A STRATIFICATION OF THE HANOI GRAPH FOR 4 PEGS

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ABSTRACT. For the Tower of Hanoi problem, it has been found that it is possible to construct a nice graph with legal states of the Tower of Hanoi as the vertices and legal moves between legal states as the edges between corresponding vertices. This "Hanoi graph" we denote by H_n^k for k pegs and n disks. Our motivation is to describe the properties of this graph for $k \geq 4$ and to describe a class of graphs called Stratified Hanoi graphs, appropriately denoted by SH_n^k, which will ultimately be useful in visualizing the Frame-Stewart algorithm.

1. Background Information

In this article, we will follow the convention of [POOLE] to label vertices. Our motivation is to describe a class of graphs called Hanoi graphs, denoted by H_n^k for k pegs and n disks, and to ultimately describe Stratified Hanoi graphs, appropriately denoted by SH_n^k for k pegs and n disks. In order to do that, in this section we will introduce some notation and definitions that we will use throughout this article.

Definition 1.1. A *legal state* is a configuration of n disks on the k pegs in which each disk is either on top of a larger disk or on an empty peg.

Let us label the pegs 0, 1, $\cdots$, $k-1$ and the disks from smallest to largest 0, 1, $\cdots$, $n-1$. We label each legal state with an n-bit k-ary string $a_{n-1}a_{n-2}...a_0$ where $a_i \in \{0, 1, \cdots, k-1\}$ and $a_i = j$ if disk i lies on peg j. It is not difficult to see how this labeling is useful, because each string corresponds to a unique legal state.

In this article, we will denote the vertex set of a graph G by $V(G)$ and the edge set of G by $E(G)$.

Definition 1.2. A *legal move* from one legal state to another legal state is achieved by moving exactly one disk from one peg to another peg under the following rules.

(1) A disk can be moved onto a larger disk.
(2) A disk can be moved onto an empty peg.
(3) A disk cannot be moved onto a smaller disk.

Definition 1.3. Let k,n be positive integers. We define H_n^k as a graph such that $V(H_n^k)$ is the set of vertices each labeled by an n-bit k-ary string which corresponds to a unique legal state, and $E(H_n^k)$ is the set of edges each of which can only exist between two vertices if there exists a legal move between their corresponding legal states.

We can see that $|V(H_n^k)| = k^n$.

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Definition 1.4. A *perfect state* is a vertex corresponding to the legal state in which all disks are on one peg. We can also denote the perfect state given by the $a_i a_i ... a_i$ where $a_i \in \{0,\ 1,\ \cdots,\ k-1\}$ in boldface by $\mathbf{a_i}$.

There are k perfect states in H_n^k.

Definition 1.5. Let $x \in \{0,\ 1,\ \cdots,\ k-1\}$ and $[x]$, called a *block*, is defined to be a subgraph of H_n^k. A vertex belongs to the vertex set of a block if its label begins with x. This vertex set corresponds to all legal states in which the largest disk is on peg x.

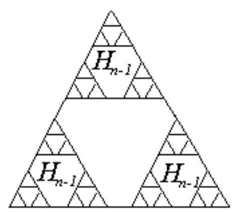


FIGURE 1. H_n^3 has a well-understood recursive structure.

Each block is isomorphic to H_{n-1}^k. Thus, H_n^k is made up of k copies of H_{n-1}^k which are interconnected by edges which we call *bridges* in 1.6. Similarly, since each block is comprised of k sub-blocks, H_n^k is made up of k^j copies of H_{n-j}^k.

Definition 1.6. An edge is called a *bridge* if one of its end-vertices is in $[x]$ and its other end-vertex is in $[y]$ where $x \neq y$.

In H_n^3, for any n, we can see that there exist 3 bridges. The recursive structure of H_n^k for $k > 3$ offers some additional complexity, as there are more bridges as k increases.

Definition 1.7. The subset of $V(H_n^k)$ which corresponds to legal states in which the largest disk is on peg y and all other disks are on pegs x_0, x_1, $\cdots$, or x_m is denoted by $yB_{x_0,x_1,\cdots,x_m}$.

