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GEODESICS WITH BOUNDED INTERSECTION NUMBER ON SURFACES ARE SPARSELY DISTRIBUTED†

JOAN S. BIRMAN and CAROLINE SERIES

INTRODUCTION

LET M be a surface of negative Euler characteristic, possibly with boundary, which is either compact or obtained from a compact surface by removing a finite set of points. Let D be the Poincaré disc. Choose any representation of M as U/Γ, where $U \subseteq D$ is the universal covering space of M and $\Gamma \subset \text{Isom}(D)$. Then the Poincaré metric on D induces a metric of constant negative curvature on M and geodesics in U project to geodesics on M. A geodesic on M is said to be *complete* if it is either closed and smooth, or open and of infinite length in both directions. Complete geodesics coincide with those which never intersect ∂M. Note that if M is obtained from a compact surface by removing a finite number of points to form cusps then a complete open geodesic on M might tend toward infinity along a cusp.

In this paper we study the family G_k of complete geodesics which have at most k transversal self-intersections, $k \geqslant 0$. Our main results are:

THEOREM I. *For each $k \geqslant 0$, the set S_k of points of M which lie on a geodesic $\gamma \in G_k$ is nowhere dense and has Hausdorff dimension one.*

THEOREM II. *Let $T_k \subseteq (\partial D \times \partial D - \text{diagonal})$ be the set of pairs of points which represent endpoints of geodesics in the Poincaré disc D whose projection on M is in G_k. Then T is nowhere dense and has Hausdorff dimension zero.*

THEOREM III. *Let U_k be the set of tangent vectors in the unit tangent bundle $T_1 M$ which project onto tangents to geodesics in G_k. Then U_k is nowhere dense and has Hausdorff dimension one, with respect to any natural choice of metric on $T_1 M$.* (For example, the metric obtained by projection to $T_1 M$ of the metric $\sqrt{d_1^2 + d_2^2}$ on $T_1 D = D \times S^1$, where d_1 is the hyperbolic metric and d_2 is arc length on S^1.)

The geodesics in G_0 are *simple.* A moment's reflection will convince one that, trivially, incomplete simple geodesics cover M, hence the restriction of Theorem I that geodesics be complete.

Note that Theorem I is in striking contrast to the analogous situation in the Euclidean case. Let T be a torus represented as $\mathbb{R}^2/Z \oplus Z$. The Euclidean metric on the universal covering space $\mathbb{R}^2$ induces a metric of curvature zero on T, and straight lines on $\mathbb{R}^2$ project to geodesics on T. If l is a line on $\mathbb{R}^2$, then its image on T will always be complete and simple, and will be closed if and only if l has rational slope. Thus $S_0 = T$, in fact through each point $x \in T$ there are infinitely many complete simple geodesics of both finite and infinite length. However, if one removes a point from T one obtains a surface of negative Euler characteristic and Theorem I applies.

Note also that $S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots$ and that $\bigcup_{k=1}^{\infty} S_k$ is a dense subset of M, in fact the union of all points which lie on a complete *closed* geodesic is dense in M.

Theorem I answers a question of Jorgensen [3] as to whether S_0 has measure zero, and also Abikoff's question [3] as to whether S_0 is dense.

A constant which depends only upon the choice of a fundamental domain for the action of Γ will be said to be *universal*. The main work in proving Theorems I, II, III is to establish:

† This paper is a substantial revision of [1].

PROPOSITION 4.1. *There exist universal constants L, C, $\alpha > 0$ and polynomials $P_k(x)$ such that for each $n \in \mathbb{N}$ there is a set F_n of simple geodesic arcs, each of length at most L, so that $card(F_n) \leq P_k(n)$ and so that*

$$S_k \subseteq \cup \{B_\varepsilon^H(\gamma)/\gamma\varepsilon F_n\}, \quad \varepsilon = ce^{-\alpha n}$$

where $B_\varepsilon^H(\gamma)$ is a tubular neighbourhood of γ of hyperbolic radius $\varepsilon > 0$.

The key idea is to parameterize simple geodesic arcs by a finite set of integers in such a way that two arcs with the same parameterization have lifts to the universal cover $U \subseteq D$ which are exponentially close together. Our parameterization is not unlike the Dehn–Thurston parametrization for simple closed curves, see [2]. However, the Dehn–Thurston parameters are not appropriate for our work, because we are interested in the unique *geodesic* representative of the isotopy class of a closed curve, whereas the Dehn–Thurston parameters require that one choose a representative which passes in a particular way through the regions of a pants-annuli decomposition of the surface. Such representatives may be very far from being geodesics.

In fact it is clear that there are a number of different but related ways to parameterize simple closed curves. In an earlier version of this paper the authors used yet another method which is uniquely adapted to the representation of a curve as a shortest word in a given set of generators of $\pi_1(M)$, [1].

