

DIVERGENCE AND QUASIMORPHISMS OF RIGHT-ANGLED ARTIN GROUPS

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ABSTRACT. We give a group theoretic characterization of geodesics with superlinear divergence in the Cayley graph of a right-angled Artin group A_Γ with connected defining graph. We use this to prove that the divergence of A_Γ is linear if Γ is a join and quadratic otherwise. As an application, we give a complete description of the cut points in any asymptotic cone of A_Γ . We also show that every non-abelian subgroup of A_Γ has an infinite-dimensional space of non-trivial quasimorphisms.

1. INTRODUCTION

The *divergence of a geodesic*, $\gamma: [-\infty, \infty] \rightarrow X$, in a metric space, can roughly be thought of as the growth rate of a function from \mathbb{N} to \mathbb{R} , which for each $N \in \mathbb{N}$ gives the length of the shortest path in $X \setminus B_N(\gamma(0))$ from $\gamma(-N)$ to $\gamma(N)$, where $B_N(\gamma(0))$ is the open ball of radius N about $\gamma(0)$. We refer to the *divergence of a finitely generated group* to mean the largest divergence over all geodesics in a given Cayley graph of G .

The divergence function has proven to be a useful tool in studying the large scale geometry of groups. Gersten classified geometric 3-manifolds by their divergence [Ger] which allows one to distinguish quasi-isometry classes of 3-manifold groups containing hyperbolic pieces from graph manifold groups [KL]. In addition, divergence functions are closely related to cut-points in the asymptotic cones of a group. Interest in the existence of such cut-points arose from Druţu–Osin–Sapir’s result that a group is relatively hyperbolic with respect to a collection of subgroups \mathcal{H} if and only if every asymptotic cone has a collection of cut-points with the property that the maximal subsets of the asymptotic cone not separated by any one of these cut-points arise from asymptotic cones of the subgroups \mathcal{H} [DS].

On the other hand, cut-points in asymptotic cones also arise in groups which are not relatively hyperbolic. To prove that any point in an asymptotic cone of a mapping class group is a cut-point, the first author showed that axes of pseudo-Anosov elements in a mapping class group have super-linear divergence. This also implies that these directions are quasi-geodesically stable, or equivalently, Morse geodesics. [Beh]. (Alternate proofs have since been obtained by [DMS] and [DR]). More recently, Druţu–Mozes–Sapir showed in [DMS] that a group has superlinear divergence if and only if its asymptotic cones contain cut-points, and that this occurs if and only if the group contains Morse geodesics.

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In this paper we discuss divergence in right-angled Artin groups. Given a finite, simplicial graph Γ , the *right-angled Artin group* A_Γ is the finitely presented group with generators corresponding to vertices of Γ and relators of the form $x^{-1}y^{-1}xy$ whenever the vertices x and y of Γ are connected by an edge. Right-angled Artin groups form a rich family of groups interpolating between \mathbb{Z}^n , the group corresponding to the complete graph on n vertices, and the free group F_n , corresponding to the graph with n vertices and no edges.

If Γ is disconnected or consists of a single vertex, then A_Γ is a free product or $A_\Gamma = \mathbb{Z}$ and the divergence in A_Γ is infinite. Thus, we assume throughout that Γ is a connected graph with at least two vertices. If Γ_1 and Γ_2 are two graphs, their *join* is the graph obtained by connecting every vertex of Γ_1 to every vertex of Γ_2 by an edge. Subgraphs of Γ that decompose as joins are central to understanding divergence of geodesics. We define a notion of *join length* of a geodesic, which measures the number of cosets of join subgroups the geodesic passes through (see Section 3 for the precise definition) and we prove,

Theorem 4.4 (Divergence and join length). *Let Γ be a connected graph with at least 2 vertices and let α be a bi-infinite geodesic in A_Γ . Then α has linear divergence if and only if the join length of α is finite.*

The proof uses the action of A_Γ on a CAT(0) cube complex, X_Γ , the universal cover of the Salvetti complex of A_Γ . We show that the join length of a geodesic α determines the behavior of the walls in X_Γ crossed by α .

From the divergence theorem, we obtain the following complete characterization of when two points in an asymptotic cone of a right-angled Artin group, A_Γ^ω , can be separated by a cut-point.

Theorem 4.6 (Classification of pieces). *Let Γ be a connected graph with at least 2 vertices. Fix a pair of points $\mathbf{a}, \mathbf{b} \in A_\Gamma^\omega$. The following are equivalent.*

- (1) *No point of A_Γ^ω separates \mathbf{a} from \mathbf{b} .*
- (2) *There exist points \mathbf{a}', \mathbf{b}' arbitrarily close to \mathbf{a}, \mathbf{b} , respectively for which the join length between \mathbf{a}', \mathbf{b}' is finite.*

In the terminology of [DS], cut-points in an asymptotic cone give rise to a *tree-grading* whose *pieces* are the maximal subsets that cannot be separated by a point. The above result gives a complete description of the pieces in A_Γ^ω . Since, except for the integers, freely indecomposable right-angled Artin groups are not relatively hyperbolic with respect to any collection of subgroups or even subsets [BDM], it follows that these pieces do not arise by taking asymptotic cones of subgroups of A_Γ [DS].

In [BDM], Behrstock–Druţu–Mosher introduce a notion of *algebraic thickness* of a group. Theorem 4.6 shows that for a connected graph Γ , A_Γ is algebraically thick of order zero if Γ is a join, and otherwise it is algebraically thick of order at least one with respect to the set of maximal join subgroups. It was established in [BDM, Corollary 10.8], that, except for \mathbb{Z} , right-angled Artin groups with connected presentation graph are thick of order at most one. Together, these two results show that if A_Γ is a join, then it is algebraically thick of order exactly zero, and otherwise it is algebraically thick of order exactly one.

As a consequence, we deduce the following result.

Corollary 4.8 (Quadratic divergence). *Let Γ be a connected graph with at least 2 vertices. A_Γ has linear divergence if and only if Γ is a join; otherwise its divergence is quadratic.*

An independent proof of this corollary appears in a recently posted preprint by A. Abrams, N. Brady, P. Dani, M. Duchin, and R. Young [ABDDY].

Right-angled Artin groups have been shown to have an extremely rich family of subgroups, cf. [BB], [HW], [CW], and [Wi]. Thus, properties which apply to arbitrary subgroups of a right-angled Artin groups are of particular interest. Recall that a rank-one isometry of a CAT(0) space is a hyperbolic isometry with an axis not bounding a half-plane. Our characterization of divergence of geodesics identifies all rank-one geodesics in X_Γ . In the case of periodic geodesics, it implies

Theorem 5.2 (Rank-one isometries). *If $G \subseteq A_\Gamma$ is any subgroup which is not contained in a conjugate of a join subgroup, then G contains an element which acts as a rank-one isometry of X_Γ .*

This is consistent with the results of Caprace and Fujiwara [CF] characterizing rank-one periodic geodesics in Coxeter groups, as well as the recent results of Caprace and Sageev [CS] giving more general criteria for a group of isometries of a CAT(0) cube complex to contain a rank-one isometry.