If $y \in \{x_0, x_1, \cdots, x_m\}$, then we instead write $B_{x_0,x_1,\cdots,x_m}$ which denotes the set of all vertices which correspond to legal states in which all disks are on pegs x_0, x_1, $\cdots$, or x_m.

It is easy to see that $|B_{x_0,x_1,\cdots,x_m}| = (m+1)^n$ and that $|yB_{x_0,x_1,\cdots,x_m}| = (m+1)^{n-1}$.

For example, $201011 \in 2B_{0,1}$ and $|2B_{0,1}| = 2^2$.

Theorem 1.8. *The total number of bridges for H_n^k is exactly $\binom{k}{2}(k-2)^{n-1}$.*

For example, for $k = 3$ the total number of bridges is 3 and for $k = 4$ the total number of bridges is $6 \times 2^{n-1}$.

Proof. An edge is a bridge between two blocks, say $[0]$ and $[1]$, if and only if it is an edge between vertices in $0B_{2,3,\cdots,k-1}$ and $1B_{2,3,\cdots,k-1}$.

$$|0B_{2,3,\cdots,k-1}| = |1B_{2,3,\cdots,k-1}| = (k-2)^{n-1}$$

Each vertex has exactly one bridge (to one vertex in the other block). Thus, given two blocks, there are exactly $(k-2)^{n-1}$ bridges between them. There are $\binom{k}{2}$ ways to choose two from k blocks, so the total number of bridges is $\binom{k}{2}(k-2)^{n-1}$. □

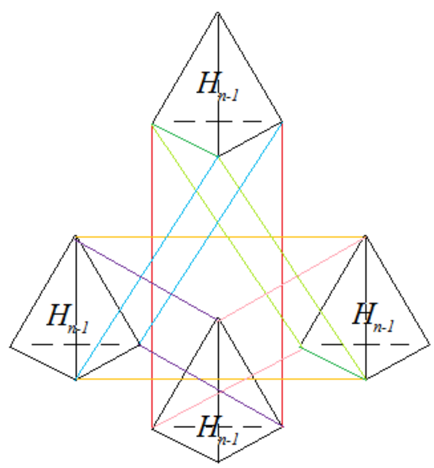


FIGURE 2. the bridges are all the colored edges. We can see the bridges between [0] and [1] as the light-green edges. We can also see that they only exist between $0B_{2,3}$ and $1B_{2,3}$, the green bottom-left sides of [0] and [1], respectively

2. Exploring the Properties of the H_n^4 Graph

Convention 2.1. Let the degree of vertex v be denoted by $\deg(v)$, the maximum degree of graph G be denoted by $\Delta(G)$, and the minimum degree be denoted by $\delta(G)$.

Theorem 2.2. *For all k and all n, $\delta(H_n^k) = k-1$. Furthermore, $\delta(H_n^k) = \deg(v)$ if and only if v is a perfect state in H_n^k.*

Proof. The minimum degree of a graph is the minimum degree of its vertices, i.e. the degree of the vertex with the fewest legal moves to and from its corresponding legal state.

First, we'll show that for any k and any n, a lower bound for the degree of any vertex in H_n^k is $k-1$. This lower bound exists, because the smallest disk is always free to move and since there are k pegs, there are exactly $k-1$ other pegs to which the smallest disk can move. Thus, there are exactly $k-1$ distinct legal moves which the smallest disk can take.

If we can show that there exists a vertex with a degree of $k-1$, then we're done. Every perfect state in H_n^k for all k and all n has a degree of $k-1$. Thus, for all k and all n, $\delta(H_n^k) = k-1$.

To prove the second part of the statement, we must show that it is true in both directions. We have already shown that if v is a perfect state in H_n^k, then $\delta(H_n^k) = \deg(v)$. All that is left is to show that if $\delta(H_n^k) = \deg(v)$, then v is a perfect state in H_n^k. If not every disk is on the same peg, which corresponds to a non-perfect state, then it is possible to move a disk other than the smallest disk, so the degree of a non-perfect state is strictly greater than $k-1$. Thus, $\delta(H_n^k) = \deg(v)$ if and only if v is a perfect state in H_n^k.

