lorenz knot is unknotted is an

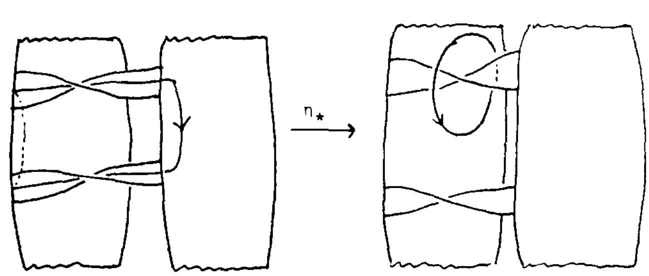


Fig. 5.3.

immediate consequence of Theorem 5.1. The converse will follow from the genus inequality (5.5). 2-trip knots may be represented as closed 2-braids, hence they are torus knots of type $(2, 2m+1)$. By the inequality (5.5), m must be ≥ 1.

To establish (5.5) let K be a Lorenz knot of trip number t, and consider the representation of K as a positive braid which is given by Theorem 5.1. From the picture in Figure 5.1, we see that the braid includes a full twist on t strings. This twist contributes $(t)(t-1)$ crossings, as one can easily check. Since the full twist induces the identity permutation on the braid strings, while the actual permutation is necessarily a t – cycle (because K is a knot) there must be at least $t-1$ additional crossings since a t – cycle cannot be written as a product of fewer than $t-1$ transpositions of adjacent symbols. Thus $c \geq t(t-1)+t-1$, *so*

$$2g = c - t + 1 \geq t(t-1).$$

Equality in (5.5) is achieved for the t trip Lorenz knot which is defined by the word $w = x(xy)^t$. The knot which is so-represented is the torus knot of type $(t, t+1)$, as will be shown in Section 6 below. Thus the bound in (5.5) is sharp. ||

Remark. For a Lorenz link, the crossing number (by the above argument) is at least $(t)(t-1)$.

COROLLARY 5.4. *Lorenz knots have positive signature.*

Proof of Corollary 5.4. Lee Rudolph [29] has proved that non-trivial closed positive braids have positive signature.

COROLLARY 5.5. *Lorenz knots are non-amphicheiral.*

Proof of Corollary 5.5. This is an immediate consequence of Corollary 5.5, because amphicheiral knots have zero signature.

We next develop a braid formula for Lorenz links. Thus let L be a Lorenz link and t its trip number. By Theorem 5.1 we may assume that the corresponding link is a braid on t strands.

PROPOSITION 5.6. *The braid word of a Lorenz link as a braid on t strands, t the trip number, has the form*

$$(a)\ \Delta^2 \prod_{i=1}^{t-1} (\sigma_1\sigma_2 \ldots \sigma_i)^{n_i} \prod_{i=t-1}^{1} (\sigma_{t-1} \ldots \sigma_i)^{m_i}.$$

The exponents can be computed as

$$\text{(b)}\ n_i = card\,\{j \mid \pi_*(j) - j = i+1 \text{ and } \pi_*(j) < \pi_*^2(j)\}$$

$$\text{(c)}\ m_i = card\,\{j \mid j - \pi_*(j) = i+1 \text{ and } \pi_*(j) > \pi_*^2(j)\}.$$

Proof of 5.6. We think of L as the union of finitely many periodic orbits of φ_t. As such it is separated into 4 groups of arcs by the branch line; respectively, those joining the right side of the branch line to the right side (RR), those joining the right to the left, $RL, \ldots LR, \ldots LL$.

From Fig. 5.1 we see that the group LR can be considered to contribute a full twist to our braid, and RL to contribute nothing. We turn to LL: the first arc of LL connects point 1 to (say) $i+1$, $i \geq 0$. Then upon straightening this arc (see Fig. 5.4) the

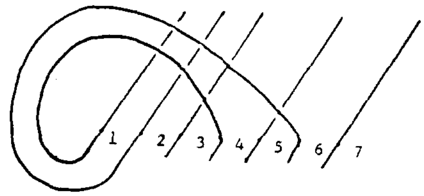


Fig. 5.4.

resulting braid moves are

$$\sigma_1\sigma_2\ldots\sigma_i$$

since now the second arc of LL becomes the first strand. (In the trivial case $i=0$, straightening only removes a trivial loop.) But note that the second arc must go *further* than the first, corresponding to the fact that the Poincaré map is order preserving. That is, it connects point 2 to $j+1$, where $j\geq i+1$. Hence it corresponds to $\sigma_1\sigma_2\ldots\sigma_j$, $j\geq i$. The same sort of inequality is forced on each succeeding arc by the fact that the Poincaré map is order preserving. Thus the arcs of LL contribute a term exactly like the second in the product (5.6a), where of course some of the n_i's can be zero.

The proof that the arcs of group RR contribute the third term is quite similar. One need only note that the arcs of RR are naturally ordered from right to left, to see the difference. This proves (a).

To prove (b), note that the first arc of LL connects 1 to $\pi_*(I)$ and in general, each such arc connects some j to $\pi_*(j)$ and that in turn $\pi_*(j)$ is connected to $\pi_*^2(j)$ by either an arc of LL *or* one of LR. In particular, $\pi_*(j)<\pi_*^2(j)$. Thus this particular arc contributes $\sigma_1\ldots\sigma_i$ where $\pi_{j+1}-\pi_j=i+1$, as required. This proves (b). The proof of (c) is similar.

COROLLARY 5.7. *With notation as above for a Lorenz knot, we have the formula*

$$2g=(t-1)^2+\Sigma\ \pi_*(j)-j-1+\Sigma j-\pi(j)-1$$

where the first summation is over all j such that $j<\pi_*(j)<\pi_*{}^2(j)$ *and the second over all j such that* $j>\pi_*(j)>\pi_*{}^2(j)$.

Formulas (b) and (c) in Proposition 5.6 suffer from being far removed from the words in x and y which form the natural ingredients here. To rectify this we give a formula for $\pi_*(j)-j$ in terms of words: continue the notation of 5.6 and recall the Algorithm 2.4.3. The link L is the union of n periodic orbits corresponding to the words $w_1,\ldots,w_n$. The points $\{1,\ldots,m\}$ correspond 1-to-1 with the truncations $\{s^\gamma\bar{w}_i|i=1,\ldots,n\}$ under the lexicographic ordering. Here $\bar{w}$ means the infinite word $www\ldots$. Let us suppose j is a point in question, say j corresponds to $s^k\bar{w}_\beta=x^\alpha yw'$. Note we must have $\alpha\geq 2$ in order to have $\pi_*(j)<\pi_*{}^2(j)$. Then

PROPOSITION 5.8. *If j corresponds to* $x^\alpha yw$, $\alpha\geq 2$ *then* $\pi_*(j)-j-1=\{($*the number of exponents of the x's which are at least* $\alpha)+($*card B*$)\}$ *where* $B=\{(\gamma,\delta)|s^\gamma\bar{w}_\delta=yw'$ *and* $s^{\gamma+1}\bar{w}_\delta=x^{\alpha-1}yw''<s^{k+1}\bar{w}_\beta$.