Bestvina and Fujiwara [BF2] have shown that for group actions on a CAT(0) space satisfying a weak discontinuity property, the existence of rank-one isometries gives rise to non-trivial quasimorphisms. A function $\phi: G \rightarrow \mathbb{R}$ is a *homogeneous quasimorphism* if $\phi(g^n) = n\phi(g)$ for all $n > 0$, and there exists a constant $D \geq 0$ such that

$$|\phi(gh) - \phi(g) - \phi(h)| \leq D$$

for every $g, h \in G$. The vector space of homogeneous quasimorphisms, modulo the subspace of true homomorphisms, is denoted $\widetilde{QH}(G)$ and is related to the bounded cohomology of G .

Corollary 5.3 (Quasimorphisms). *If G is any non-abelian subgroup of A_Γ , then $\widetilde{QH}(G)$ is infinite dimensional.*

This corollary can also be derived from the analogous theorem of Bestvina-Fujiwara [BF1] for mapping class groups by embedding the right-angled Artin group into a mapping class group using results of Crisp and Paris [CP]. On the other hand, the action of A_Γ on its Salvetti complex is a much more direct way to obtain quasimorphisms.

Quasimorphisms also give rise to rigidity theorems. By Burger and Monod [BM1, BM2, Mon1], if Λ is an irreducible lattice in a connected semisimple Lie group with finite center, no compact factors, and rank at least 2, then Λ has no non-trivial quasimorphisms. Thus, Corollary 5.3 implies that for such a Λ , every homomorphism $\rho: \Lambda \rightarrow A_\Gamma$ is trivial. This statement, however, also follows from another well-known property of right-angled Artin groups: a theorem of Duchamp and Krob [DK] states that A_Γ is residually torsion-free nilpotent, from which it follows that every non-trivial subgroup of A_Γ has infinite abelianization.

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2. WALLS

Let Γ be a finite, simplicial graph with vertex set V . The *right-angled Artin group* associated to Γ is the group A_Γ with presentation

$$A_\Gamma = \langle V \mid vw = wv \text{ if } v \text{ and } w \text{ are connected by an edge in } \Gamma \rangle.$$

Associated to any right-angled Artin group A_Γ is a CAT(0) cubical complex X_Γ with a free action of A_Γ . In this section we describe X_Γ and investigate the structure of walls in this complex.

Let T be a torus of dimension $|V|$ with edges labeled by the elements of V . Let \overline{X}_Γ denote the subcomplex of T consisting of all faces whose edge labels span a complete subgraph in Γ (or equivalently, mutually commute in A_Γ). \overline{X}_Γ is called the *Salvetti complex* for A_Γ . It is easy to verify that the Salvetti complex has fundamental group A_Γ and that the link of the unique vertex is a flag complex. It follows that its universal cover, X_Γ , is a CAT(0) cube complex with a free, cocompact action of A_Γ .

If Γ' is a full subgraph of Γ , then the inclusion $\Gamma' \rightarrow \Gamma$ induces an injective homomorphism $A_{\Gamma'} \rightarrow A_\Gamma$ and an embedding $X_{\Gamma'} \rightarrow X_\Gamma$. This embedding is locally geodesic, and hence (since X_Γ is CAT(0)) it is globally geodesic. We may thus view $X_{\Gamma'}$ as a convex subspace of X_Γ .

We now recall some basic facts about walls in a CAT(0) cube complex X . Consider the equivalence relation on the set of midplanes of cubes generated by the rule that two midplanes are related if they share a face. A *wall* (or *hyperplane*), H , is the union of the midplanes in a single equivalence class. Every wall H is a geodesic subspace of X which separates X into two components. Moreover, if H contains a (positive length) segment of a geodesic γ , then it contains the entire geodesic γ .

In the discussion that follows, we are interested in the relation between non-intersecting pairs of walls. The following terminology will be convenient.

Definition 2.1. Two walls H_1, H_2 in a CAT(0) cube complex are *strongly separated* if $H_1 \cap H_2 = \emptyset$ and no wall intersects both H_1 and H_2 .

Consider this definition in the context of a right-angled Artin group A_Γ and the cube complex X_Γ . For example, suppose Γ consists of two disjoint edges, so A_Γ is the free product $\mathbb{Z}^2 * \mathbb{Z}^2$. In this case, the Salvetti complex is the wedge of two tori, and its universal cover X_Γ consists of flats which pairwise intersect in at most one vertex. Since walls never contain vertices of X_Γ , they remain entirely in one flat. It follows that two walls are strongly separated if and only if they lie in different flats.

At the other extreme, suppose Γ is a square, in which case $A_\Gamma = F_2 \times F_2$, the product of two free groups of rank 2, and X_Γ is the product of two trees $T_1 \times T_2$. The walls consist of trees of the form $T_1 \times \{y\}$ and $\{x\} \times T_2$ where x and y are midpoints of edges in T_1 and T_2 respectively. It is now easy to see that no two walls are strongly separated.

Now let A_Γ be an arbitrary right-angled Artin group and let H_1 and H_2 be two walls in X_Γ . Consider the set of all minimal length geodesics from H_1 to H_2 . It follows from [BH, Proposition II.2.2] that the union of all such paths forms a convex subspace of X_Γ which we call the *bridge* between H_1 and H_2 .

Lemma 2.2. *If H_1 and H_2 are strongly separated, then the bridge B between them consists of a single geodesic from H_1 to H_2 .*

Proof. It suffices to show that $B \cap H_1$ (and by symmetry $B \cap H_2$) is a single point. We first show that $B \cap H_1$ does not intersect any other wall H . For suppose $x \in B \cap H_1 \cap H$. Since $x \in B$, it is the initial point of some minimal length geodesic γ from H_1 to H_2 . The initial segment of γ lies in some cube σ of X_Γ which contains midplanes in both H_1 and H . These midplanes span σ , hence the initial segment of γ , which is orthogonal to H_1 , must lie in H . It follows that all of γ lies in H and hence $H \cap H_2 \neq \emptyset$. This contradicts the assumption that H_1 and H_2 are strongly separated.

