

# QUASI-ISOMETRIC CLASSIFICATION OF NON-GEOMETRIC 3-MANIFOLD GROUPS

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ABSTRACT. We describe the quasi-isometric classification of fundamental groups of irreducible non-geometric 3-manifolds which do not have “too many” arithmetic hyperbolic geometric components, thus completing the quasi-isometric classification of 3-manifold groups in all but a few exceptional cases.

## 1. INTRODUCTION

In [1] we discussed the quasi-isometry classification of fundamental groups of 3-manifolds (which coincides with the bilipschitz classification of the universal covers of three-manifolds). This classification reduces easily to the case of irreducible manifolds. Moreover, no generality is lost by considering only orientable manifolds. So from now on we only consider compact connected orientable irreducible 3-manifolds of zero Euler characteristic (i.e., with boundary consisting only of tori and Klein bottles) since these are the orientable manifolds which, by Perelman’s Geometrization Theorem [9, 10, 11], decompose along tori and Klein bottles into geometric pieces (this decomposition removes the boundary tori and a family of embedded tori, so the pieces of the decomposition are without boundary). The minimal such decomposition is what is called the *geometric decomposition*.

We described (loc. cit.) the classification for geometric 3-manifolds as well as for non-geometric 3-manifolds with no hyperbolic pieces in their geometric decomposition (i.e., graph-manifolds). For geometric manifolds this was a summary of work of others; our contribution was in the non-geometric case. In this paper we extend to allow hyperbolic pieces. However, our results are still not quite complete: at present we exclude manifolds with “too many” arithmetic hyperbolic pieces<sup>1</sup>, and some of our results are only proved using the “cusp covering conjecture” in dimension 3 (see below and Section 5).

In the bulk of this paper we restrict to non-geometric manifolds, all of whose geometric components are hyperbolic and at least one of which is non-arithmetic (we will call these *NAH-manifolds* for short). In the final section 8 we extend to the case where Seifert fibered pieces are also allowed.

The classification for graph-manifolds in [1] was in terms of finite labelled graphs; the labelling consisted of a color black or white on each vertex and the classifying objects were such two-colored graphs which are minimal under a relation called *bisimilarity*. For NAH-manifolds the classification is again in terms of finite labelled

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<sup>1</sup>The essential remaining case to address is when *all* pieces are arithmetic hyperbolic. These behave rather differently from the other cases—more like arithmetic hyperbolic manifolds.

graphs, and the the classifying objects are again given by labelled graphs that are minimal in a similar sense. The labelling is more complex: each vertex is labelled by the isomorphism type of a hyperbolic orbifold<sup>2</sup> and each edge is labelled by a linear isomorphism between certain 2-dimensional  $\mathbb{Q}$ -vectorspaces. We will call these graphs *NAH-graphs*. The ones that classify are the ones that are *minimal* and *balanced*. All these concepts will be defined in Section 3.

Finally, when both Seifert fibered and hyperbolic pieces occur the classifying graphs are a hybrid of the two-colored graphs and the NAH-graphs, so we call them *H-graphs*. In this case we need that every component of the manifold obtained by removing all Seifert fibered pieces is NAH. We will call irreducible non-geometric manifolds of this type *good*. For example, 3-manifolds which contain a hyperbolic piece but no arithmetic hyperbolic piece provide a large family of good manifolds.

The following three theorems summarize our main results. The second two theorems complement the first, by making the classification effective, and then relating the quasi-isometric and commensurability classification for some NAH-manifolds.

**Classification Theorem.** *Each good 3-manifold has an associated minimal H-graph and two such manifolds have quasi-isometric fundamental groups (in fact, bilipschitz equivalent universal covers) if and only if their minimal H-graphs are isomorphic.*

**Realization Theorem.** *The minimal H-graph associated to a good 3-manifold is balanced. Moreover, the converse is true, namely if a minimal H-graph is balanced, then it is the minimal H-graph for some quasi-isometry class of good 3-manifold groups.*

**Commensurability Theorem.** *If two NAH-manifolds have quasi-isometric fundamental groups and their common minimal NAH-graph is a tree with manifold labels then they (and in particular, their fundamental groups) are commensurable.*

We do not know to what extent our restrictive conditions on the minimal NAH-graph in the Commensurability Theorem are needed; whether “NAH-manifolds are quasi-isometric if and only if they are commensurable” holds in complete generality remains a very interesting open question.

The converse part of the Realization theorem and the Commensurability Theorem both depend on the “Cusp Covering Conjecture” in dimension 3, which we explain in Section 5. Since the original version of this paper, Dani Wise has pointed out that this conjecture follows from work in his preprint [13], and has now included a proof as Corollary 18.11 of [13], see also [14]. Although not relevant to the present results, we note that for dimension  $\geq 4$  the Cusp Covering Conjecture remains open.

A slightly surprising byproduct of this investigation are the *minimal orbifolds* of section 2, which play a role similar to commensurator quotients but exist also for cusped arithmetic hyperbolic orbifolds. Although their existence is easy to prove, they were new to us. Minimal orbifolds are precisely the orbifolds that can appear as vertex labels of minimal NAH-graphs.

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<sup>2</sup>“Hyperbolic orbifold” always means an orientable complete hyperbolic 3-orbifold of finite volume.

2. MINIMAL ORBIFOLDS

We consider only orientable manifolds and orbifolds. Let  $N$  be a hyperbolic 3-orbifold (not necessarily non-arithmetic) with at least one cusp. Then each cusp of  $N$  has a smallest cover by a toral cusp (one with toral cross section) and this cover has cyclic covering transformation group  $F_C$  of order 1, 2, 3, 4, or 6. We call this order the “orbifold degree of the cusp.”

**Proposition 2.1.** *Among the orbifolds  $N'$  covered by  $N$  having the same number of cusps as  $N$  with each cusp of  $N'$  having the same orbifold degree as the cusp of  $N$  that covers it, there is a unique one,  $N_0$ , that is covered by all the others. We call this  $N_0$  a minimal orbifold.*

*More generally, for each cusp of  $N$  specify a “target” in  $\{1, 2, 3, 4, 6\}$  that is a multiple of the orbifold degree and ask that the corresponding cusp of  $N'$  have orbifold degree dividing this target. The same conclusion then holds.*

*Proof.* We first “neuter”  $N$  by removing disjoint open horoball neighborhoods of the cusps to obtain a compact orbifold with boundary which we call  $N^0$ . If we prove the proposition for  $N^0$ , with boundary components interpreted as cusps, then it holds for  $N$ .

Let  $\tilde{N}^0$  be the universal cover of  $N^0$ . Any boundary component  $C$  of  $N^0$  is isomorphic to the quotient of a euclidean plane  $\tilde{C}$  by the orbifold fundamental group  $\pi_1^{orb}(C)$ , which is an extension of a lattice  $\mathbb{Z}^2$  by the cyclic group  $F_C$  of order 1, 2, 3, 4, or 6. We choose an oriented foliation of this plane by parallel straight lines and consider all images by covering transformations of this foliation on boundary planes of  $\tilde{N}^0$ . Each boundary plane of  $\tilde{N}^0$  that covers  $C$  will then have  $|F_C|$  oriented foliations, related by the action of the cyclic group  $F_C$ . If the target degree  $n_C$  for the given cusp is a proper multiple of  $|F_C|$ , we also add the foliations obtained by rotating by multiples of  $2\pi/n_C$ .

