

WEAKLY PROPER MODULI STACKS OF CURVES

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ABSTRACT. This is the first in a projected series of three papers in which we construct the second flip in the log minimal model program for \overline{M}_g . We introduce the notion of a weakly proper algebraic stack, which may be considered as an abstract characterization of those mildly non-separated moduli problems encountered in the context of Geometric Invariant Theory (GIT), and develop techniques for proving that a stack is weakly proper without the usual semistability analysis of GIT. We define a sequence of moduli stacks of curves involving nodes, cusps, tacnodes, and ramphoid cusps, and use the aforementioned techniques to show that these stacks are weakly proper. This will be the key ingredient in forthcoming work, in which we will prove that these moduli stacks have projective good moduli spaces which are log canonical models for \overline{M}_g .

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1. INTRODUCTION

In [Has05], Hassett proposed the problem of studying log canonical models of \overline{M}_g . For any $\alpha \in \mathbb{Q} \cap [0, 1]$ such that $K_{\overline{M}_g} + \alpha\Delta$ is big, Hassett and Keel define

$$\overline{M}_g(\alpha) := \mathbf{Proj} \bigoplus_{m \geq 0} H^0(\overline{\mathcal{M}}_g, [m(K_{\overline{M}_g} + \alpha\Delta)]),$$

and ask whether the spaces $\overline{M}_g(\alpha)$ admit a modular interpretation. In [HH09, HH08], Hassett and Hyeon carried out the first two steps of this program by showing that there is a contraction $\overline{M}_g \rightarrow \overline{M}_g(9/11)$ and a flip $\overline{M}_g(7/10+\epsilon) \leftarrow \overline{M}_g(7/10) \rightarrow \overline{M}_g(7/10-\epsilon)$. Moreover, $\overline{M}_g(9/11) = \overline{M}_g(7/10 + \epsilon)$ parameterizes pseudostable curves with cusps and nodes, and $\overline{M}_g(7/10)$ and $\overline{M}_g(7/10 - \epsilon)$ parameterize certain equivalence classes of curves with nodes, cusps, and tacnodes.

In [HH09, HH08], these alternate birational models of \overline{M}_g are constructed using Geometric Invariant Theory (GIT). Indeed, one of the most appealing features of the Hassett-Keel program is the way it ties together the different compactifications of M_g obtained by varying the parameters implicit in Mumford's classical GIT construction of \overline{M}_g . In this paper, however, we will outline a program to construct modular interpretations for the spaces $\overline{M}_g(\alpha)$ without GIT. The program has three steps.

- (1) Define a *weakly proper* moduli stack $\overline{\mathcal{M}}_g(\alpha)$ parameterizing certain singular curves.
- (2) Construct a *good moduli space* $\overline{\mathcal{M}}_g(\alpha) \rightarrow X$.
- (3) Show that some multiple of the \mathbb{Q} -line bundle $K_{\overline{\mathcal{M}}_g(\alpha)} + \alpha\delta$ on $\overline{\mathcal{M}}_g(\alpha)$ descends to an ample line bundle on X . Use a discrepancy calculation to conclude that $X = \overline{M}_g(\alpha)$.

Let us elaborate on each of these steps.

- (1) The notion of a *weakly proper* algebraic stack is introduced in Section 2, and is the key definition of this paper. Roughly speaking, weak properness is an abstract characterization of those mildly non-separated moduli problems encountered in the context of GIT, which nevertheless possess a proper moduli space. More precisely, the statement that $\overline{\mathcal{M}}_g(\alpha)$ is weakly proper translates as: given a family of curves $\mathcal{C}^* \rightarrow \Delta^*$ over a punctured disc, then (1) after a base change, there is an extension to a family of curves $\mathcal{C} \rightarrow \Delta$ in $\overline{\mathcal{M}}_g(\alpha)$, and (2) given two such extensions $\mathcal{C} \rightarrow \Delta$ and $\mathcal{C}' \rightarrow \Delta$ with both central fibers C_0 and C'_0 closed in $\overline{\mathcal{M}}_g(\alpha)$, there is an isomorphism $C_0 \cong C'_0$. In other words, *closed limits are unique*.
- (2) Good moduli spaces are introduced and studied in [Alp08], and they should be considered as an abstract version of the quotients produced by GIT. One essential difference however is that, whereas GIT quotients are automatically projective, good moduli spaces are *a priori* only algebraic spaces. This step can be viewed as an analogue of the Keel-Mori theorem guaranteeing the existence of a coarse moduli space for separated Deligne-Mumford stacks.
- (3) Under mild hypotheses, $\overline{\mathcal{M}}_g(\alpha)$ possesses tautological line-bundles

$$\lambda_m := \wedge^m \pi_* \omega_{\mathcal{C}/\overline{\mathcal{M}}_g(\alpha)}.$$

Using the Grothendieck-Riemann-Roch formula, one can express $K_{\overline{\mathcal{M}}_g(\alpha)} + \alpha\delta$ as a linear combination of λ and λ_2 . As we explain in [AFS10], it is relatively simple to understand what linear combinations of K and δ (or equivalently λ and λ_2) descend to the good moduli space. Thus, using Kleiman's criterion, the problem of proving that $K_{\overline{\mathcal{M}}_g(\alpha)} + \alpha\delta$ defines an ample divisor class reduces to showing that a certain linear combination of tautological classes is positive on one parameter families of singular curves (i.e., the curves contained in $\overline{\mathcal{M}}_g(\alpha)$). Finally, assuming that sections of $K_{\overline{\mathcal{M}}_g(\alpha)} + \alpha\delta$ lift to sections of $K_{\overline{\mathcal{M}}_g} + \alpha\delta$ (which can be verified by a straightforward discrepancy calculation), it follows that the good moduli space associated to $\overline{\mathcal{M}}_g(\alpha)$ is $\overline{M}_g(\alpha)$.

This is the first in a projected series of papers in which we will follow this approach to construct the second flip in the Hassett-Keel log minimal model program for \overline{M}_g . In addition, we will recover the results of Hassett and Hyeon, and extend their constructions to the case of $\overline{M}_{g,n}$ with $n > 0$. We should emphasize the point that our methods make no use of GIT. Indeed, once completed, our construction should provide the first examples of moduli spaces associated to non-separated moduli functors, for which no GIT-construction is known.

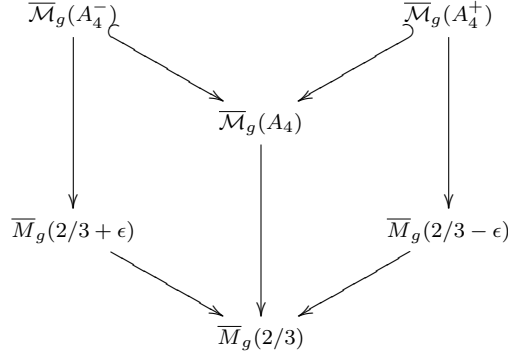
The main result. In the present paper, we accomplish the first of the three steps outlined above. For $k = 2, 3, 4$, we define moduli stacks $\overline{\mathcal{M}}_g(A_k^-)$, $\overline{\mathcal{M}}_g(A_k)$, $\overline{\mathcal{M}}_g(A_k^+)$ parameterizing certain curves with A_k -singularities, which we call A_k^- -stable, A_k -stable, and A_k^+ -stable curves respectively (Definition 4.9). We should warn the reader that our notation has built-in redundancy: A_k^+ -stability is precisely the same as A_{k+1}^- -stability. Our main result is the following.

Theorem 1.1.

- (1) $\overline{\mathcal{M}}_g(A_k^-)$, $\overline{\mathcal{M}}_g(A_k)$, $\overline{\mathcal{M}}_g(A_k^+)$ are weakly proper algebraic stacks.
- (2) These stacks fit into the following diagram, where the horizontal arrows are open immersions.

$$\begin{array}{ccccccc}
 \overline{\mathcal{M}}_g & & \overline{\mathcal{M}}_{g,n}(A_2^+) & & \overline{\mathcal{M}}_{g,n}(A_3^+) & & \\
 \parallel & \hookrightarrow & \parallel & \hookrightarrow & \parallel & \hookrightarrow & \\
 \overline{\mathcal{M}}_{g,n}(A_2^-) & & \overline{\mathcal{M}}_{g,n}(A_3^-) & & \overline{\mathcal{M}}_{g,n}(A_4^-) & & \\
 & & \parallel & & \parallel & & \\
 & & \overline{\mathcal{M}}_{g,n}[A_3] & & \overline{\mathcal{M}}_{g,n}(A_4) & & \overline{\mathcal{M}}_{g,n}(A_4^+)
 \end{array}$$

The main result is proved in Corollary 4.11 and Theorem 9.1. In forthcoming papers in the series, we will use this result to prove that these stacks have projective good moduli spaces. Once this program is complete, we will have the following modular interpretation of the second flip,



in which the locus of curves containing a genus 2 tail attached at a Weierstrass point is flipped to the locus of curves containing a ramphoid cusp.

Remark. It was our original hope that the main theorem would generalize to all k providing weakly proper moduli stacks $\overline{\mathcal{M}}_{g,n}(A_k)$ parameterizing curves with A_2, \dots, A_k singularities. However, using the natural extension of the definitions, the deformation openness for $\overline{\mathcal{M}}_{g,n}(A_5^+)$ fails. At this stage, one expects to replace genus 2 bridges attached nodally at Weierstrass conjugate points with oscnodes ($y^2 = x^6$). The difficulty is that in our definition of A_5^+ -stability, curves with tacnodally attached genus 2 curves are A_5^+ -stable. However, there exists degenerations $C_t \rightsquigarrow C_0$ of curves admitting genus 2 bridges attached nodally at Weierstrass conjugate points degenerating (as the two nodes collide) to a curve with a tacnodally attached genus 2 tail. To remedy this, one needs to first replace genus 2 tails attached nodally to a curve K at an non-Weierstrass point with “dangling” A_5 -singularities (obtained by the oscnodal union of K and a rational curve). In fact, [AFS10] gives precise predictions for the moduli interpretations of $\overline{\mathcal{M}}_g(\alpha)$ and the methods of this paper should extend to prove weak properness of these moduli stacks.

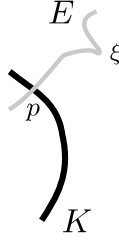
Local variation of GIT. One of the key ideas in this paper is that étale locally around any closed point $[C] \in \overline{\mathcal{M}}_g(A_k)$, the open inclusions

$$\overline{\mathcal{M}}_g(A_k^-) \subseteq \overline{\mathcal{M}}_g(A_k) \supseteq \overline{\mathcal{M}}_g(A_k^+)$$

correspond to the open chambers

$$X^- \subseteq X = \widehat{\text{Def}}(C) \supseteq X^+$$

given by applying variation of GIT to the action of $\text{Aut}(C)$ on the miniversal deformation space $\widehat{\text{Def}}(C)$. This gives a powerful tool for analyzing the local geometry of the stacks $\overline{\mathcal{M}}_g(A_k^-) \subseteq \overline{\mathcal{M}}_g(A_k) \supseteq \overline{\mathcal{M}}_g(A_k^+)$, and is essential in the proof of Theorem 1.1. To emphasize how this works, let us compute the X^-/X^+ -chambers for the action of $\text{Aut}(C)$ on $X = \widehat{\text{Def}}(C)$, where C is the union of a smooth genus $g - 2$ curve K with a nodally attached rational ramphoid cuspidal curve E (see Figure 1). Let $p \in C$

FIGURE 1. The curve $C = K \cup E$.

be the attaching node, $\xi \in E$ be the ramphoid cusp ($y^2 = x^5$), and $\nu : \mathbb{P}^1 \rightarrow E$ the normalization of E with $\nu(0) = \xi, \nu(\infty) = p$.