Now every wall H has an open neighborhood $N(H)$ isometric to $H \times (0, 1)$, namely the neighborhood consisting of the interiors of all cubes containing a midplane in H . Then the same argument as above (using parallel copies of H in $N(H)$) shows that $B \cap H_1 \cap N(H)$ must also be empty for all $H \neq H_1$. The only convex subsets of H_1 disjoint from every $N(H)$ are single vertices, so it follows that $B \cap H_1$ is a single point. \square

Lemma 2.3. *There are universal constants $C, D > 1$ (depending only on the dimension of X_Γ) such that if H_1 and H_2 are strongly separated and B is the bridge between them, then*

- (1) *for any $x \in H_1$ and $y \in H_2$,*

$$d(x, y) \geq C^{-1}(d(x, B) + d(y, B)) - d(H_1, H_2) - 4$$

- (2) *for any geodesic α in X_Γ , and any point c on α , if H_1 and H_2 intersect α inside the ball of radius r about c , then the bridge B is contained in the ball of radius Dr about c .*

Proof. (1) For any two points x, y in X_Γ , define $d_H(x, y)$ to be the number of walls separating x and y , or equivalently, the number of walls crossed by a geodesic from x to y . This distance function is quasi-isometric to the geodesic metric in X_Γ . More precisely,

$$d(x, y) - C \leq d_H(x, y) \leq Cd(x, y) + C$$

where C is the diameter of a maximal cube.

By Lemma 2.2, B consists of a single geodesic γ from H_1 to H_2 . Let h_1, h_2 be the endpoints of γ . Let α be the geodesic from h_1 to x , and β be the geodesic from y to h_2 . Note that α lies in H_1 and β lies in H_2 . Since the strongly separated hypothesis guarantees that no wall crosses both α and β , the path $\alpha\gamma\beta$ crosses any given wall at most twice and $d_H(x, y)$ is the number of walls it crosses exactly once. It follows that

$$d_H(x, y) \geq d_H(x, h_1) + d_H(y, h_2) - d_H(h_1, h_2).$$

Applying the inequalities above, we obtain

$$\begin{aligned} d(x, y) &\geq C^{-1}d_H(x, y) - 1 \\ &\geq C^{-1}(d_H(x, h_1) + d_H(y, h_2) - d_H(h_1, h_2)) - 1 \\ &\geq C^{-1}(d(x, h_1) + d(y, h_2) - 2C) - d(h_1, h_2) - 2 \\ &\geq C^{-1}(d(x, B) + d(y, B)) - d(H_1, H_2) - 4. \end{aligned}$$

(2) Suppose $x = H_1 \cap \alpha$ and $y = H_2 \cap \alpha$ are in the ball of radius r about c . Then every point in B is within $k = \frac{1}{2}(d(x, B) + d(y, B)) + d(H_1, H_2)$ of either x or y and hence within $k + r$ of c . By part (1), $d(x, B) + d(y, B)$ is bounded by a linear function of $d(x, y)$, and by hypothesis, $d(H_1, H_2) \leq d(x, y) \leq 2r$ so k is bounded by a linear function of r . \square

We now introduce the notion of *divergence* for bi-infinite geodesics and discuss how the existence of strongly separated walls affects the divergence.

Definition 2.4. Let X be a geodesic metric space. Let $\alpha: \mathbb{R} \rightarrow X$ be a bi-infinite geodesic in X and let ρ be a linear function $\rho(r) = \delta r - \lambda$ with $0 < \delta < 1$ and $\lambda \geq 0$. Define $\text{div}(\alpha, \rho)(r) = \text{length of the shortest path from } \alpha(-r) \text{ to } \alpha(r) \text{ that stays outside the ball of radius } \rho(r) \text{ about } \alpha(0)$ (or $\text{div}(\alpha, \rho)(r) = \infty$ if no such path exists). We say α has *linear divergence* if for some choice of ρ , $\text{div}(\alpha, \rho)(r)$ is bounded by a linear function of r , and say α has *super-linear divergence* otherwise.

It is not difficult to verify that the definition of linear divergence is independent of the choice of basepoint $\alpha(0)$. We leave this as an exercise for the reader.

Theorem 2.5. *Let α be a bi-infinite geodesic in X_Γ and suppose α crosses an infinite sequence of walls (H_0, H_1, \dots) such that for all i , H_i is strongly separated from H_{i+1} . Then H_i is strongly separated from H_j for all $i \neq j$ and α has superlinear divergence.*

Proof. Let α^+ and α^- denote the limit points of α in ∂X_Γ . Since H_i is disjoint from H_{i+1} , the half spaces H_i^+ containing α^+ form a directed set $H_0^+ \subset H_1^+ \subset \dots$. Hence no two of these walls intersect and if some wall H intersects both H_i and H_j , $i < j$, then it must cross H_{i+1} , contradicting the strong separation of H_i from H_{i+1} . It follows that H_i and H_j are strongly separated for any $i < j$.

Let $r' = \rho(r)$ and consider the ball $B_{r'}$ of radius r' about $\alpha(0)$. Let $x_i = H_i \cap \alpha$. By Lemma 2.3, for any n , we can choose r large enough so that $B_{r'/2}$ contains x_i for all $i \leq n$, as well as the bridge between H_{i-1} and H_i . Let β be any path from $\alpha(-r)$ to $\alpha(r)$ which stays outside $B_{r'}$. Then β must cross H_0, H_1, \dots, H_n in a sequence of points y_0, y_1, \dots, y_n . Note that each y_i is distance at least $r'/2$ from the bridges to the adjacent walls, hence by Lemma 2.3, there is a universal constant C such that

$$\begin{aligned} d(y_{i-1}, y_i) &\geq \frac{r'}{C} - (d(H_{i-1}, H_i) - 4) \\ &\geq \frac{r'}{C} - d(x_{i-1}, x_i) - 4. \end{aligned}$$

It follows that the length of β satisfies

$$\begin{aligned} |\beta| &\geq \sum d(y_{i-1}, y_i) \\ &\geq \frac{nr'}{C} - 4n - d(x_0, x_n) \\ &\geq \frac{nr'}{C} - 4n - r'. \end{aligned}$$

Since $n \rightarrow \infty$ as $r \rightarrow \infty$, this proves the superlinear divergence of α . \square

The following example shows that the converse of the above theorem does not hold in complete generality. However, when Γ is a connected graph, we will give a complete characterization of geodesics with superlinear divergence in Theorem 4.4 below.

Example 2.6. Suppose Γ is disconnected then A_Γ splits as a free product and every vertex of X_Γ is a global cut-point. Take any vertex $p \in X_\Gamma$, and any pair of geodesic rays γ_1 and γ_2 emanating from p for which $\gamma_1 \setminus \{p\}$ and $\gamma_2 \setminus \{p\}$ are in distinct components of $X_\Gamma \setminus \{p\}$. Then the union of γ_1 and γ_2 is a bi-infinite geodesic with super-linear divergence (indeed infinite divergence, since γ_1 and γ_2 can not be connected in the complement of any ball around p). Assuming Γ has two connected components consisting of more than one vertex, we can choose each of the γ_i to lie in the cube neighborhood of a wall. Then $\alpha = \gamma_1 \cup \gamma_2$ yields a geodesic with

superlinear divergence such that no three walls crossed by α are pairwise strongly separated.