We give the boundary planes of  $\tilde{N}^0$  different labels according to which boundary component of  $N^0$  they cover, and we construct foliations on them as above. So each boundary plane of  $\tilde{N}^0$  carries a finite number (1, 2, 3, 4, or 6) of foliations and the collection of all these oriented foliations and the labels on boundary planes are invariant under the covering transformations of the covering  $\tilde{N}^0 \rightarrow N^0$ . Let  $G_N$  be the group of all orientation preserving isometries of  $\tilde{N}^0$  that preserve the labels of the planes and preserve the collection of oriented foliations. Note that  $G_N$  does not depend on choices: The only relevant choices are the size of the horoballs removed when neutering and the direction of the foliation we first chose on a boundary plane of  $\tilde{N}^0$ . If we change the size of the neutering and rotate the direction of the foliation then the size of the neutering and direction of the image foliations at all boundary planes with the same label change the same amount, so the relevant data are still preserved by  $G_N$ . It is clear that  $G_N$  is discrete (this is true for any group of isometries of  $\mathbb{H}^3$  which maps a set of at least three disjoint horoballs to itself).  $N_0^0 := \tilde{N}^0/G_N$  is thus an oriented orbifold; it is clearly covered by  $N^0$ , has the same number of boundary components as  $N^0$ , and the boundary component covered by a boundary component  $C$  of  $N^0$  has orbifold degree dividing the chosen target degree  $n_C$ . Moreover if  $N_1^0$  is any cover of  $N_0^0$  with the same property, then  $\tilde{N}_1^0 = \tilde{N}^0$  and the labellings and foliations on  $\tilde{N}_1^0$  and  $\tilde{N}^0$  can be chosen the same, so  $N_0^0$  is the minimal orbifold also for  $N_1^0$ . The proposition thus follows.  $\square$

Note that a minimal orbifold may have a non-trivial isometry group (in contrast with commensurator quotients of non-arithmetic hyperbolic manifolds).

### 3. NAH-GRAPHS

The graphs we need will be finite, connected, undirected graphs. We take the viewpoint that an edge of an undirected graph consists of a pair of oppositely directed edges. The reversal of a directed edge  $e$  will be denoted  $\bar{e}$  and the initial and terminal vertices of an edge will be denoted  $\iota e$  and  $\tau e (= \iota \bar{e})$ .

We will label the vertices of our graph by hyperbolic orbifolds so we first introduce some terminology for these. A horosphere section  $C$  of a cusp of a hyperbolic orbifold  $N$  will be called the *cuspid orbifold*. Although the position of  $C$  as a horosphere section of the cusp involves choice, as a flat 2-dimensional orbifold,  $C$  is canonically determined up to similarity by the cusp. We thus have one cusp orbifold for each cusp of  $N$ .

Since a cusp orbifold  $C$  is flat, its tangent space  $TC$  is independent of which point on  $C$  we choose (up to the action of the finite cyclic group  $F_C$ ).  $TC$  naturally contains the maximal lattice  $\mathbb{Z}^2 \subset \pi_1^{orb}(C)$ , so it makes sense to talk of a linear isomorphism between two of these tangent spaces as being *rational*, i.e., given by a rational matrix with respect to oriented bases of the underlying integral lattices. Moreover, such a rational linear isomorphism will have a well-defined determinant.

**Definition 3.1** (NAH-graph). An *NAH-graph* is a finite connected graph with the following data labelling its vertices and edges:

- (1) Each vertex  $v$  is labelled by a hyperbolic orbifold  $N_v$  plus a map  $e \mapsto C_e$  from the set of directed edges  $e$  exiting that vertex to the set of cusp orbifolds  $C_e$  of  $N_v$ . This map is injective, except that an edge which begins and ends at the same vertex may have  $C_e = C_{\bar{e}}$ .
- (2) The cusp orbifolds  $C_e$  and  $C_{\bar{e}}$  have the same orbifold degree.
- (3) Each directed edge  $e$  is labelled by a rational linear isomorphism  $\ell_e : TC_e \rightarrow TC_{\bar{e}}$ , with  $\ell_{\bar{e}} = \ell_e^{-1}$ . Moreover  $\ell_e$  reverses orientation (so  $\det(\ell_e) < 0$ ).
- (4) We only need  $\ell_e$  up to right multiplication by elements of  $F_e := F_{C_e}$ ; in other words, the relevant datum is really the coset  $\ell_e F_e$  rather than  $\ell_e$  itself. This necessitates:
- (5)  $\ell_e$  conjugates the cyclic group  $F_e$  to the cyclic group  $F_{\bar{e}}$ , i.e.,  $\ell_e F_e = F_{\bar{e}} \ell_e$  (this holds automatically if the orbifold degree is  $\leq 2$ ; otherwise it is equivalent to saying that  $\ell_e$  is a similarity for the euclidean structures).
- (6) At least one vertex label  $N_v$  is a non-arithmetic hyperbolic orbifold.

**Definition 3.2.** We call an NAH-graph *balanced* if the product of the determinants of the linear maps on edges around any closed directed path is equal to  $\pm 1$ .

We call the NAH-graph *integral* if each  $\ell_e$  is an integral linear isomorphism (i.e., an isomorphism of the underlying  $\mathbb{Z}$ -lattices). Integral clearly implies balanced.

**Definition 3.3.** An integral NAH-graph contains precisely the information to specify how to glue neutered versions  $N_v^0$  of the orbifolds  $N_v$  together along their boundary components to obtain a NAH-orbifold  $M$ , called the *associated orbifold*. Conversely, any NAH-orbifold  $M$  has an *associated integral NAH-graph*  $\Gamma(M)$ , which encodes its decomposition into geometric pieces.

We next want to define morphisms of NAH-graphs.

**Definition 3.4** (Morphism). A *morphism* of NAH-graphs,  $\Gamma \rightarrow \Gamma'$ , consists of the following data:

- (1) an abstract graph homomorphism  $\phi: \Gamma \rightarrow \Gamma'$ , and
- (2) for each vertex  $v$  of  $\Gamma$ , a covering map  $\pi_v: N_v \rightarrow N_{\phi(v)}$  of the orbifolds labelling  $v$  and  $\phi(v)$  which respects cusps (so for each departing edge  $e$  at  $v$  one has  $\pi_v(C_e) = C_{\phi(e)}$ ), subject to the condition:
- (3) for each directed edge  $e$  of  $\Gamma$  the following diagram commutes

$$\begin{array}{ccc} TC_e & \xrightarrow{\ell_e} & TC_{\bar{e}} \\ \downarrow & & \downarrow \\ TC_{\phi(e)} & \xrightarrow{\ell_{\phi(e)}} & TC_{\phi(\bar{e})} \end{array}$$

Here the vertical arrows are the induced maps of tangent spaces and commutativity of the diagram is up to the indeterminacy of item (4) of Definition 3.1.

Morphisms compose in the obvious way, so a morphism is an *isomorphism* if and only if  $\phi$  is a graph isomorphism and each  $\pi_v$  has degree 1.

**Definition 3.5.** An NAH-graph is *minimal* if every morphism to another NAH-graph is an isomorphism.

The relationship of existence of a morphism between NAH-graphs generates an equivalence relation which we call *bisimilarity*.

**Theorem 3.6.** *Every bisimilarity class of NAH-graphs contains a unique (up to isomorphism) minimal member, so every NAH-graph in the bisimilarity class has a morphism to it.*

NAH-graphs play an analogous role to the two-colored graphs that we used in [1] to classify graph-manifold groups up to quasi-isometry. For those graphs the morphisms were called “weak coverings”—they were color preserving open graph homomorphisms (i.e., open as maps of 1-complexes). The term “bisimilarity” was coined in that context, since the concept is known to computer science under this name. Theorem 3.6 has a proof analogous to the proof we gave in [1] for two-colored graphs. We give a different proof, which we postpone to the next section, since it follows naturally from the discussion there (an analogous proof works in the two-colored graph case).

**Theorem 3.7.** *If  $\Gamma \rightarrow \Gamma'$  is a morphism of NAH-graphs and  $\Gamma$  is balanced, then so is  $\Gamma'$ . In particular, a bisimilarity class contains a balanced NAH-graph if and only if its minimal NAH-graph is balanced.*

*Proof.* We will denote the negative determinant of the linear map labelling an edge  $e$  of an NAH-graph by  $\delta_e$ , so  $\delta_e = \delta_{\bar{e}}^{-1} > 0$ .