Proof of Proposition 5.8. For $\pi_*(j)-j-1$ is the number of the points $\{I,\ldots,m\}$ which are between j and $\pi_*(j)$. Thus we want to compute the number of truncations

$s^{\gamma}\bar{w}_{\delta}$ between $s^k\bar{w}_{\beta}$ and $s^{k-1}\bar{w}_{\beta}$ as these last correspond to j and $\pi_*(j)$. Assume $s^{\gamma}\bar{w}_{\delta} = x^a yw'$; then exactly one of the two truncations $s^{\gamma}\bar{w}_{\delta}$ and $s^{\gamma+1}\bar{w}_{\delta}$ is between $s^k\bar{w}_{\beta}$ and $s^{k+1}\bar{w}_{\beta}$. This accounts for the first summand in the formula 5.8. Similarly, if $(\gamma, \delta) \in B$, then $s^k\bar{w}_{\beta} < s^{\gamma}\bar{w}_{\delta} < s^{k-1}\bar{w}_{\beta}$. These cover the only possible cases for truncations to lie between these 2 and hence completes the proof of this formula. ||

Example. The linking number of the orbits corresponding to $x^a y^b$ and $x^c y^d$, is at least 2, for $a, c \geq 2$. To see this, note that this is a link on 2 strands and formula (a) of Prop. 5.6 applies. Then $\Delta^2 = \sigma_1^2$ contributes 1 to the linking number. Now use 5.8 (ignoring card B) to see that there is another contribution via the fact that some $n_i > 0$.

We close this section with the statement of a partial result toward the conjecture that trip number is in fact the braid index of a Lorenz knot, i.e. the smallest integer m such that K may be represented by an m-braid. To explain what we can prove, we note that by a theorem due to Markov (see [1]) if $\gamma \in B_t$ and $\gamma' \in B_{t'}$ are two braid representations of a knot K, then there is a finite sequence of pairs:

$$(\gamma, t) = (\gamma_2, t_1) \to (\gamma_2, t_2) \to \dots \to (\gamma_s, t_s) = (\gamma', t')$$

such that each $\gamma_i \in B_{t_i}$ represents K and such that for each adjacent pair $(\gamma_i, t_i) \to (\gamma_{i+1}, t_{i+1})$ one of the following holds:

$$\text{(i)}\ \ t_{i+1} = t_i \text{ and } \gamma_{i+1} \text{ is conjugate to } \gamma_i \text{ in } B_{t_i}$$

or

$$\text{(ii)}\ \ t_{i+1} = t_i + 1 \text{ and } \gamma_{i+1} = \gamma_i \sigma \frac{+1}{t_i}$$

or

$$\text{(iii)}\ \ t_{i+1} = t_i - 1 \text{ and } \gamma_i = \gamma_{i+1} \sigma \frac{+1}{t_{i+1}}$$

If (ii) occurs, we say that γ_i is *reducible* to γ_{i+1}.

THEOREM 5.9. *If K is a Lorenz knot of trip number t and if $\gamma \in B_t$ is the positive braid representative produced by the method of Theorem* 5.1, *then γ is not conjugate to a reducible braid.*

Remark. This does not necessarily imply that t is the braid index because it is conceivable that by a sequence of operations of types (i)–(iii) one might achieve a reduction which cannot be achieved by a single conjugation and reduction. However, this is highly unlikely because of the fact that K is fibered and γ is positive.

We omit the proof of Theorem 5.9 because it requires the explanation of techniques not noted elsewhere in this paper, and it is only a partial result.

§6. LORENZ LINKS, ALGEBRAIC LINKS AND ITERATED TORUS LINKS

In this section we discuss the relationship between Lorenz knots and links, algebraic knots and links and iterated torus knots and links. Algebraic knots and links were defined in §1. They will be characterized below as a proper subclass of iterated torus knots and links. If L is an iterated torus link with Alexander polynomial $\Delta(t)$, then the roots of $\Delta(t)$ are roots of unity, however there are links with this property which are not algebraic. The principle results of this section are:

THEOREM 6.1. *Every torus knot is a Lorenz knot.*

THEOREM 6.2. *Let K be a Lorenz knot with crossing number c. Let (a, b') be arbitrary coprime positive integers. Then the type $(a, b' + ac)$-cable on K is a Lorenz knot.*

THEOREM 6.3. *Every algebraic knot is a Lorenz knot. Some algebraic links are Lorenz links.*

THEOREM 6.4. *There are algebraic links (in fact torus links) which are not Lorenz links.*

THEOREM 6.5. *There are Lorenz knots which are iterated torus knots but which are not algebraic. Also, there are Lorenz knots which have Alexander polynomials with roots which are not roots of unity. Such a knot cannot be an iterated torus knot, and in particular it cannot be algebraic.*

Our discussion begins with a review of definitions and the adoption of conventions with regard to the "type" of an iterated torus knot or link. Caution: different authors have used all possible permutations of all possible conventions and there does not seem to be any preferred convention. We follow conventions in Ref. [9].

Let V_1 be a standard solid torus in R^3 with core K_0 and let $l_1 \subset \partial V_1$ be a preferred longitude for v_1, i.e. l_1 bounds a disc in the complement of V_1. A simple closed curve $K_1 \subset \partial V_1$ is a *torus knot* of type (a, b) if K_1 winds a times longitudinally and b times meridionally about V_1, so that the linking number of K_1 with K_0 is b. Let K_2 be a knot in R^3 and let V_2 be a solid torus neighborhood of K_2. Let $l_2 \subset \partial V_2$ be a preferred longitude for K_2. Let h: $V_1 \to V_2$ be a homeomorphism which takes l_1 onto l_2. Then $K_2(a, b) = h(K_1)$ is a *type (a, b) cable about* K_2. Note that if K_2 is oriented, there is a natural induced orientation on $K_2(a, b)$.

For each integer $i = 1, \ldots, r$ let (a_i, b_i) be coprime integers and let K_1 be a torus knot of type (a_i, b_1). An *iterated torus knot K of type* $((a_1, b_1), \ldots, (a_r, b_r))$ is defined inductively by $K_j = K_{j-1}(a_j, b_j)$.

For iterated torus knots, it will be convenient to introduce the *adjusted type* (a_j, b'_j), $1 \le j \le r$, where the b'_j are defined iteratively by

$$b'_1 = b_1, b'_j = b_j - a_j a_{j-1} b_{j-1}, 2 \le j \le r. \tag{6.1}$$

It is well-known that an iterated torus knot is an *algebraic knot* if and only if the adjusted type numbers a_j, b'_j satisfy:

$$a_j > 0, b'_j > 0 \text{ for each } j = 1, 2, \ldots, r. \tag{6.2}$$

Let L be a link with component L_i and let (a, b) be an arbitrary pair of coprime integers. The link $L \cup L_i(a, b)$ is a *toral iteration* of type (a, b) on $L_i \subset L$. The new component $L_i(a, b)$ will lie on the boundary of that component of a solid torus neighborhood of L which contains L_i in its interior. This process may be repeated to give *consecutive toral iterations* on L_i, with the second solid torus neighborhood of L

lying inside the first, also a third inside the second, and so forth. The most general *iterated torus link* is obtained by the following procedure:

(1) Starting with the unknot, perform toral iterations with arbitrary coprime (a, b) at each stage, using any link component previously constructed.

(2) Delete an arbitrary subset of the components.

(3) Reverse the orientation of any subset of the remaining components.

Let $L = \cup_{j=1}^{t} L_j$ be an iterated torus link. Then each component L_j is an iterated torus knot of type $((a_{1j}, b_{1j}), (a_{2j}, b_{2j}), \ldots, (a_{r_j j} b_{r_j j}))$, where the type numbers are determined by the ordered set of toral iterations which led to the construction of L_j. In this scheme it may happen that two of the components, say L_j and L_s, are obtained by steps which coincide up to, say, the kth toral iteration, in which case we will have

$$a_{ij} = a_{is},\ b_{ij} = b_{is},\ 1 \le i \le k.$$

In view of this remark we may associate to each L_j an unambiguous *adjusted type symbol* $((a_{ij}, b_{1j}), (a_{2j}, b_{2j}), \ldots, (a_{r_j j}, b'_{r_j j}))$ where as before the integers b_{ij} are defined iteratively by:

$$b'_{1j} = b_{1j},\ b'_{ij} = b_{ij} - a_{ij}a_{i-1,j}b_{i-1,j},\ 2 \le i \le r_j,\ 1 \le j \le t.$$

Following[9] we have:

(i) Every algebraic link is an iterated torus link.