$\square$

Theorem 2.3. *For $n \geq k-1$, $\Delta(H_n^k) = \deg(d) = \sum_{i=1}^{k-1} i = \frac{k(k-1)}{2}$.*

We say *geodesic* to mean the shortest path between two vertices.

Definition 2.4. Let the *distance* between vertex α and vertex β, denoted by $d(\alpha, \beta)$, be defined as the number of edges in a geodesic between α and β.

Conjecture 2.5. *In H_n^4, given $\alpha \in B_{1,2}$, $d(\alpha, \mathbf{0})$ is maximized when $\alpha = \mathbf{1}$ or $\mathbf{2}$.*

Remark 2.6. In H_n^4, there exist two vertices $\alpha \in [0]$, $\beta \in [1]$, in which the shortest path between them passes through [2] or [3]. This means that the largest disk moves twice in a geodesic.

For example, the shortest path from $\alpha = 0111$ to $\beta = 1000$ passes through [2] and [3].

Open Question 2.7. *In H_n^4, do there exist two vertices $\alpha \in [0]$, $\beta \in [1]$, in which the shortest path between them passes through [2] and [3]? This means that the largest disk moves three times in a geodesic.*

3. Graphing the Frame-Stewart Solution

The Frame-Stewart algorithm is conjectured to give the shortest path from one perfect state to another on 4 pegs. The Frame-Stewart solution from perfect state $\mathbf{0}$ to perfect state $\mathbf{1}$ is given as follows.

(1) Using all pegs, move the smallest $n - l_n$ disks from peg 0 to peg 2, where l_n is the smallest integer such that $n \leq \frac{l_n(l_n+1)}{2}$ (cf. [MR])
(2) Using the remaining pegs 0, 1 and 3, move the largest l_n disks from peg 0 to peg 1.
(3) Using all pegs, move the smallest $n - l_n$ disks from peg 2 to peg 1.

A table of values for l_n, $n - l_n$ for low values of n is given below. A series of legal moves from one state to another using all pegs is denoted by $\longrightarrow$ and a series of legal moves using only pegs x, y and z is denoted by $\xrightarrow{x,y,z}$.

n	l_n	$n - l_n$	sequence of legal moves
1	1	0	$0 \longrightarrow 1$
2	2	0	$00 \xrightarrow{0,1,3} 11$
3	2	1	$000 \longrightarrow 002 \xrightarrow{0,1,3} 112 \longrightarrow 111$
4	3	1	$0000 \longrightarrow 0002 \xrightarrow{0,1,3} 1112 \longrightarrow 1111$
5	3	2	$00000 \longrightarrow 00022 \xrightarrow{0,1,3} 11122 \longrightarrow 11111$
6	3	3	$000000 \longrightarrow 000222 \xrightarrow{0,1,3} 111222 \longrightarrow 111111$

We will introduce a new class of graphs in which H_n^k is stratified into *layers* (with illustrations for the specific case $k = 4$).

Definition 3.1. We say a peg m is *fixed* if we create a restriction on H_n^k in which disks may not be moved to or from peg m.

For example, if peg 0 is fixed in H_2^k, an edge does not exist between 22 and 20, but an edge exists between 20 and 10.

Thus, we can rewrite step 2 of the Frame-Stewart algorithm as: "Fixing peg 2, move the largest l_n disks from peg 0 to peg 1".

Definition 3.2. A *layer* is a subgraph of H_n^k defined as follows. Given a vertex α and fixing peg m, a vertex β belongs to the same layer as α if β can be reached from α through a series of legal moves without using peg m. An edge is in the edge set of a layer if it exists between two vertices in the vertex set of the layer.