Assume  $\Gamma$  is balanced. We will show  $\Gamma'$  is balanced. We only need to show that the product of the  $\delta_e$ 's along any simple directed cycle in  $\Gamma'$  is 1. Let  $\mathcal{C} = (e'_0, e'_1, \dots, e'_{n-1})$ , be such a cycle, so  $\tau e'_i = \iota e'_{i+1}$  for each  $i$  (indices modulo  $n$ ). Denote

$$(1) \quad D := \prod_{i=0}^{n-1} \delta_{e'_i},$$

so we need to prove that  $D = 1$ .

From now on we will restrict attention to one connected component  $\Gamma_0$  of the full inverse image of  $\mathcal{C}$  under the map  $\Gamma \rightarrow \Gamma'$ .

Let  $e$  be an edge of  $\Gamma_0$  which maps to an edge  $e'$  of  $\mathcal{C}$  ( $e'$  may be an  $e'_i$  or an  $\bar{e}'_i$ ). Its start and end correspond to cusp orbifolds which cover the cusp orbifolds corresponding to start and end of  $e'$  with covering degrees that we shall call  $d_e$  and  $d_{\bar{e}}$  respectively. We claim that

$$(2) \quad d_e \delta_{e'} = \delta_e d_{\bar{e}}.$$

Indeed, if the cusp orbifolds are all tori, this equation (multiplied by  $-1$ ) just represents products of determinants for the two ways of going around the commutative diagram in part (3) of Definition 3.4. In general,  $d_e$  and  $d_{\bar{e}}$  are the determinants multiplied by the order of the cyclic group  $F_e$ , but this is the same factor for each of  $d_e$  and  $d_{\bar{e}}$  so the equation remains correct.

Let  $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$  be the infinite cyclic cover of  $\mathcal{C}$ , and  $\tilde{\Gamma}_0 \rightarrow \Gamma_0$  the pulled back infinite cyclic cover of  $\Gamma_0$ . Define  $d_e$  or  $\delta_e$  for an edge of either of these covers as the value for the image edge.

If  $(e_1, e_2, \dots, e_m)$  is a directed path in  $\tilde{\Gamma}_0$  then the product of the equations (2) over all edges of this path gives  $\prod_{i=1}^m d_{e_i} \delta_{\pi e_i} = \prod_{i=1}^m \delta_{e_i} d_{\bar{e}_i}$ , so

$$\prod_{i=1}^m \frac{\delta_{\pi e_i}}{\delta_{e_i}} = \prod_{i=1}^m \frac{d_{\bar{e}_i}}{d_{e_i}}.$$

Since  $\tilde{\Gamma}_0$  and  $\tilde{\mathcal{C}}$  are both balanced, it follows that  $\prod_{i=1}^m \frac{d_{\bar{e}_i}}{d_{e_i}}$  only depends on the start and end vertex of the path. Thus, if we fix a base vertex  $v_0$  of  $\tilde{\Gamma}_0$  and define

$$(3) \quad c(v) := \prod_{i=1}^m \frac{\delta_{\pi e_i}}{\delta_{e_i}} = \prod_{i=1}^m \frac{d_{\bar{e}_i}}{d_{e_i}} \quad \text{for any path from } v_0 \text{ to } v,$$

we get a well defined invariant of the vertices of  $\tilde{\Gamma}_0$ . For an edge  $e$  this invariant satisfies

$$\frac{c(\iota \bar{e})}{c(\iota e)} = \frac{c(\tau e)}{c(\tau \bar{e})} = \frac{d_{\bar{e}}}{d_e},$$

whence

$$\zeta(e) := \frac{d_e}{c(\iota e)}$$

is an invariant of the undirected edge:  $\zeta(e) = \zeta(\bar{e})$ .

Denote the map  $\tilde{\Gamma}_0 \rightarrow \tilde{\mathcal{C}}$  by  $\pi$ . Number the vertices sequentially along  $\tilde{\mathcal{C}}$  by integers  $i \in \mathbb{Z}$ , and for each vertex  $v$  of  $\tilde{\Gamma}_0$  let  $i(v) \in \mathbb{Z}$  be the index of  $\pi(v)$ . Let  $h: \tilde{\Gamma}_0 \rightarrow \tilde{\Gamma}_0$  be the covering transformation:  $h(v)$  is the vertex of  $\tilde{\Gamma}_0$  with the same image in  $\Gamma_0$  as  $v$  but with  $i(h(v)) = i(v) + n$ .

Note that the equations (1) and (3) and the fact that  $\Gamma_0$  is balanced implies that  $c(h(v)) = Dc(v)$  for any vertex  $v$  of  $\tilde{\Gamma}_0$ , so

$$(4) \quad \zeta(h(e)) = D^{-1}\zeta(e).$$

From now on consider the edges of  $\tilde{\Gamma}_0$  directed only in the direction of increasing  $i(v)$ . Denote by  $Z(j)$  the sum of  $\zeta(e)$  over all edges with  $i(\tau e) = j$ .

The sum of the  $d_e$ 's over outgoing edges at a vertex  $v$  of  $\tilde{\Gamma}_0$  equals the sum of  $d_{\bar{e}}$ 's over incoming edges at  $v$ , since each sum equals the degree of the covering map from the orbifold labelling  $v$  to the one labelling vertex  $\pi(v)$  of  $\tilde{\mathcal{C}}$ . Since  $\zeta(e) = d_e/c(v)$

for an outgoing edge and  $\zeta(\bar{e}) = d_{\bar{e}}/c(v)$  for an incoming one, the sum of  $\zeta(e)$  over outgoing edges at  $v$  equals the sum of  $\zeta(e)$  for incoming ones. Thus  $Z(j)$  is the sum of  $\zeta(e)$  over all edges with  $i(\tau e) = j$ . These are the edges with  $i(\tau e) = j + 1$ , so  $Z(j) = Z(j + 1)$ . Thus  $Z(j)$  is independent of  $j$ . Clearly  $Z(j) > 0$ . Equation (4) implies  $Z(j + n) = D^{-1}Z(j)$ , so  $D = 1$ , as was to be proved. (We are grateful to Don Zagier for help with this proof.)  $\square$

We close this section with an observation promised in the introduction.

**Proposition 3.8.** *An orbifold can be a vertex label in a minimal NAH-graph if and only if it is a minimal orbifold.*

*Proof.* We first prove “only if.” In condition (3) of the definition of a morphism,  $\ell_{\phi(e)}$  is determined by  $\ell_e$  since the vertical arrows are isomorphisms (but  $\ell_e$  may not be determined by  $\ell_{\phi(e)}$ , since the indeterminacy  $F_{\phi(e)}$  of  $\ell_{\phi(e)}$  may be greater than the indeterminacy  $F_e$  of  $\ell_e$ ). Thus for any NAH-graph there is an outgoing morphism to a new NAH-graph for which the underlying graph homomorphism is an isomorphism and every orbifold vertex label is simply replaced in the new graph by the corresponding minimal orbifold; the new edge labels are then as just described.

For the converse, if  $N$  is a minimal orbifold, any NAH-graph which is star-shaped, with  $N$  labelling the middle vertex and with one-cusp non-arithmetic commensurator quotients labelling the outer ones, is minimal. One needs a one-cusp commensurator quotient for each cusp degree in  $\{1, 2, 3, 4, 6\}$  for this construction; these are not hard to find.  $\square$

#### 4. MINIMAL NAH-GRAPHS CLASSIFY FOR QUASI-ISOMETRY

Let  $M = M^3$  be a NAH-manifold. For simplicity of exposition we first discuss the case that  $M$  has no arithmetic pieces, and then discuss the modifications needed when arithmetic pieces also occur.