(ii) Let $L = \cup_{j=1}^{t} L_j$ be an iterated torus link, where the jth component L_j has adjusted type symbol $((a_{1j}, b'_{1j}), (a_{2j}, b'_{2j}), \ldots, (a_{r_j j}, b'_{r_j j}))$. Then L is algebraic if and only if:

(a) the components L_j are coherently oriented.

(b) $a_{ij} > 0, b'_{ij} > 0$ for each $1 \le i < r_j, 1 \le j \le t$.

(c) Suppose two consecutive toral iterations of type (a_{ij}, b'_{ij}), $(a_{i,j+1}, b'_{i,j+1})$ respectively are on a single component of the link at some intermediate stage in the construction. Assume that these have been labeled so that the iteration of type $(a_{i,j+1}, b'_{i,j+1})$ occured first, and the iteration of type $(a_{i,j}, b'_{i,j})$ occurred second (on a concentric torus of smaller radius). Then for any such pair:

$$a_{i,j}/b'_{i,j} \ge a_{i,j+1}/b'_{i,j+1}.$$

Having established conventions, we are now ready to prove Theorems 6.1–6.6, stated earlier. Theorem 6.1 asserts that torus knots are Lorenz.

Proof of Theorem 6.1. Let (p, q) be arbitrary coprime integers. We claim that the permutation π_* on $p + q$ symbols which is defined by

$$\pi_*(i) = i + q, \quad 1 \le i \le p;$$

$$\pi_*(i) = i - p, \quad p + 1 \le i \le p + q.$$

defines a Lorenz knot having the knot type of a torus knot of type (p, q). First note that π_* is a cycle. For, suppose that π_* is a product of k disjoint cycles ($k \ge 1$). Then any one of these cycles contains, say, p' (respectively q') distinct symbols with indices between 1 and p (respectively $p + 1$ and $p + q$), and π_* acts by increasing (respectively decreasing) these indices by q (respectively p). In a cycle the sum of all successive differences is necessarily 0, hence $q'p = p'q$. Since p and q are coprime, this means that $p|p'$ and $q|q'$. Since $p' \leq p, q' \leq q$ we then have $p' = p, q' = q$, hence π_* is a cycle.

To see that π_* defines a type p, q torus knot, examine Fig. 6.1. The first picture shows the defining braid. The second shows the closed braid, embedded on a torus minus a disk, wrapping p times longitudinally and q times medidionally. ||

Remark. The defining word may be recovered from π_* by the method used in the proof of Proposition 3.2. For example $w = x^2yxy$ gives a type $(2, 3)$ torus knot and $w = x^2yx^2yxy$ gives a knot of type $(3, 5)$. A general rule is: to obtain a torus knot of type (p, q), use the Lorenz word w which contains x p times and y q times, spaced out as "evenly" as possible. Thus, the trip number will be the smaller of p, q.

We now show that the class of Lorenz knots is closed under a rather special kind of cabling. (Theorem 6.2). This generalizes Theorem 6.1 because the torus knot of type (p, q) where $q > p$, can be obtained by choosing $a = p, b' = q - p$ and cabling on the unknot, defined by the Lorenz word $w = xy$.

Proof of Theorem 6.2. Let K be an arbitrary Lorenz knot, with defining Lorenz braid β of string index n and crossing number c. Let a_1, b_1' and a_2, b_2' be arbitrary

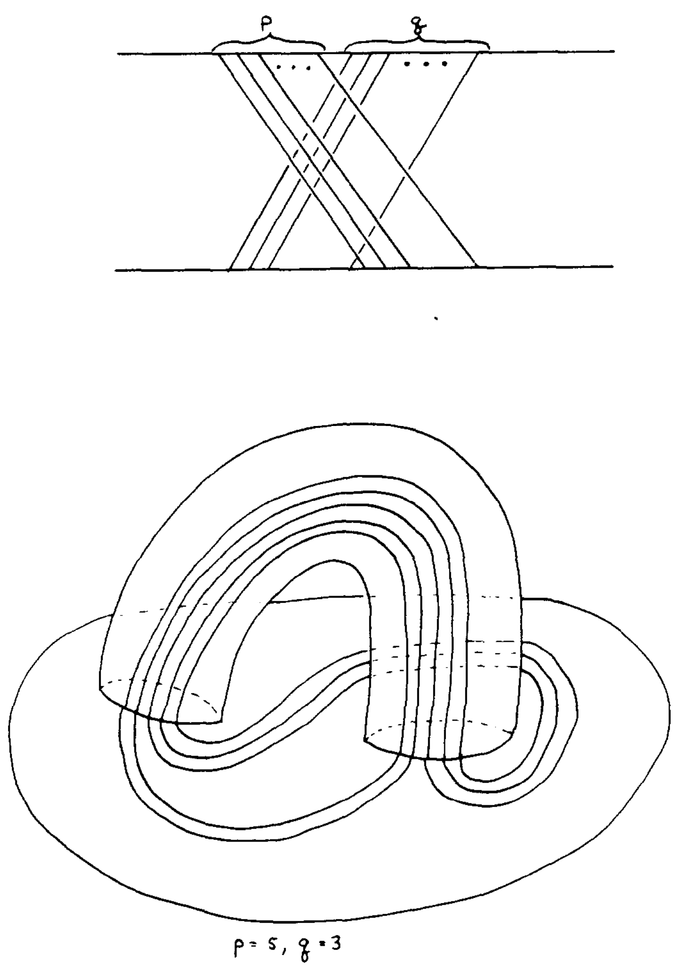


Fig. 6.1.

pairs of coprime positive integers. We will define two new Lorenz braids $\hat{\beta}$ and $\tilde{\beta}$, on $a_1n + b_1'$ and $a_2n + b_2'$ strings respectively, and will show that;

(1) $\tilde{\beta} \cup \beta \cup \hat{\beta}$ is a Lorenz braid,

(2) the associated link $\tilde{K} \cup K \cup \hat{K}$ is Lorenz,

(3) $\hat{K}$ is a type $(a_1, b_1' + a_1c)$-cable on K. $\tilde{K}$ is a type $(a_2, b_2' + a_2c)$-cable on K.

This is somewhat stronger than the result needed to establish Theorem 6.2, but it will be useful later.

It will be helpful to think of β as colored red. We will augment the braid β by adding strings colored green for $\hat{\beta}$ and blue for $\tilde{\beta}$. To obtain $\beta \cup \hat{\beta}$, augment the red braid β as follows:

Step 1. To the right of each red strand of β put down $a = a_1$ parallel green strands. Do this in such a way that each time the ith red strand of β crosses over the jth, the parallel green strands of $\hat{\beta}$ associated to the ith red strand cross over the parallel green strands of $\hat{\beta}$ associated to the jth red strand. (See Fig. 6.2.)

Step 2. At the extreme right of the braid obtained by Step 1 above, add $b' = b_1'$ green strands, all parallel, in such a way that this new group of b' strands crosses under the right-most group of a green strands once, at the bottom of the braid, but does not cross over or under any other green or red strands (see Fig. 6.3).

Thus, if $\pi = (\pi_1, \ldots, \pi_p)$ is the Lorenz permutation which defines β then the Lorenz permutation $\hat{\pi}$ defining $\hat{\beta}$ will have pa entries, given by

$$(a\pi_1 - a + 1, a\pi_1 - a + 2, \ldots, a\pi_1, a\pi_2 - a + 1,$$
$$a\pi_2 - a + 2, \ldots, a\pi_2, \ldots, a\pi_{p-1}, a\pi_p + b' - a + 1, a\pi_p + b' - a + 2, \ldots, a\pi_p + b').$$

We now show that $\hat{\pi}$ satisfies the condition of proposition 3.2. Condition (a) is clearly satisfied; to show that (b) is satisfied it will be adequate to show that $\hat{\pi}$ defines a knot rather than a link, i.e. that $\hat{\pi}$ is a cycle.