3. JOINS

In this section we give a group-theoretic interpretation of Theorem 2.5. We assume throughout the section on that Γ is connected and contains at least two vertices.

Choosing a vertex x_0 in X_Γ as a basepoint, we may identify the 1-skeleton of X_Γ with the Cayley graph of A_Γ so that vertices are labeled by elements of A_Γ and edges by elements of the standard generating set (namely the vertex set of Γ). For a generator v , let e_v denote the edge from the basepoint 1 to the vertex v . Any edge in X_Γ determines a unique wall, namely the wall containing the midpoint of that edge. Denote by H_v the wall containing the midpoint of e_v .

For a cube in X_Γ , all of the parallel edges are labeled by the same generator v . It follows that all of the edges crossing a wall H have the same label v , and we call this a wall of *type* v . Since A_Γ acts transitively on edges labeled v , a wall is of type v if and only if it is a translate of the standard wall H_v .

Let $lk(v)$ denote the subgraph of Γ spanned by the vertices adjacent to v and let $st(v)$ denote the subgraph spanned by v and $lk(v)$. The stabilizer of the wall H_v is the group generated by $lk(v)$, which we denote by L_v and call a *link subgroup*. To see this, note that in any cube containing the edge e_v , all other edges labeled v are of the form ge_v for some $g \in L_v$. An induction on the number of cubes between e_v and e now shows that the same holds for any edge e which crosses H_v .

In what follows, for two subgroups K and L of A_Γ , we consider the double coset, denoted KL , which is the set of elements of A_Γ which can be written as a product kl for some $k \in K, l \in L$. In general, KL is not a subgroup.

Lemma 3.1. *Let $H_1 = g_1H_v$ and $H_2 = g_2H_w$. Then*

- (1) H_1 intersects $H_2 \iff v, w$ commute and $g_1^{-1}g_2 \in L_vL_w$.
- (2) $\exists H_3$ intersecting both H_1 and $H_2 \iff \exists u \in st(v) \cap st(w)$ such that $g_1^{-1}g_2 \in L_vL_uL_w$.

Here (2) includes the case in which H_3 is equal to H_1 or H_2 , hence H_1 and H_2 are strongly separated if and only if the conditions in (2) are not satisfied.

Proof. Without loss of generality, we may assume that $H_1 = H_v$ and $H_2 = gH_w$.

(1) If v, w commute, they span a cube in X_Γ , hence H_v and H_w intersect. Suppose $g = ab$, with $a \in L_v, b \in L_w$. Then $H_v = a^{-1}H_v$ and $H_w = bH_w$, so translating by a , we see that H_v intersects gH_w .

Conversely, suppose H_v intersects gH_w in a cube C . Then C contains edges of type v and of type w hence v and w must commute. Moreover, C is a translate $C = hC'$ of a cube C' at the basepoint containing the edges e_v and e_w . Since e_v and he_v both intersect H_v , h lies in L_v . Since ge_w and he_w both intersect gH_w , $h^{-1}g$ lies in L_w . Thus, $g \in L_vL_w$.

(2) If $u \in st(v) \cap st(w)$ and $g = abc \in L_vL_uL_w$, then H_v and $bH_w = bcH_w$ both intersect $H_u = bH_u$. Translating by a , we see that H_v and gH_w both intersect aH_u .

Conversely, suppose that $H_3 = hH_u$ intersects both H_1 and H_2 . By part (1), u must commute with both v and w , so $u \in st(v) \cap st(w)$. Also by part (1), $h \in L_vL_u$ and $h^{-1}g \in L_uL_w$, so $g \in L_vL_uL_w$. \square

For two walls H_v and gH_w to satisfy the conditions of (2), both w and the letters in g must lie in a 2-neighborhood of v . The converse is not true. Consider for example the case of the Artin group associated to a pentagon Γ with vertices labeled (in cyclic order) a, b, c, d, e . Every vertex lies in a 2-neighborhood of a , but we claim that H_a and daH_c are strongly separated. This follows from the lemma since $st(a) \cap st(c) = \{b\}$ and da does not lie in $L_a L_b L_c = \langle e, b \rangle \langle a, c \rangle \langle b, d \rangle$.

To guarantee that no two walls in X_Γ are strongly separated, we need a stronger condition. Let Γ_1 and Γ_2 be (non-empty) graphs. The *join* of Γ_1 and Γ_2 is the graph formed by joining every vertex of Γ_1 to every vertex of Γ_2 by an edge. The Artin group associated to such a graph splits as a direct product, $A_\Gamma = A_{\Gamma_1} \times A_{\Gamma_2}$ and X_Γ splits as a metric product $X_\Gamma = X_{\Gamma_1} \times X_{\Gamma_2}$. The walls in X_Γ are thus of the form $H_1 \times X_{\Gamma_2}$ or $X_{\Gamma_1} \times H_2$ for some wall H_i in X_{Γ_i} . Clearly, every wall of the first type intersects every wall of the second type, and it follows that no two walls are strongly separated.

We now introduce two *a priori* different ways to measure how an edge path travels through X_Γ , these will then be related via Lemmas 3.2 and 3.3.

Let $g \in A_\Gamma$ and let $v_1 v_2 \dots v_k$ be a minimal length word representing g . For $i < k$, set $g_i = v_1 v_2 \dots v_i$. Then the set of walls crossed by the edge path in X_Γ from x_0 to $g x_0$ labeled $v_1 v_2 \dots v_k$ is given by

$$\mathcal{H} = \{H_{v_1}, g_1 H_{v_2}, g_2 H_{v_3}, \dots, g_{k-1} H_{v_k}\}.$$

A different choice of minimal length word gives the same set of walls, changing only the order in which they are crossed. Define the *separation length* of g to be

$$\ell_S(g) = \max\{k \mid \mathcal{H} \text{ contains } k \text{ walls which are pairwise strongly separated}\}.$$

If J is a complete subgraph of Γ which decomposes as a non-trivial join, then we call A_J a *join subgroup* of A_Γ . Define the *join length* of g to be

$$\ell_J(g) = \min\{k \mid g = a_1 \dots a_k \text{ where } a_i \text{ lies in a join subgroup of } A_\Gamma\}.$$