To understand the action of $\text{Aut}(C)$ on $T^1(C)$, the space of first order deformations of C , recall that $T^1(C)$ maps surjectively onto the product of the spaces of first order deformations of the singularities p and ξ with kernel given by the space of first order locally trivial deformations $\text{LT}^1(C)$. Moreover, a locally trivial deformation of C induces a deformation of (K, p) and the map $\text{LT}^1(C) \rightarrow T^1(K, p)$ is surjective with kernel $\text{LT}^1(E, p)$, parameterizing how $0 \in \mathbb{P}^1$ is “crimped” to a ramphoid cusp. Thus, we obtain a diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \text{LT}^1(E, p) & & & & \\
 & & \downarrow & & & & \\
 0 & \longrightarrow & \text{LT}^1(C) & \longrightarrow & T^1(C) & \longrightarrow & T^1(\widehat{\mathcal{O}}_{C, \xi}) \times T^1(\widehat{\mathcal{O}}_{C, p}) \rightarrow 0 \\
 & & \downarrow & & & & \\
 & & T^1(K, p) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

where the column and row are exact sequences of $\text{Aut}(C)$ -representations. Let us now explicitly describe each representation. We have

$$\begin{aligned}
 T^1(\widehat{\mathcal{O}}_{C, \xi}) &= \{y^2 = x^5 + s_3x^3 + s_2x^2 + s_1x + s_0 : s_i \in \mathbb{C}\} \\
 T^1(\widehat{\mathcal{O}}_{C, p}) &= \{y^2 = x^2 + n : n \in \mathbb{C}\}
 \end{aligned}$$

Furthermore, the first order crimping space $\text{LT}^1(E, p)$ parameterizes subrings of the power series ring $\mathbb{C}[[z]]$ abstractly isomorphic to $\mathbb{C}[[z^2, z^5]]$. One easily sees that this

space is parameterized by a parameter $c \in \mathbb{C}$ so that

$$\mathrm{LT}^1(E, p) = \{\mathbb{C}[(z + cz^2)^2, (z + cz^2)^5] \subseteq \mathbb{C}[[z]] : c \in \mathbb{C}\}$$

Therefore, we can write

$$\widehat{\mathrm{Def}}(C) = \widehat{\mathrm{Def}}(K, p) \times \mathrm{Spf} \mathbb{C}[[\{s_i\}, n, c]]$$

If we fix an isomorphism $\mathrm{Aut}(C)^\circ \cong \mathbb{G}_m = \mathrm{Spec} \mathbb{C}[t]_t$ which acts on a local coordinate z around $0 \in \mathbb{P}^1$ by $z \mapsto tz$, then the action of $\mathrm{Aut}(C)^\circ$ on $\widehat{\mathrm{Def}}(C)$ is given by

$$s_i \mapsto t^{-(10-2i)} s_i, \quad n \mapsto tn, \quad c \mapsto tc$$

and is trivial on $\widehat{\mathrm{Def}}(K, p)$. Now, in this simple case, the chambers X^- and X^+ are defined as the non-vanishing locus of functions of negative and positive weight respectively (for the general definition of local variation of GIT chambers, see Section 7). Thus, $X^- \subseteq X = \widehat{\mathrm{Def}}(C) \supseteq X^+$ are defined by the closed loci:

$$X \setminus X^- = V(\{s_i\}) \quad \text{and} \quad X \setminus X^+ = V(n, c)$$

The locus $V(\{s_i\})$ precisely corresponds to deformations in which the ramphoid cusp singularity $\xi \in C$ is preserved, and the locus $V(n, c)$ corresponds to deformations preserving the node $p \in C$ and with the node being a Weierstrass point of the genus 2 curve E . Since $\overline{\mathcal{M}}_g(A_k) \setminus \overline{\mathcal{M}}_g(A_k^-)$ is the locus of curves with a ramphoid cusp, and $\overline{\mathcal{M}}_g(A_k) \setminus \overline{\mathcal{M}}_g(A_k^+)$ corresponds to the locus of curves containing a genus 2 tail attached at a Weierstrass point, we see that the GIT chambers do cut out the inclusions $\overline{\mathcal{M}}_g(A_k^-) \subseteq \overline{\mathcal{M}}_g(A_k) \supseteq \overline{\mathcal{M}}_g(A_k^+)$ in a neighborhood of C . The bulk of this paper (Sections 5, 6, 7 and 8) is devoted to generalizing this description of the local structure of the inclusions $\overline{\mathcal{M}}_g(A_k^-) \subseteq \overline{\mathcal{M}}_g(A_k) \supseteq \overline{\mathcal{M}}_g(A_k^+)$ to arbitrary closed points of $\overline{\mathcal{M}}_g(A_k)$, where combinatorial considerations require a fairly extensive case-by-case analysis.

Roadmap. In Section 2, we introduce the key definition of this paper, namely weakly proper morphisms, and systematically develop their properties. In particular, we show that if an algebraic stack \mathcal{X} admits a good moduli space $\mathcal{X} \rightarrow Y$, then under mild hypotheses, \mathcal{X} is weakly proper if and only if Y is proper (Proposition 2.11).

In Section 3, we introduce the algebraic stacks $\overline{\mathcal{H}}_{m,1}$ (resp., $\overline{\mathcal{H}}_{m,2}$) parameterizing genus m hyperelliptic curves with a marked Weierstrass point (resp., genus m hyperelliptic curves with two marked Weierstrass conjugate points). The algebraic stack $\overline{\mathcal{H}}_{m,1}$ (resp., $\overline{\mathcal{H}}_{m,2}$) is a birational model of the variety of “stable tails” associated to an A_{2m} -singularity (resp. A_{2m+1} -singularity). We also introduce the algebraic stacks $\overline{\mathcal{S}}_{m,1}$ (resp., $\overline{\mathcal{S}}_{m,2}$) parameterizing genus m curves obtained by imposing an A_{2m} -singularity on a smooth rational curve (resp., an A_{2m+1} -singularity on two smooth rational curves). The algebraic stack $\overline{\mathcal{S}}_{m,1}$ (resp., $\overline{\mathcal{S}}_{m,2}$) is a birational model of the variety of “crimping deformations” associated to an A_{2m} -singularity (resp. A_{2m+1} -singularity). We give explicit quotient presentations for the stacks $\overline{\mathcal{H}}_{m,1}$, $\overline{\mathcal{H}}_{m,2}$, $\overline{\mathcal{S}}_{m,1}$, $\overline{\mathcal{S}}_{m,2}$ (Proposition 3.9 and Proposition 3.4) from which we obtain the key fact that will be used repeatedly

throughout the sequel, namely that any $H_{m,1}$ or $S_{m,1}$ -curve (resp. $H_{m,2}$ or $S_{m,2}$ -curve) admits an isotrivial degeneration to the *monomial $H_{m,1}$ -curve* (resp., *monomial $H_{m,2}$ -curve*), which is the curve obtained by imposing a monomial A_{2m} -singularity at $0 \in \mathbb{P}^1$ and marking ∞ (resp., imposing a monomial A_{2m+1} -singularity at the two origins in \mathbb{P}^1 with marked points at the ∞ 's); see Definition 3.10.

In Section 4, we define the moduli stacks $\overline{\mathcal{M}}_{g,n}(A_k)$ (resp., $\overline{\mathcal{M}}_{g,n}(A_k^+)$, $\overline{\mathcal{M}}_{g,n}(A_k^-)$) parameterizing A_k -stable (resp., A_k^+ -stable, A_k^- -stable) curves; see Definition 4.7. The definitions are inductive so that A_k^+ -stability is equivalent to A_{k+1}^- -stability. The main result of this section is Theorem 4.10 which states that these stability conditions are deformation open for $k \leq 4$, from which it follows immediately that $\overline{\mathcal{M}}_{g,n}(A_k^-)$, $\overline{\mathcal{M}}_{g,n}(A_k)$, $\overline{\mathcal{M}}_{g,n}(A_k^+)$ are algebraic stacks. The proof is surprisingly subtle, due to the intricate combinatorics of the loci being added and removed. After this section, we always assume $k \leq 4$.

In Section 5, we give a geometric characterization of the closed points of $\overline{\mathcal{M}}_g(A_k)$. We show that for k even (resp., k odd), any curve C in $\overline{\mathcal{M}}_g(A_k)$ has a canonical decomposition $C = K \cup E_1 \cup \dots \cup E_r$ where the “core” K is an A_k -stable curve containing no $H_{m,1}$ -tails and each E_i, \dots, E_r is a nodally attached $H_{m,1}$ -tail (resp., $C = K \cup E_1 \cup \dots \cup E_r$ where the “core” K is an A_k -stable curve containing no $H_{m,2}$ -chains and each E_i is a nodally attached $H_{m,2}$ -link (i.e., a nodal union of $H_{m,2}$ -chains)). See Definitions 5.2 and 5.5 for precise details, as well as descriptions of the canonical decomposition in the more general case of marked points. The main result of this section is Theorem 5.14 which states that an A_k -stable curve $(C, \{p_i\})$ is a closed point of $\overline{\mathcal{M}}_{g,n}(A_k)$ if and only if the core K is a closed point in $\overline{\mathcal{M}}_{h,m}(A_k^-)$ and every nodally attached $H_{m,1}$ -tail/ $H_{m,2}$ -bridge is monomial.

In Section 6, we give an explicit description, around any closed point $[C, \{p_i\}] \in \overline{\mathcal{M}}_{g,n}(A_k)$, of the action of the automorphism group $\text{Aut}(C, \{p_i\})$ on the deformation space of a marked curve $(C, \{p_i\})$. This makes essential use of the geometric characterization of closed points in Section 5, as well as the description of the deformation theory of $H_{m,1}/H_{m,2}$ -curves and $S_{m,1}/S_{m,2}$ -curves contained in Section 3. First, we describe the action of $\text{Aut}(C, \{p_i\})$ on the first order deformation space $T^1(C, \{p_i\})$ (see Propositions 6.2 and 6.3). We then describe how one can choose geometric coordinates for the miniversal deformation space $\widehat{\text{Def}}(C, \{p_i\})$ which diagonalize the action of $\text{Aut}(C, \{p_i\})$ and have the property that the vanishing of certain subsets of these coordinates cuts out the closed loci $\overline{\mathcal{S}}_{g,n}(A_k) := \overline{\mathcal{M}}_{g,n}(A_k) \setminus \overline{\mathcal{M}}_{g,n}(A_k^-)$ and $\overline{\mathcal{H}}_{g,n}(A_k) := \overline{\mathcal{M}}_{g,n}(A_k) \setminus \overline{\mathcal{M}}_{g,n}(A_k^+)$ (see Propositions 6.5 and 6.7).

In Section 7, we calculate the plus/minus-chambers obtained by variation of GIT for the action of the automorphism group $\text{Aut}(C, \{p_i\})$ on the deformation space $\text{Def}(C, \{p_i\})$ of a closed point $[C, \{p_i\}] \in \overline{\mathcal{M}}_{g,n}(A_k)$. The main result is Theorem 7.9, which asserts that the closed loci $V^- \subseteq \widehat{\text{Def}}(C) \supseteq V^+$ coming from variation of GIT cut out

the closed loci $\overline{\mathcal{S}}_{g,n}(A_k) \subseteq \overline{\mathcal{M}}_{g,n}(A_k) \supseteq \overline{\mathcal{H}}_{g,n}(A_k)$ corresponding to the complements of $\overline{\mathcal{M}}_{g,n}(A_k^-)$ and $\overline{\mathcal{M}}_{g,n}(A_k^+)$ respectively.

Section 8 handles the technicalities needed to jump from the formal local description of the inclusions $\overline{\mathcal{M}}_{g,n}(A_k^-) \subseteq \overline{\mathcal{M}}_{g,n}(A_k) \supseteq \overline{\mathcal{M}}_{g,n}(A_k^+)$ given in Theorem 7.9 to the étale local description given in Theorem 8.3.

Section 9 proves the main result of this paper, namely that the stacks $\overline{\mathcal{M}}_{g,n}(A_k^-)$, $\overline{\mathcal{M}}_{g,n}(A_k)$ and $\overline{\mathcal{M}}_{g,n}(A_k^+)$ are weakly proper. Since $\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n}(A_2^-)$ is proper, it suffices to show: (1) $\overline{\mathcal{M}}_{g,n}(A_k^-)$ is weakly proper implies $\overline{\mathcal{M}}_{g,n}(A_k)$ weakly proper, and (2) $\overline{\mathcal{M}}_{g,n}(A_k)$ weakly proper implies $\overline{\mathcal{M}}_{g,n}(A_k^+)$ weakly proper. For (1), given a family $\mathcal{C}^* \rightarrow \Delta^*$ of smooth curves, one obtains the unique A_k -stable limit as follows: first, take the unique A_k^- -stable limit, then degenerate all $H_{m,1}/H_{m,2}$ -curves to monomial $H_{m,1}/H_{m,2}$ -curves. Since this procedure is canonical, the uniqueness of A_k -stable limits follows from the uniqueness of A_k^- -stable limits. For (2), we do not give an explicit construction of the limiting process but instead deduce it from the étale local description in Theorem 8.3 of the inclusion $\overline{\mathcal{M}}_{g,n}(A_k^+) \subset \overline{\mathcal{M}}_{g,n}(A_k)$ as the variation of GIT locus $\text{Def}(C, \{p_i\})^+ \subseteq \text{Def}(C, \{p_i\})$, using a purely formal diagram chase.

Conventions. The symbol \mathbb{C} will denote a fixed algebraically closed field of characteristic zero. All schemes, algebraic spaces and algebraic stacks are assumed to be quasi-separated over \mathbb{C} . We use the notation $\Delta = \text{Spec } R$ and $\Delta^* = \text{Spec } K$, where R is a valuation ring with fraction field K ; we set 0 , η and $\overline{\eta}$ to be the closed point, the generic point and the geometric generic point respectively of Δ . A *curve* is a connected reduced proper \mathbb{C} -scheme of dimension one. An *n -pointed curve* $(C, \{p_i\}_{i=1}^n)$ is a curve C with n distinct smooth marked points $p_i \in C$.

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2. WEAK PROPERNESS

Let G be a reductive group acting on a projective scheme X with an ample G -linearization \mathcal{L} . Consider the quotient stack $\mathcal{X} = [X_{\mathcal{L}}^{\text{ss}}/G]$ where $X_{\mathcal{L}}^{\text{ss}}$ is the semistable locus. If there exists strictly semi-stable points in X , then \mathcal{X} is necessarily non-separated (and therefore not proper) as a strictly semi-stable point $x \in X$ has a non-finite affine stabilizer G_x . However, there is a good moduli space $[X^{\text{ss}}/G] \rightarrow X^{\text{ss}}/G := \text{Proj } \bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}^d)^G$ where the GIT quotient $X^{\text{ss}}//G$ is projective.