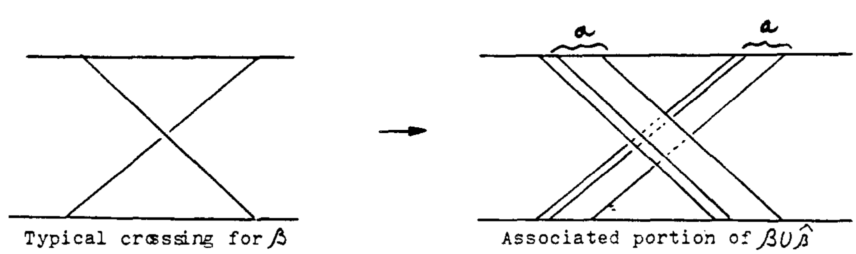


Fig. 6.2.

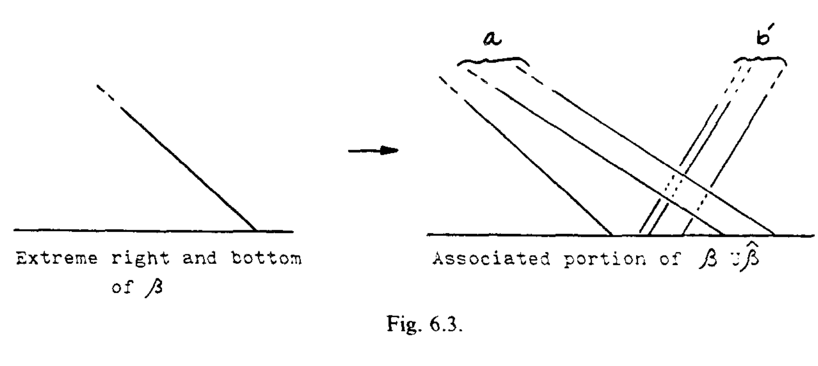


Fig. 6.3.

Focus on the local picture near the right-most two groups of green strands (containing $a + b'$ strands).

First, we examine the right-most group of $a + b'$ strands of $\hat{\beta}$. If the right-most b' strands are identified top and bottom, as in Fig. 6.4, a "curl" is introduced into the braid. The curl causes a permutation in the a strands, and the permutation must be an a-cycle because a and b' are coprime.

Now, the permutation π_* associated to the original braid β was an n-cycle because K is a knot. From this it follows that the n groups of a parallel strands are permuted by $\hat{\pi}_*$ in the same way as were the n strands of β by π. Hence the orbit of any individual group of a parallel green strand (e.g. the left-most group) under $\hat{\pi}_*$ is any other such group. The only place where permutations are introduced within such a group is at the extreme right, and from earlier observations an a-cycle is introduced. Putting these together, we can see that $\hat{\pi}_*$ is a cycle, i.e. the orbit of any one symbol under $\hat{\pi}_*$ is the entire set of $na + b'$ symbols. Hence $\hat{K}$ is a knot.

We next show that $\hat{K}$ can be embedded on the boundary of a solid torus neighborhood X of a knot which has the type of K, winding a times longitudinally and $b' + ac$ times meridionally about the core, where c is the crossing number of K. Figure 6.5 shows $T = (\partial X - \text{disc})$ near the curls of Fig. 6.4 in the case $a = 2, b' = 5$. The remainder of $\mathcal{L}$ embeds in an obvious way as a set of parallel strands on two ribbons. Note that ∂T bounds a disc in $S^3 - \hat{K}$, even when K is knotted. The longitudinal winding number of $\hat{K}$ with respect to X is clearly a and the meridional winding number is $b' + ac$.

Now the entire procedure may be repeated, adding a third braid $\tilde{\beta}$ to the *left* of the original braid β. To do this, observe that the knot holder H in Fig. 2.2 is invariant under a rotation of 180° about the z axis. We construct $\tilde{\beta}$ by rotating H, doing the

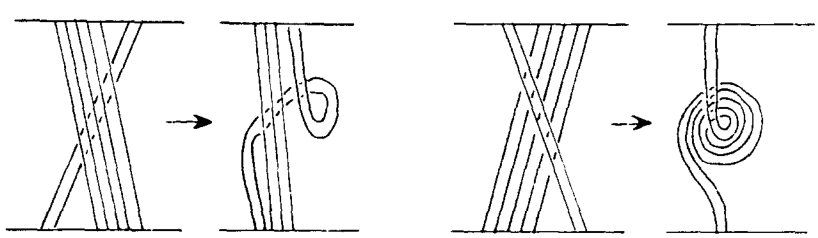


Fig. 6.4.

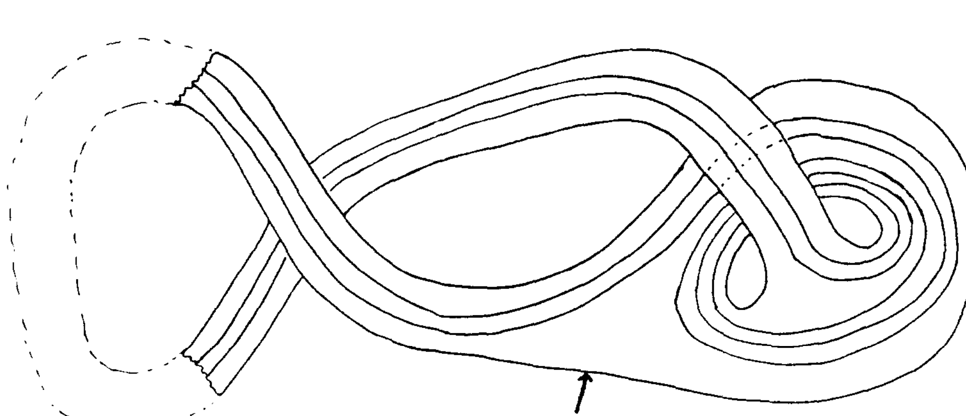


Fig. 6.5.

earlier construction with new coprime integers a_2, b_2', and then rotating back. The braid $\tilde{\beta}$ contains a_2 parallel blue strands to the *left* of each red strand, and an additional group of b_2' parallel strands to the left of all strands already in place, which cross over the left-most group of a_2 blue strands once but do not cross any red or green strands. As before, the associated closed braid $\tilde{K}$ will be a knot. It will be a type $(a_2, b_2' + a_2c)$ cable on K. This completes the proof of Theorem 6.2. ||

To prove that algebraic knots are Lorenz, we next show that the restriction on the cabling coefficients in Theorem 6.2 are in fact precisely those which occur in algebraic knots:

Proof of Theorem 6.3. Let N_r be algebraic, of type $((a_1, b_1), \ldots (a_r, b_r))$ and adjusted type $((a_1, b_1'), \ldots, (a_r, b_r'))$. If $r = 1$ then N_1 is a torus knot, hence by Theorem 6.1 it is Lorenz. Its crossing number is $a_1 b_1' = a_1 b_1$. Assume, inductively, that N_{r-1} is Lorenz, with crossing number $a_{r-1}b_{r-1}$. Since N_r is algebraic, $b_r' > 0$. By Theorem 6.2 the $(a_r, b_r' + a_r a_{r-1} b_{r-1}) = (a_r, b_r)$ cable on N_{r-1} is Lorenz, with crossing number $a_r b_r$. Hence N_r is Lorenz.