If α is a (finite) geodesic in A_Γ , we can approximate α by an edge path p which crosses the same set of walls as α . The word labeling this edge path determines an element $g_\alpha \in A_\Gamma$ which is independent of the choice of p . We define $\ell_S(\alpha) = \ell_S(g_\alpha)$ and $\ell_J(\alpha) = \ell_J(g_\alpha)$. If α is a bi-infinite geodesic, and α_n denotes the restriction of α to the interval $[-n, n]$, we define the separation and join lengths of α to be

$$\ell_S(\alpha) = \lim_{n \rightarrow \infty} \ell_S(\alpha_n) \quad \ell_J(\alpha) = \lim_{n \rightarrow \infty} \ell_J(\alpha_n).$$

Lemma 3.2. *A bi-infinite geodesic α in X_Γ has finite join length if and only if both the positive and negative rays of α eventually stay in a single join, i.e., the ray minus some compact set is completely contained in a translate of X_J for some join $J \subset \Gamma$. If every bi-infinite periodic geodesic in X_Γ has finite join length, then Γ is itself a join.*

Proof. For any join J in Γ , X_J is a convex subspace of X_Γ so once α leaves X_J , it will never return, and similarly for translates of X_J . If α has finite join length it lies entirely in some finite set of these join subspaces and hence each ray must eventually remain in a single join. The reverse implication is obvious.

For the second statement, suppose Γ is not a join. Let J be a maximal join in Γ and let v be a vertex not in J . Let $g \in A_J$ be the product of all the vertices in J and consider the bi-infinite geodesic $\alpha = \dots g v g v g v g v \dots$. Note that no vertex $w \in J \cup v$ commutes with both J and v since otherwise, we would have $J \cup v$

contained in the join $st(w)$, contradicting the maximality of J . It follows that the tails of α must involve every vertex of $J \cup v$, hence by the first statement of the lemma, α has infinite join length. \square

In the proof of the previous lemma, we used the fact that for any vertex v of Γ , $st(v)$ is always a join, namely it is the join of $\{v\}$ and $lk(v)$. This fact plays a crucial role in the next lemma as well as in the proof of Theorem 4.4 below.

Lemma 3.3. *For any $g \in A_\Gamma$,*

$$\ell_S(g) \leq \ell_J(g) \leq 2\ell_S(g) + 1.$$

Thus a bi-infinite geodesic has infinite join length if and only if it has infinite separation length.

Proof. The first inequality follows from the observation above that no two walls in a join are strongly separated. For the second inequality, fix a minimal word for g and let \mathcal{H} be the sequence of walls crossed by the corresponding edge path as listed above. Set $H = H_{v_1}$ and let $H' = g_i H_{v_{i+1}}$ be the first wall in the sequence strongly separated from H . Then by Lemma 3.1, g_i lies in the product of three link subgroups, $L_{v_1} L_{u_1} L_{v_{i+1}}$, for some u_1 , hence $g_{i+1} = g_i v_{i+1}$ lies in a product of the three join subgroups generated by $st(v_1)$, $st(u_1)$, and $st(v_{i+1})$. Now repeat this argument starting with $H = g_i H_{v_{i+1}}$ and taking $H' = g_j H_{v_{j+1}}$ to be the next strongly separated wall (or $H' =$ the last wall in \mathcal{H} if no more strongly separated walls exist), to conclude that g_{j+1} lies in the product of join subgroups

$$\langle st(v_1) \rangle \langle st(u_1) \rangle \langle st(v_{i+1}) \rangle \langle st(u_2) \rangle \langle st(v_{j+1}) \rangle.$$

Continuing this process, each new strongly separated wall adds two star subgroups. Since we encounter at most $\ell_S(g)$ strongly separated walls, the inequality follows. \square

4. THE ASYMPTOTIC CONE

The goal of this section is to understand the structure of the asymptotic cones of A_Γ . We begin by recalling some preliminaries on asymptotic cones, tree graded spaces, and divergence; we refer the reader to [dDW], [DS], and [DMS] for more details.

Let (X, d) be a geodesic metric space. Let ω be a non-principal ultrafilter on the natural numbers, (o_n) a sequence of observation points in X , and (d_n) a sequence of scaling constants such that $\lim_\omega d_n = \infty$. Then the asymptotic cone, $\text{Cone}_\omega(X, (o_n), (d_n))$, is the metric space consisting of equivalence classes of sequences (a_n) satisfying $\lim_\omega d(o_n, a_n)/d_n < \infty$, where two such sequences $(a_n), (a'_n)$ represent the same point \mathbf{a} if and only if $\lim_\omega d(a_n, a'_n)/d_n = 0$, and the metric is given by $d_\omega(\mathbf{a}, \mathbf{b}) = \lim_\omega d(a_n, b_n)/d_n$.

We will assume the observation points and scaling constants are fixed and write X^ω for $\text{Cone}_\omega(X, (o_n), (d_n))$. In general, X^ω is a complete geodesic metric space. In the case where X has a cocompact group action, X^ω is independent of choice of observation points (but not, in general, of scaling constants) and is homogeneous.

A complete geodesic metric space is *tree graded* if it contains a collection of closed subsets, \mathcal{P} , called *pieces* such that the following three properties are satisfied: in each $P \in \mathcal{P}$, every pair of points is connected by a geodesic in P ; any simple geodesic triangle is contained in some $P \in \mathcal{P}$; and each distinct pair $P, P' \in \mathcal{P}$ is

either disjoint or intersects in a single point. Druţu–Osin–Sapir proved that a group is relatively hyperbolic if and only if all of its asymptotic cones are tree-graded with respect to pieces obtained by taking asymptotic cones of the peripheral subgroups. On the other hand, Behrstock–Druţu–Moshier proved that right-angled Artin groups are relatively hyperbolic if and only if their defining graph is disconnected [BDM]. In this section, we show that for connected defining graphs, although the Artin group A_Γ is not relatively hyperbolic, the asymptotic cones of A_Γ still have a non-trivial tree grading provided Γ is not a join. Moreover, although the pieces do not come from asymptotic cones of subgroups, they can be characterized group-theoretically (see Theorem 4.6).

We begin by recalling the work of Druţu–Mozes–Sapir [DMS] on divergence and cut-points.

Definition 4.1. Let $\rho(k) = \delta k - \lambda$ with $0 < \delta < 1$ and $\lambda \geq 0$. For points $a, b, c \in X$, set $k = d(c, \{a, b\})$ and define $\text{div}(a, b, c; \rho)$ to be the length of the shortest path in X from a to b which lies outside the ball of radius $\rho(k)$ about c . The *divergence of X with respect to ρ* is the function

$$\text{Div}(X, \rho)(r) = \sup\{\text{div}(a, b, c; \rho) \mid a, b, c \in X, d(a, b) \leq r\}.$$

We say X has *linear divergence* if $\text{Div}(X, \rho)(r)$ is linear for some ρ .