$M$  is then pasted together from pieces  $M_i$ , each of which is a neutered non-arithmetic hyperbolic manifold. We can choose neuterings in a consistent way, by choosing once and for all a neutering for the commensurator quotient in each commensurability class of non-arithmetic manifolds, and choosing each  $M_i$  to cover one of these “standard neutered commensurator quotients.”

The pasting identifies pairs of flat boundary tori with each other by affine maps which may not be isometries. To obtain a smooth metric on the result we glue a toral annulus  $T^2 \times I$  between the two boundaries with a metric that interpolates between the flat metrics at the two ends in a standard way (if  $g_0$  and  $g_1$  are the flat product metrics induced on  $T^2 \times I$  by the flat metric on its left and right ends we use  $(1 - \rho(t))g_0 + \rho(t)g_1$  where  $\rho: [0, 1] \rightarrow [0, 1]$  is some fixed smooth bijection with derivatives 0 at each end). This specifies the metric on  $M$  up to rigid translations of the gluing maps, so we get a compact family of different metrics on  $M$ .

The universal cover  $\widetilde{M}$  is glued from infinitely many copies of the  $\widetilde{M}_i$ 's with “slabs”  $\mathbb{R}^2 \times I$  interpolating between them. Each slab admits a full  $\mathbb{R}^2$  of isometric translations. The pieces  $\widetilde{M}_i$  will be called “pieces” and each boundary component of a piece will be called a “flat” and will be oriented as part of the boundary of the piece it belongs to.

Let  $M'$  be another such manifold and  $\widetilde{M}'$  its universal cover, metrized as above. Kapovich and Leeb [5] show that any quasi-isometry  $f: \widetilde{M} \rightarrow \widetilde{M}'$  is a bounded

distance from a quasi-isometry that maps slabs to slabs and geometric pieces to geometric pieces. Then a theorem of Schwartz [12] says that  $f$  is a uniformly bounded distance from an isometry on each piece  $\widetilde{M}_i$ . Our uniform choice of neuterings assures that we can change  $f$  by a bounded amount to be an isometry on each piece and an isometry followed by a shear map on each slab (a “shear map”  $\mathbb{R}^2 \times I \rightarrow \mathbb{R}^2 \times I$  will mean one of the form  $(x, t) \mapsto (x + \rho(t)v, t)$  with  $v \in \mathbb{R}^2$  and  $\rho: I \rightarrow I$  as described earlier). We then say  $f$  is *straightened*.

Consider now the group<sup>3</sup>

$$\mathcal{I}(\widetilde{M}) := \{f: \widetilde{M} \rightarrow \widetilde{M} \mid f \text{ is a straightened quasi-isometry}\}.$$

$\mathcal{I}(\widetilde{M})$  acts on the set of pieces of  $\widetilde{M}$ ; the pieces  $\widetilde{M}_i$  that cover a given piece  $M_i$  of  $M$  are all in one orbit of this action, but the orbit may be larger. The subgroup  $\mathcal{I}_{\widetilde{M}_i}(\widetilde{M})$  of  $\mathcal{I}(\widetilde{M})$  that stabilizes a fixed piece  $\widetilde{M}_i$  of  $\widetilde{M}$  acts discretely on  $\widetilde{M}_i$ , so  $\widetilde{M}_i/\mathcal{I}_{\widetilde{M}_i}(\widetilde{M})$  is an orbifold (which clearly is covered by  $M_i$ )

The subgroup  $\mathcal{I}_S(\widetilde{M}) \subset \mathcal{I}(\widetilde{M})$  that stabilizes a slab  $S \subset \widetilde{M}$  acts on  $S$  by isometries composed with shear maps. It acts discretely on each boundary component of  $S$  but certainly not on  $S$ . But an element that is a finite order rotation on one boundary component must be a similar rotation on the other boundary component (and is in fact finite order on the slab; the group  $\mathcal{I}_S(\widetilde{M})$  is abstractly an extension of  $\mathbb{Z}^2 \times \mathbb{Z}^2$  by a finite cyclic group  $F_S$  of order 1, 2, 3, 4, or 6; the two  $\mathbb{Z}^2$ 's are the translation groups for the two boundaries of  $S$ ).

We form an NAH-graph  $\Omega(\widetilde{M})$  as follows. The vertices of  $\Omega(\widetilde{M})$  correspond to  $\mathcal{I}(\widetilde{M})$ -orbits of pieces and the edges correspond to orbits of slabs—the edge determined by a slab connects the vertices determined by the abutting pieces. We label each vertex by the corresponding orbifold  $\widetilde{M}_i/\mathcal{I}_{\widetilde{M}_i}(\widetilde{M})$  and each edge by the derivative of the affine map between the flats that bound a corresponding slab  $S$  (this map is determined up to the cyclic group  $F_S$ ).

**Proposition 4.1.** *The NAH-graph  $\Omega(\widetilde{M})$  constructed above is determined up to isomorphism by  $\widetilde{M}$ . The manifold  $\widetilde{M}$  is determined up to bilipschitz diffeomorphism by  $\Omega(\widetilde{M})$ .*

*Proof.* The first sentence of the proposition is true by construction.

We describe how to reconstruct  $\widetilde{M}$  from  $\Omega(\widetilde{M})$ .  $\Omega(\widetilde{M})$  gives specifications for inductively gluing together pieces that are the universal covers of the orbifolds corresponding to its vertices, and slabs between these pieces, according to a tree: we start with a piece  $\widetilde{N}$  corresponding to a vertex  $v$  of  $\Gamma$  and glue slabs and adjacent pieces on all boundary components as specified by the outgoing edges at  $v$  in  $\Omega(\widetilde{M})$ , and repeat this process for each adjacent piece, and continue inductively. (There is an underlying tree for this construction which is the universal cover of the graph obtained by replacing each edge of  $\Gamma$  by a countable infinity of edges.) The construction involves choices, since each gluing map is only determined up to a group of isometries of the form  $\mathbb{R}^2 \rtimes F_S$ , where  $S$  is the slab. We need to show that the resulting manifold is well defined up to bilipschitz diffeomorphism.

<sup>3</sup>Note that  $\mathcal{I}(\widetilde{M}) \rightarrow \mathcal{QI}(\widetilde{M})$  is an isomorphism, where  $\mathcal{QI}$  denotes the group of quasi-isometries (defined by identifying maps that differ by a bounded distance).

The constructed manifold  $X$  and the original  $\widetilde{M}$  can both be constructed in the same way from  $\Omega(\widetilde{M})$ , but they potentially differ in the choices just mentioned. We can construct a bilipschitz diffeomorphism  $f: X \rightarrow \widetilde{M}$  inductively, starting with an isometry from one piece of  $X$  to one piece of  $\widetilde{M}$  and extending repeatedly over adjacent pieces. At any point in the induction, when extending to an adjacent piece across an adjacent slab  $S$ , we use an isometry of the adjacent piece  $X_i$  of  $X$  to the adjacent piece  $M_i$  of  $\widetilde{M}$  that takes the boundary component  $X_i \cap S$  of  $X_i$  to the boundary component  $M_i \cap S$  of  $M_i$ . Restricted to this boundary component  $E$ , this isometry is well defined up to the action of a lattice  $\mathbb{Z}^2$ , so there is a choice that can be extended across the slab  $S$  with an amount of shear bounded by the diameter of the torus  $E/\mathbb{Z}^2$ . Since only finitely many isometry classes of such tori occur in the construction, we can inductively construct the desired diffeomorphism using a uniformly bounded amount of shear on slabs. This diffeomorphism therefore has a uniformly bounded bilipschitz constant, as desired.  $\square$

We now describe how the above arguments must be modified if  $M$  has arithmetic pieces. Let  $M_1$  be an arithmetic hyperbolic piece which is adjacent to a non-arithmetic hyperbolic piece  $M_2$ . We have already discussed how  $M_2$  is neutered; we take an arbitrary neutering of  $M_1$  (and any other arithmetic pieces) which we will adjust later. As before, the notation  $M_i$  refers to the neutered pieces and we glue  $M$  from these pieces mediating with toral annuli  $T^2 \times I$  between them.