To see that some algebraic links are Lorenz links note that in the proof of Theorem 6.2 we not only showed that for each Lorenz knot K the type $(a, b' + ac)$-cable on K was Lorenz, but even more that there is a Lorenz link having 3 components $\tilde{K} \cup K \cup \hat{K}$ where $\tilde{K}$ and $\hat{K}$ are type $(a_1, b_1' + a_1c)$ and type $(a_2, b_2' + a_2c)$ cables on K. Choose $b_1' = b_2' = 1$ and $a_1 = a_2 = 1$. Then $\tilde{K}$ and $\hat{K}$ may be constructed on concentric tori with K as core. If K was algebraic, then $\tilde{K}$ and $\hat{K}$ are also algebraic. Now use Theorem 6.3 to cable further on any of $\tilde{K}$, K, $\hat{K}$. In this way we obtain a family of rather special algebraic links which are Lorenz. ||

Proof of Theorem 6.4. We will show that if $n \geq 4$ the torus link of type (n, n) is not a Lorenz link.

Note that the n components in such a link are all unknotted and that any two link one-another with linking number 1. Our task will be to show that there is no Lorenz link with these properties if $n \geq 4$.

For assume the contrary. Recall that the trip number is the number of syllables in a Lorenz word. Then the four (or more) orbits correspond to words of the form $x^{a_i}y^{b_i}$ as these are the only unknotted orbits other than the trivial ones, since by Corollary 5.3 each componant has trip number 1, but the trivial ones link nothing. Then either (1) $a_i \geq 2$ for two i's or (2) $b_i \geq 2$ for two i's; by symmetry in x and y we need only test case 1. But in this case the corresponding orbits have linking number ≥ 2 by the example after the proof of Proposition 5.8. ■

Example. The only triples corresponding to the (3.3) torus link are of the form $x^a y$, xy and xy^b, with $a, b \geq 2$.

Proof of Theorem 6.5. Let K_1 be the Lorenz knot which is defined by the Lorenz permutation $\pi = (4, 6, 7, 8, 9, 10)$, or alternatively by the Lorenz word $w = x^2yx^2yxyxy$. It has genus 7 and it may be identified as a type (2.3, 2.11) interated torus knot. The adjusted type symbol b_2' defined by eqn (6.1) does not satisfy the inequality of eqn (6.2), hence K_1 is not algebraic.

To establish the second assertion of Theorem 6.5 let K_2 be the Lorenz knot which is defined by the Lorenz permutation (6, 9, 10) or by the Lorenz word $w = xyxy^3xy^3$. Its Alexander polynomial is $p(t) = 1 - t + t^3 - t^4 + t^5 - t^6 + t^7 - t^9 + t^{10}$. Then $p(\mathcal{O}) = 1$, $p(-1) = -1$; hence $p(t)$ has a root between $t = 0$ and $t = -1$. Such a root cannot be a root of unity.

Remark 1. The knot K_2 which is defined by $\pi = (6, 9, 10)$ has a Lorenz projection defined by the Lorenz braid in Fig. 3.1. (Compare Fig. 1.1) A different projection is given in Figure 6.6. It belongs to a well-known family of so-called "pretzel" knots. This one has type $(-3, -7, +2)$. Other pretzel knots which are Lorenz are of type $(-3, -2n-1, 2)$ determined by $\pi = (2n, 2n+3, 2n+4)$ or $w = xyxy^nxy^n$.

Remark 2. John Morgan has asked whether hyperbolic knots occur in the flow associated to eqns (1–3)? It is known (see [3]) that a pretzel knot of type (a, b, c) is hyperbolic provided $1/a + 1/b + 1/c \neq 1/m$ for some integer m. The pretzel knots of type $(-3, -2n-1, 2)$ then include many examples of hyperbolic knots which are Lorenz knots.

§7. LORENZ LINKS ⊂ POSITIVE BRAIDS

We have already observed (and made use of) the fact that every Lorenz link may be represented as a closed positive braid. This fact enabled us, for example, to prove that all Lorenz links are fibered and to derive a formula for the genus (Theorem 5.2). A natural question to ask then, is whether every knot defined by a closed positive braid is a Lorenz knot? We answer this in the negative by establishing:

PROPOSITION 7.1. *The granny knot is a closed positive braid, but is not a Lorenz knot.*

Proof of Proposition 7.1. Figure 7.1 shows the granny knot, represented by the closed positive 3-braid $\beta = \sigma_1^3\sigma_2^3$. Its genus is 2, by a well-known formula or by theorem 6.2, with $c = 6$, $n = 3$.

However, a Lorenz knot of genus 2 must have trip number 2 by the formula $2g \leq t(t-1)$. By Corollary 5.3 such knots are torus knots of type $(2, 2m+1)$. As the genus of such a knot is m, we have $m = 2$ which proves that a Lorenz knot of genus 2 can only be a type $(2, 5)$ torus knot, distinct from the granny.

§8. ZETA-FUNCTIONS

In an earlier paper[36, 37] a type of non-abelian zeta-function was introduced, and it was used to distinguish (topologically) different Lorenz attractors. These occur if the coefficients in eqn (1,3) are changed. Since we are concerned with the boundary

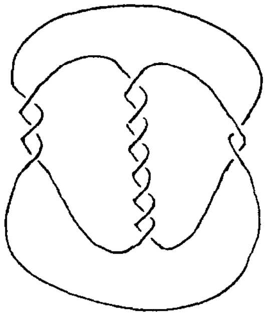


Fig. 6.6.

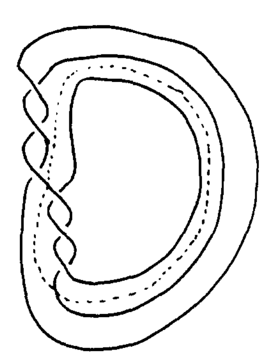


Fig. 7.1.

line case here, this function is very simple:

$$\eta(x, y) = \sum_{i=1}^{\infty} \frac{tr\begin{pmatrix} x & x \\ y & y \end{pmatrix}^i}{i}$$

$$= x + y + \frac{x^2 + xy + yx + y^2}{2} + \dots.$$

If one abelianizes one gets

$$\zeta(x, y) = \exp \eta(xy) = \frac{1}{1 - x - y}$$

and

$$\zeta(t, t) = \frac{1}{1 - 2t}.$$

This last is familiar as the zeta function of the full 2-shift. We want to include more knot-theoretic information in such "zeta-type" functions. We would really like to formulate a result which would tell us as much about what knots occur (say for Bowen–Parry flows which lie in S^3–see below) as the

THEOREM (*Bowen–Lanford* [2], *Parry* [27]: *The zeta-function of the subshift of finite type determined by a* 0–1 *matrix* A *is* $[det(I - tA)]^{-1}$. *That is to say, for such a subshift* s *the only sequences* $(N_1, N_2, N_3, \dots)$, $N_i = \text{card}(\text{fix } s^i)$, *which occur, are those of the form* $(tr\,A,\ Tr\,A^2,\ tr\,A^3, \dots)$ *for some square integral matrix* A.

We introduced the term *Bowen–Parry flow* in §2 to mean a *suspension of a sub-shift of finite type.* Much is known about these flows, and Rufus Bowen and Bill Parry are responsible for a large part of this. There is also the pretty result of Franks [11] which relates the Alexander polynomial of a periodic attracting orbit to other dynamical information about flows on S^3.

Knot Problem. Classify the families K of knots which occur as the set of all periodic orbits of a Bowen–Parry flow in S^3.

We note that the set of all Bowen–Parry flows in S^3 is countable, so that there certainly *are* restrictions on such families (reminiscent of the fact that there are only countably many rational zeta-functions.) The first step toward codifying such information is to find formulas for such things as N_{ig} (say), where

$$N_{ig} = \neq \text{ of periodic orbits of word-length } i \text{ and genus } g.$$

Along these lines we have two contributions. First, if we count the trip number t instead of genus g, we have a good theory (Theorem 5.2). This is related to computing N_{ig} as $2g \geq t(t-1)$; furthermore, we conjecture (§5, just before the proof of 5.1) that the trip number t is a knot invariant. Secondly, we have an exact formula (see Prop 5.6) for computing the knot or link as a braid on t-strands. This yields in particular a formula for the genus. We proceed to formulate the trip number-period "zeta" functions.