Theorems I and II show that the points in S_0 are special points of M. A slight extension of a result of Jorgensen [4] shows that some points in S_0 are *very* special:

THEOREM IV. *Let α, β be closed simple geodesics on M which intersect exactly once, at $x \in M$. Then infinitely many closed simple geodesics pass through x.*

Here is an outline of the paper. In §1 we establish some estimates which relate the Euclidean and hyperbolic metrics. These will be needed later in the paper. The next two sections, §2, §3, are directed at the proof of Proposition 4.1, which is established in §4. In §5 we prove Theorems I, II and III. In §6 we prove Theorem IV. The final section, §7, concerns open questions.

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§1. HYPERBOLIC GEOMETRY

Through sections 1–4, M is a fixed compact Riemann surface of curvature -1. The universal cover U of M is contained in the Poincaré disc

$$D = \{z \in \mathbb{C} | \quad |z| < 1\}$$

with the hyperbolic metric $ds = 2|dz|/1 - |z|^2$. The covering group $\Gamma = \pi_1(M)$ acts by isometries of D and p is the projection $U \to U/\Gamma = M$. The hyperbolic length of an arc γ on M or D will be written $l(\gamma)$; d and d^H denote respectively the Euclidean and hyperbolic metrics on D and $B_\varepsilon(P)$, $B_\varepsilon^H(P)$ are Euclidean and hyperbolic balls of radius ε in D centered on $P \in D$. The Euclidean metric extends naturally to $\bar{D}$. If γ is a complete geodesic in D we write γ^+, γ^- for the endpoints of γ on ∂D. The symbols $B_\varepsilon(\gamma)$ and $B_\varepsilon^H(\gamma)$ denote Euclidean and hyperbolic ε-tubular neighborhoods of γ.

We shall need some simple estimates from hyperbolic geometry.

LEMMA 1.1. (a) *Suppose that $P \in D$ and that $d^H(P, 0) \geq n$. Then, given $L > 0$, with $L < n$, there is a constant $c > 0$, depending only on L, such that*

$$B_L^H(P) \subseteq B_{ce^{-n}}(P).$$

(b) *Suppose that γ is a complete geodesic in D. Let $Q, P \in \gamma$ and let γ^+ be the endpoint of γ in the direction from Q to P. Suppose that $d^H(Q,0) \leq L'$ and $d^H(Q,P) > n$. Then there exists $c' > 0$, depending only on L', such that $d(P,\gamma^+) < c'e^{-n}$.*

(c) *Let γ, η be complete geodesics in D with $d(\gamma^+, \eta^+) < \varepsilon$ and $d(\gamma^-, \eta^-) < \varepsilon$. Let $K \subset D$ be compact. Then there exists a constant $c'' > 0$, depending only on K, such that $\eta \cap K \subset B^H_{c''\varepsilon}(\gamma \cap K)$.*

Proof. (a) The proof is given in [5]; we repeat it here for convenience. Choose $Q \in B^H_L(P)$. Since $d^H(P,0) \geq n$, $d^H(Q,P) \leq L$, it follows that $d^H(Q,0) \geq n - L$. Therefore $d(Q,0) \geq r = \tanh((n-L)/2)$, because $d(Q,0) = \tanh(d^H(Q,0)/2)$ for every $Q \in D$, and $\tanh x$ is a monotonic increasing function.

Now, the ball $B^H_L(P)$ is an off-center Euclidean ball, which is a convex set in both metrics, so the hyperbolic line joining P to Q lies inside $B^H_L(P)$ and hence outside $B_r(0)$. The formula relating the Euclidean and hyperbolic metrics then gives that

$$d(P,Q) \leqslant \left(\frac{(1-r^2)}{2}\right)(d^H(P,Q)) \leqslant \frac{L}{2}\operatorname{sech}^2\left(\frac{n-L}{2}\right)$$

$$= \frac{2L}{e^{n-L} + e^{-n+L} + 2} < \frac{2L}{e^{n-L}} = (2Le^L)e^{-n}.$$

Therefore $B^H_L(P) \subset B_\varepsilon(P)$ where $\varepsilon = ce^{-n}$, $c = 2Le^L$.

(b) The hyperbolic map ρ which fixes the diameter through Q and which moves Q to 0 has a derivative bounded in terms of L' so that $d(P,\gamma^+) \leq kd(\rho P, \rho\gamma^+)$ for some $k > 0$. Thus without loss of generality we may assume that $Q = 0$.

Let $P_0 = P, P_1, P_2, \ldots$ be points on γ with $d^H(P_i, P_{i+1}) = 1$ for each i, so that $d^H(P_i, 0) > n + i$. Applying part (a) with $L = 1$ we obtain $d(P_i, P_{i+1}) < ce^{-(n+i)}$ for $i = 0, 1, 2, \ldots$. Since $\gamma^+ = \lim_{i\to\infty} P_i$ it follows that

$$d(P,\gamma^+) \leq \sum_{i=0}^{\infty} d(P_i, P_{i+1}) < c'e^{-n}.$$

(c) Let γ_1 and γ_2 be the geodesics whose endpoints lie at distance ε from those of γ and which are on either side of γ. The tubular region T between γ_1 and γ_2 is geodesically convex so that $\eta \subset T$.