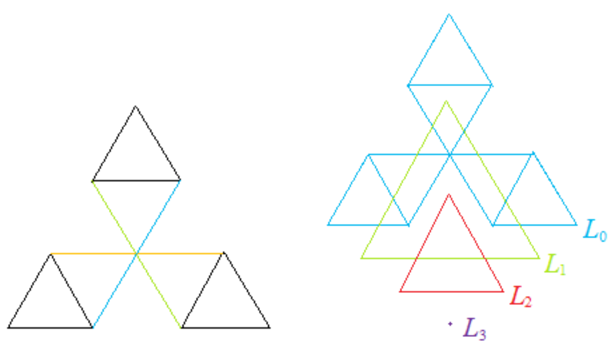
In other words, layers are subgraphs of H_n^k that are apparent when a peg m is fixed.

Finally, we define the Stratified Hanoi graph $SH_n^k(m)$.

Definition 3.3. The (pairwise disjoint) union of all layers in H_n^k when peg m is fixed is denoted by $SH_n^k(m)$.

Theorem 3.4. *Each layer of $SH_n^k(m)$ is isomorphic to H_ν^{k-1} where $\nu \in \{0, 1, \cdots n\}$.*

Proof. Since there are k pegs and exactly one peg is fixed (peg m), there are $k - 1$ non-fixed pegs. If i disks are on peg m, since disks cannot be moved onto or off peg m, there will always be i disks on peg m in the layer. Thus, all possible legal moves in this layer will only involve $n - i$ disks on $k - 1$ pegs, so the layer is isomorphic to H_{n-i}^{k-1}. □



(a) $B_{0,1,3}$ for H_2^4 (b) layers in SH_2^4

FIGURE 3. $B_{0,1,3}$ and $SH_2^4(2)$

We can see that $B_{0,1,3}$ is isomorphic to layer L_0 when peg 2 is fixed. To label these layers, we need to find a set of vertices with the property that each vertex belongs to a unique layer.

Lemma 3.5. *Each vertex in $B_{x,m}$ where $x, m \in \{0,\ 1,\ \cdots,\ k-1\}$ and $x \neq m$ belongs to a unique layer in $SH^k_n(m)$.*

Vertices in $B_{x,m}$ where $x \neq m$ can be ordered as follows:

$$xx...xxx,\ xx...xxm,\ xx...xmx,\ xx...xmm, \cdots,\ mm...mmm$$

Proof. Every vertex of $B_{x,m}$ is in $SH^k_n(m)$, so each vertex of $B_{x,m}$ is in a layer. If peg m is fixed, it is impossible for there to be edges between any two vertices in $\{xx...xxx,\ xx...xxm,\ xx...xmx,\ xx...xmm,\ \cdots,\ mm...mmm\}$. Thus, there cannot be two vertices from $B_{x,m}$ in the same layer. □

Convention 3.6. We can give each layer in $SH^k_n(m)$ a unique label L_λ. Each vertex in $B_{x,m}$ where $x, m \in \{0,\ 1,\ \cdots,\ k-1\}$ and $x \neq m$ lies in a unique layer (3.5). We can convert the label of each vertex $a_{n-1}a_{n-2}...a_0$ where $a_i \in \{x,\ m\}$ and $x \neq m$ into a unique integer under the following rules.

(1) $a_i = x \Rightarrow c_i = 0$
(2) $a_i = m \Rightarrow c_i = 1$
(3) We can convert the n-bit binary string $c_{n-1}c_{n-2}...c_0$ to a decimal λ:

$$\sum_{i=0}^{n-1} c_i 2^i = \lambda$$

For example, $xx...xmm$ becomes 00...011 and the binary string 00...011 converts 3, so $xx...xmm$ lies in layer L_3 in $SH^k_n(m)$.

There are k ways to partition H^k_n, and each way corresponds to fixing a different peg. Figure 3(b) shows only one of the four ways to partition H^4_2.