Let $N_{n,i} = \text{card}\ \{\Lambda | \Lambda \text{ is a periodic orbit of word-length } n \text{ and trip number } i\}$. Then for the (full) Lorenz flow the following formulas are proved below:

$$(*)\quad \sum_i N_{n,m}t^n s^i = tr\begin{pmatrix} t & t \\ ts & t \end{pmatrix}^n$$

and more generally

$$(**)\quad \sum W_{n,m}s^i = tr\begin{pmatrix} x & y \\ xs & y \end{pmatrix}^n$$

where $W_{n,m} \in Z[x, y]$ is the "sum" of all words w in the free monoid in x and y such that the corresponding orbit $\Lambda = \Lambda(w)$ has trip number i.

There are several conventions: frst, though x and y don't commute, s commutes with both of them. Secondly, for n composite, we include in $W_{n,m}$ all ("periodic") words of the form w^α where $\alpha(\text{degree } w) = n$.

Thus

$$\eta(x, y, s) = \sum_{n=1}^{\infty} \frac{1}{n}\, tr\begin{pmatrix} x & y \\ xs & y \end{pmatrix}^n.$$

In particular, the fifth term of this series includes (1/5) times the following $x^5, x^2yx^2s, x^3y^2s, x^2yxys^2, xyxyxs^2$, etc. Here s measures the number of syllables (counted/cyclically) or equivalently, the trip number of the corresponding Lorenz orbit.

THEOREM 8.1. *The period-trip number η function for the (full) Lorenz system is given by*

$$\eta(x, y, s) = \sum_{n=1}^{\infty} \frac{1}{n}\, tr\begin{pmatrix} x & y \\ xs & y \end{pmatrix}^n; \text{ or abelianizing,}$$

$$\zeta_H(x, y, s) = \exp \eta_H(x, y, s) = 1/\det\begin{pmatrix} 1-x & -y \\ -xs & 1-y \end{pmatrix} = 1/[1 - x - y + xy(1-s)].$$

$$\zeta(t, s) = \zeta(t, t, s) = 1/[1 - 2t + t^2(1-s)].$$

$$\zeta(t) = \zeta(t, 1) = \frac{1}{1-2t}.$$

Proof. The only things new here are the formulas (*) and (**) and the latter in turn implies the former. But (**) is conceptually just like many symbolic dynamics formulas: there are 2 "windows" or partitions, one labeled x, the other y. The matrix (with composition on the left) says that we can go from either window to the other, and record the window we have just left. In addition, one s is included for each passage from x to y. Then a monomial $x^{n_1}y^{m_1}\ldots x^{n_i}y^{m_i}s^t, x^{n_1}y^{m_1}\ldots x^{n_i}y^{m_i}x^a s^t$, or $y^{m_1}x^{n_2}\ldots x^{n_i}y^{m_i}x^{n_1}y^b s^t$ which shows up in $tr\begin{pmatrix} x & y \\ xs & y \end{pmatrix}^n$ will have to have $i = t$, as a factor of s is added only upon transition from x to y. One can check this in each of the three possible types of monomials above. Note that counted cyclically, each of these has exactly t syllables, i.e. t subwords of the form $x^{n_i}y^{m_i}$, since $i = t$. ■

Remarks 1. This trip-number counter s can just as easily be built into the η

functions of [36, 37]; a typical finite matrix would be

$$\begin{pmatrix} 0 & xs & 0 & 0 \\ 0 & 0 & xs & xs \\ y & 0 & 0 & 0 \\ 0 & y & y & 0 \end{pmatrix}$$

Remark 2. We conjecture this result holds for any Bowen–Parry flow. More precisely, let φ_t be such a flow on S^3 and let S^1 be any chosen braid axis. For Λ a periodic orbit of φ_t we attach two numbers, $n(\Lambda)$ = the linking number of Λ with the braid axis and $b(\Lambda)$ = the braid index of Λ. Let $N_{n,b} = \text{card}\ \{\Lambda | n(\Lambda) = n,\ b(\Lambda) = b\}$, where appropriate conventions for retracing orbits are followed. Then

CONJECTURE.

$$\zeta(t, s) = \exp \sum_{n=1}^{\infty} \frac{1}{n} N_{n,b}\, t^n s^b$$

is a rational function of s and t.

We expect similar results for $\eta(x, y, s)$ and $\eta_H(x, y, s)$.

§9. KNOT GROUPS AND ALEXANDER INVARIANTS.

In this section we will give an algorithm for calculating a presentation for $\pi_1(S^3 - L)$ when L is a Lorenz link. We also give an algorithm for computing an Alexander matrix for L when L is a knot (a slight modification giving a similar result for links). The Alexander polynomial and also the generator for the Alexander ideals can be calculated without difficulty from this matrix. A second method is given in §10.

Both of these algorithms start with a defining permutation and proceed mechanically to enable the rapid calculation of individual examples, although the more general question of characterizing the groups or the polynomials seems very difficult and beyond the scope of this paper. Indeed, since Lorenz knots are a generalization of algebraic knots, it seems fairly clear that this is a highly non-trivial problem.

Remark. The formulas which define Lorenz knots as closed t-braids may be useful here.

Let L be a Lorenz link, and suppose that L is represented by a Lorenz braid with n strings, p of which are overcrossing strings and q of which are undercrossing strings. Let $\pi = (\pi_1, \ldots, \pi_p)$ by the defining Lorenz permutation and let π_* be the associated permutation of the symbols $(1, 2, \ldots, n)$. Then π_* has the form:

$$i \to \pi_i\ (1 = 1, \ldots, p);\ p + j \to \delta_j\ (j = 1, \ldots, q).$$

Here $n = \pi_p = p + q$. Let $\delta = ((\delta_1, \ldots, \delta_q)$.

THEOREM 9.1. *For each integer j ($j = 1, \ldots, q$) let μ_j denote the smallest integer among the set of integers $\{\pi_1, \ldots, \pi_p\}$ which is larger than δ_j. Such an integer always exists because $\pi_p = p + q > \delta_j$ for all $j = 1, \ldots, q$. Let $G = \pi_1(S^3 - L)$. Then G admits the presentation:*

$$\text{generators: } x_1, x_2, \ldots, x_{p+q}$$

defining relations:

$$x_i = x_{\pi_i}, i = 1, \ldots, p$$

$$x_{p+j} = (x_{\pi_p}^{-1} x_{p-1}^{-1} \ldots x_{\mu_j}^{-1}) x_{\sigma_j} (x_{\pi_{\mu_j}} \ldots x_{p-1} x_{\pi_p})$$

$$j = 1, \ldots, q.$$

In this presentation, any one relation is a consequence of the others.

Example. $\pi = (4, 5, 7, 8, 9)$. Here $p = 5$, $q = 4$, $n = 9$. Then $\delta = (1, 2, 3, 6)$, so that $\mu_1 = 44$, $\mu_2 = 4$, $\mu_3 = 4$, $\theta_4 = 7$. The group G has the presentation

generators: $x_1, \ldots, x_9$

defining relations:

$$\begin{array}{ll} x_1 = x_4 & x_6 = x_9^{-1} x_8^{-1} x_7^{-1} x_5^{-1} x_4^{-1} x_1 x_4 x_5 x_7 x_8 x_9 \\ x_2 = x_5 & x_7 = x_9^{-1} x_8^{-1} x_7^{-1} x_5^{-1} x_4^{-1} x_2 x_4 x_5 x_7 x_8 x_9 \\ x_3 = x_7 & x_8 = x_9^{-1} x_8^{-1} x_7^{-1} x_5^{-1} x_4^{-1} x_3 x_4 x_5 x_7 x_8 x_9 \\ x_4 = x_8 & \\ x_5 = x_9 & x_9 = x_9^{-1} x_8^{-1} x_7^{-1} x_6 x_7 x_8 x_9. \end{array}$$

Proof of Theorem 9.1. It will be assumed that the reader is familiar with methods for finding presentations of knot groups. A good reference is [10 or 28]. The generators $x_1, \ldots, x_{p+q}$ in the presentation of Theorem 9.1 are represented by loops which encircle and seperate the braid strings, in order, at the top of the braid. The first p of these generators may be slid along the overpasses to the bottom of the braid without interference. After identifications are completed we then obtain the relations $x_1 = x_{\pi_i}$ ($i = 1, \ldots, p$), which simply say that we named too many generators.