Choose a disc D_0 of radius r_0, with centre 0, so that $K \subset D_0$. Let l be the Euclidean perpendicular bisector of the arc $\gamma \cap D_0$. The centres of the circles defining γ_1 and γ_2 lie on l, so by symmetry there is a Euclidean circle C_i through γ^+, γ^- and the two points $\gamma_i \cap \partial D_0$, $i = 1, 2$. Choose C to be the one of C_1, C_2 making a maximum angle θ with γ, and let C' be the circle through γ^+, γ^- making the same angle on the opposite side. The region S between C and C' is a hyperbolic tubular neighborhood of γ and it is clear that its width δ is a smooth function of θ and hence of ε. Thus by the mean value theorem there is a constant $c'' > 0$ so that $\delta < c''\varepsilon$.

Now clearly $T \cap D_0 \subset S$ and hence $\eta \cap K \subset T \cap K \subset S = B^H_{c''\varepsilon}(\gamma)$, as required.

§2. PARAMETRIZING GEODESIC ARCS

In this section we develop a method for parametrizing finite geodesic arcs with a given number of self-intersections on the surface M. Our goal will be twofold. First, we will show that if γ, γ' are two arcs of the same length n (see below) which have the same parameters, then γ and γ' have lifts to the universal covering space U of M which lie in the same sequence of n copies of the fundamental domain. The precise statement of this assertion is in Lemmas 2.1 and 2.4. Secondly we produce the polynomial $P_k(x)$ of Proposition 4.1. This is accomplished in Lemma 2.5.

We assume in this section and in sections 3 and 4 below that M is compact. The case when M is non-compact (i.e. has finitely many points removed) is treated separately, in §5. For simplicity of exposition, we shall first consider simple arcs and then generalize to the case of arcs with at most k self-intersections.

To begin, fix a fundamental domain $R \subset U$ for the action of Γ on U. Assume R chosen so that the origin 0 of the Poincaré disc D lies in R, and also so that the sides of R are a finite number of geodesic arcs.

Some technical difficulties occur in dealing with geodesics which either pass through vertices of $p(\partial R)$ or have self-intersections on $p(\partial R)$. Such geodesics we call *exceptional* relative to R. To avoid these difficulties we note that we can certainly choose three different fundamental regions so that any geodesic in G_k is non-exceptional relative to at least one of the regions. Thus it is enough to prove Proposition 4.1 for the non-exceptional geodesics relative to any given fundamental region. In this section, therefore, we parameterize only non-exceptional geodesics relative to R.

Let $A = \{a_1, \ldots, a_m\}$ denote the ordered set of oriented sides of R with anti-clockwise ordering, with some arbitrary but henceforth fixed initial side a_1. A *simple diagram* on R is a collection of finitely many pairwise disjoint arcs joining pairs of distinct elements of A. We regard two simple diagrams as being identical if they agree up to isotopy supported on each side of R. For $a_i, a_j \in A$, $i \neq j$, let n_{ij} denote the number of arcs joining a_i to a_j. The *length* n of a simple diagram is Σn_{ij}, $1 \leq i < j \leq m$.

Let J_0 be the set of oriented simple non-exceptional geodesic arcs γ on M such that $\partial\gamma \subseteq p(\partial R)$. Choose $\gamma \in J_0$. Lifting the components of $\gamma \cap \operatorname{Int}(pR)$ to R and taking closures one obtains a simple diagram on R.

We refer to the components of $\gamma \cap \operatorname{Int}(pR)$ as the *segments* of γ and the points of $\gamma \cap p(\partial R)$ as the *partition points* of γ. We label the partition points $t_0, \ldots, t_n$ in the order in which they occur along γ and we set $||\gamma|| = n$. The partition points divide γ into subarcs $\gamma_1, \ldots, \gamma_n$ with γ_i joining t_{i-1} to t_i. These subarcs are covered by subarcs δ_i in R, $i = 1, \ldots, n$.

Our parametrization of elements of J_0 will consist of two sets of data. The first is defined by a map $h_1: J_0 \to \mathbb{Z}^p$, $p = m(m-1)/2$, with $h_1(\delta) = \{n_{12}, n_{13}, \ldots, n_{m-1,m}\}$ which records for each pair of distinct sides a_i, a_j of R the number of n_{ij} of segments which join a_i to a_j. Our second set of data records information about the position of the initial and final points t_0, t_n of γ. Let $a(t_i)$ be the element of A containing t_i and let $j(t_i) \in \mathbb{N}$ be the position of t_i among the partition points of γ which lie along $a(t_i)$ counting in the anticlockwise direction round ∂R. Now define h_2: $J_0 \to (A \times Z)^2$, $h_2(\gamma) = (a(t_0), j(t_0), a(t_n), j(t_n))$.