The GIT quotient stack $[X^{\text{ss}}/G]$ satisfies the well-known semistable replacement property: if $0 \in \Delta$ is a pointed disc and $f : \Delta^* \rightarrow [X^{\text{ss}}/G]$ is a morphism from the punctured disc, then there exists a covering $\Delta' \rightarrow \Delta$ branched over 0 and a morphism

$\Delta' \rightarrow [X^{\text{ss}}/G]$ extending f . In other words, $[X^{\text{ss}}/G]$ satisfies the valuative criterion for universally closedness. One can always choose an extension $\tilde{f} : \Delta' \rightarrow [X^{\text{ss}}/G]$ such that $\tilde{f}(0')$ is a closed point. Furthermore, the closed point $\tilde{f}(0')$ is unique; as $X^{\text{ss}}//G$ is proper there exists a unique extension $h : \Delta' \rightarrow X^{\text{ss}}//G$ and $\tilde{f}(0')$ corresponds to the unique closed point in the stack $[X^{\text{ss}}/G]$ (i.e., the unique closed orbit in X^{ss}) over $h(0')$.

The guiding principle is to introduce a notion of weakly separated (resp., weak properness) such that:

- (1) the property can be checked in practice on moduli problems,
- (2) given the existence of a good moduli space, the property is equivalent to the separation (resp., properness) of the good moduli space, and
- (3) the property is useful for establishing the existence of a good moduli space.

(Unfortunately, the notion we introduce only satisfies the condition (2) if the good moduli space is known a priori to be locally separated (i.e., the diagonal is an immersion (eg. a scheme)); see Example 2.8.)

Definition 2.1. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of algebraic stacks.

- (1) We say that f is *weakly separated* if for every valuation ring R with fraction field K , and 2-commutative diagrams

$$(2.1) \quad \begin{array}{ccc} \Delta^* = \text{Spec } K & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow^{h_1} & \downarrow f \\ \Delta = \text{Spec } R & \longrightarrow & \mathcal{Y} \end{array}$$

such that $h_1(0)$ and $h_2(0)$ are closed in $|\mathcal{X} \times_{\mathcal{Y}} \Delta|$, then $h_1(0) = h_2(0) \in |\mathcal{X} \times_{\mathcal{Y}} \Delta|$.

- (2) We say that f is *weakly proper* if f is weakly separated, finite type and universally closed.

Remark 2.2. While the valuative criterion for properness can be checked for valuation rings where the image of the generic point lies in a fixed dense open, the analogue for weak properness is not true.

Remark 2.3. We stress that in (1) we are only requiring that the images of the points $h_1(0)$ and $h_2(0)$ agree as points in the topological space $|\mathcal{X} \times_{\mathcal{Y}} \Delta|$. If one was to require in (1) above the existence of an isomorphism $h_1 \xrightarrow{\sim} h_2$ (not necessarily extending the given isomorphism $h_1|_{\text{Spec } K} \xrightarrow{\sim} h_2|_{\text{Spec } K}$), the resulting definition is too strong for our purposes as GIT quotient stacks $[X^{\text{ss}}/G]$ with projective good moduli spaces will not necessarily be weakly separated. For instance, consider $[(\mathbb{P}^1)^4, \text{ss}/\text{PGL}_2]$ with the symmetric linearization. Consider the two families $(0, [t^2, 1], [1, t^2], \infty)$ and $(0, [t^3, 1], [1, t], \infty)$ over $\text{Spec } \mathbb{C}[[t]]$. Over the generic fiber $\text{Spec } \mathbb{C}((t))$, the families are isomorphic by $x \mapsto tx$. The central fibers both correspond to the closed point $(0, 0, \infty, \infty)$ but there is no isomorphism of the families over $\text{Spec } \mathbb{C}[[t]]$.

We now give a different interpretation of this definition. Let Δ be the spectrum of a valuation ring R with fraction field K . Consider a 2-commutative diagram \mathfrak{D} :

$$(2.2) \quad \begin{array}{ccc} \Delta^* & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ \Delta & \longrightarrow & \mathcal{Y} \end{array}$$

Define the set $\Sigma_{\mathfrak{D}} \subseteq |\mathcal{X} \times_{\mathcal{Y}} \Delta|$ to consist of points x' such that:

- (1) There exists an extension K' of K and a valuation ring R' for K' dominating R .
- (2) There is an extension of \mathfrak{D} to a 2-commutative diagram \mathfrak{D}'

$$(2.3) \quad \begin{array}{ccccc} \Delta'^* = \mathrm{Spec} K' & \longrightarrow & \Delta^* = \mathrm{Spec} K & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & \nearrow h' & \downarrow f \\ \Delta' = \mathrm{Spec} R' & \longrightarrow & \Delta = \mathrm{Spec} R & \longrightarrow & \mathcal{Y} \end{array}$$

such that $x' = h'(0')$ where $0' \in \Delta'$ is the closed point.

Lemma 2.4. *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of algebraic stacks. Let \mathfrak{D} be a commutative diagram as in Equation (2.2). Let $\xi \in |\mathcal{X} \times_{\mathcal{Y}} \Delta|$ be the image of $\Delta^* \rightarrow \mathcal{X}$. Let $x \in |\mathcal{X} \times_{\mathcal{Y}} \Delta|$. Then the following are equivalent:*

- (1) $x \in \Sigma_{\mathfrak{D}}$.
- (2) There is a specialization $\xi \rightsquigarrow x$ in $|\mathcal{X} \times_{\mathcal{Y}} \Delta|$ over $\eta \rightsquigarrow 0$.
- (3) There exists an extension to a diagram \mathfrak{D}' as in Diagram 2.3 where $K \hookrightarrow K'$ is a finite, separable extension and $x = h'(0')$.

In particular, the topological space $\Sigma_{\mathfrak{D}}$ is closed under specialization.

Proof. It is clear that (3) \implies (1) \implies (2). Suppose $\xi \rightsquigarrow x$ is a specialization in $|\mathcal{X} \times_{\mathcal{Y}} \Delta|$ over $\eta \rightsquigarrow 0$. If $U \rightarrow \mathcal{X} \times_{\mathcal{Y}} \Delta$ is a smooth presentation with U a scheme, then there exists a specialization $u \rightsquigarrow u_0$ over $\xi \rightsquigarrow x$. We may assume that u and ξ are generic points. After slicing \mathcal{O}_{U, u_0} , we may find a valuation ring R' and a morphism $\mathrm{Spec} R' \rightarrow U$ with the fraction field K' of R' a finite, separable extension of K . \square

Lemma 2.5. *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of algebraic stacks. Then*

- (1) f is weakly separated if and only if for all diagrams \mathfrak{D} as in Equation (2.2), $\Sigma_{\mathfrak{D}}$ has at most one closed point.
- (2) If f is quasi-compact, then f is universally closed if and only if for all diagrams \mathfrak{D} as in Equation (2.2), $\Sigma_{\mathfrak{D}}$ is non-empty.
- (3) If f is finite type, then f is weakly proper if and only if for all diagrams \mathfrak{D} as in Equation (2.2), $\Sigma_{\mathfrak{D}}$ has a unique closed point.

Furthermore, in (1), (2) and (3) we may restrict to diagrams \mathfrak{D} as in Equation (2.2) where R is a complete valuation ring with algebraically closed residue field. If $f : \mathcal{X} \rightarrow$

\mathcal{Y} is a locally of finite type morphism with \mathcal{Y} locally noetherian, then in (1) we may restrict to diagrams \mathfrak{D} as in Equation (2.2) where R is a complete discrete valuation ring with algebraically closed residue field.

Proof. For (1), the “if” direction is clear. Conversely, suppose there is a commutative diagram \mathfrak{D} as in Equation (2.2) and two extensions as in Equation (2.3)

$$\begin{array}{ccccc} \Delta_i^* = \mathrm{Spec} K_i & \longrightarrow & \Delta^* & \longrightarrow & \mathcal{X} \\ \downarrow & & \swarrow h_i & \downarrow & \downarrow f \\ \Delta_i = \mathrm{Spec} R_i & \longrightarrow & \Delta & \longrightarrow & \mathcal{Y} \end{array}$$

for $i = 1, 2$ with both $h_1(0_1)$ and $h_2(0_2)$ distinct closed points of $|\mathcal{X} \times_{\mathcal{Y}} \Delta|$, where $0_i \in \mathrm{Spec} R_i$ is the closed point. There is a field extension $K \hookrightarrow K'$ containing both K_1 and K_2 and a valuation ring $R' \subseteq K'$ dominating both R_1 and R_2 giving a 2-commutative diagram

$$\begin{array}{ccc} \Delta^* = \mathrm{Spec} K' & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow h'_1 & \downarrow f \\ \Delta' = \mathrm{Spec} R' & \longrightarrow & \mathcal{Y} \end{array}$$

with two lifts h'_1 and h'_2 such that $h'_1(0')$ and $h'_2(0')$ are closed and distinct. This contradicts f being weakly separated.

Statements (2) and (3) follow from [LMB00, Theorem 7.3]. The refinements follow from [LMB00, 7.2.1-7.2.3].

□

Remark 2.6. It is not clear whether the valuative criterion for universally closed (and hence also weakly proper) can be checked on *discrete valuation rings* for finite type morphisms of noetherian algebraic stacks. For instance, it is easy to see that if the valuative criterion for universally closed holds for DVRs for a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of noetherian algebraic stacks, then for any *noetherian* \mathcal{Y}' and morphism $\mathcal{Y}' \rightarrow \mathcal{Y}$, the base change $\mathcal{X}' \rightarrow \mathcal{Y}'$ is closed. In the case of separated schemes or separated algebraic stacks, one shows that this implies that any base change is closed by using Chow’s lemma (see [DG67, II.5.6.3] and [LMB00, Prop 7.12]).

Proposition 2.7.

- (1) *If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a locally separated and representable morphism of algebraic stacks, then f is separated if and only if f is weakly separated. In particular, a morphism of schemes is separated if and only if it is weakly separated.*
- (2) *Weakly separated (resp., weakly proper) morphisms are stable under base change.*

Proof. For (1), it is clear that if f is separated, then f is weakly separated. Conversely, let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a locally separated and weakly separated morphism of algebraic stacks. Let $Y \rightarrow \mathcal{Y}$ be a smooth presentation with Y a scheme. Then $f' : \mathcal{X} \times_{\mathcal{Y}} Y \rightarrow Y$ is a locally separated and weakly separated morphism of algebraic spaces. If f' is separated, then by descent f is separated. So we can assume that $f : X \rightarrow Y$ is a locally separated and weakly separated morphism of algebraic spaces. Suppose we have a commutative diagram

$$\begin{array}{ccc} \Delta^* & \longrightarrow & X \\ \downarrow & \nearrow h_1 & \downarrow f \\ \Delta & \longrightarrow & Y \end{array}$$

with two lifts h_1 and h_2 . This gives a commutative diagram

$$\begin{array}{ccc} \Delta^* & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{(h_1, h_2)} & X \times_Y X \end{array}$$

Since $X \rightarrow X \times_Y X$ is an immersion, $X \times_{X \times_Y X} \Delta \rightarrow \Delta$ is an immersion. Since f is weakly separated, $h_1(0) = h_2(0)$ in $X \times_Y \Delta$ and it follows that $X \times_{X \times_Y X} \Delta \rightarrow \Delta$ is an isomorphism. Therefore $h_1 = h_2$ and f is separated. Part (2) is clear. \square

Example 2.8. A weakly separated morphism of algebraic spaces need not be separated. For example, take the bug-eyed cover of \mathbb{A}^1 over \mathbb{C} obtained by taking the quotient X/\mathbb{Z}_2 of the non-separated affine line $X = \mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus \{0\}} \mathbb{A}^1$ by the action of \mathbb{Z}_2 which acts by $x \mapsto -x$ and flips the origins. Let $h_1, h_2 : \text{Spec } \mathbb{C}[[x]] \rightarrow X/\mathbb{Z}_2$ be the two morphisms obtained by mapping to the two origins in X . This gives a diagram

$$\begin{array}{ccc} \Delta^* & \longrightarrow & X/\mathbb{Z}_2 \\ \downarrow i & \nearrow h_1 & \downarrow f \\ \Delta & \longrightarrow & \text{Spec } \mathbb{C} \end{array}$$

with $h_1 \circ i = h_2 \circ i$ and $h_1(0) = h_2(0) \in |X/\mathbb{Z}_2|$. However $h_1 \neq h_2$.

Example 2.9. A Deligne-Mumford stack with non-finite inertia may be weakly separated. For an example, let \mathcal{X} be the Deligne-Mumford locus of $[(\mathbb{P}^1)^4/S_4 \times PGL_2]$ over \mathbb{C} consisting of points with finite stabilizer groups. Then $\mathcal{X} \rightarrow \text{Spec } \mathbb{C}$ is weakly separated but is not separated.