We aim to show that we can adjust the neutering of the arithmetic pieces so that any quasi-isometry of  $\widetilde{M}$  can be straightened as in the beginning of this section to be an isometry on pieces and a shear map on slabs.

Consider  $\widetilde{M}_1$  as a subset of  $\mathbb{H}^3$  obtained by removing interiors of infinitely many disjoint horoballs. Schwartz [12] shows that any quasi-isometry of  $\widetilde{M}_1$  is a bounded distance from an isometry of  $\widetilde{M}_1$  to a manifold obtained by changing the sizes of the removed horoballs by a uniformly bounded amount.

In the universal cover  $\widetilde{M}$  choose lifts  $\widetilde{M}_1$  and  $\widetilde{M}_2$  glued to the two sides  $\partial_1 S$  and  $\partial_2 S$  of a slab  $S \cong \mathbb{R}^2 \times I$ . Consider a quasi-isometry  $f$  of  $\widetilde{M}$  which maps  $S$  to a bounded Hausdorff distance from itself. We can assume that  $f$  is an isometry on  $\widetilde{M}_2$ . By the previous remarks, the map  $f$  restricted to  $\widetilde{M}_1$  is a bounded distance from an isometry of  $\widetilde{M}_1$  which moves its boundary component  $\partial_1 S$  to a parallel horosphere (if we consider  $\widetilde{M}_1$  as a subset of  $\mathbb{H}^3$ ); by inverting  $f$  if necessary, we can assume the diameter of the horosphere has not decreased. Thus  $\widetilde{M}_1$  can be positioned in  $\mathbb{H}^3 = \{(z, y) \in \mathbb{C} \times \mathbb{R} : y > 0\}$  so that  $\partial_1 S$  is the horosphere  $y = 1$  and  $f(\partial_1 S)$  is the horosphere  $y = \lambda$  for some  $\lambda \leq 1$ . Using a smooth isotopy of  $f$  which is supported in an  $\epsilon$ -neighborhood of the region between  $f(\partial_1 S)$  and  $\partial_1 S$ , and which moves  $f(\partial_1 S)$  to  $\partial_1 S$ , we can adjust  $f$  to map  $\partial_1 S$  to itself. This moves each point of  $f(\partial_1 S)$  to its closest point on  $\partial_1 S$  by a euclidean similarity, scaling distance uniformly by a factor of  $\lambda$ . The resulting adjusted  $f$  is still a quasi-isometry, so restricted to  $S$  we then have a quasi-isometry which scales metric on  $\partial_1 S$  by  $\lambda$  and is an isometry on  $\partial_2 S$ . This is only possible if  $\lambda = 1$ , so  $f$ , once straightened on  $\widetilde{M}_1$ , maps  $\partial_1 S$  to itself.

Now consider the subgroup of the group of quasi-isometries of  $\widetilde{M}$  which takes  $\widetilde{M}_1$  to itself, and just consider its restriction to  $\widetilde{M}_1$ , which we can think of as embedded in  $\mathbb{H}^3$ . By straightening, we have a group of isometries of  $\mathbb{H}^3$  which preserves a family of disjoint horoballs (the ones that are bounded by images of  $\partial_1 S$ ). Any

subgroup of  $\text{Isom}(\mathbb{H}^3)$  which preserves an infinite family of disjoint horoballs is discrete. Thus, from the point of view of the construction above,  $M_1$  behaves like a non-arithmetic piece. By repeating the argument, this behavior propagates to any adjacent arithmetic pieces, hence, so long as at least one piece is non-arithmetic, the construction of the graph  $\Omega(\widetilde{M})$  goes through as before and the proof of Proposition extends.

**Proposition 4.2.**  *$\Omega(\widetilde{M})$  is balanced, and is the minimal NAH-graph in the bisimilarity class of the NAH-graph  $\Gamma(M)$  associated with  $M$  (Definition 3.3).*

*Proof.* By construction, there is a morphism  $\Gamma(M) \rightarrow \Omega(\widetilde{M})$ . In particular,  $\Omega(\widetilde{M})$  is in the bisimilarity class of  $\Gamma(M)$ . It is balanced by Theorem 3.7, since  $\Gamma(M)$  is integral. It remains to show that it is the minimal NAH-graph in its class.

We have not yet proved Theorem 3.6, which says that there is a unique minimal graph in each bisimilarity class. The proof that  $\Omega(\widetilde{M})$  is minimal will follow from that proof, so we do that first.

*Proof of Theorem 3.6.* The construction of the proof of Proposition 4.1 works for any NAH-graph  $\Gamma$ , gluing together infinitely many copies of the universal covers  $\widetilde{N}_i$  of the orbifolds that label the vertices, with slabs between them, according to an infinite tree (the universal cover of the graph obtained by replacing each edge of  $\Gamma$  by countable-infinitely many). We get a simply connected riemannian manifold  $X(\Gamma)$  which, as long as  $\Gamma$  is finite, is well defined up to bilipschitz diffeomorphism by the same argument as before.

The group of straightened self-diffeomorphisms  $\mathcal{I}(X(\Gamma))$ , when restricted to a piece  $\widetilde{N}_i$ , includes the covering transformations for the covering  $\widetilde{N}_i \rightarrow N_i$ . It follows that the construction of a NAH-graph from  $X(\Gamma)$ , as given in the first part of this section, yields an NAH-graph  $\Omega(X(\Gamma))$  together with a morphism  $\Gamma \rightarrow \Omega(X(\Gamma))$ .

If  $\Gamma \rightarrow \Gamma'$  is a morphism of NAH-graphs, then  $X(\Gamma)$  and  $X(\Gamma')$  are bilipschitz diffeomorphic, since the instructions for assembling them are equivalent. Hence  $\Omega(X(\Gamma)) = \Omega(X(\Gamma'))$ ; call this graph  $m(\Gamma)$ . Since  $m(\Gamma) = m(\Gamma')$  and the existence of a morphism generates the relation of bisimilarity of NAH-graphs,  $m(\Gamma)$  is the same for every graph in the bisimilarity class. It is also the target of a morphism from every  $\Gamma$  in this class. Thus it must be the unique minimal element in the class, so Theorem 3.6 is proved.  $\square$

If  $M$  is an NAH-orbifold and  $\Gamma = \Gamma(M)$  its associated integral NAH-graph, then  $X(\Gamma)$  reconstructs  $\widetilde{M}$ , so  $\Omega(\widetilde{M}) = \Omega(X(\Gamma)) = m(\Gamma)$ , and is hence minimal by the previous proof, so Proposition 4.2 now follows.  $\square$

*Proof of Classification Theorem for NAH-manifolds.* Since  $\widetilde{M}$  is quasi-isometric to  $\pi_1(M)$ , a quasi-isometry between fundamental groups of  $M$  and  $M'$  induces a quasi-isometry  $\widetilde{M} \rightarrow \widetilde{M}'$ . We have already explained how this can then be straightened and thus give an isomorphism of the corresponding minimal NAH-graphs.  $\square$

For the Classification Theorem to be a complete classification of quasi-isometry types of fundamental groups of good manifolds we will need to know that every minimal balanced NAH-graph is realized by an NAH-manifold. This is the content of the Realization Theorem which we prove later.

## 5. COVERS AND RFCH

For our realization theorem, and also for our commensurability theorem, we need to produce covers of 3-manifolds with prescribed boundary behavior. The covers we need can be summarized by the following purely topological conjecture, which we find is of independent interest.