LEMMA 2.1. *Suppose that $\gamma, \gamma' \in J_0$ and that $h_1(\gamma) = h_1(\gamma')$, $h_2(\gamma) = h_2(\gamma')$. Let $t_0, \ldots, t_n$ and $t'_0, \ldots, t'_n$ be the partition points of γ, γ',. Then $a(t_i) = a(t'_i)$ for each $i = 0, \ldots, n$.*

Proof. Suppose that we are given any simple diagram on R with parameters $(n_{ij})_{1 \leq i < j \leq m}$ and $(a(t_0), j(t_0), a(t_n), j(t_n))$ equal to the parameters of γ. Let the collection of unoriented and unordered arcs on this diagram be $\{\delta_{\mu_1}, \ldots, \delta_{\mu_n}\}$. It is clear that we can find an isotopy in R supported on each side of R which moves the segments δ_{μ_i} onto the segments γ_i of γ. Notice that there is only one way to order the arcs incident on a_i so that they are disjoint. Notice also, that since the arcs γ_i link to form γ, that excluding the initial and final points on $a(t_0)$ and $a(t_n)$ the same number of arcs δ_i is incident on any two paired sides of R. Moreover the relative position of the initial and final points of the diagram is specified by the value of $h_2(\gamma)$. There is a unique way to join up the arcs $\delta_{\mu_1}, \ldots, \delta_{\mu_n}$ so that the union of their images under p is simple. There is also a unique way to orient them from the initial point to the final point. Now this whole process could equally well have been carried out for γ'. Since $h_1(\gamma) = h_1(\gamma')$, $h_2(\gamma) = h_2(\gamma')$, the lemma follows.

LEMMA 2.2. *Let $J_0(n) = \{\gamma \in J_0 : ||\gamma|| = n\}$. There is a polynomial $P_0(n)$ such that*

$$\text{card}\,\{(h_1(\gamma), h_2(\gamma)) : \gamma \in J_0(n)\} \leq P_0(n).$$

Proof. Observe that if $\gamma \in J_0(n)$, then n is the sum of the entries in $h_1(\gamma)$, so that in particular each individual term in $h_1(\gamma)$ is bounded by n. Since there are $(m(m-1)/2) < m^2$ entries in $h_1(\gamma)$, it follows that the number of distinct arrays $h_1(\gamma)$, $\gamma \in J_0(n)$, is bounded by n^{m^2}. Also, there are at most n choices for $j(t_0)$ and n for $j(t_n)$, and m choices for $a(t_0)$ and $a(t_n)$, hence the number of distinct arrays $h_2(\gamma)$, $\gamma \in J(n)$, is bounded by m^2n^2. Therefore $P_0(n) = m^2n^{m^2+2}$ is an upper bound for the number of arcs $\gamma \in J(n)$ with distinct parameters $h_1(\gamma)$, $h_2(\gamma)$. □

We now generalize Lemmas 2.1 and 2.2 to non-exceptional geodesic arcs on M which have length n and have at most k self-intersections, $k > 0$. Let R and A be defined as before. A *diagram* (no longer necessarily simple) on R is a collection of finitely many arcs in R, with each arc in the collection joining a pair of distinct elements of A, and any two arcs in the collection either disjoint or intersecting once, transversally, in the interior of R. The diagram is an *r-diagram* if there are r intersections between pairs of arcs of the diagram. Two r-diagrams are identical if they agree up to isotopy supported on each component of A. By analogy with the set J_0 defined earlier, let J_k be the set of oriented non-exceptional geodesic arcs on M which begin and end on $p(\partial R)$, and which have at most k self-intersections. For $\gamma \in J_k$, define $||\gamma||$, t_i, $a(t_i)$, $J(t_i)$, n_{ij}, n_i, γ_i, $h_1(\gamma)$, $h_2(\gamma)$ exactly as before. Then $p^{-1}(\{\gamma_1, \ldots, \gamma_n\})$ is an r-diagram, $r \le k$.

Note that if δ_ρ, δ_τ are components of $\rho^{-1}(\{\gamma_1, \ldots, \gamma_n\})$, then δ_ρ, δ_τ intersect once or not at all, because $R \subset D$ and two geodesics in the Poincaré disc are either disjoint or intersect once, transversally. If $\delta_\rho \cap \delta_\tau \neq \emptyset$, and if δ_ρ joins $a, a' \in A$ and δ_τ joins $b, b' \in A$, then we say the intersection is *type* 1 if a, a', b, b' are all distinct, in which case a, a' necessarily separate b, b' on ∂R. Otherwise, the intersection is *type* 2. An r-diagram on R is *type* 1 if all of its intersections are *type* 1. The first step in generalizing Lemma 2.1 is:

LEMMA 2.3. *If $\gamma \in J_k$, with $h_1(\gamma) = (n_{12}, n_{13}, \ldots, n_{m-1,m})$, then there is a type 1 diagram in R having n_{ij} arcs joining a_i to a_j for each $1 \le i \le j \le m$, which is unique up to isotopy supported on each side a of R.*

