Sample Final Spring 2005  
Calculus IV

Name:  
ID #:  

Please show your work and provide full solutions, not just answers. Clearly state any theorems which you use.
Cross out anything the grader should ignore.

(1) (25 pts) Evaluate \( \int_\gamma 3y \, dr \) where \( \gamma \) is a curve in \( \mathbb{R}^2 \) parametrized as \( r(t) = (t, t^3) \) for \( 0 \leq t \leq 1 \).

(2) (20 pts) Set up (but do not evaluate!) a triple integral for the volume of the solid given by intersecting \( x^2 + y^2 + z^2 = 4 \) with the positive quadrant (i.e., assuming that \( x \geq 0, y \geq 0, \) and \( z \geq 0 \)).

(3) (25 pts) Let \( F = (y + e^{\sin(x)}, y^2) \) and let \( \Delta \) be the triangle with vertices \((0,0), (0,2), \) and \((1,0)\).
   (a) (10 pts) Compute \( \int_{\partial \Delta} F \cdot dr \). (Don’t forget the orientation convention for boundaries!)
   (b) (10 pts) Let \( \gamma \) denote the oriented curve consisting of the horizontal line from \((0,0)\) to \((1,0)\), followed by the straight line from \((1,0)\) to \((0,2)\). Compute \( \int_\gamma F \cdot dr \).

(4) (30 pts) Verify the Divergence Theorem for the vector field \( F(x, y, z) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) and the unit ball \( x^2 + y^2 + z^2 \leq 1 \) by explicitly computing appropriate integrals over this region and over its boundary.

(5) (35 pts) Evaluate \( \int_S \text{curl}(F) \cdot dS \) where
\[
F(x, y, z) = (e^{x^2 + y^2} z) \mathbf{i} + y^2 \mathbf{j} + z^3 e^{xy} \mathbf{k}
\]
and \( S \) is the part of the sphere \( x^2 + y^2 + z^2 = 10 \) that lies above the plane \( z = 1 \), and \( S \) is oriented upwards.
 Hint: Consider ways to compute this integral by computing over other domains...with a clever choice you can simplify the calculation dramatically!

(6) (30 pts)
   (a) Write down the key equation for Generalized Stokes' Theorem (just the equation, you don’t need to write down any hypothesis, this should be less than 10 symbols).
   (b) Now, let \( \omega = P \, dy \, dz + Q \, dz \, dx + R \, dx \, dy \) represent the differential 2-form which when integrated over a surface \( S \) gives \( \int_S F \cdot dS \) for the smooth vector field \( F = Pi + Qj + Rk \).

   Assume that \( S \) is the boundary of a set \( E \subset \mathbb{R}^3 \) (i.e., \( S = \partial E \)).
   Compute \( d\omega \) and write out part (a) using the result of your computation.
   (c) What special case of Generalized Stokes' Theorem did you write in part (b).

(7) (35 pts) Let \( c \) be a positive number. A differential equation of the form
\[
\frac{dy}{dt} = k y^{1+c}
\]
where \( k \) is a positive constant, is called a doomsday equation.
   (a) Determine the solution that satisfies the initial condition \( y(0) = y_0 \).
   (b) Show that there is a finite time \( t = T \) (the doomsday) such that \( \lim_{t \to T^-} y(t) = \infty \).