Example 2.10. Consider the \mathbb{G}_m -action on the nodal cubic X in \mathbb{P}^2 . Let $\mathcal{X} = [X/\mathbb{G}_m]$. Then \mathcal{X} is weakly proper but does not admit a good moduli space; a good moduli space would necessarily be $\text{Spec } \mathbb{C}$ which would imply that \mathcal{X} is cohomologically affine and therefore that X is affine, a contradiction. Moreover, consider the \mathbb{G} -action on the normalization $\tilde{X} \cong \mathbb{P}^1$ fixing 0 and ∞ . Consider the composition $[\tilde{X}/\mathbb{G}_m] \xrightarrow{f} [X/\mathbb{G}_m] \xrightarrow{g}$

Spec \mathbb{C} . Then g is weakly proper and f is finite, but the composition $g \circ f$ is not weakly separated.

The following proposition is the main result of this section and justifies the introduction of the weakly separated/properness condition.

Proposition 2.11. *Suppose \mathcal{X} is an algebraic stack over a scheme S . Let $\phi : \mathcal{X} \rightarrow Y$ be a good moduli space.*

- (1) *ϕ is weakly separated. If ϕ is also finite type (e.g., if $\mathcal{X} \rightarrow S$ is finite type), then ϕ is weakly proper.*

Suppose in addition that Y is locally separated. Then

- (2) *$\mathcal{X} \rightarrow S$ is weakly separated if and only if $Y \rightarrow S$ is separated.*
 (3) *If \mathcal{X} and Y are finite type over S , then $\mathcal{X} \rightarrow S$ is weakly proper if and only if $Y \rightarrow S$ is proper.*

Remark 2.12. If S is noetherian and $\mathcal{X} \rightarrow S$ is finite type, then $Y \rightarrow S$ is also finite type.

Proof. For (1), suppose we have a commutative diagram \mathfrak{D} as in Diagram 2.1 for the morphism $\mathcal{X} \rightarrow Y$ with two lifts $h_1, h_2 : \Delta \rightarrow \mathcal{X}$. Then the base change $\mathcal{X} \times_Y \Delta \rightarrow \Delta$ is a good moduli space and therefore there is a unique closed point in $\mathcal{X} \times_Y \Delta$ over the closed point of $0 \in \Delta$ (see [Alp08, Prop. 4.7, Thm. 4.16]). Therefore $h_1(0) = h_2(0) \in |\mathcal{X} \times_Y \Delta|$. The second statement in (1) follows because ϕ is universally closed ([Alp08, Thm. 4.16]).

For (2), suppose $Y \rightarrow S$ is separated. As the composition of a weakly separated morphism followed by a separated morphism is weakly separated, it follows from (1) that $\mathcal{X} \rightarrow S$ is weakly separated. Conversely, suppose that $Y \rightarrow S$ is not separated. Then there exists a commutative diagram \mathfrak{D} as in Diagram 2.1 for the morphism $Y \rightarrow S$ with two distinct lifts $h_1, h_2 : \Delta \rightarrow Y$. Since $\mathcal{X} \rightarrow Y$ is universally closed, there exists an extension $K \hookrightarrow K'$ and valuation ring $R' \subseteq K'$ dominating R , a lift $\Delta'^* = \text{Spec } K' \rightarrow \mathcal{X}'$ and two lifts $\tilde{h}_1, \tilde{h}_2 : \Delta' = \text{Spec } R' \rightarrow \mathcal{X}$ such that $\tilde{h}_1(0')$ and $\tilde{h}_2(0')$ are closed in $|\mathcal{X} \times_S \Delta'|$, where $0' \in \Delta'$ is the closed point. But since $\mathcal{X} \rightarrow S$ is weakly separated, $\tilde{h}_1(0') = \tilde{h}_2(0')$ in $|\mathcal{X} \times_S \Delta|$ so $h_1(0) = h_2(0)$ in $|Y \times_S \Delta|$. Therefore, $Y \rightarrow S$ is weakly separated and by Proposition 2.7(1), $Y \rightarrow S$ is separated.

For (3), the morphism $\mathcal{X} \rightarrow S$ is universally closed if and only if $Y \rightarrow S$ is universally closed as ϕ is surjective. \square

3. $S_{m,1}/S_{m,2}$ -CURVES AND $H_{m,1}/H_{m,2}$ -CURVES

Definition 3.1 (A_k -singularities). We say that a curve C has an A_k -singularity at a point p if $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[x, y]]/(y^2 - x^{k+1})$. An A_1 - (resp., A_2 -, A_3 -, A_4 -) singularity is also called a *node* (resp., cusp, tacnode, ramphoid cusp).

3.1. $S_{m,1}/S_{m,2}$ -curves.

Definition 3.2 ($S_{m,1}/S_{m,2}$ -curves).

- (1) An $S_{m,1}$ -curve is a 1-pointed curve of arithmetic genus m obtained by taking a smooth rational curve with a labeled point and imposing an A_{2m} -singularity on it at a point distinct from the labeled point.
- (2) An $S_{m,2}$ -curve is a 2-pointed curve of arithmetic genus m obtained by taking two smooth rational components, each with a labeled point, and identifying them at points distinct from the labeled points to form an A_{2m+1} -singularity.

A family of curves $(\mathcal{C} \xrightarrow{\pi} B, \sigma)$ (resp., $(\mathcal{C} \xrightarrow{\pi} B, \sigma_1, \sigma_2)$) is a *family of $S_{m,1}$ - (resp., $S_{m,2}$ -) curves* if its geometric fibers are $S_{m,1}$ - (resp., $S_{m,2}$ -) curves, the relative singular locus Σ of π maps isomorphically to B and π has an A_{2m} - (resp., A_{2m+1} -) singularity along Σ . Note that this is not just a condition on fibers.

Notation 3.3. Denote by $\overline{\mathcal{S}}_{m,1}$ (resp., $\overline{\mathcal{S}}_{m,2}$) the category fibered in groupoids of $S_{m,1}$ - (resp., $S_{m,2}$ -) curves.

Proposition 3.4. $\overline{\mathcal{S}}_{m,1}$ is isomorphic to the stack $[\mathbb{A}^{m-1}/\mathbb{G}_m]$, where for $m \geq 2$ \mathbb{G}_m acts with weights $1, 3, \dots, 2m-3$. $\overline{\mathcal{S}}_{m,2}$ is isomorphic to the stack $[\mathbb{A}^{m-1}/\mathbb{G}_m]$, where for $m \geq 2$ \mathbb{G}_m acts with weights $1, 2, \dots, m-1$.

Proof. This is proved carefully in [vdW10, Examples 1.111 and 1.112]. Let us sketch the idea of the isomorphism $\overline{\mathcal{S}}_{m,1} \simeq [\mathbb{A}^{m-1}/\mathbb{G}_m]$. To specify the isomorphism class of an $S_{m,1}$ -curve is equivalent to specifying a \mathbb{C} -subalgebra $R \subset \mathbb{C}[[t]] \simeq \widehat{\mathcal{O}}_{\mathbb{P}^1,0}$ which is abstractly isomorphic to $\mathbb{C}[[x, y]]/(y^2 = x^{2m+1})$. It is not difficult to see that any such subalgebra is generated by t^{2m+1} and $t^2 + a_1 t^3 + a_2 t^5 + \dots + a_{m-1} t^{2m-1}$, with $a_i \in \mathbb{C}$, and that two such subalgebras are isomorphic iff $(a_1, \dots, a_{m-1}) = (\lambda a'_1, \dots, \lambda^{m-1} a'_{m-1})$, $\lambda \in \mathbb{G}_m$. □

3.2. $H_{m,1}/H_{m,2}$ -curves.

Definition 3.5 ($H_{m,1}/H_{m,2}$ -curves).

- (1) An $H_{m,1}$ -curve is a 1-pointed curve (E, q) of arithmetic genus m which admits a finite, surjective, degree two map $\phi : E \rightarrow \mathbb{P}^1$ such that $\phi^{-1}(\{\infty\}) = \{q\}$ and q is a smooth point.
- (2) An $H_{m,2}$ -curve is a 2-pointed curve (E, q_1, q_2) of arithmetic genus m which admits a finite, surjective, degree two map $\phi : E \rightarrow \mathbb{P}^1$ such that $\pi^{-1}(\{\infty\}) = \{q_1 + q_2\}$ and q_1, q_2 are smooth points.

Remark 3.6. It will occasionally be useful to use the notation H_k -curve, defined as follows: if $k = 2m$ is even, then an H_k -curve is an $H_{m,1}$ -curve; if $k = 2m + 1$ is odd, then an H_k curve is an $H_{m,2}$ -curve.

Lemma 3.7.

- (1) *If (E, q) is an $H_{m,1}$ -curve, then E is irreducible.*
- (2) *If (E, q_1, q_2) is an $H_{m,2}$ curve, then E has at most two irreducible components. Furthermore, if E has two components, each component is a smooth rational curve.*

Proof. In case (1), $\mathcal{O}_E(q)$ is ample, since $\mathcal{O}_E(2q) = \phi^* \mathcal{O}_{\mathbb{P}^1}(1)$. In particular, $\mathcal{O}_{E_1}(q)$ has positive degree on every irreducible component of E , which forces E to be irreducible. Similarly, in case (2), $\mathcal{O}_E(q_1 + q_2)$ has positive degree on each irreducible component, which means that E has at most two components, one containing q_1 and one containing q_2 . Finally, if E has two components, say $E = E_1 \cup E_2$, then each composition $E_1 \hookrightarrow E \rightarrow \mathbb{P}^1$ is finite, surjective of degree one, so $E_i \simeq \mathbb{P}^1$. \square

A family of 1-pointed curves $(\mathcal{C} \xrightarrow{\pi} B, \sigma)$ is a *family of $H_{m,1}$ -curves* if there is a map ϕ from \mathcal{C} to a \mathbb{P}^1 -bundle \mathcal{P} over B , such that ϕ is a uniform cyclic cover of degree 2 and branch degree $m + 1$ which is simply ramified along σ .

A family of 2-pointed curves $(\mathcal{C} \xrightarrow{\pi} B, \sigma_1, \sigma_2)$ is a *family of $H_{m,2}$ -curves* if there is a map ϕ from \mathcal{C} to a \mathbb{P}^1 -bundle \mathcal{P} over B , such that ϕ is a uniform cyclic cover of degree 2 and branch degree $m + 1$ and $\phi\sigma_1 = \phi\sigma_2$.

Notation 3.8. Denote by $\overline{\mathcal{H}}_{m,1}$ (resp., $\overline{\mathcal{H}}_{m,2}$) the category fibered in groupoids of $H_{m,1}$ - (resp., $H_{m,2}$ -) curves.

Proposition 3.9. $\overline{\mathcal{H}}_{m,1}$ is isomorphic to the stack $[\mathbb{A}^{2m}/\mathbb{G}_m]$ where \mathbb{G}_m acts with weights $-4, -6, \dots, -(4m + 2)$. $\overline{\mathcal{H}}_{m,2}$ is isomorphic to the stack $[\mathbb{A}^{2m+1}/\mathbb{G}_m]$, where \mathbb{G}_m acts with weights $-2, -3, \dots, -(2m + 2)$.

Proof. It follows as in the proof of [AV04, Theorem 4.1] that $\overline{\mathcal{H}}_{m,1}$ is isomorphic to $[\mathbb{A}_{2m+2}/(\mathrm{Tri}_2/\mu_{m+1})]$, where

$$\mathbb{A}_{2m+2} := \mathrm{Spec} \mathbb{C}[a_{2m+1}, a_{2m+1}^{-1}, a_{2m}, \dots, a_0]$$

and Tri_2 is the subgroup scheme of GL_2 of upper-triangular matrices and μ_{m+1} is the group scheme of $(m + 1)$ th roots of unity, embedded in Tri_2 as multiples of the identity matrix. For any ring A , we regard $\mathbb{A}_{2m+2}(A)$ as the following set of homogeneous polynomials in the variables x, z :

$$\{a_{2m+1}x^{2m+1}z + a_{2m}x^{2m}z^2 + \dots + a_0z^{2m+2} : a_{2m+1} \in A^\times; a_{2m}, \dots, a_0 \in A\}.$$

The right action of Tri_2/μ_{m+1} on \mathbb{A}_{2m+2} is given in functorial notation by

$$f\left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right) \cdot [M] = f\left(M\left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right)\right).$$

This is clearly well-defined. Note that the absence of an x^{2m+2} term in elements of $\mathbb{A}_{2m+2}(A)$ corresponds to the requirement that $\phi : \mathcal{C} \rightarrow \mathcal{P}$ be ramified along σ and the condition that the coefficient of $x^{2m+1}z$ is a unit in A corresponds to the requirement that \mathcal{C} be smooth along σ .