Proof. For each pair i, j with $1 \le i < j \le m$ construct n_{ij} parallel arcs joining a_i to a_j in R. Let these be $\{\sigma_1, \ldots, \sigma_n\}$, $n = \sum_{j=2}^{m} \sum_{i=1}^{j-1} n_{ij}$. Choose $\sigma_1, \ldots \sigma_n$ so that if σ_ρ joins a to a' and σ_τ joins b to b', then $\sigma_\rho \cap \sigma_\tau$ is either one point or empty, with one point if and only if a, a' separate b, b' on ∂R. This gives the desired type 1 diagram. □

The diagram $\{\sigma_1, \ldots, \sigma_n\}$ constructed in Lemma 2.3 will not coincide with the diagram $p^{-1}(\{\gamma_1, \ldots, \gamma_n\})$ if γ contains type 2 intersections. We wish to compare these two diagrams. It is clear that one may be obtained from the other by permuting the relative positions of the endpoints of the segments $\delta_i = p^{-1}(\gamma_i)$ on each side of R. We claim that this permutation can be effected by a product of at most k transpositions.

Suppose that δ_ρ intersects δ_τ with an intersection of type 2. Then δ_ρ joins some pair of components a_i, a_j of A and δ_τ joins a_i, a_q, where possibly $j = q$. We say that the intersection occurs *at* a_i if $j \neq q$ and *at* a_s if $j = q$ and $s = \min(i, j)$.

Suppose that there are r_i intersections of type 2 which occur at a_i. By moving the arcs δ_j slightly by isotopy if necessary, we may suppose that no two of these intersection points coincide and that they all lie at different distances from a_i. Order these intersection points by their distance from a_i. By transposing in turn the relative positions of the endpoints on a_i of pairs of adjacent segments δ_i, δ_k which intersect at a_i, starting with the intersection closest to a_i, we obtain a permutation π_i of the n_i points $\{\delta_1, \ldots, \delta_n\} \cap a_i$ which is a product of exactly r_i transpositions, which uncrosses the δ_i intersecting at a_i and which does not create any new intersections on the diagram. Letting $\pi = \pi_1 \circ \pi_2 \circ \ldots \circ \pi_m$ we obtain a permutation which uncrosses all the type 2 intersections and which is the product of at most $r_1 + r_2 + \ldots + r_m \le k$ transpositions.

In order to recover a curve from its diagram we need to add the permutation π to our list of parameters. Let Σ_{2n} be the set of permutations on the $2n$ symbols $x_1, \ldots, x_{2n}$. For fixed $\gamma \in J_k(n)$ identify the $2n$ endpoints of $\{\delta_1, \ldots, \delta_n\}$ with the symbols $x_1, \ldots, x_{2n}$ by using the anticlockwise ordering on ∂R, beginning with a_1. Let $h_3: J_k \to \Sigma_{2n}$ be defined by $h_3(\gamma) = \pi^{-1}$, where π is as described above. Let Ω be the image of J_k in $Z^p \times (A \times Z)^2 \times \Sigma_{2n}$ under the map $H = h_1 \times h_2 \times h_3: J_k \to \Omega$, $H(\gamma) = (n_{12}, n_{13}, n_{23}, \ldots, n_{m-1,m}, a(t_0), j(t_0), a(t_n), j(t_n), \pi^{-1})$.

LEMMA 2.4. *Suppose that $\gamma, \gamma' \in J_k$, with $H(\gamma) = H(\gamma')$. Let $t_0, \ldots, t_n$ and $t_0', \ldots, t_n'$ be the partition points of γ, γ' respectively. Then $a(t_i) = a(t_i')$ for each $i = 0, \ldots, n$.*

Proof. By Lemma 2.3 there is a unique type 1 diagram determined by $h_1(\gamma)$. By the discussion above, this diagram can be altered by the permutation $h_3(\gamma)$ in a unique way to an r-diagram, $r \leq k$, which covers γ. By an argument exactly like that used in the proof of Lemma 2.1, the data in $h_2(\gamma)$ determines a unique order in which to join the components to form a connected curve with the given initial and final points. Thus $a(t_i) = a(t_i')$ for each $i = 0, 1, \ldots, n$. □

LEMMA 2.5. *Let $J_k(n) = \{\gamma \in J_k: \|\gamma\| = n\}$. Then there is a polynomial $P_k(n)$ such that*

$$Card\ \{H(\gamma): \gamma \in J_k(n)\} < P_k(n).$$

Proof. The permutation $h_3(\gamma)$ is a permutation of $2n$ points which is a product of at most k transpositions, hence there are at most $[2n(2n-1)]^k$ distinct ways to choose $h_3(\gamma)$. By Lemmas 2.2 and 2.3 there are at most $P_0(n)$ ways to choose $h_1(\gamma) \times h_2(\gamma)$. Hence $P_k(\gamma) = (2n)^{2k} P_0(n)$ is a suitable bound. □

§3. DISTANCE ESTIMATES

We now show that arcs with the same parametrization lie exponentially close in M.

LEMMA 3.1. *There exists a universal constant $\alpha > 0$ so that*

$$l(\gamma) \geq \alpha \|\gamma\|$$

for $\gamma \in J_k$ with $\|\gamma\|$ sufficiently large.