Theorem 3.7. *There are 2^n layers in $SH^k_n(m)$.*

Proof. Each unique configuration of disks on peg m corresponds to a layer of $SH^k_n(m)$. The total number of unique configurations of disks on peg m is equal to

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

□

Proof. Another proof is that since each vertex in $B_{x,m}$ where $x, m \in \{0,\ 1,\ \cdots,\ k-1\}$ and $x \neq m$ lies in a unique layer in $SH^k_n(m)$ (proved in 3.5), $|B_{x,m}|$ is equal to number of layers in $SH^k_n(m)$. There are 2^n vertices in $B_{x,m}$, so the total number of unique configurations of disks on peg m is 2^n. □

Theorem 3.8. *Given an arbitrary vertex $\alpha = a_{n-1}a_{n-2}...a_0$ where $a_i \in \{0,\ 1,\ \cdots,\ k-1\}$ in $SH^k_n(m)$ we can find its corresponding layer, i.e. in which layer this vertex exists.*

(1) $a_i \neq m \Rightarrow c_i = 0$
(2) $a_i = m \Rightarrow c_i = 1$
(3) *We can convert the n-bit binary string $c_{n-1}c_{n-2}...c_0$ to a decimal λ:*

$$\sum_{i=0}^{n-1} c_i 2^i = \lambda$$

Vertex α belongs to layer L_λ in $SH^k_n(m)$.

For example, the vertex labeled by 12302 in $SH_5^4(2)$ lies in layer L_{17}, because 12302 becomes 01001 and the binary string 01001 becomes 17.

Proof. Given two vertices in H_n^k, respectively labeled $\alpha = a_{n-1}b_{n-2}...a_0$ and $\beta = b_{n-1}b_{n-2}...b_0$, if $\forall i\ a_i = m \Leftrightarrow b_i = m$, then vertices α and β are in the same layer in $SH_n^k(m)$. Thus, we may generalize 3.6 into 3.8. □

Theorem 3.9. *In $SH_n^k(m)$, layer L_λ is isomorphic to $H_{n-\mu}^{k-1}$. $\mu = \sum_{i=0}^{n} a_i$ where $a_i \in \{0,1\}$ such that $\sum_{i=0}^{n} a_i 2^i = \lambda$.*

Proof. μ is the number of 1s in the binary expression of λ, and that gives us the the number of m's in a vertex lying in layer L_λ (cf.). Thus, μ gives us the number of disks on peg m, and $n - \mu$ gives us the number of disks on the remaining $k-1$ pegs. □

The Frame-Stewart algorithm tells us after moving the smallest $n - l_n$ disks from peg 0 to peg 2, to travel along the layer $L_{2^{n-l_n}}$, which is a $H_{l_n}^3$ graph. Thus, the three steps of the Frame-Stewart algorithm can be thought of as three graphs, $H_{n-l_n}^4$, $H_{l_n}^3$ and $H_{n-l_n}^4$, respectively, and the Frame-Stewart solution splices these graphs together.

Proposition 3.10. *Let $x, y, z \in \{0,\ 1,\ 3,\ 4\}$ where $x \neq y \neq z$. If the Frame-Stewart algorithm gives the optimal solution from one perfect state to another on 4 pegs, then a vertex on $B_{x,y}$ that is the shortest distance away from a perfect state, say* **z** *in H_n^4, exists on layer $L_{2^{n-l_{n+1}}}$ in $SH_n^4(m)$ with either $m = x$ or $m = y$.*

Proof. We assume that the shortest path between two perfect states, say **0** and **1**, only travels along one bridge. Since the only bridges between [0] and [1] lie between $0B_{2,3}$ and $1B_{2,3}$ and since the number of legal moves made in [0] are equal to the number of legal moves made in [1] (say x legal moves), the shortest path takes $2x + 1$ legal moves. Then, the shortest path between perfect states **0** and **1** must minimize x. In order to minimize x, the shortest path between perfect states **0** and **1** must visit the vertex on $0B_{2,3}$ which is the shortest distance away from **0**. The Frame-Stewart algorithm tells us to travel along the layer $L_{2^{n-l_n}}$. The bridge we take between [0] and [1] must in the layer $L_{2^{n-l_n}}$ and thus the bridge must be between the vertices in $0B_{2,3}$ and $1B_{2,3}$ which are in the layer $L_{2^{n-l_n}}$. Thus the vertex in $0B_{2,3}$ which is the shortest distance away from **0** lies in the layer $L_{2^{n-l_n}}$. Similarly, the vertex in $B_{2,3}$ which is the shortest distance away from **0** lies in the layer $L_{2^{n-l_{n+1}}}$. □

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