It follows similarly that $\overline{\mathcal{H}}_{m,2}$ is isomorphic to $[\mathbb{A}_{2m+3}/(\text{Tri}_2/\mu_{m+1})]$, where

$$\mathbb{A}_{2m+3} := \text{Spec } \mathbb{C}[b, b^{-1}, a_{2m+1}, a_{2m}, \dots, a_0].$$

We regard $\mathbb{A}_{2m+3}(A)$ as the following set of homogeneous polynomials in the variables x, z .

$$\{bx^{2m+2} + a_{2m+1}x^{2m+1}z + a_{2m}x^{2m}z^2 + \dots + a_0z^{2m+2} : b \in A^\times; a_{2m+1}, \dots, a_0 \in A\}.$$

The right action of Tri_2/μ_{m+1} on \mathbb{A}_{2m+3} is now defined as follows. The coefficients of $x^{2m+1}z, \dots, z^{2m+2}$ in $f(\frac{x}{z}).[M]$ are determined via the formula

$$f(\frac{x}{z}).[M] = f(M(\frac{x}{z}))$$

but the coefficient of x^{2m+2} in $f(\frac{x}{z}).[M]$ is $(M_{11})^{m+1}b$, where b is the coefficient of x^{2m+2} in $f(\frac{x}{z})$. This action is clearly well-defined. Note that the condition on elements of $\mathbb{A}_{2m+3}(A)$ that the coefficient of x^{2m+2} be a unit corresponds to the requirement that \mathcal{C} is not ramified along σ_1, σ_2 . The fact that $[M]$ acts on the coefficient of x^{2m+2} by multiplication by $(M_{11})^{m+1}$ rather than $(M_{11})^{2m+2}$ corresponds to the requirement that σ_1, σ_2 have been given an ordering, and that automorphisms respect the ordering.

Return to the the stack $\mathcal{H}_{m,1}$ and consider the map $\pi : \mathbb{A}^{2m} \rightarrow [\mathbb{A}_{2m+2}/(\text{Tri}_2/\mu_{m+1})]$ induced by the map $\iota : \mathbb{A}^{2m} \rightarrow \mathbb{A}_{2m+2}$, $(a_{2m-1}, \dots, a_0) \mapsto x^{2m+1}z + a_{2m-1}x^{2m-1}z^3 + \dots + a_0z^{2m+2}$. We will check that π is smooth and surjective and that the associated groupoid scheme is isomorphic to the groupoid scheme $\mathbb{A}^{2m} \times \mathbb{G}_m \rightrightarrows \mathbb{A}^{2m}$ associated to the specified action of \mathbb{G}_m on \mathbb{A}^{2m} . Indeed, consider the 2-commutative diagram

$$\begin{array}{ccccc} R & \longrightarrow & R_2 & \longrightarrow & \mathbb{A}^{2m} \\ \downarrow & & \downarrow & & \downarrow \iota \\ R_1 & \longrightarrow & \mathbb{A}_{2m+2} \times (T_2/\mu_{m+1}) & \xrightarrow{t} & \mathbb{A}_{2m+2} \\ \downarrow & & \downarrow s & & \downarrow \\ \mathbb{A}^{2m} & \xrightarrow{\iota} & \mathbb{A}_{2m+2} & \longrightarrow & [\mathbb{A}_{2m+2}/(T_2/\mu_{m+1})] \end{array}$$

where t and s denote the action and projection maps respectively and R_1, R_2 and R are defined to make the relevant squares Cartesian. It follows immediately from the definitions that we can identify R with (using functorial notation)

$$\{f, [M] : f_{2m+1} = 1, f_{2m} = 0, (f.[M])_{2m+1} = 1, (f.[M])_{2m} = 0\}.$$

But this is equal to

$$\{f, [M] : f_{2m+1} = 1, f_{2m} = 0, M_{22} = (M_{11})^{-2m-1}, M_{12} = 0\},$$

using the fact that $2m + 1$ is not a zero-divisor since $\text{char}(\mathbb{C}) = 0$. Under this identification, the horizontal map $R \rightarrow \mathbb{A}^{2m}$ is

$$(a_{2m-1}, \dots, a_0), \left[\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-2m-1} \end{pmatrix} \right] \mapsto (\dots, \alpha^{i+(2m+2-i)(-2m-1)} a_i, \dots).$$

Now the map of group schemes

$$\{[M] \in T_2/\mu_{m+1} : M_{22} = (M_{11})^{-2m-1}, M_{12} = 0\} \rightarrow \mathbb{G}_m \quad [M] \mapsto (M_{11})^{m+1}$$

is an isomorphism, so we can identify R with $\mathbb{A}^{2m} \times \mathbb{G}_m$. Under this identification, the horizontal map $R \rightarrow \mathbb{A}^{2m}$ is

$$(a_{2m-1}, \dots, a_0), t \mapsto (\dots, t^{2i-4m-2} a_i, \dots).$$

and the vertical map $R \rightarrow \mathbb{A}^{2m}$ is projection onto the first factor. This proves the desired isomorphism of groupoid schemes. It remains to check that the map $R_2 \rightarrow \mathbb{A}_{2m+2}$ is smooth and surjective. This is immediate from the fact that R_2 can be identified with $\mathbb{A}_{2m+2} \times (\mathbb{G}_m/\mu_{m+1})$ (using as above that $\text{char} \mathbb{C} = 0$) in such a way that the map to \mathbb{A}_{2m+2} is projection onto the first factor.

For $\overline{\mathcal{H}}_{m,2}$, we find, using analogous notation, that R can be identified with

$$\{f, [M] : f_{2m+1} = 1, f_{2m} = 0, M_{11}^{m+1} = 1, M_{12} = 0\}.$$

Now using the isomorphism of group schemes

$$\{[M] \in T_2/\mu_{m+1} : (M_{11})^{m+1} = 1, M_{12} = 0\} \rightarrow \mathbb{G}_m \quad [M] \mapsto M_{11}/M_{22},$$

we can identify R with $\mathbb{A}^{2m+1} \times \mathbb{G}_m$. Under this identification, the horizontal map $R \rightarrow \mathbb{A}^{2m+1}$ is

$$(\dots, a_i, \dots), t \mapsto (\dots, t^{i-2m-2} a_i, \dots)$$

and the vertical map $R \rightarrow \mathbb{A}^{2m+1}$ is projection onto the first factor. This proves the desired isomorphism of groupoid schemes. It is easy to complete the proof by checking as for $\overline{\mathcal{H}}_{m,2}$ that the map $R_2 \rightarrow \mathbb{A}_{2m+1}$ is smooth and surjective. \square

3.3. Monomial $H_{m,1}/H_{m,2}$ -curves.

Definition 3.10 (Monomial $H_{m,1}/H_{m,2}$ -curves).

- (1) The *monomial $H_{m,1}$ -curve* is the following 1-pointed curve (E, q) : E is obtained by identifying $\text{Spec } \mathbb{C}[x, y]/(y^2 - x^{2m+1})$ and $\text{Spec } \mathbb{C}[s]$ along $D(x)$ and $D(s)$ via $x = s^{-2}, y = s^{-(2m+1)}$; q is the point $s = 0$.
- (2) The *monomial $H_{m,2}$ -curve* is the following 2-pointed curve (E, q_1, q_2) : E is obtained by identifying $\text{Spec } \mathbb{C}[x, y]/(y^2 - x^{2m+2})$ and $\text{Spec } \mathbb{C}[s_1] \sqcup \text{Spec } \mathbb{C}[s_2]$ along $D(x)$ and $D(s_1) \sqcup D(s_2)$ via $x = s_1^{-1} \oplus s_2^{-1}, y = s_1^{-(m+1)} \oplus -s_2^{-(m+1)}$; q_1 is the point $s_1 = 0$ of $\text{Spec } \mathbb{C}[s_1]$ and q_2 is the point $s_2 = 0$ of $\text{Spec } \mathbb{C}[s_2]$.

The monomial $H_{m,1}$ - (resp., $H_{m,2}$ -) curve is also an $H_{m,1}$ - (resp., $H_{m,2}$ -) curve. We will denote by p its singular point.

The automorphism group scheme of the monomial $H_{m,1}$ -curve (E, q) is isomorphic to \mathbb{G}_m . We fix once and for all the following isomorphism $\mathbb{G}_m \xrightarrow{\sim} \text{Aut}(E, q)$:

$$A^\times \rightarrow \text{Aut}(E, q)(A) \quad a \mapsto (x \mapsto a^2x, y \mapsto a^{2m+1}y, s \mapsto a^{-1}s).$$

The automorphism group scheme of the monomial $H_{m,2}$ -curve (E, q_1, q_2) is isomorphic to \mathbb{G}_m . We fix once and for all the following isomorphism $\mathbb{G}_m \xrightarrow{\sim} \text{Aut}(E, q_1, q_2)$:

$$\begin{aligned} A^\times &\rightarrow \text{Aut}(E, q_1, q_2)(A) \\ a &\mapsto (x \mapsto ax, y \mapsto a^{m+1}y, s_1 \mapsto a^{-1}s_1s_2 \mapsto a^{-1}s_2) \end{aligned}$$

Proposition 3.11. *Every $S_{m,1}$ - (resp., $S_{m,2}$ -) curve and every $H_{m,1}$ (resp., $H_{m,2}$ -) curve admits an isotrivial specialization to the monomial $H_{m,1}$ - (resp., $H_{m,2}$ -) curve, which is the unique closed point of $\overline{S}_{m,1}$ (resp., $\overline{S}_{m,2}$) and $\overline{\mathcal{H}}_{m,1}$ (resp., $\overline{\mathcal{H}}_{m,2}$).*

Proof. It is immediate from the descriptions of $\overline{S}_{m,1}$ (resp., $\overline{S}_{m,2}$) and $\overline{\mathcal{H}}_{m,1}$ (resp., $\overline{\mathcal{H}}_{m,2}$), Proposition 3.4 and Proposition 3.9, that the point corresponding to the monomial $H_{m,1}$ - (resp., $H_{m,2}$ -) curve is the unique closed point of both stacks and that every point contains this point in its closure. \square

Corollary 3.12. *$\overline{\mathcal{H}}_{m,1}$, $\overline{\mathcal{H}}_{m,2}$, $\overline{S}_{m,1}$, $\overline{S}_{m,2}$ are weakly proper.* \square

4. $A_k^-/A_k/A_k^+$ -STABILITY

In this section, we define $A_k^-/A_k/A_k^+$ -stability for $2 \leq k \leq 4$, and show that these are deformation open conditions. In particular, we show that corresponding moduli stacks $\overline{\mathcal{M}}_{g,n}(A_k^-)$, $\overline{\mathcal{M}}_{g,n}(A_k)$, and $\overline{\mathcal{M}}_{g,n}(A_k^+)$ are algebraic stacks of finite type over \mathbb{C} .

4.1. Definition of $A_k^-/A_k/A_k^+$ -stability.

Definition 4.1 (Gluing morphism). If $(E, \{q_i\}_{i=1}^m)$ is an m -pointed curve and C is any complete curve, a *gluing morphism* $i : (E, \{q_i\}_{i=1}^m) \hookrightarrow C$ is a morphism $E \rightarrow C$, which is an open immersion when restricted to $E - \{q_1, \dots, q_m\}$.

Remark 4.2.

- (1) We do not require the points $\{i(q_i)\}_{i=1}^m$ to be distinct.
- (2) Locally around $i(q_j)$, i is the normalization of one branch of $i(q_j) \in C$.

Definition 4.3. We say that an n -pointed curve $(C, \{p_i\}_{i=1}^n)$ contains an $H_{m,1}$ -tail (resp., contains an $H_{m,2}$ -bridge) if there is a gluing morphism

$$i : (E, q) \hookrightarrow C \text{ (resp., } i : (E, q_1, q_2) \hookrightarrow C),$$

where (E, q) is an $H_{m,1}$ -curve (resp., (E, q_1, q_2) is an $H_{m,2}$ -curve). In this case we say also that (E, q) is an $H_{m,1}$ -tail (resp., (E, q_1, q_2) is an $H_{m,2}$ -bridge) of $(C, \{p_i\}_{i=1}^n)$.

Definition 4.4. We say that an n -pointed curve $(C, \{p_i\}_{i=1}^n)$ contains an $H_{m,2}$ -chain if there exists a morphism

$$i : \prod_{i=1}^r (E_i, q_{2i-1}, q_{2i}) \hookrightarrow C,$$

where each $i|_{(E_i, q_{2i-1}, q_{2i})}$ is a gluing morphism satisfying:

- (E_i, q_{2i-1}, q_{2i}) is an $H_{m,2}$ -curve for $i = 1, \dots, r$;
- $i(q_{2i}) = i(q_{2i+1})$ is an A_{2m+1} -singularity of C for $i = 1, \dots, r-1$.

In this case, we say also that $(\bigcup_{i=1}^r E_i, q_1, q_{2r})$ is an $H_{m,2}$ -chain of $(C, \{p_i\}_{i=1}^n)$

Remark 4.5. An $H_{m,2}$ -bridge is the same thing as an $H_{m,2}$ -chain of length one.

Definition 4.6 (Destabilizing Tails/Chains).

- (1) We say that an $H_{m,1}$ -tail $i : (E, q) \hookrightarrow C$ is *destabilizing* if $i(q)$ is a node, a marked point, or an A_l -singularity with $l \geq 2m+1$.
- (2) We say that an $H_{m,2}$ -chain $i : (E, q_1, q_2) \hookrightarrow C$ is *destabilizing* if $i(q_1), i(q_2)$ are nodes, marked points, or A_l -singularities with $l \geq 2m+2$.