Proof. Let V denote the projection of the vertices of ∂R on M and let X be the set of projections of the sides of ∂R. Let q be the maximum number of elements of X which meet at any vertex $v \in V$. Choose $\varepsilon > 0$ so that the hyperbolic discs $B_\varepsilon^H(v)$ of radius ε about $v \in V$ are disjoint and so that any segment of any geodesic arc which does not intersect $\cup \{B_\varepsilon^H(v) | v \in V\}$ has length at least ε.

Let $\gamma \in J_k$, $v \in V$ and consider a component t of $\gamma \cap B_\varepsilon^H(v)$. Since both t and the arcs in X are geodesics and since $B_\varepsilon^H(v)$ is simply connected, t intersects each curve in X at most once. Therefore at most $q-1$ consecutive segments of γ can intersect $B_\varepsilon^H(v)$. Hence, in any q consecutive segments of γ, at least one has length at least ε, which gives the result. □

LEMMA 3.2. *Let $\gamma, \gamma' \in J_k(2n+1)$ and suppose that $H(\gamma) = H(\gamma')$. Let $\delta \subset \gamma$, $\delta' \subset \gamma'$ denote the segments of γ and γ' lying between the partition points t_n, t_{n+1}* and t'_m, t'_{m+1}. *Let $\tilde{\gamma}, \tilde{\gamma}'$* be lifts of γ, γ' *so that the lifts $\tilde{\delta}, \tilde{\delta}'$ of δ, δ' lie in $\tilde{R}$. Then $\delta' \subset B^H_{c'e^{-\alpha n}}(\delta)$ where $c', \alpha > 0$ are universal constants.*

Proof. Let $\tilde{t}_i$, $\tilde{t}_i'$ denote the lifts of the partition points t_i, t_i' of γ, γ'. By Lemma 3.1, the hyperbolic distance between each of the pairs $(\tilde{t}_0, \tilde{t}_n)$, $(\tilde{t}_0', \tilde{t}_n')$, $(\tilde{t}_{n+1}, \tilde{t}_{2n+1})$, and $(\tilde{t}_{n+1}', \tilde{t}_{2n+1}')$ is at least αn.

By Lemma 2.4 t_q and t_q' lie in the same element of A for $q = 0, \ldots, 2n+1$, and by construction $\tilde{t}_n, \tilde{t}_n', \tilde{t}_{n+1}, \tilde{t}_{n+1}' \in \partial R$. It is easy to see by an inductive argument that this forces $\tilde{t}_q, \tilde{t}_q'$ to lie in the same side or vertex of the same copy of R for $q = 0, \ldots, 2n+1$. In particular $d^H(\tilde{t}_0, \tilde{t}_0') \leq L = \text{diam } R$ and $d^H(\tilde{t}_{2n+1}, \tilde{t}_{2n+1}') \leq L$.

By Lemma 1.1 (a) one sees that there is a universal constant $c > 0$ so that $d(\tilde{t}_0, \tilde{t}_0') \leq ce^{-\alpha n}$

and $d(\tilde{t}_{2n+1}, \tilde{t}'_{2n+1}) \leq ce^{-\alpha n}$. Using part (b) one obtains

$$d(\tilde{\gamma}^+, \tilde{\gamma}'^+) \leq c' e^{-\alpha n}, d(\tilde{\gamma}^-, \tilde{\gamma}'^-) \leq c' e^{-\alpha n}$$

for some other universal $c' > 0$, where $\tilde{\gamma}^+, \ldots, \tilde{\gamma}'^-$ are the positive and negative endpoints of the extensions of $\tilde{\gamma}, \tilde{\gamma}'$ to ∂D. Finally the result follows from part (c) of the same Lemma. □

§4. PROOF OF PROPOSITION 4.1

Proof. Choose an arbitrary but henceforth fixed non-negative integer k. As remarked at the beginning of section 2, it is enough to prove the proposition for the set of geodesics in G_k which are non-exceptional relative to some given fundamental region R. We denote this set by $\hat{G}_k$ and the corresponding points on M by $\hat{S}_k$.

Recall that for each geodesic arc $\gamma \in J_k$ we defined a set of parameters which were described by a surjective map $H: J_k \to \Omega$. For $\omega \in \Omega$, choose any $\gamma \in H^{-1}(\omega)$ and define $\|\omega\| = \|\gamma\|$. For $q \in N$ let $\Omega(q) = \{\omega \in \Omega : \|\omega\| = q\}$, $\Gamma_k(q, \omega) = \{\gamma \in H^{-1}(\omega) | \gamma \subset \beta \in \hat{G}_k$ and $\omega \in \Omega(q)\}$. For each $\omega \in \Omega(q)$ and $\Gamma_k(q, \omega) \neq \emptyset$ pick some fixed representative $\gamma' \in \Gamma_k(q, \omega)$. Thus γ' has length q, and parameters $H(\gamma') = \omega$, and is a subarc of some non-exceptional complete geodesic $\beta' \in \hat{G}_k$.