For a bi-infinite geodesic α , the divergence function introduced in Section 2 can be written as,

$$\text{div}(\alpha, \rho)(r) = \text{div}(\alpha(-r), \alpha(r), \alpha(0); \rho).$$

In particular, if X has linear divergence, then every bi-infinite geodesic in X has linear divergence.

Druţu–Mozes–Sapir establish the following correspondence between cut-points and divergence functions [DMS, Lemma 3.14].

Proposition 4.2 ([DMS]). *Let $\mathbf{a} = (a_n)$, $\mathbf{b} = (b_n)$, $\mathbf{c} = (c_n)$ be three points in X^ω , and let $k = d_\omega(\mathbf{c}, \{\mathbf{a}, \mathbf{b}\})$. Then \mathbf{c} is a cut-point separating \mathbf{a} from \mathbf{b} if and only if for some ρ ,*

$$\frac{\lim_\omega \text{div}(a_n, b_n, c_n; \frac{\rho}{k})}{d_n} = \infty.$$

In particular, for a bi-infinite geodesic α in X , taking $a_n = \alpha(-d_n)$, $b_n = \alpha(d_n)$, and $c_n = \alpha(0)$, the proposition implies that \mathbf{c} is a cut-point separating \mathbf{a} from \mathbf{b} if and only if α has super-linear divergence.

We say that X is *wide* if no asymptotic cone of X has cut-points. In the case that X is the Cayley graph of a group G , the proposition above leads to the following criterion for G to be wide (see [DMS, Proposition 1.1]).

Proposition 4.3 ([DMS]). *A group G is wide if and only if $\text{Div}(G, \rho)(r)$ is linear for $\rho(r) = \frac{1}{2}r - 2$.*

In the case of A_Γ , the divergence of a bi-infinite geodesic is determined by its join length.

Theorem 4.4 (Divergence and join length). *Let Γ be a connected graph with at least 2 vertices and let α be a bi-infinite geodesic in X_Γ . Then α has linear divergence if and only if the join length of α is finite.*

Proof. If the join length of α is infinite, then by Lemma 3.3, so is the separation length. By Theorem 2.5, it follows that α has super-linear divergence.

Now suppose that the join length of α is finite. We will show that α lies in a subspace of X_Γ whose image in any asymptotic cone X_Γ^ω has no cut-points. It then follows from the remarks following Proposition 4.2 that α has linear divergence.

By Lemma 3.2, α lives entirely in a finite union of join subspaces, that is, subspaces which are translates of X_J for some join J . Since A_J decomposes as a direct product of infinite groups, X_J is wide. Hence in any asymptotic cone X_Γ^ω , the cone on gX_J gives rise to a subspace with no cut-points. If $g'X_{J'}$ is another join subspace which shares a geodesic line with gX_J , then the union of their asymptotic cones contains a line in X_Γ^ω hence also has no cut-points.

Thus, it suffices to show that any two join subspaces gX_J and $g'X_{J'}$ are connected by a sequence of join subspaces such that consecutive subspaces share a line. We will call this a *connecting sequence*. By hypothesis, the graph Γ is connected, so we can find a sequence of joins beginning at J and ending at J' such that consecutive joins in the sequence share at least one vertex in Γ . (For example, take a path from J to J' and take the sequence of stars of the vertices along this path.) For $g = g'$, it follows that there is a connecting sequence from gX_J to $gX_{J'}$.

For the general case, we may assume without loss of generality that $g = 1$. Say $g' = a_1 \dots a_k$ where each a_i lies in some join J_i . Then the observation above shows that there are connecting sequences from X_J to $X_{J_1}(= a_1X_{J_1})$, from $a_1X_{J_1}$ to $a_1X_{J_2}(= a_1a_2X_{J_2})$, and so on to $g'X_{J_k}$, and finally, from $g'X_{J_k}$, to $g'X_{J'}$. \square

We now generalize the notion of join length to points in the asymptotic cone A_Γ^ω . For two points $\mathbf{a}, \mathbf{b} \in A_\Gamma^\omega$, we say that the pair \mathbf{a}, \mathbf{b} (or the geodesic between them) has *finite join length* if there exist representative sequences $(a_n), (b_n)$ for which $\lim_\omega \ell_J(a_n^{-1}b_n) < \infty$.

Lemma 4.5. *Let \mathbf{a}, \mathbf{b} be distinct points in A_Γ^ω and let α be a geodesic between them. If \mathbf{c} is a point in the interior of α which does not separate \mathbf{a} from \mathbf{b} , then there exists a neighborhood of \mathbf{c} in α such that any two points in this neighborhood have finite join length. Moreover, the union of any two such neighborhoods of \mathbf{c} also has this property.*

Proof. By Proposition 4.2, the hypotheses imply that for some choice of linear function ρ , $\text{div}(a_n, b_n, c_n; \rho/k)$ is bounded by a linear function of d_n . Let $D > 0$ be as in Lemma 2.3 and let $k_n = d(a_n, b_n)$. Since $\lim_\omega \frac{k_n}{d_n} > 0$, we can choose $\epsilon > 0$ such that $D\epsilon < \frac{\rho(k_n)}{2d_n}$ for ω -almost every n . Consider two sequences $(a'_n), (b'_n)$ lying within ϵd_n of (c_n) . For any two strongly separated walls between a'_n and b'_n , Lemma 2.3 (2) implies that the bridge between them lies in the ball of radius $\frac{\rho(k_n)}{2}$ about c_n . The number of such walls must be bounded (independent of n), since if not, then arguing as in Theorem 2.5, we would deduce that $\text{div}(a_n, b_n, c_n; \rho/k)$ was super-linear. It follows from Lemma 3.3 that the join length is also bounded.

To prove the last statement of the lemma, suppose \mathbf{c} lies in two such intervals and let \mathbf{a}', \mathbf{b}' be a point in each. Then there exist two representative sequences $(c_n), (c'_n)$ for \mathbf{c} such that the sequences $(a_n'^{-1}c_n)$ and $(c_n'^{-1}b'_n)$ have bounded join length. If $(c_n'^{-1}c'_n)$ has infinite join length, then $\text{div}(a_n', b_n', c_n; \rho/k)/d_n$ is unbounded for every ρ , so by Proposition 4.2, \mathbf{c} is a cut-point. This contradicts the hypotheses of the lemma, so we conclude that $(c_n'^{-1}c'_n)$, and hence $(a_n'^{-1}b_n')$, has bounded join length. \square

Theorem 4.6 (Classification of pieces). *Let Γ be a connected graph with at least 2 vertices. Fix a pair of points $\mathbf{a}, \mathbf{b} \in A_\Gamma^\omega$. The following are equivalent.*

- (1) *No point of A_Γ^ω separates \mathbf{a} from \mathbf{b} .*
- (2) *There exist points \mathbf{a}', \mathbf{b}' arbitrarily close to \mathbf{a}, \mathbf{b} , respectively for which the join length between \mathbf{a}', \mathbf{b}' is finite.*

Proof. Suppose (1) holds. Let α be the geodesic from \mathbf{a} to \mathbf{b} . Then by Lemma 4.5, every point on α is contained in an open interval in which any two points have finite join length. Moreover, if two such intervals intersect, then their union also has this property. It now follows easily that the maximal open interval of α such that any two points have finite join length is the entire interior of α .