Now we may define our stability conditions:

Definition 4.7 ($A_k^-/A_k/A_k^+$ -stability). An n -pointed curve $(C, \{p_i\}_{i=1}^n)$ is $A_k^-/A_k/A_k^+$ -stable if $\omega_C(\sum_i p_i)$ is ample, and

- (1) (A_k^- -stability)
 - (a) C has only A_l -singularities, $l < k$,
 - (b) C contains no destabilizing $H_{m,1}$ -tails ($2m-1 < k$) or destabilizing $H_{m,2}$ -chains ($2m < k$).
- (2) (A_k -stability)
 - (a) C has only A_l -singularities, $l \leq k$,
 - (b) C contains no destabilizing $H_{m,1}$ -tails ($2m-1 < k$) or destabilizing $H_{m,2}$ -chains ($2m < k$).
- (3) (A_k^+ -stability)
 - (a) C has only A_l -singularities, $l \leq k$,
 - (b) C contains no destabilizing $H_{m,1}$ -tails ($2m-1 \leq k$) or destabilizing $H_{m,2}$ -chains ($2m \leq k$).

Remark 4.8. A_k^+ -stability is the same as A_{k+1}^- -stability.

A family of $A_k^-/A_k/A_k^+$ -stable curves is defined in the usual way to be a flat, proper, finitely-presented morphism, together with n sections, whose geometric fibers are $A_k^-/A_k/A_k^+$ -stable curves of arithmetic genus g . Evidently, families of $A_k^-/A_k/A_k^+$ -stable curves form a stack, so we may make the following definition.

Definition 4.9. Let $\overline{\mathcal{M}}_{g,n}(A_k^-)$, $\overline{\mathcal{M}}_{g,n}(A_k)$, $\overline{\mathcal{M}}_{g,n}(A_k^+)$ denote the moduli stacks of families of $A_k^-/A_k/A_k^+$ -stable curves.

4.2. Deformation-Openness of Moduli Functors. The purpose of this section is to prove Theorem 4.10 stating that the stability conditions introduced in the previous section are deformation-open conditions.

Theorem 4.10. *For $k \leq 4$, A_k -stability and A_k^+ -stability are deformation open conditions.*

Corollary 4.11. *$\overline{\mathcal{M}}_{g,n}(A_k^-), \overline{\mathcal{M}}_{g,n}(A_k), \overline{\mathcal{M}}_{g,n}(A_k^+)$ are algebraic stacks of finite-type over $\text{Spec } \mathbb{C}$. Moreover, that natural inclusions,*

$$\overline{\mathcal{M}}_{g,n}(A_k^-) \hookrightarrow \overline{\mathcal{M}}_{g,n}(A_k) \hookrightarrow \overline{\mathcal{M}}_{g,n}(A_k^+).$$

are open immersions.

Proof. Families of $A_k^-/A_k/A_k^+$ -stable curves satisfy étale descent since they are canonically polarized. Once we know these are deformation-open conditions, we can use an open subset of a suitable Hilbert scheme to give an atlas. \square

Definition 4.12. Let $\mathcal{U}_{g,n}(A_k)$ be the moduli stack of families of curves $(\pi : \mathcal{C} \rightarrow T, \{\sigma_i\}_{i=1}^n)$ satisfying:

- (1) $\omega_{\mathcal{C}/T}$ is relatively ample.
- (2) The sections $\{\sigma_i\}_{i=1}^n$ are distinct and lie in the smooth locus of π
- (3) The geometric fibers of π are connected, reduced curves of arithmetic genus g .
- (4) The only singularities of the geometric fibers of π are of type A_1 - A_k .

Since $\mathcal{U}_{g,n}(A_k)$ parameterizes canonically polarized curves, $\mathcal{U}_{g,n}(A_k)$ is obviously an algebraic stack of finite type over \mathbb{C} .

Definition 4.13. If $(C, \{p_i\}_{i=1}^n)$ is a curve containing an $H_{m,1}$ -tail (resp., $H_{m,2}$ -chain) $i : (E, \{q_i\}_{i=1}^m) \hookrightarrow C$, we say that this $H_{m,1}$ -tail (resp., $H_{m,2}$ -chain) has A_k -*attaching* (resp., A_{k_1, k_2} -*attaching*) if $i(q_1)$ is an A_k -singularity (resp., $i(q_1), i(q_2)$ are A_{k_1}, A_{k_2} -singularities). Note that we allow k, k_1, k_2 to be zero, with the understanding that in this case $i(q)$ (resp., $i(q_1), i(q_2)$) is one of the marked points of C . We may then define the following constructible subsets of $\mathcal{U}_{g,n}(A_l)$:

$$\begin{aligned} \mathcal{T}_m^k &:= \text{Locus of curves admitting an } H_{m,1}\text{-tail with } A_k\text{-attaching,} \\ \mathcal{B}_m^{k_1, k_2} &:= \text{Locus of curves admitting an } H_{m,2}\text{-chain with } A_{k_1}/A_{k_2}\text{-attaching.} \end{aligned}$$

Each of our stability conditions is defined by removing loci of the form \mathcal{T}_m^k and $\mathcal{B}_m^{k_1, k_2}$ from $\mathcal{U}_{g,n}(A_k)$. More precisely, we have (as sets):

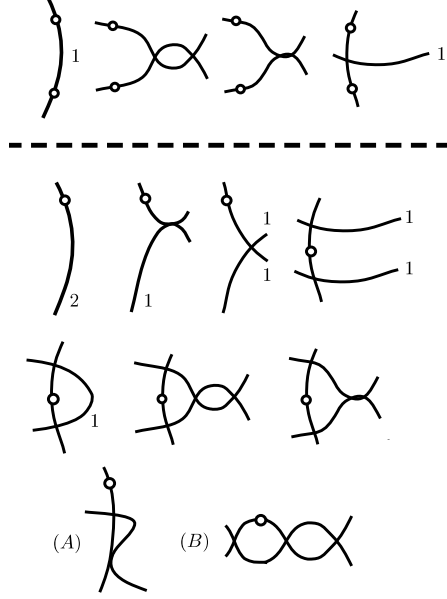


FIGURE 2. Topological types of curves in $\mathcal{U}_{1,2}(A_4)$ and $\mathcal{U}_{2,1}(A_4)$. For convenience, we have suppressed the data of singularities internal to each component, and we record only: the arithmetic genus of each component, and the singularities where two components meet (which are either nodes or tacnodes, as indicated by the picture). Components without a label have arithmetic genus zero.

$$\begin{aligned}
 \overline{\mathcal{M}}_{g,n}(A_1) &= \mathcal{U}_{g,n}(A_1) & \overline{\mathcal{M}}_{g,n}(A_1^+) &= \overline{\mathcal{M}}_{g,n}(A_1) \\
 \overline{\mathcal{M}}_{g,n}(A_2) &= \mathcal{U}_{g,n}(A_2) & \overline{\mathcal{M}}_{g,n}(A_2^+) &= \overline{\mathcal{M}}_{g,n}(A_2) - T_1^1 \\
 \overline{\mathcal{M}}_{g,n}(A_3) &= \mathcal{U}_{g,n}(A_3) - \bigcup_{j \in \{1,3\}} T_j^1 & \overline{\mathcal{M}}_{g,n}(A_3^+) &= \overline{\mathcal{M}}_{g,n}(A_3) - \bigcup_{i,j \in \{0,1\}} B_1^{i,j} \\
 \overline{\mathcal{M}}_{g,n}(A_4) &= \mathcal{U}_{g,n}(A_4) - \bigcup_{j \in \{1,3\}} T_j^1 - \bigcup_{i,j \in \{0,1,4\}} B_1^{i,j} & \overline{\mathcal{M}}_{g,n}(A_4^+) &= \overline{\mathcal{M}}_{g,n}(A_4) - T_2^1,
 \end{aligned}$$

To show that our stability conditions are open, we must show that at each stage the collection of loci \mathcal{T}_m^k and $\mathcal{B}_m^{k_1, k_2}$ that we excise are closed. For this, we must analyze degenerations of curves with $H_{m,1}$ -tails and $H_{m,2}$ -chains. We break this analysis into two stages: In Lemma 4.14, we analyze degenerations of a single $H_{m,1}$ -tail or $H_{m,2}$ -bridge, and in Lemma 4.15, we analyze how the attaching singularities of an $H_{m,1}$ -tail or $H_{m,2}$ -chain may degenerate. Combining these results will allow us to prove the desired statement (Proposition 4.16)

Lemma 4.14 (Limits of $H_{m,1}/H_{m,2}$ -curves).

- (1) Let $(\mathcal{H} \rightarrow \Delta, \tau_1)$ be a family in $\mathcal{U}_{1,1}(A_4)$ whose generic fiber is a smooth $H_{1,1}$ -curve. Then the special fiber C is an $H_{1,1}$ -curve.
- (2) Let $(\mathcal{H} \rightarrow \Delta, \tau_1, \tau_2)$ be a family in $\mathcal{U}_{1,2}(A_4)$ whose generic fiber is a smooth $H_{1,2}$ -curve. Then the special fiber C satisfies one of the following conditions:
 - (a) C is an $H_{1,2}$ -curve.
 - (b) C contains an $H_{1,1}$ -tail.
- (3) Let $(\mathcal{H} \rightarrow \Delta, \tau_1)$ be a family in $\mathcal{U}_{2,1}(A_4)$ whose generic fiber is a smooth $H_{2,1}$ -curve. Then the special fiber C satisfies one of the following conditions:
 - (a) C is an $H_{2,1}$ -curve.
 - (b) C contains an $H_{1,1}$ -tail or an $H_{1,2}$ -bridge.

Proof. For (1), the special fiber (H, p) is necessarily a curve of arithmetic genus one with $\omega_H(p)$ ample. Since $\omega_H(p)$ has degree one, H must be irreducible. It follows immediately (by Riemann-Roch) that $|2p|$ gives a degree two map to \mathbb{P}^1 , so (H, p) is an $H_{1,1}$ -curve.

For (2), the special fiber (H, p_1, p_2) is a curve of arithmetic genus one with $\omega_H(p_1 + p_2)$ ample. Since $\omega_H(p_1 + p_2)$ has degree two, H has at most two components. The possible topological types of H are listed in the top row of Figure 2. We see immediately that any curve with one of the first three topological types is an $H_{2,1}$ -curve, while any curve with the last topological type has an $H_{1,1}$ -tail.

Finally, for (3), the special fiber (H, p) is a curve of arithmetic genus two with $\omega_H(p)$ ample. Since $\omega_H(p)$ has degree three, H has at most three components, and the possible topological types of H are listed in the bottom three rows of Figure 2. One sees immediately that if H does not have an $H_{1,1}$ -tail or an $H_{1,2}$ -bridge, there are only three possibilities for the topological type of H : either H is irreducible or H has topological type (A) or topological type (B). Thus, it suffices to show that if H is irreducible, then (H, p) is an $H_{2,1}$ -curve, and that curve of type (A) and type (B) cannot arise as limits of a family of smooth $H_{2,1}$ -curves. The first claim is easy; we only need to know that $\omega_H \sim 2p$, but this follows from the corresponding linear equivalence on the general fiber.

It remains to show that topological types (A) and (B) cannot occur as the special fiber of a family of $H_{1,2}$ -tails. For this, it suffices to prove that if $\mathcal{H} \rightarrow \Delta$ is any family of genus two curves with smooth general fiber and special fiber of topological type (A) or (B), then the limits of the 6 Weierstrass points of the general fiber lie in the singular locus of the special fiber. More precisely, we claim that in a curve of type (A), two Weierstrass points are absorbed into the node and four are absorbed into the tacnode, while in the curve of type (B), two Weierstrass points are absorbed into each node. To see this, one simply observes that the unique isomorphism class of curve of type (A) and (B) can each be expressed as a double cover of \mathbb{P}^1 branched over $2(0) + 4(\infty)$ and $2(0) + 2(1) + 2(\infty)$ respectively, and that all deformations of these curves are obtained by deforming the branch divisor (along with the corresponding cover). \square

Next, we must consider how the singularities being used to attach a hyperelliptic bridge or tail may degenerate.

Lemma 4.15 (Limits of Attaching Singularities). *Suppose $\mathcal{C} \rightarrow \Delta$ is a flat, proper family of curves whose geometric fibers have only A_i -singularities. Suppose this family is endowed with k sections τ_1, \dots, τ_k satisfying:*

- (1) $\tau_i(\bar{\eta}) \in \mathcal{C}_{\bar{\eta}}$ is an A_{2m_i-1} -singularity.
- (2) The normalization of $\mathcal{C}_{\bar{\eta}}$ along $\bigcup_{i=1}^k \tau_i(\bar{\eta})$ consists of two connected components, and that $\pi^{-1}(\tau_i(\bar{\eta}))$ consists of two points $\alpha_i(\bar{\eta})$ and $\beta_i(\bar{\eta})$, with $\{\alpha_i(\bar{\eta})\}_{i=1}^k$ lying in the first component and $\{\beta_i(\bar{\eta})\}_{i=1}^k$ in the second.

Then the normalization $\pi : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ of \mathcal{C} along $\bigcup_{i=1}^k \tau_i$ consists of two connected components and (after a finite base-change) we may assume that $\pi^{-1}(\tau_i)$ splits into two sections α_i and β_i , with $\{\alpha_i\}_{i=1}^k$ lying in the first component and $\{\beta_i\}_{i=1}^k$ in the second. We claim that

- (1) If the limit points $\{\alpha_i(0)\}_{i=1}^n$ are distinct, then each limit point $\tau_i(0)$ remains on A_{2m_i} -singularity.
- (2) If any subset of limit points $\{\alpha_i(0)\}_{i \in S}$ coincide, then the subset $\{\beta_i(0)\}_{i \in S}$ also coincides, and the limit point $\tau_i(0)$ (for any $i \in S$) is an $A_{\sum_{j \in S} 2m_j - 1}$ -singularity.