Choose $x \in \hat{S}_k$. Then $x \in \beta$, where $\beta \in \hat{G}_k$. The complete geodesic β is partitioned into infinitely many segments by $p(\partial R)$. Let δ be the segment which contains x (if $x \in p(\partial R)$ either choice will do) and let γ be the segment δ together with the n segments of β on either side of δ. Thus $\|\gamma\| = 2n + 1$. Then $\gamma \subset \beta \in \hat{G}_k$, $\gamma \in J_k(2n+1)$, and $H(\gamma) = \omega \in \Omega(2n+1)$. Therefore $\Gamma_k(2n+1, \omega) \neq \emptyset$. Let γ' be the representative of $\Gamma_k(2n+1, \omega)$ chosen earlier. Then $H(\gamma') = H(\gamma)$, so $\|\gamma'\| = \|\gamma\| = 2n+1$. Let δ' be the central segment of γ'. Lift γ, γ' to geodesic arcs $\tilde{\gamma}, \tilde{\gamma}' \subset D$, chosen so that the lifts $\tilde{\delta}, \tilde{\delta}'$ of δ, δ' lie in $\bar{R}$. By Lemma 3.2, there are universal constants $c, \alpha > 0$ so that $\delta \subset B^H_{ce^{-\alpha n}}(\delta')$.

We have shown that any $x \in \hat{S}_k$ lies on an arc $\delta \subset B^H_{ce^{-\alpha n}}(\delta')$, where δ' is the central segment of the representative arc of $\Gamma_k(2n+1, \omega)$ for some $\omega \in \Omega(2n+1)$. Denote the collection of such δ' by F_n. By Lemma 2.5, card (F_n) is bounded by a polynomial $P_k(n)$. This proves Proposition 4.1.

§5. PROOFS OF THEOREMS I, II AND III

Proof of Theorem I, M compact

We first show that S_k is nowhere dense. Let $V \subseteq M$ be open. By Proposition 4.1, for each n the set $S_k \cap V$ is covered by $P_k(n)$ bands of length at most diam (R) and width $2ce^{-\alpha n}$. Each band has hyperbolic area at most $c'e^{-\alpha n}$ for some universal constant c', so the total area occupied by the bands is bounded by $c'P_k(n)e^{-\alpha n}$ which becomes arbitrarily small as $n \to \infty$. In particular $V - S$ contain non-empty open sets, which proves the result.

The proof that S_k has Hausdorff dimension one is similar. For $\gamma \in F_n$, the tubular neighbourhood $B^H_{ce^{-\alpha n}}(\gamma)$ is covered by (diam $R)/(2ce^{-\alpha n})$ balls of radius $2ce^{-\alpha n}$ (see Fig. 1). Thus S_k is covered by at most $c'P_k(n)e^{\alpha n}$ balls of radius $2ce^{-\alpha n}$. Suppose the balls in this cover have radii $r_1, \ldots, r_m$. Then $\sum_{i=1}^{m} r_i^\delta \leqslant$ const. $P_k(n)e^{\alpha n}e^{-\delta n}$. For any $\delta > 1$ the term on the right

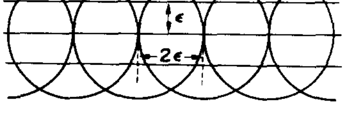

Fig. 1.

converges to zero as $n \to \infty$, so that the Hausdorff dimension of $S_k \le 1$. Since S_k certainly contains one dimensional sets the other inequality is trivial.

Extension to the non-compact case

Suppose now that M is obtained from a compact surface by removing a finite set of points to form cusps. We set up our parameters as before, except that we must now allow geodesics which go to infinity in one of the cusps. We therefore add the vertices of R at infinity, which correspond to the cusps, to the collection of sides of R, and allow in our parameter diagrams strands which run from one side of R to a vertex at infinity, or which join two of these vertices. We record the numbers of such strands in our parameter set. Notice that two strands which end at the same vertex at infinity necessarily meet at infinity, so that we do not need to record a permutation corresponding to type two intersections at the cusp.

The proof of Theorem I will work as before provided we have the distance estimate 3.1. Thus we need to consider strands which join adjacent sides of R that meet in a vertex at infinity.

LEMMA 5.1. *Let γ be a geodesic on M containing a segment β which joins two adjacent sides of R which meet in a vertex at infinity. Then β contains a self-intersection point of γ.*