**Cusp Covering Conjecture 5.1** ( $CCC_n$ ). *Let  $M$  be a hyperbolic  $n$ -manifold. Then for each cusp  $C$  of  $M$  there exists a sublattice  $\Lambda_C$  of  $\pi_1(C)$  such that, for any choice of a sublattice  $\Lambda'_C \subset \Lambda_C$  for each  $C$ , there exists a finite cover  $M'$  of  $M$  whose cusps covering each cusp  $C$  of  $M$  are the covers determined by  $\Lambda'_C$ .*

We only need this conjecture for  $n = 3$ , which, as we mention in the introduction, follows from work of Dani Wise [13, Lemma 18.9].

We now show that the general version of the conjecture follows from the well-known residual finiteness conjecture for hyperbolic groups (RFCH).

**Theorem 5.2.** *The residual finiteness conjecture for hyperbolic groups (RFCH) implies  $CCC_n$  for all  $n$ .*

*Proof.* The Dehn surgery theorem for relatively hyperbolic groups of Osin [8] and Groves and Manning [2] guarantees the existence of sublattices  $\Lambda_C$  of  $\pi_1(C)$  for each cusp so that, given any subgroups  $\Lambda'_C \subset \Lambda_C$  for each  $C$ , the result of adding relations to  $\pi_1(M)$  which kill each  $\Lambda'_C$  gives a group  $G$  into which the groups  $\pi_1(C)/\Lambda'_C$  inject and which is relatively hyperbolic relative to these subgroups. If the  $\Lambda'_C$  are sub-lattices, then  $G$  is relatively hyperbolic relative to finite (hence hyperbolic) subgroups, and is hence itself hyperbolic. By RFCH we may assume  $G$  is residually finite, so there is a homomorphism of  $G$  to a finite group  $H$  such that each of the finite subgroups  $\pi_1(C)/\Lambda'_C$  of  $G$  injects. The kernel  $K$  of the composite homomorphism  $\pi_1(M) \rightarrow G \rightarrow H$  thus intersects each  $\pi_1(C)$  in the subgroup  $\Lambda'_C$ . The covering of  $M$  determined by  $K$  therefore has the desired property.  $\square$

Let  $\Gamma$  be a NAH-graph. For an edge  $e$  of  $\Gamma$  let  $T_e$  be the tangent space of the cusp orbifold corresponding to the start of  $e$ . We can identify  $T_e$  with  $T_{\bar{e}}$  using the linear map  $\ell_e$ , so we will generally not distinguish  $T_e$  and  $T_{\bar{e}}$ . Then  $T_e$  contains two  $\mathbb{Z}$ -lattices, the underlying lattices for the orbifolds at the two ends of  $e$ , and we will denote their intersection by  $\Lambda_e$ . Thus, a torus  $T_e/\Lambda$  is a common cover of the cusp orbifolds at the two ends of  $e$  if and only if  $\Lambda \subset \Lambda_e$ .

From  $CCC_3$ , we can now choose a sublattice  $\Lambda'_e$  of  $\Lambda_e$  for each edge  $e$  of  $\Gamma$  such that for each vertex  $v$  of  $\Gamma$  the corresponding orbifold  $N_v$  has a cover  $M_v$  with the following property:

**Property 5.3.** For each cusp of  $N_v$  corresponding to an edge  $e$  departing  $v$ , all cusps of  $M_v$  which cover it are of type  $T_e/\Lambda'_e$ .

This property is precisely the consequence of the  $CCC_3$  that we use in our proofs of the Realization and Commensurability Theorems.

## 6. REALIZING GRAPHS

*Proof of Realizability Theorem for NAH-manifolds.* Let  $\Gamma$  be a balanced NAH-graph. We want to show there is some NAH-manifold which realizes a graph in its bisimilarity class. As pointed out in the previous section, this is equivalent to finding an integral NAH-graph in the bisimilarity class.

Using  $\text{CCC}_3$ , after we choose a sublattice  $\Lambda'_e$  of  $\Lambda_e$  for each edge  $e$  of  $\Gamma$ , we may assume Property 5.3 holds.

Let  $d_v$  be the degree of the cover  $M_v \rightarrow N_v$ . For an edge  $e$  departing  $v$  let  $d_e$  be the degree of the corresponding cover of cusp orbifolds of  $M_v$  and  $N_v$ , i.e., the index of the lattice  $\Lambda'_e$  in the fundamental group of the boundary component  $C_e$  of  $N_v$  corresponding to  $e$  (this is slightly different from the usage in the proof of Theorem 3.7). Since the cusps of the  $M_v$  corresponding to the two ends of  $e$  are equal, we have

$$(5) \quad d_e \delta_e = d_{\bar{e}}.$$

(Recall that  $\delta_e$  denotes the determinant of the linear map  $\ell_e$ .)

Since  $\Gamma$  is balanced, we can assign a positive rational number  $m(v)$  to each vertex with the property that for any edge  $e$  one has  $m(\tau e) = \delta_e m(\iota e)$ . Thus, by (5),

$$(6) \quad \frac{m(\tau e)}{d_{\bar{e}}} = \frac{m(\iota e)}{d_e}.$$

Choose a positive integer  $b$  such that  $n(v) := \frac{bm(v)}{d_v}$  is integral for every vertex of  $\Gamma$  (so  $b$  is some multiple of the lcm of the denominators of the numbers  $\frac{m(v)}{d_v}$ ). Let  $M'_v$  be the disjoint union of  $n(v)$  copies of  $M_v$ , so  $M'_v$  is a  $bm(v)$ -fold cover of  $N_v$ . Let  $\pi_v: M'_v \rightarrow N_v$  be the covering map.

For an edge  $e$  of  $\Gamma$  from  $v = \iota e$  to  $w = \tau e$ , the number of boundary components of  $M'_v$  covering the boundary component  $C_e$  of  $N_v$  corresponding to  $e$  is  $bm(v)/d_e$ . By (6) this equals  $bm(w)/d_{\bar{e}}$ , which is the number of boundary components of  $M'_w$  covering the boundary component  $C_{\bar{e}}$  of  $N_w$ . Thus  $\pi_v^{-1}C_e$  and  $\pi_w^{-1}C_{\bar{e}}$  have the same number of components, and each component is  $T_e/\Lambda'$ , so we can glue  $M'_v$  to  $M'_w$  along these boundary components using any one-one matching between them. Doing this for every edge gives a manifold  $M$  whose NAH-graph has a morphism to  $\Gamma$ ; if  $M$  is disconnected, replace it by a component (but one can always do the construction so that  $M$  is connected). This proves the theorem.  $\square$

## 7. COMMENSURABILITY

*Proof of Commensurability Theorem.* Let  $M_1$  and  $M_2$  be two NAH-manifolds whose NAH-graphs are bisimilar. Assume that their common minimal NAH-graph is a tree and all the vertex labels are manifolds. We want to show that  $M_1$  and  $M_2$  are commensurable.

Let  $\Gamma$  be the common minimal NAH-graph for  $M_1$  and  $M_2$ . Using  $\text{CCC}_3$  we may assume Property 5.3 holds. Accordingly, for each vertex of  $\Gamma$  we may take a common cover  $N'_v$  of all the pieces of  $M_1$  that cover  $N_v$  and then choose the lattices  $\Lambda'_e$ , as in the previous section, to be subgroups of the cusp groups of these manifolds  $N'_v$ . In this way we arrange that the pieces  $M_v$  of the manifold  $M$  constructed in that section are covers of  $N'_v$ , and hence of the pieces of  $M_1$ . Elementary arithmetic as in the proof of the Realization Theorem shows that there is a  $b_0$  so that if the number  $b$  of that proof is a multiple of  $b_0$  then we can choose the glueing in that proof to make  $M$  a covering space of  $M_1$ . Call the resulting manifold  $M'_1$ . If we initially choose  $N'_v$  to also cover the type  $v$  pieces of  $M_2$  then we can also construct a covering space  $M'_2$  of  $M_2$  out of copies of pieces  $M_v$ . The decompositions of  $M'_1$  and  $M'_2$  then give NAH-graphs whose vertices are labelled by  $M_v$ 's and whose edges are labelled by  $\mathbb{Z}$ -isomorphisms. It suffices to show that  $M'_1$  and  $M'_2$  are commensurable.