The remaining relations arise at the crossing encountered as one attempts to slide along an underpass. At each crossing-point there is a relation which corresponds to conjugarion by the generator associated to the overpass. The generator x_{p+j} ($j = 1, \ldots, q$) at the top becomes $(x_{\pi_p}) x_{p+j} (x_{\pi_p}^{-1})$ after passing the first crossing, $(x_{\pi_{p-1}} x_{\pi_p}) x_{p+j} (x_{\pi_p}^{-1} x_{p-1}^{-1})$ after passing the second crossing and so forth down to the bottom of the braid, the last conjugation being by x_{π_j}. In this way we obtain, at the very bottom, the relation

$$x_{\delta_j} = (x_{\mu_j} \ldots x_{\pi_{p-1}} x_{\pi_p}) x_{p+j} (x_{\pi_p}^{-1} x_{\pi_{p-1}}^{-1} \ldots x_{\mu_j}^{-1}),$$

because after identifying the top and bottom of the braid strings the loop representing x_{δ_j} may be slid around the loop from the top to the bottom of the braid, whence it is seen to be equal to the loop obtained by the repeated conjugation. ||

THEOREM 9.2. *Let* $\pi = (\pi_1, \ldots, \pi)$ be a Lorenz permutation which defines a Lorenz knot K. Construct the following $\pi_p \times \pi_p$ *matrix* $B(t)$ *over the group ring* ZT (*of Laurant polynomials over the integers* $\mathbb{Z}$):

(1) *For each* $i = 1, \ldots, p$ *enter* $t^{\pi_i - i}$ *in the* ith *row*, π_ith *column.*

(2) *For each* $i = 1, \ldots, p$ *enter* 0 *in the* ith *row*, jth *column for each* $j > \pi_i$.

(3) *For each* $i = 1, \ldots, p$ *complete the* π_ith *column by filling in zeros in every empty space.*

(4) *For each* $i = 1, \ldots, p$ complete the *i*th *row by inserting the entries* $(1-t)$, $(t^2-t^3), \ldots, t^{\pi_{\Gamma}-i-1} - t^{\pi_{\Gamma}-i}$, *in order, starting at the left, and spacing out to avoid any zeros already present from steps* 1–3.

(5) For each $i = p+1, \ldots, \pi_p$ *enter a unit vector in the* ith *row, with the unit entry in row* $p+j$ *in the* δ_jth *column.*

Let $B(t)$ *be the matrix so-obtained.*

The Alexander polynomial of K is the determinant of any $(\pi_p - 1) \times (\pi_p - 1)$ minor of the matrix $B(t) - I$ constructed above.

Examples. We give two examples.

Example 1. Let $\pi = (3, 4, 5)$. (The Lorenz knot is the trefoil.) Then $\delta = (1, 2)$. The matrix $B(t) - I$ is the 5×5 matrix:

$-t$	$t-t^2$	t^2	0	0
$1-t$	$t-t^2-1$	0	t^2	0
$1-t$	$t-t^2$	-1	0	t^2
1	0	0	-1	0
0	1	0	0	-1

The Alexander polynomial is the determinant of any 4×4 minor, e.g. delete the third row and second column to obtain

$$\Delta_K(t) = \begin{vmatrix} -t & t^2 & 0 & 0 \\ 1-t & 0 & t^2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = \begin{vmatrix} -t & t^2 & 0 \\ 1-t & 0 & t^2 \\ 1 & 0 & -1 \end{vmatrix} = t^2(1-t+t^2)$$

Example 2. This example illustrates all of the essential features of the algorithm. We take $\pi = (4, 6, 9, 11)$. Then $\delta = (1, 2, 3, 5, 7, 8, 10)$. The matrix $B(t) - I$ is

$-t$	$t-t^2$	t^2-t^3	t^3	0	0	0	0	0	0	0
$1-t$	$t-t^2-1$	t^2-t^3	0	t^3-t^4	t^4	0	0	0	0	0
$1-t$	$t-t^2$	t^2-t^3-1	0	t^3-t^4	0	t^4-t^5	t^5-t^6	t^6	0	0
$1-t$	$t-t^2$	t^2-t^3	-1	t^3-t^4	0	t^4-t^5	t^5-t^6	0	t^6-t^7	t^7
1	0	0	0	-1	0	0	0	0	0	0
0	1	0	0	0	-1	0	0	0	0	0
0	0	1	0	0	0	-1	0	0	0	0
0	0	0	0	1	0	0	-1	0	0	0
0	0	0	0	0	0	1	0	-1	0	0
0	0	0	0	0	0	0	1	0	-1	0
0	0	0	0	0	0	0	0	0	1	-1

Proof of Theorem 9.2. The matrix $B(t) - I$ which we have calculated is the matrix $\|(\partial r_i/\partial x_j)^{\varphi}\|$, where the x_j's and r_i's are generators and relations in the presentation of G given in Theorem 9.2, and $(\partial r_i/\partial x_j)^{\varphi}$ is the "free derivative" (see [10]) of r_i with respect to x_j, evaluated at the abelianizer of the knot group G. ||

§10. SEIFERT SURFACES, SEIFERT MATRICES, AND SIGNATURES OF LORENZ KNOTS

Recall that if K is a knot in S^3, a *Seifert surface* for K is an orientable surface $M \subset S^3$ with $\partial M = K$. Let $\dot{M} \times [1, 1]$ be a bicollar for $\dot{M}$ in $S^3 - K$. Let $x \in H_1(\dot{M})$ be represented by a 1-cycle in $\dot{M}$, and let x^+ denote the 1-cycle which is carried by $x \times \{1\}$ in the bicollar. Let f: $H_1(\dot{M}) \times H_1(\dot{M}) \to Z$ be defined by $f(x, y) = 1k(x^+, y)$. Let $e_1, \ldots, e_{2g}$ be a basis for $H_1(\dot{M})$ as a Z-module. Then the $2g \times 2g$ integer matrix $V = \|f(e_i, e_j)\|$ is a *Seifert matrix* for K. The signature $\sigma(K)$ of K is the signature of $(V + V^{tr})$. It is a well-known knot invariant, in particular, any knot which is null-cobordant has signature 0.

The purpose of this section is to give a procedure which allows the calculation of a Seifert matrix for any Lorenz knot.

Using this construction, Rudolph[29] has proved the interesting result that closed positive braids have positive signature. As a consequence,

THEOREM 10.1 [29]. *Non-trivial Lorenz links have positive signature. In particular, this implies that Lorenz links are non-amphicheiral, i.e. there is no orientation-reversing homeomorphism of S^3 which preserves any Lorenz link setwise.*

Let K be a Lorenz knot, and let γ be the positive braid in the t-string braid group, where t = trip number, which represents K (see §5 and Proposition 5.6). We construct a Seifert surface for K.

Consider a collection of n discs which are arranged in a stack as in Fig. 10.1 and joined by half-twisted bands as in Fig. 5.1 or equivalently by hooks as in Fig. 10.1, where a hook joining the ith and $(i+1)$st disc is associated to the appearance of the braid generator σ_i in γ, also distinct hooks appear in distinct horizontal layers and are ordered from top to bottom to correspond to the ordering of the σ_i's in γ. The collection of discs with connecting hooks is clearly a Seifert surface for K. We denote it by the symbol M.