Proof. Note that since $\tilde{\mathcal{C}}$ is S_2 , the special fiber $\tilde{\mathcal{C}}|_0$ is reduced and $\pi|_0 : \tilde{\mathcal{C}}|_0 \rightarrow \mathcal{C}|_0$ is a partial normalization of $C_0 := \mathcal{C}|_0$. We will show that it is actually the full normalization of C_0 .

First, suppose that the limit points $\{\alpha_i(0)\}_{i=1}^k$ remain distinct. We claim that the limits $\{\beta_i(0)\}_{i=1}^k$ remain distinct as well. If $\beta_i(0)$ and $\beta_j(0)$ coincide, then $\pi|_0$ maps $\alpha_i(0)$, $\alpha_j(0)$, and $\beta_i(0) = \beta_j(0)$ to the same point $p \in C_0$. Since $\pi|_0$ is a partial normalization map, this implies that $p \in C$ has at least three branches. This is impossible since A_k -singularities have only two branches. Next, we claim that each of the limit points $\{\alpha_i(0)\}_{i=1}^k$ and $\{\beta_i(0)\}_{i=1}^k$ is smooth. If not, then one of the limit points $\alpha_i(0)$ or $\beta_i(0)$ has either two branches or a singular branch. In the first case, the limit point $\tau_i(0)$ would have at least three branches (since $\alpha_i(0)$ and $\beta_i(0)$ map to the same point of C_0). In the second case, one branch of $\tau_i(0)$ would necessarily be singular. Both are impossible since we are assuming the special fiber has only A_k -singularities. Obviously, $\tilde{\mathcal{C}}|_0 \rightarrow \mathcal{C}|_0$ is finite and surjective, and we have just shown that there are two smooth points lying above each of the points $\tau_i(0)$. It follows that $\tilde{\mathcal{C}}|_0$ is in fact the normalization of $\mathcal{C}|_0$ at $\{\tau_i(0)\}_{i=1}^k$. Thus,

$$\sum_{i=1}^k \delta(\tau_i(0)) - 1 = p_a(\tilde{\mathcal{C}}|_0) - p_a(\mathcal{C}|_0) = p_a(\tilde{\mathcal{C}}|_\eta) - p_a(\mathcal{C}|_\eta) = \sum_{i=1}^k m_i - 1.$$

Obviously, $\delta(\tau_i(0)) \geq m_i$ since the δ -invariant of a singularity can only increase under specialization, so the above inequality forces $\delta(\tau_i(0)) = m_i$ for each i . Since $\tau_i(0)$ is

an A_k -singularity with two branches, we conclude $\tau_i(0)$ is an A_{2m_i-1} -singularity as desired.

The second case is argued similarly. Suppose that S_1, \dots, S_l is a partition of $[k]$, and that two limit points $\alpha_i(0)$ and $\alpha_j(0)$ coincide iff i, j lie in the same subset of the partition. Arguing exactly as above, we conclude that the limit points $\beta_i(0)$ coincide according to the same partition, and that each of the limit points $\{\alpha_i(0)\}_{i=1}^k$ and $\{\beta_i(0)\}_{i=1}^k$ is smooth. Thus, $\tilde{\mathcal{C}}|_0 \rightarrow \mathcal{C}|_0$ is a full normalization of $\mathcal{C}|_0$ at $\{\tau_i(0)\}_{i=1}^k$, and genus considerations force $\tau_i(0)$ to be an $A_{2\sum_{j \in S} m_j - 1}$ -singularity (where S is the unique subset containing i). \square

Proposition 4.16.

- (1) $T_1^j \subset \mathcal{U}_{g,n}(A_4)$ is closed for any $j \in \{1, 3\}$.
- (2) $B_1^{i,j} \subset \mathcal{U}_{g,n}(A_4) - \bigcup_{i \in \{1,3\}} T_1^i$ is closed for any $i, j \in \{0, 1, 4\}$.
- (3) $T_2^1 \subset \mathcal{U}_{g,n}(A_4) - \bigcup_{i \in \{1,3\}} T_1^i - \bigcup_{i,j \in \{0,1,4\}} B_1^{i,j}$ is closed.

Proof. The loci T_m^k and $\mathcal{B}_m^{k_1, k_2}$ are obviously constructible, so it suffices in each case to show that they are closed under specialization.

For (1), let $(\pi : \mathcal{C} \rightarrow \Delta, \{\sigma_i\}_{i=1}^n)$ be the family in $\mathcal{U}_{g,n}(A_4)$ such that the generic fiber lies in T_1^k with $k \in \{1, 3\}$. We must show that special fiber lies in T_1^k . Possibly after a finite base change, π admits a section τ_1 picking out the attaching A_k -singularity of an $H_{1,1}$ -tail in the generic fiber. By Lemma 4.15, the limit point $\tau_1(0)$ is still an A_k -singularity and the normalization of $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ induces a simultaneous normalization of the family. Let $\mathcal{H} \subset \tilde{\mathcal{C}}$ be the component whose generic fiber is a smooth $H_{m,1}$ -curve, and let α_1 be the section on \mathcal{H} lying over τ_1 . We may consider $(\mathcal{H} \rightarrow \Delta, \alpha_1)$ as a family in $\mathcal{U}_{1,1}(A_4)$ whose generic fiber is a smooth $H_{1,1}$ -curve. By Lemma 4.14 (1), $(H_0, \alpha_1(0))$ is an $H_{1,1}$ -curve, so C_0 contains an $H_{1,1}$ -tail with A_k -attaching, as desired. The proof for (3) is identical, using Lemma 4.14 (3) instead of Lemma 4.14 (1).

For (2), let $(\pi : \mathcal{C} \rightarrow \Delta, \{\sigma_i\}_{i=1}^n)$ be a family in $\mathcal{U}_{g,n}(A_4)$ such that the general fiber lies in $B_1^{1,1}$, i.e. contains an elliptic chain. (The cases where the general fiber lies in $B_1^{i,j}$ with $i, j \in \{0, 1, 4\}$ are essentially identical, and we leave the details to the reader.) Possibly after a finite base change, there exist sections τ_1, τ_2 picking out the attaching nodes of the elliptic chain in the general fiber. We claim that $\tau_1(0)$ and $\tau_2(0)$ are distinct, so that the normalization $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ along τ_1 and τ_2 gives a simultaneous normalization of the fibers. By Lemma 4.15, it suffices to check that if α_1 and α_2 are the sections of $\tilde{\mathcal{C}}$ lying on the connected component of $\tilde{\mathcal{C}}$ representing the elliptic chain, then $\alpha_1(0) \neq \alpha_2(0)$. If α_1 and α_2 lie on different irreducible components of the general fiber, then this is clear; otherwise, the entire elliptic chain is a single irreducible curve of arithmetic genus one and $\alpha_1(0) = \alpha_2(0)$ would force the special fiber to have an elliptic curve meeting the rest of the fiber in a single point - a contradiction, since

we are assuming the special fiber contains no elliptic bridges. Thus, $\alpha_1(0) \neq \alpha_2(0)$ as desired.

Now, let $(\mathcal{H} \rightarrow \Delta, \alpha_1, \alpha_2)$ be the connected component of $\tilde{\mathcal{C}}$ whose generic fiber is the given elliptic chain; we must show that the special fiber is an elliptic chain. If the chain has length r , then there exist sections $\gamma_1, \dots, \gamma_{r-1}$ picking out the tacnodes in the general fiber at which the sequence of $H_{1,2}$ -curves are attached to each other. Applying Lemma 4.15 to each of the sections γ_i individually, we conclude that the limits $\gamma_1(0), \dots, \gamma_{r-1}(0)$ remain tacnodes, so the normalization of \mathcal{H} along $\gamma_1, \dots, \gamma_{r-1}$ induces a simultaneous normalization of the fibers, and we obtain r distinct flat families whose generic fiber is a $H_{1,2}$ -curve. It suffices to see that all these remain $H_{1,2}$ -curves in the special fiber. This follows immediately from Lemma 4.14 (3), since we are assuming the special fiber has no elliptic tails. \square

Proof of Theorem 4.10. Using the description of $\overline{\mathcal{M}}_{g,n}(A_k)$ and $\overline{\mathcal{M}}_{g,n}(A_k^+)$ given in the discussion following Definition 4.13, we see that (1) implies $\overline{\mathcal{M}}_{g,n}(A_2^+)$ and $\overline{\mathcal{M}}_{g,n}(A_3)$ are obtained by excising closed subsets of $\mathcal{U}_{g,n}(A_2)$ and $\mathcal{U}_{g,n}(A_3)$ respectively. Similarly, (2) implies that $\overline{\mathcal{M}}_{g,n}(A_3^+)$ and $\overline{\mathcal{M}}_{g,n}(A_4)$ are obtained by excising closed subsets of $\mathcal{U}_{g,n}(A_3)$ and $\mathcal{U}_{g,n}(A_4)$. Finally, (3) implies that $\overline{\mathcal{M}}_{g,n}(A_4^+)$ is obtained by excising a closed subset from $\mathcal{U}_{g,n}(A_4)$. \square

5. CLOSED POINTS OF $\overline{\mathcal{M}}_{g,n}(A_k)$

5.1. The canonical decomposition of an A_k -stable curve.

Lemma 5.1. *Suppose that $(C, \{p_i\}_{i=1}^n)$ is an A_k -stable curve.*

- (1) *If $k = 2m$ is even and $i_1(E_1), i_2(E_2) \subset C$ are the images of two distinct destabilizing $H_{m,1}$ -tails, then $i_1(E_1)$ and $i_2(E_2)$ have no component in common.*
- (2) *If $k = 2m + 1$ is odd and $i_1(E_1), i_2(E_2) \subset C$ are the images of two distinct destabilizing $H_{m,2}$ -chains, then $i_1(E_1)$ and $i_2(E_2)$ have no component in common.*

Proof. Case (1) is obvious, since any $H_{m,1}$ -tail is irreducible. For case (2), consider two distinct destabilizing $H_{m,2}$ -chains:

$$\begin{aligned} i_1 &: (E_1, q_1, q_2) \hookrightarrow C, \\ i_2 &: (E_2, r_1, r_2) \hookrightarrow C. \end{aligned}$$

Note that since C contains no A_l -singularities with $l > 2m + 1$, the attaching points $i_1(q_1), i_1(q_2), i_2(r_1), i_2(r_2)$ are either nodes or marked points of C . We claim that if $i_1(E_1) \subset C$ and $i_2(E_2) \subset C$ share a common component, we may assume, without loss of generality, that $i_1(q_1)$ is a node internal to E_2 . To see this, consider cases:

- (1) If $i_2(r_1) = i_2(r_2) \in C$ is a node, then $C \simeq E_2$ and C has no marked points. Now if E_1 and E_2 are not identical, then one of the attaching points $i_1(q_1), i_2(q_2)$

must be a node internal to E_2 . Without loss of generality, we may assume this attaching point is $i_1(q_1)$.

- (2) If $i_2(r_1)$ and $i_2(r_2)$ are marked points of C , then C has exactly two marked points and $(C, p_1, p_2) \simeq (E_2, r_1, r_2)$. Obviously, if E_1 and E_2 are distinct, then one of the attaching points of E_1 must be a node, and this node is then internal to E_2 .
- (3) If $i_2(r_1)$ and $i_2(r_2)$ are distinct nodes of C or a node and a marked point, then either E_1 contains a nodal attaching point internal to $i_2(E_2)$, or else $i_1(E_1)$ contains $i_2(E_2)$ entirely, in which case we can just switch the role of E_1 and E_2 .

Now let $p := i_1(q_1)$. Evidently, $p \in C$ must be a node adjacent to two irreducible components, say $Z_1, Z_2 \in C$, both of which are contained in E_2 . Since the only nodes internal to an $H_{m,2}$ -chain are contained within a single $H_{m,2}$ -bridge, we must have $Z_1 \cup Z_2$ constituting an $H_{m,2}$ -bridge inside E_2 . By Corollary 3.7, each of Z_1 and Z_2 is smooth rational. Furthermore, the fact that $p \in C$ is in attaching point for $H_{m,2}$ -chain E_1 implies that Z_1 meets Z_2 only at p . We conclude that $Z_1 \cup Z_2$ has arithmetic genus zero, a contradiction since the genus of an $H_{m,2}$ -bridge is $m > 0$. We conclude that $i_1(E_1)$ and $i_2(E_2)$ may not have components in common. \square

Using Lemma 5.1, we may define a decomposition of an A_k -stable curve into its $H_{m,1}$ -tails/ $H_{m,2}$ -chains and a complementary subcurve which we shall call the core. The k even case is considerably easier than the k odd case, so we state them separately.