We can encode the information needed to construct  $M'_1$  in a simplified version  $\Gamma_0(M'_1)$  of its NAH-graph. The underlying graph is still just the graph describing the decomposition of  $M'_1$  into pieces, but the labelling is simplified as follows. Each vertex  $w$  of  $\Gamma_0(M'_1)$  corresponds to a copy of some  $M_v$ , which is a normal covering of the orbifold  $N_v$ . We say vertex  $w$  is of *type*  $v$ . The covering transformation group for the covering  $M_v \rightarrow N_v$  induces a permutation group  $P_v$  on the edges of  $\Gamma_0(M'_1)$  exiting  $w$ . We record the type and permutation action for each vertex of  $\Gamma_0(M'_1)$ . It is easy to see that the graph  $\Gamma_0(M'_1)$  with these data records enough information to reconstruct  $M'_1$  up to diffeomorphism from its pieces. A covering of such a graph induces a covering of the same degree for the manifolds they encode.

A graph with fixed permutation actions at vertices as above is called *symmetry restricted* in [7]. The graphs  $\Gamma_0(M'_1)$  and  $\Gamma_0(M'_2)$  have the same universal covering as symmetry restricted graphs.

So we would like to know that when two such graphs have a common universal covering, then they have a common finite covering. A generalization, proved in [7] of Leighton's theorem [6] (which deals with graphs without the extra structure) shows that this is true if the underlying graph is a tree, so we are done.  $\square$

The results of [7] allow one to carry out the above proof under slightly weaker assumptions on the minimal graph than being a tree, but not sufficiently general to warrant going into details.

## 8. ADDING SEIFERT FIBERED PIECES

We will modify Definition 3.1 to allow the inclusion of Seifert fibered space pieces. The graphs we use are called H-graphs, and we define them below.

We will need to consider Seifert fibered orbifolds among the pieces, so we first describe a coarse classification into types. As always, the manifolds and orbifolds we consider are oriented. We will distinguish two types: the oriented Seifert fibered orbifold  $N$  is type “o” or “n” according as the Seifert fibers can be consistently oriented or not.  $N$  is type “o” if and only if the base orbifold  $S$  of the Seifert fibration is orientable. This can fail in two ways: the topological surface underlying  $S$  may be non-orientable, or  $S$  may be non-orientable because it has mirrors. The latter arises when parts of  $N$  look locally like a Seifert fibered solid torus  $D^2 \times S^1$  factored by the involution  $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$  (using coordinates in  $\mathbb{C}^2$  with  $|z_1| \leq 1$  and  $|z_2| = 1$ ). The fibers with  $z_1 \in \mathbb{R}$  in this local description are intervals (orbifolds of the form  $S^1/(\mathbb{Z}/2)$ ), and the set of base points of such interval fibers of  $N$  form mirror curves in  $S$  which are intervals and/or circles embedded in the topological boundary of  $S$ . If any of these mirror curves are intervals, so they merge with part of the true boundary of  $S$  (image of boundary of  $N$ ), then the corresponding boundary component of  $N$  is a pillow orbifold (topologically a 2-sphere, with four 2-orbifold points).

If two type “o” Seifert fibered pieces in the decomposition of  $M$  are adjacent along a torus, and we have oriented their Seifert fibers, then there is a sense in which these orientations are compatible or not. We say the orientation is *compatible* or *positive* if, when viewing the torus from one side, the intersection number in the torus of a fiber from the near side with a fiber from the far side is positive. Note that this is well defined, since if we view the torus from the other side, both its orientation and the order of the two curves being intersected have changed, so the intersection number is unchanged.

We define H-graphs below, with the geometric meanings of the new ingredients in square brackets. But there is a caveat to these descriptions. Just as an NAH-graph can be associated to the geometric decomposition of an NAH-manifold, an H-graph can be associated with the decomposition of a good manifold or orbifold  $M$  (see the Introduction for the definition of “good”). However, the NAH-graph associated with a geometric decomposition is a special kind of NAH-graph (it is “integral” in the terminology of Section 3), and an H-graph coming from the geometric decomposition of a good manifold is similarly special. The geometric explanations therefore only match precisely for special cases of H-graphs.

**Definition 8.1.** An  $H$ -graph  $\Gamma$  is a finite connected graph with decorations on its vertices and edges as follows:

- (1) Vertices are partitioned into two types: *hyperbolic vertices* and *Seifert vertices*. The full subgraphs of  $\Gamma$  determined by the hyperbolic vertices, respectively the Seifert vertices, are called the *hyperbolic subgraph*, respectively the *Seifert subgraph*.
- (2) The hyperbolic subgraph is labelled as in Definition 3.1, so that each of its components is an NAH-graph. In particular, each hyperbolic vertex  $v$  is labelled by a hyperbolic orbifold  $N_v$  and there is a map  $e \mapsto C_e$  from the set of directed edges  $e$  exiting that vertex to the set of cusp orbifolds  $C_e$  of  $N_v$ . This map is defined on the set of *all* edges exiting  $e$ , not just the edges in the hyperbolic subgraph. As before, it is injective, except that an edge which begins and ends at the same vertex may have  $C_e = C_{\bar{e}}$ .
- (3) Each Seifert vertex is labelled by one of two colors, black or white [for an H-graph coming from a geometric decomposition this encodes whether the Seifert fibered piece in the geometric decomposition of  $M$  contains boundary components of the ambient 3-manifold or not, as in [1]]. It is also labelled by a Seifert fibration type “o” or “n”, as described above.
- (4) For an edge  $e$  starting at a Seifert vertex the group  $F_e = F_{\bar{e}}$  is  $\{1\}$  or  $\{\pm 1\}$ . If  $F_e$  is  $\{\pm 1\}$  then the Seifert vertex is type “n”.
- (5) For each edge  $e$  from a hyperbolic vertex to a Seifert vertex the group  $F_e$  associated to the cusp section  $C_e$  is either trivial or  $\{\pm 1\}$ . The edge is labelled by a non-zero rational vector in  $TC_e$  called the *slope*  $s_e$ , determined up to the action of  $F_e$ . [ $s_e$  encodes the direction and length of the fibers of the adjacent Seifert fibered piece.]
- (6) Each edge  $e$  connecting a type “o” Seifert vertex with a type “o” Seifert vertex or a hyperbolic vertex has a *sign label*  $\epsilon_e = \pm 1$  with  $\epsilon_e = \epsilon_{\bar{e}}$  [this describes compatibility of orientations of Seifert fibers of adjacent pieces or—if the edge connects a Seifert and a hyperbolic vertex—of Seifert fiber and slope].
- (7) The data described in items (5) and (6) are subject to the equivalence relation generated by the following moves:
  - (a) For any type “o” Seifert vertex the signs at all edges adjacent to it may be multiplied by  $-1$  [reversal of orientation of the Seifert fibers].
  - (b) The slope  $s_e$  of item (5) can be multiplied by  $-1$  while simultaneously multiplying  $\epsilon_e$  by  $-1$ .
  - (c) For any Seifert vertex, the slopes at all adjacent hyperbolic vertices may be multiplied by a fixed non-zero rational number.

Note that the data encoded by the sign weights modulo the equivalence relation of item (7a) are equivalent to an element of  $H^1(\Gamma \setminus \Gamma_n, \Gamma_h; \mathbb{Z}/2)$ , where  $\Gamma_h$  is the hyperbolic subgraph and  $\Gamma_n$  the full subgraph on Seifert vertices of type “n”.