Now suppose (2) holds. By hypothesis, for every $\epsilon > 0$ there exist points \mathbf{a}' and \mathbf{b}' with representatives (a'_n) and (b'_n) whose distances in A_Γ^ω are less than ϵ from \mathbf{a} and \mathbf{b} , respectively, and the join length between \mathbf{a}' and \mathbf{b}' is finite, that is, there exists a constant M such that ω -almost every $a_n'^{-1}b_n'$ is a product of at most M subwords, each contained in a join subgroup. Hence the corresponding geodesic is contained in a finite sequence of join cosets. As in the proof of Theorem 4.4 there is a connecting sequence, that is, a finite sequence of additional join cosets which we may add between any two of these to get an ordered sequence of join cosets where each intersects the next in an infinite diameter set. Starting with M join cosets, the construction in Theorem 4.4 yields a connecting sequence whose length is at most $M \cdot \text{diam}(\Gamma)$, where $\text{diam}(\Gamma)$ is the diameter of Γ . Denote this sequence by \mathcal{S}_n .

We will say that a coset of A_J has join type J . Since there are only finitely many joins in Γ , the sequence of join types in \mathcal{S}_n is the same for ω -almost every n . Any two cosets of the same join type are isometric, so for each i , the subspace of points $(x_n) \in A_\Gamma^\omega$ such that x_n lies in the i^{th} term of \mathcal{S}_n is isometric to $A_{J_i}^\omega$ for some join J_i . In particular, this subspace has no cut-points. Moreover, the intersection of any two consecutive subspaces in this sequence has infinite diameter. It follows that their union, which contains \mathbf{a}' and \mathbf{b}' , has no cut-points.

Hence, \mathbf{a} and \mathbf{b} can be approximated arbitrarily closely by points \mathbf{a}' and \mathbf{b}' which cannot be separated by a point. Since not being separated by a point is a closed condition, this completes the proof that (2) implies (1). \square

The following example shows that one cannot replace condition (2) by the simpler statement that the geodesic from \mathbf{a} to \mathbf{b} has finite join length.

Example 4.7. Suppose x, y are two vertices in Γ that are not contained in a join. For simplicity, take the scaling constants for A_Γ^ω to be $d_n = n$. Let $a_n = 1$ for all n and let

$$b_n = x^{\lfloor \frac{n}{2} \rfloor} y^{\lfloor \frac{n}{4} \rfloor} x^{\lfloor \frac{n}{8} \rfloor} y^{\lfloor \frac{n}{16} \rfloor} \dots$$

Then the join length of the geodesic from \mathbf{a} to \mathbf{b} is infinite. However, if we truncate each b_n after k terms, setting

$$b_n^{(k)} = x^{\lfloor \frac{n}{2} \rfloor} y^{\lfloor \frac{n}{4} \rfloor} \dots z^{\lfloor \frac{n}{2^k} \rfloor}$$

where $z = x, y$ depending on whether k is odd or even, we obtain a point $\mathbf{b}^{(k)}$ in the asymptotic cone whose distance from \mathbf{b} is $\frac{1}{2^k}$ while the join length from \mathbf{a} to $\mathbf{b}^{(k)}$ is k . It follows from the theorem above that \mathbf{a} and \mathbf{b} lie in the same piece of the asymptotic cone, despite the fact that the join length between them is infinite.

Combining the results above, we obtain

Corollary 4.8 (Quadratic divergence). *Let Γ be a connected graph with at least 2 vertices. Then the following are equivalent.*

- (1) Γ is a join,
- (2) A_Γ is wide,
- (3) A_Γ has linear divergence.

If Γ is not a join, then A_Γ has quadratic divergence.

Proof. (1) implies (2) by Theorem 4.6, (2) implies (3) by Theorem 4.3, and (3) implies (1) by Theorem 4.4.

For the last statement, assume Γ is not a join. Then A_Γ contains a cyclically reduced element g that does not lie in any join subgroup. It is easy to see that for such an element, the join length of g^n grows linearly with n , and hence by Lemma 3.3, so does the separation length. It follows from the proof of Theorem 2.5 that the divergence of g^n is at least quadratic. (One can also deduce this from a more general result of Kapovich and Leeb [KL, Proposition 3.3] which implies that a complete periodic geodesics in a locally-compact CAT(0) space has divergence which is either linear or at least quadratic.)

On the other hand, for any finite length geodesic β , the argument in Theorem 4.4 shows that β is contained in a connecting sequence of join cosets whose length is at most linear in the length of β . Since divergence in each join coset is linear, it follows that the divergence of β is at most quadratic in the length of β . \square

The last argument in the proof uses the fact that A_Γ is algebraically thick of order 1 in the sense of [BDM]. A generalization of this argument for metric spaces which are thick of arbitrary degree will appear in a forthcoming paper of the first author and C. Druţu.

5. RANK-ONE ISOMETRIES AND QUASIMORPHISMS

Recall that a hyperbolic isometry of a proper CAT(0) space is called a *rank-one isometry* if some axis of that isometry does not bound a half-plane. Bestvina and Fujiwara [BF1] show that for a group acting properly discontinuously on a CAT(0) space, rank-one isometries give rise to quasimorphisms. A *quasimorphism* on a group G is a function $\phi: G \rightarrow \mathbb{R}$ for which there exists a constant $D(\phi) \geq 0$ such that

$$|\phi(gh) - \phi(g) - \phi(h)| \leq D(\phi)$$

for every $g, h \in G$. A quasimorphism is *homogeneous* if for each $g \in G$ and each $n \in \mathbb{N}$, we have $\phi(g^n) = n\phi(g)$. The set of homogeneous quasimorphisms on a given group G form a vector space. The quotient of this vector space by homomorphisms from G to \mathbb{R} is denoted $\widetilde{QH}(G)$ and is isomorphic to the kernel of the map from the second bounded cohomology of G (with \mathbb{R} coefficients) to the second cohomology of G . (For details, see [Gro] and [Cal].)

Burger and Monod proved that there are no non-trivial homogeneous quasimorphisms on any irreducible lattice in a connected semisimple Lie group of rank at least 2 with finite center and no compact factors [BM1, BM2, Mon1]. On the other hand, several interesting families of groups, including non-elementary hyperbolic groups [EF] and mapping class groups [BF1], have been shown to have infinite dimensional \widetilde{QH} .