Proof. Without loss of generality, we may work in the upper half plane with the cusp C at ∞ and with the two sides of R in question two vertical lines L_1, L_2 at distance λ apart. Choose a lift $\tilde{\gamma}$ of γ so that $\tilde{\gamma}$ intersects the strip between L_1 and L_2 exactly in a lift $\tilde{\beta}$ of β. Let P_i be the point $\tilde{\gamma} \cap L_i$; $i = 1, 2$. Think of L_1, L_2 as chords of the circle K centered on $\mathbb{R}$ whose upper half is $\tilde{\gamma}$. Consider all the vertical chords of K spaced at distance λ, starting from L_1 and L_2. For definiteness, suppose $\operatorname{Im} P_1 \le \operatorname{Im} P_2$ and $\operatorname{Re} P_1 < \operatorname{Re} P_2$. It is clear that we can find chords M_1, M_2 in our set intersecting $\tilde{\gamma}$ in Q_1, Q_2 so that $\operatorname{Re} Q_1 < \operatorname{Re} Q_2$ and $\operatorname{Im} Q_1 \ge \operatorname{Im} P_2$, $\operatorname{Im} Q_2 < \operatorname{Im} P_2$. Let $\tilde{\beta}'$ be the segment of $\tilde{\gamma}$ joining Q_1 to Q_2. Translating $\tilde{\beta}'$ by a suitable integer multiple of λ, we obtain an arc lying in the vertical strip between L_1 and L_2 which intersects $\tilde{\beta}$. Since translation by λ is an element of $\Gamma = \pi_1(M)$, this arc projects to the same arc as $\tilde{\beta}'$ on M; hence we have found a self intersection point of γ on β as required.

COROLLARY 5.2. *In a parameter diagram corresponding to a sub-arc of a geodesic $\gamma \in G_k$, the number of strands joining adjacent sides of R which meet at infinity is bounded by $4k$.*

Proof. By the above Lemma, every such strand contains at least one self-intersection point of γ. Each intersection point corresponds to at most four strands in the diagram. (This allows for intersections at the endpoints of a strand. If several strands go through the same point we count multiplicities.) Since there are at most k self-intersection points on γ, we obtain the required bound.

COROLLARY 5.3. (*Length estimate.*) *The estimate of* 3.1 *still holds when M has cusps.*

Proof. This is an easy consequence of 5.2.

Proof of Theorem II. Using the methods of Proposition 4.1 one can find a polynomial bound for the number of squares of side $ce^{-\alpha n}$ needed to cover T in $\partial D \times \partial D$ – diagonal. The result follows by the method of Theorem I.

Proof of Theorem III. The universal cover of $T_1 M$ is $D \times \mathbb{R}$, and the product metric on $D \times \mathbb{R}$ projects to a natural metric on $T_1 M$. Use this metric on $T_1 M$. In the proof of Theorem 1 we actually showed that any $u \in U_k$ lies along a segment δ of geodesic arc which is ε close to one of the representative segments δ' not only in position, but also in direction. Therefore U_k is contained in the union of the ε-tubular neighborhoods of the lifts of the representative segments δ' to $T_1 M$. The same reasoning as before now completes the proof.

§6. PROOF OF THEOREM IV

If γ is a closed curve on M, let $[\gamma]$ denote its free homotopy class. Then the universal covering space projection p induces a map $p_*: \Gamma \to$ free homotopy classes on M, as follows: if $G \in \Gamma$ has axis $\tilde{\gamma}$, define $p_*(G) = [p(\tilde{\gamma})]$.

Let $\tilde{\alpha}, \tilde{\beta}, \tilde{x}$ be lifts of α, β, x to D, chosen so that $\tilde{\alpha} \cap \tilde{\beta} = \tilde{x}$. Let A, B be the elements of Γ having axes $\tilde{\alpha}, \tilde{\beta}$. Then $p_*(A) = [\alpha]$, $p_*(B) = [\beta]$. Also, if t_α denotes a Dehn twist about α, then $p_*(A^m B) = [t_\alpha^m(\beta)]$ for each $m \in \mathbb{Z}$. Note that $t_\alpha^m(\beta)$ is a simple closed curve for each $m \in \mathbb{Z}$ because β is a simple closed curve; also $[t_\alpha^m(\beta)] \neq [t_\alpha^k(\beta)]$ if $m \neq k$.

Let $\tilde{\gamma}_n$ be the axis of $A^n B A^n$, $n \in \mathbb{Z}$. Let $\gamma_n = \rho(\tilde{\gamma}_n)$. Jorgensen has shown in [4] that the smooth closed geodesics γ_n pass through x for each $n \in \mathbb{N}$. By construction, $[\gamma_n] = p_*(A^n B A^n) = p_*(A^{2n} B) = [t_\alpha^{2n}(\beta)]$, and our proof is complete. □

§7. REMARKS

There are many interesting open problems about the sets S_k and G_k. Here are just a few.

7.1. The proof of Theorem I shows that S_k is a very "thin" set, that is the geodesics in G_k travel together for long distances as essentially parallel curves. One expects that the result "Hausdorff dimension 1" could be improved.

7.2. One can study the growth function for the number of complete simple geodesics of length n or $< n$. In the case of the once-punctured torus one can easily show that the number of closed smooth simple geodesics of length $n > 4$ is $2\phi(n)$, where ϕ is the Euler function. In fact the degree of the polynomial $P_0(n)$ bounding the number of simple geodesics of length n is at most $6g + 2b - 6$ where g is the genus and b the number of boundary components of M. It is not coincidental that this number is also the dimension of the Thurston parameterization of the space of measured geodesic laminations on M. This polynomial estimate is very crude as is apparent even from the example of the punctured torus above. In general the precise nature of the bound seems to be a very interesting number theoretic question.

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