We define a *morphism of H-graphs*,  $\pi: \Gamma \rightarrow \Gamma'$ , to be an open graph homomorphism which restricts to a NAH-graph morphism of the hyperbolic subgraphs (Definition 3.4), preserves the black/white coloring on the Seifert vertices, and on the edges between hyperbolic and Seifert vertices preserves the slope (in the sense that the slope at the image edge is the image under the tangent map on cusp orbifolds of the slope at the source edge). Moreover, it must map type “n” vertices to type “n” vertices, and when an “o” vertex  $v$  is mapped to an “n” vertex  $w$ , then the preimage of each edge at  $w$  must either include edges of different signs, or an edge terminating in a type “n” Seifert vertex.

As in Section 3, the existence of morphisms between H-graphs generates an equivalence relation which we call *bisimilarity*.

*Proof of Classification Theorem.* To prove the Classification Theorem we will show that each bisimilarity class of H-graphs has a minimal element, and if a H-graph comes from a non-geometric manifold  $M$  then the minimal H-graph determines and is determined by  $\widetilde{M}$  up to quasi-isometry.

We first explain why the qi-type of the universal cover  $\widetilde{M}$  (or equivalently of  $\pi_1(M)$ ) determines a minimal H-graph  $\Omega(\widetilde{M})$ .

As in Section 4, we can straighten any quasi-isometry  $\widetilde{M} \rightarrow \widetilde{M}'$  and assume it takes geometric pieces to geometric pieces and slabs to slabs. We may also assume it is an isometry on hyperbolic pieces. A Seifert piece in  $\widetilde{M}$  is bi-Lipschitz homeomorphic to a fattened tree times  $\mathbb{R}$ , so the fibration by  $\mathbb{R}$  fibers is coarsely preserved, and we can straighten it so it is actually preserved. Moreover, if an adjacent piece is hyperbolic, then the straightened quasi-isometry is an isometry on the corresponding flat, so the affine structure on fibers of the Seifert piece is coarsely preserved, and we can straighten so that it is actually preserved. However, where a Seifert piece is adjacent to a Seifert piece the  $\mathbb{R} \times \mathbb{R}$  product structure on the corresponding flat (given by Seifert fibers on the two sides) is coarsely preserved, but the affine structures on the  $\mathbb{R}$  fibers need only be preserved up to quasi-isometry.

Considering straightened quasi-isometries in the above sense, we denote again

$$\mathcal{I}(\widetilde{M}) := \{f: \widetilde{M} \rightarrow \widetilde{M} \mid f \text{ is a straightened quasi-isometry}\}.$$

As in Section 4, the underlying graph for our minimal H-graph  $\Omega(\widetilde{M})$  has a vertex for each orbit of the action of  $\mathcal{I}(\widetilde{M})$  on the set of pieces of  $\widetilde{M}$  and edge for each orbit of the action on the set of slabs. The labelling of the hyperbolic subgraph is as before; in particular, any vertex corresponding to an orbit of hyperbolic pieces is labelled by the hyperbolic orbifold obtained by quotienting a representative piece in the orbit by its isotropy subgroup in  $\mathcal{I}(\widetilde{M})$ . A Seifert vertex of the H-graph is of type “n” if some element of  $\mathcal{I}(\widetilde{M})$  takes a corresponding Seifert fibered piece to itself reversing orientations of fibers, and is otherwise of type “o”. For each type “o” vertex we choose an orientation of the fibers of one piece in the corresponding orbit and then extend equivariantly to the other pieces in the orbit.

For a Seifert vertex adjacent to at least one hyperbolic vertex the fibers of the corresponding pieces in  $\widetilde{M}$  carry an affine structure which is defined up to affine scaling. We choose a specific scale for each such vertex, so we can speak of length

along fibers, and then the *slope* of item (5) of Definition 8.1 is given by a tangent vector of unit length, viewed in the adjacent cusp.

The sign weights of item (6) of Definition 8.1 are then defined, and item (7) of that definition reflects the choices of orientation and scale which were made.

By construction, the isomorphism type of  $\Omega(\widetilde{M})$  is determined by  $\widetilde{M}$  and thus two manifolds with quasi-isometric fundamental group have the same associated graph. It remains to show that the isomorphism type of  $\Omega(\widetilde{M})$  determines the bilipschitz homeomorphism type of  $\widetilde{M}$ .

Construct a labelled graph  $\widetilde{\Omega}$  from  $\Omega = \Omega(\widetilde{M})$  by first replacing each edge of  $\Omega$  by infinitely many edges, keeping the weights on edges, but adding sign weights to edges at type “n” vertices with infinitely many of each sign, and then taking the universal cover of the resulting weighted graph. Finally, “o” and “n” labels are now irrelevant and can be removed.

To associate a manifold  $X = X(\widetilde{\Omega})$  to this labelled graph, we must glue together appropriate pieces according to the tree  $\widetilde{\Omega}$ , with appropriate choices for the gluing between slabs. The pieces for hyperbolic vertices will be universal covers of the hyperbolic orbifolds which label them, while for a Seifert vertex we take the universal cover of some fixed Seifert fibered manifold with base of hyperbolic type and having a boundary component for each incident edge in  $\Omega$  and—if the vertex is a black vertex—an additional boundary component to contribute to boundary of  $X$ . (The universal cover  $Y$  of this Seifert piece is then a fattened tree times  $\mathbb{R}$ , and, as in [1], it is in fact only important that there is a bound  $B$  such that for each boundary component of  $Y$  there are boundary components of all “types” within distance  $B$  of the give boundary component.)

The choices in gluing depend on the types of the abutting pieces: Between hyperbolic pieces, the gluing map is, as before, determined up to a group of isometries of the form  $\mathbb{R}^2 \rtimes F_S$ , where  $S$  is the slab. Between Seifert fibered pieces the gluing will be an affine map such that the fibers from the two pieces then intersect in the intervening flat with sign given by the sign label of the edge. Finally, between a Seifert fibered and a hyperbolic piece the gluing will be an affine map matching unit tangent vector along fibers with the slope vector for the hyperbolic piece. For each edge of  $\Omega$  we make a fixed choice of how to do the gluing subject to the above constraints and do it this way for every corresponding edge of  $\widetilde{\Omega}$ .

To complete the proof, it remains to show that, independent of these choices, there exists a bilipschitz homeomorphism from  $\widetilde{M}$  to  $X$ .

As in the proof of Proposition 4.1, the desired bilipschitz homeomorphism is built inductively, starting with a homeomorphism from one piece of  $\widetilde{M}$  to a piece of  $X$  and then extending via adjacent slabs to adjacent pieces. There are four cases: (1), when both adjacent pieces are hyperbolic, this is exactly the case of Proposition 4.1; (2), when both pieces are Seifert fibered; (3), extending from a hyperbolic piece to an adjacent Seifert fibered piece; and (4), extending from a Seifert fibered piece to an adjacent hyperbolic piece. For Case (2) we use [1, Theorem 1.3] (as in the proof of [1, Theorem 3.2]) to extend over the adjacent Seifert fibered piece, respecting the “types” of boundary components (i.e., belonging to boundary of  $M$  or not, and if not, then the “type” is given by the edge of  $\Omega$  that the boundary component corresponds to). Case (3) is essentially the same argument, and Case (4) is immediate.

Thus we obtain the desired bilipschitz homeomorphism, completing the proof.  $\square$

*Proof of Realization Theorem.* The construction of  $\Omega(\widetilde{M})$ , given above, has built in a morphism from the H-graph associated to  $M$ . Since the graph associated to  $M$  is balanced, it follows from Proposition 4.2 that  $\Omega(\widetilde{M})$  is balanced.

The balanced condition is only a constraint on NAH-graph components of an H-graph. Thus, from the case of NAH-graphs which we established in Section 6, we may conclude that every balanced minimal H-graph is the minimal H-graph of some quasi-isometry class of good 3-manifold group.  $\square$

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