We will show that for a right-angled Artin group A_Γ acting on its Salvetti complex X_Γ , rank-one isometries are prevalent. We begin with some basic facts about normal forms in right-angled Artin groups. We refer the reader to [Lau] for details.

Let V be the generating set for A_Γ and let g be an element of A_Γ . A *reduced word* for g is a minimal length word in the free group $F(V)$ representing g . Given an arbitrary word representing g , one can obtain a reduced word by a process of “shuffling” (i.e. interchanging commuting elements) and canceling inverse pairs. Any two reduced words for g differ only by shuffling. In particular, the *support* of g , that is the set $\text{Supp}(g) \subseteq V$ of letters appearing in a reduced word for g , is independent of choice of reduced word.

For an element $g \in A_\Gamma$, a *cyclic reduction* of g is a minimal length element of the conjugacy class of g . If w is a reduced word representing g , then we can find a cyclic reduction \bar{g} by shuffling commuting generators in w to get a maximal length word u such that $w = u\bar{w}u^{-1}$. In particular, g itself is cyclically reduced if and only if every shuffle of w is cyclically reduced as a word in the free group $F(V)$. Moreover, any other cyclic reduction of g can be obtained by shuffling and cyclically permuting \bar{w} so $\text{Supp}(\bar{g})$ is independent of the choice of cyclic reduction.

Lemma 5.1. *Let $g = \bar{g}$ be a cyclically reduced element of A_Γ . Then the following are equivalent.*

- (1) g is contained in a join subgroup.
- (2) The centralizer of g is non-cyclic.
- (3) The centralizer of g is contained in a join subgroup.

Proof. (1) implies (2) since the centralizer of any element (g_1, g_2) of a direct product $G_1 \times G_2$ is the product of the centralizers $C_{G_1}(g_1) \times C_{G_2}(g_2)$. (3) implies (1) is obvious, so it remains only to prove that (2) implies (3).

For any subset $S \subset V$, let $lk(S)$ denote the (possibly empty) set of vertices at distance 1 from every vertex of S . It follows from Servatius’ Centralizer Theorem [Ser] (see also Thm 1.2 of [Lau]) that the centralizer of a cyclically reduced element g lies in the subgroup generated by $\text{Supp}(g) \cup lk(\text{Supp}(g))$ and that the centralizer is cyclic unless either $lk(\text{Supp}(g))$ is nonempty or $\text{Supp}(g)$ decomposes as a join. In either case, $\text{Supp}(g) \cup lk(\text{Supp}(g))$ spans a join in Γ . \square

Theorem 5.2 (Rank-one isometries). *If $G \subseteq A_\Gamma$ is any non-cyclic subgroup which is not contained in a conjugate of a join subgroup, then G contains an element which acts as a rank-one isometry of X_Γ .*

Proof. Let g be an element of A_Γ and let \bar{g} be a cyclic reduction of g . Then \bar{g}^k is geodesic for all k . If \bar{g} does not lie in a join subgroup, then the axis for \bar{g} has infinite join length, hence by Theorem 4.4, it has super-linear divergence. It follows that the axis for \bar{g} cannot bound a half-flat and the same holds for the axis of g since it is a translate of the axis for \bar{g} . Thus to prove the first statement of the theorem, it suffices to show that G contains an element g whose cyclic reduction \bar{g} does not lie in a join subgroup, that is, $\text{Supp}(\bar{g})$ is not contained in any join.

Choose an element $c \in G$ such that $\text{Supp}(\bar{c})$ is maximal, that is, if $g \in G$ satisfies $\text{Supp}(\bar{c}) \subseteq \text{Supp}(\bar{g})$, then $\text{Supp}(\bar{c}) = \text{Supp}(\bar{g})$. Conjugating G if necessary, we may assume without loss of generality that $c = \bar{c}$. If c is not contained in a join subgroup, we are done. So suppose c is contained in a join subgroup, then by Lemma 5.1 the centralizer of c is also contained in a join subgroup, A_J .

By hypothesis, G does not lie in a join subgroup, so there is some element $h \in G$ whose support is not contained in J . Consider an element of the form $x = c^k h c^k \in G$. We claim that for sufficiently large k , $Supp(\bar{x}) \supsetneq Supp(c)$ contradicting the maximality assumption on $Supp(c)$. To see this, note that since c is cyclically reduced, cancellations can only occur between generators in c and generators in h . It follows that repeatedly multiplying h on the left or right by c , can result in at most finitely many cancellations and all canceled letters must lie in $Supp(c) \cap Supp(h)$. Thus, for k sufficiently large, a reduced word for x is of the form $c^i u c^j$ for some $i, j > 0$ and some reduced word u . In particular, the support of x satisfies $Supp(x) = Supp(c^i u c^j) = Supp(c) \cup Supp(h)$. If x is cyclically reduced, we are done.

Suppose x is not cyclically reduced. Write $u = a\bar{u}a^{-1}$ where \bar{u} is a cyclic reduction of u , so $x = c^i a\bar{u}a^{-1} c^j$. Since c is cyclically reduced, the only way x can fail to be cyclically reduced is if some initial subword a' of a commutes with c . But in this case, we can conjugate G by a'^{-1} and repeat the argument replacing h by $h' = a'^{-1} h a'$ and u by $u' = a'^{-1} u a'$ to conclude that $x' = c^i u' c^j$ is cyclically reduced. Since a' lies in the centralizer of c , it lies in A_J , whereas $h \notin A_J$. It follows that $h' \notin A_J$, hence $Supp(\bar{x}') = Supp(x') = Supp(c) \cup Supp(h') \supsetneq Supp(c)$, as claimed. \square

As a corollary, we obtain a new proof of the following facts.

Corollary 5.3 (Quasimorphisms). *If G is any non-abelian subgroup of A_Γ , then $\widetilde{QH}(G)$ is infinite dimensional.*

Proof. We will prove the corollary by induction on the number of vertices in Γ . Since A_Γ is torsion-free, virtually cyclic subgroups are cyclic. In particular, if G is non-abelian, then it is not virtually cyclic. Suppose G does not lie in the conjugate of a join subgroup. Since the action of G on X_Γ is properly discontinuous and by Theorem 5.2, it contains a rank-one isometry, the Main Theorem of [BF2] implies that $\widetilde{QH}(G)$ is infinite dimensional.

If G can be conjugated into a join subgroup $A_{\Gamma_1} \times A_{\Gamma_2}$, then the projection of G on one of the factors must also be non-abelian. By induction, this projection has an infinite dimensional space of quasimorphisms, hence so does G . \square

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