Now choose a base point p_i, $1 \le i \le n$, at the top of each disc as in Fig. 10.1. Then a 1-dimensional spine for M has vertices $p_1, \ldots, p_n$ and oriented arcs $\alpha_1, \ldots, \alpha_r$ which join adjacent p_i's. If $= \sigma_{\mu_1}\sigma_{\mu_2} \ldots \sigma_{\mu_r}$, then the oriented arc α_j associated to σ_{μ_j} joins the point p_{μ_j} to p_{μ_j+1} after passing over the hook associated to σ_{μ_j} once. There is a unique way to do this, constructing the α_j once. There is a unique way to do this, constructing the α_j's in order, so that distinct α_j's intersect only in the vertices (see Fig. 10.1).

From the construction for M, one sees immediately that a basis for $H_1(\dot{M})$ is the set of those differences $\alpha_i - \alpha_j$ for which $j > i$, $\mu_j = \mu_i$ and if $i < k < j$ then $\mu_i \neq \mu_k$.

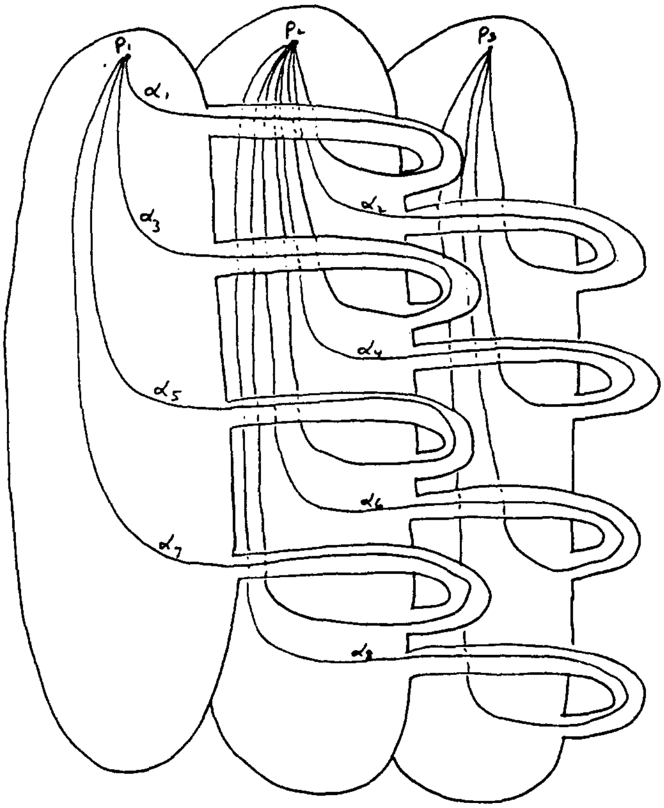
Label the basis elements $e_1, \ldots, e_{2g}$ in any order. Each e_i is a difference $\alpha_u - \alpha_v$ as noted above. Define

$$\alpha_i \cdot \alpha_j = 1 \text{ if } i = j \text{ or } i > j \text{ and } \mu_i = \mu_j$$

$$= -1 \text{ if } i > j \text{ and } \mu_i = \mu_j + 1$$

$$= 0 \text{ otherwise.} \tag{10.1}$$

$$f(e_i, e_j) = (\alpha_u - \alpha_v) \cdot (\alpha_w - \alpha_z). \tag{10.2}$$



$\gamma = (\sigma_1 \sigma_2)^4$

$$\alpha_i \cdot \alpha_j = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

$$\begin{aligned} e_1 &= \alpha_1 - \alpha_3 \\ e_2 &= \alpha_3 - \alpha_5 \\ e_3 &= \alpha_5 - \alpha_7 \\ e_4 &= \alpha_2 - \alpha_4 \\ e_5 &= \alpha_4 - \alpha_6 \\ e_6 &= \alpha_6 - \alpha_8 \end{aligned}$$

$$V = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & +1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Fig. 10.1.

Then a Seifert matrix for K is seen to be:

$$V = \|f(e_i, e_j)\|. \tag{10.3}$$

The signature of K (see Definition 4, p. 217 of [28]) is:

$$\mathcal{O}(K) = \text{Signature of } (V + V^{tr}). \tag{10.4}$$

Remark. In §9 we gave an algorithm for the computation of the Alexander polynomial $\Delta_K(x)$ of a Lorenz knot K. We now observe that K can also be computed from the matrix V of formula (10.3) above, since (see Theorem 3, p. 207 of [28]):

$$\Delta_K(x) = \det(V^{tr} - xV). \tag{10.5}$$

The matrix of formula 10.5 above is smaller than that obtained from Theorem 9.2, but the latter is sparser.

An example of the calculation is given in Fig. 10.1.

§11. QUESTION, CONJECTURES, SPECULATIONS.

The following proposition will be proved in our next paper:

PROPOSITION. *Given a flow ϕ_t on a 3-manifold M having a hyperbolic structure on the chain recurrent set, there is a branched 2-manifold H with semi-flow ϕ_t, $t > 0$, on H embedded in M such that the periodic orbits of $\bar{\phi}_t$ correspond (with a few specified exceptions) 1-to-1 with those of ϕ_t. On any finite subset of the periodic orbits the correspondence can be taken to be via isotopy.*

Anyone familiar with the techniques of [12, or 38] will see how this is proved. Meanwhile, we will call H, ϕ_t a knot holder for M, ϕ_t. The next figure is a knot holder for the most obvious suspension of the Smale horseshoe [39].

We introduced the term *Bowen–Parry flow* earlier to mean a suspension of a subshift of finite type which has a dense orbit.

CONJECTURE 11.1. *Let M, ϕ_t satisfy the condition of the proposition just stated. Then*

(1) *with finitely many exceptional ("trivial") orbits deleted, the remaining periodic orbits are unsplittable.*

(2) *ϕ_t has infinitely many distinct knot types as periodic orbits.*

A bit more vaguely, we conjecture that every ergodic flow in S^3 has knotted periodic orbits.

Question 11.2. The Lorenz equations, as noted in section 1, have been associated to the phenomenon of turbulence, an apparent chaotic movement of a viscous incompressible fluid. What, if anything, is the physical significance of the knotting of the periodic orbits?

Question 11.3. We may define infinitely many simple variations on the Lorenz attractor: (a) replace positive crossings with negative. (b) Add some number of full twists in either band. Do these modified attractors occur in other, closely related equations? What must be done to the flow pattern to change the vector field in this way? The knot holder for a suspension of the Smale horseshoe noted earlier is a related phenomenon.

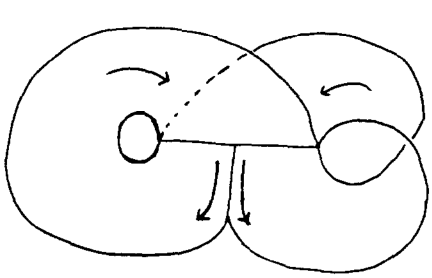


Fig. 11.1.

Problem 11.4. Characterize the groups of Lorenz links; characterize the polynomials of Lorenz knots. (These are probably very difficult problems.)

Problem 11.5. Is there some natural meaning to the fibrations associated to Lorenz links? This is a very difficult question, because the knot-holder H, which was crucial to our analysis, is a tool for the study of the periodic orbits rather than an actual geometric subset of R^3 which is associated to the flow pattern. It would seem as if there is a "fibration" of $(S^3 - H)$ which is obtained as a limit of the infinitely many fibrations of $S^3 - L$, where L ranges over the finite sublinks of L^*, however we have been unable to pursue this idea in view of the difficulties noted above.

CONJECTURE 11.6. *The trip number t of a Lorenz knot is equal to its braid index.* (*see Theorem* 5.6 *and the remarks which follow it*).

CONJECTURE 11.7. *Lorenz knots are prime.*

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