Definition 5.2 (Canonical Decomposition - k even). Suppose that $(C, \{p_i\}_{i=1}^n)$ is A_k -stable ($k = 2m$). Then one of the following holds:

Case I: The $H_{m,1}$ -tails of C are disjoint, and we have a decomposition

$$C = K \cup E_1 \cup \dots \cup E_r,$$

where E_1, \dots, E_r are the nodally-attached $H_{m,1}$ -tails of C , $K := \overline{C \setminus (E_1 \cup \dots \cup E_r)}$, and each E_i meets K at a single node $q_i \in C$. Furthermore, if we consider q_1, \dots, q_r as marked points on K , then $(K, \{p_i\}_{i=1}^n, \{q_i\}_{i=1}^r)$ is an A_k -stable curve with no destabilizing $H_{m,1}$ -tails. We call $(K, \{p_i\}_{i=1}^n, \{q_i\}_{i=1}^r)$ the *core of* $(C, \{p_i\}_{i=1}^n)$. In cases I' and II' below, we say that the core is empty.

Case I': $C = E_1 \cup E_2$, where (E_1, q_1) and (E_2, q_2) are each $H_{m,1}$ -curves, attached nodally at $q_1 = q_2 \in C$.

Case II': $(C, p_1) = (E, q_1)$ where (E, q_1) is an $H_{m,1}$ -curve.

To define the canonical decomposition in the k odd case, we need a preliminary definition:

Definition 5.3 ($H_{m,2}$ -link). We say that C contains an $H_{m,2}$ -link of length r ($k = 2m + 1$) if there exists a gluing morphism

$$i : \bigcup_{i=1}^r (E_i, q_{2i-1}, q_{2i}) \hookrightarrow C,$$

satisfying:

- (1) (E_i, q_{2i-1}, q_{2i}) is an $H_{m,2}$ -chain for $i = 1, \dots, r$.
- (2) $i(q_{2i}) = i(q_{2i+1})$ is a node for $i = 1, \dots, r - 1$.
- (3) $i(q_1), i(q_{2r})$ are nodes, marked points, or A_l -singularities with $l \geq 2m + 2$.

Remark 5.4. An $H_{m,2}$ -chain is the same things as an $H_{m,2}$ -link of length one.

Now observe that if $Z \subset C$ is the union of all $H_{m,2}$ -chains, then each connected component of Z is an $H_{m,2}$ -link. This allows us to make the following definition.

Definition 5.5 (Canonical Decomposition - k odd). Suppose that $(C, \{p_i\}_{i=1}^n)$ is A_k -stable with $k = 2m + 1$. Then one of the following holds:

Case II: We have a decomposition

$$C = K \cup E_1 \cup \dots \cup E_r \cup E_{r+1} \cup \dots \cup E_{r+s}$$

where each E_i is an $H_{m,2}$ -link meeting K at two distinct nodes, say $\{q_{2i-1}, q_{2i}\}$, and each F_i is an $H_{m,2}$ -link, one of whose endpoints meets K in a node, say q_{2r+i} , and one of whose endpoints is a marked point. If we consider q_1, \dots, q_{2r+s} as marked points on K , then the connected components of $(K, \{p_i\}_{i=1}^{n-s}, \{q_i\}_{i=1}^{2r+s})$ are A_k -stable curves with no destabilizing $H_{m,2}$ -chains. We call $(K, \{p_i\}_{i=1}^{n-s}, \{q_i\}_{i=1}^{2r+s})$ the *core of* $(C, \{p_i\}_{i=1}^n)$. We will abuse notation by referring to the core $(K, \{p_i\}_{i=1}^{n-s}, \{q_i\}_{i=1}^{2r+s})$ as an A_k -stable curve even though it may have several connected components. In cases II' and II'' below we say that the core of $(C, \{p_i\}_{i=1}^n)$ is empty.

Case II': (C, p_1, p_2) is an $H_{m,2}$ -link, whose endpoints are marked points, i.e.

$$(C, p_1, p_2) = (E_1 \cup \dots \cup E_r, q_1, q_{2r}),$$

where each (E_i, q_{2i-1}, q_{2i}) is an $H_{m,2}$ -chain, attached to each other nodally via $q_{2i} \sim q_{2i+1}$.

Case II'': C is an $H_{m,2}$ -link, whose endpoints are identified in a single node, i.e. $C = E_1 \cup \dots \cup E_r$, where each (E_i, q_{2i-1}, q_{2i}) is an $H_{m,2}$ -chain, attached to each other nodally via $q_{2i} \sim q_{2i+1}$ and $q_1 \sim q_{2r}$.

A useful fact, which will be used repeatedly in the following section, is that one decompose specializations of A_k -stable curves, according to the canonical decomposition of the generic fiber.

Lemma 5.6. *Let $(C \rightarrow \Delta, \{\sigma_i\}_{i=1}^n)$ be a family of the A_k -stable curves, and let*

$$C_{\bar{\eta}} = K \cup E_1 \cup \dots \cup E_r \text{ (resp. } K \cup E_1 \cup \dots \cup E_s)$$

be the canonical decomposition of the geometric generic fiber of C in the case k even (resp. odd). Possibly after a finite base change, we may assume there exist sections $\{\tau_i\}_{i=1}^r$ (resp. $\{\tau_i\}_{i=1}^{2r+s}$) which pick out the attaching nodes $E_i \cap K$ of the canonical decomposition of the generic fiber. We claim that the limits $\tau_i(0)$ are all nodes, and consequently we may consider the pointed normalization $\tilde{C} \rightarrow C$ of C along the union of the τ_i and \tilde{C} decomposes as

$$\tilde{C} = \mathcal{K} \cup \mathcal{E}_1 \cup \dots \cup \mathcal{E}_r \text{ (resp. } \mathcal{K} \cup \mathcal{E}_1 \cup \dots \cup \mathcal{E}_s)$$

where $\mathcal{K}, \mathcal{E}_1, \dots, \mathcal{E}_r$ (resp. $\mathcal{K} \cup E_1 \cup \dots \cup \mathcal{E}_s$) are families of A_k -stable curves with generic fibers K, E_1, \dots, E_r (resp. $K \cup E_1 \cup \dots \cup E_s$).

Proof. In the case k even, the fact that the limit $\tau_i(0)$ is a node is an immediate application of Lemma `refL:LimitNode`. In the case k odd, if E_i is an $H_{m,2}$ -link attached to K at a single node at τ_i picks out this attaching node in the general fiber, then the fact that $\tau_i(0)$ is a node is again an immediate application of Lemma `refL:LimitNode`.

It only remains to consider the case where E_i is an $H_{m,2}$ -link attached to K at two nodes, picked out by τ_i and τ_{i+1} . Let $\tilde{C} \rightarrow C$ be the normalization of C along τ_i and τ_{i+1} , let \mathcal{E} be the connected component of \tilde{C} with E_i in the general fiber and let α_i and α_{i+1} be the preimages of τ_i and τ_{i+1} on \mathcal{E} . According to Lemma `refL:LimitNode`, we only need to show that α_i and α_{i+1} do not collide in the special fiber. If E_i consists of more than two elliptic bridges, then α_i and α_{i+1} lie on non-adjacent irreducible components of \mathcal{E} and hence cannot possibly collide. If E_i consists of two elliptic bridges meeting at a tacnode, then α_i and α_{i+1} can collide only if they collide with the limit of the tacnode; but this is impossible since the limit of the tacnode must be a tacnode. Finally, if E_i consists of a single elliptic bridge and α_i and α_{i+1} collide, then the special fiber of \mathcal{E} would be an arithmetic genus one curve attached to the rest of the special fiber of C at a single point, a contradiction since the special fiber may not contain elliptic tails. This completes the proof of the lemma. \square

5.2. Characterization of closed points of $\overline{\mathcal{M}}_g(A_k)$.

Definition 5.7. We say that an $A_k^-/A_k/A_k^+$ -stable curve $(C, \{p_i\}_{i=1}^n)$ is *closed* if $(C, \{p_i\}_{i=1}^n)$ is a closed point of $\overline{\mathcal{M}}_{g,n}(A_k^-)/\overline{\mathcal{M}}_{g,n}(A_k)/\overline{\mathcal{M}}_{g,n}(A_k^+)$.

Definition 5.8. We say that an A_k -stable curve $(C, \{p_i\}_{i=1}^n)$ with $k = 2m$ (resp. $k = 2m + 1$) is *maximally degenerate* if the following conditions hold:

- (1) Every $H_{m,1}$ -tail (resp. $H_{m,2}$ -bridge) of C is monomial.
- (2) Every A_k -singularity of C lies on a monomial $H_{m,1}$ -tail (resp. $H_{m,2}$ -bridge).
- (3) The core of $(C, \{p_i\}_{i=1}^n)$ is a closed A_k^- -stable curve.

Remark 5.9.

- (1) If $(C, \{p_i\}_{i=1}^n)$ satisfies the first two conditions, then the core of C contains no A_k -singularities. Thus, the core of a maximally degenerate A_k -stable curve is A_k^- -stable, and it makes sense to require the third condition.
- (2) In an almost maximally degenerate A_k -stable curve C ($k = 2m + 1$), condition (2) implies that two $H_{m,2}$ -bridges can only meet in nodes. In particular, C only admits $H_{m,2}$ -chains of length one.

The goal of this section is to prove that an A_k -stable curve is closed iff it is maximally degenerate. One direction, namely that a closed A_k -stable curve must be maximally degenerate, is contained in the following lemma.

Lemma 5.10. *Every A_k -stable curve admits an isotrivial specialization to a maximally degenerate A_k -stable curve.*

Proof. First, let us suppose that $k = 2m$ is even. Given any A_k -stable curve $(C, \{p_i\}_{i=1}^n)$, we claim that it is sufficient to exhibit an isotrivial specialization $C \rightsquigarrow C'$, where C' is an A_k -stable curve satisfying (1) and (2). Indeed, if C' satisfies (1) and (2), then we construct an isotrivial specialization $C' \rightsquigarrow C''$ where C'' satisfies (1), (2), and (3) as follows: If the core K' of C' is not closed, then there exists an isotrivial specialization $\mathcal{K} \rightarrow \Delta$ with general fiber K' and special fiber a closed A_k^- -stable curve. Gluing this specialization to trivial families of monomial $H_{m,1}$ -curves gives the desired isotrivial specialization $C' \rightsquigarrow C''$.

To construct $C \rightsquigarrow C'$, let $q_1, \dots, q_r \in C$ be the A_k -singularities of C which are not contained on an $H_{m,1}$ -curve, and let E_1, \dots, E_s be the nodally-attached $H_{m,1}$ -curves of C . We define a curve C' as follows: First, normalize C at q_1, \dots, q_r and attach a monomial $H_{m,1}$ -curve at each of the r points lying above q_1, \dots, q_r . Second, replace each of the $H_{m,1}$ tails E_1, \dots, E_r by a monomial tail. Note that since C' is naturally isomorphic to C in a neighborhood of the marked points, we may consider $(C', \{p_i\}_{i=1}^n)$ as an n -pointed curve. Evidently, $(C', \{p_i\}_{i=1}^n)$ is A_k -stable, and we claim that there exists an isotrivial specialization $(C, \{p_i\}_{i=1}^n) \rightsquigarrow (C', \{p_i\}_{i=1}^n)$.

We will construct the desired isotrivial specialization in two steps. First, let q_1, \dots, q_r be the A_k -singularities of C . Consider the trivial family $C \times \Delta$ and let $\mathcal{C} \rightarrow C \times \Delta$ denote the normalization of $C \times \Delta$ along $\cup_{i=1}^r (q_i \times \Delta)$ and let $\{\sigma_i\}_{i=1}^n$ denote the sections of $\mathcal{C} \rightarrow \Delta$ lying above the A_k -singular locus. Now let $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ denote the blow-up of \mathcal{C} at the smooth points $\{\sigma_i(0)\}_{i=1}^n$, and let $\tilde{\sigma}_i$ denote the strict transforms of the sections. Note that the special fiber $\tilde{\mathcal{C}}$ now decomposes as $C \cup Z_1 \cup \dots \cup Z_r$, where each Z_i is a smooth \mathbb{P}^1 meeting C in a single node, with $\tilde{\sigma}_i(0) \in Z_i$. Now let $\tilde{\mathcal{C}} \rightarrow \mathcal{C}^\dagger$ denote the map obtained by crimping the sections $\{\tilde{\sigma}_i\}_{i=1}^n$ back to A_k -singularities. (For the fact that the limit singularity is again an A_k -singularity, cf. the proof of Proposition 9.6.)

Now the family \mathcal{C}^\dagger is an isotrivial specialization $C \rightsquigarrow C^\dagger$ in which the curve C^\dagger has sprouted $H_{m,1}$ -curves at q_1, \dots, q_r . It only remains to construct an isotrivial specialization $C^\dagger \rightsquigarrow C'$. For this, it suffices to isotrivially specialize all the H_k -tails of C^\dagger to monomial $H_{m,1}$ -tails. More precisely, if

$$C^\dagger = K^\dagger \cup E_1 \cup \dots \cup E_{r+s}$$

is the core decomposition of C^\dagger , then by Proposition 3.11, there exist isotrivial families $\mathcal{Z}_i \rightarrow \Delta$ with generic fiber isomorphic to Z_i and special fiber isomorphic to the monomial $H_{m,1}$ -curve. Gluing these to the trivial family $K \times \Delta$ gives the desired isotrivial specialization.

The case $k = 2m + 1$ odd is essentially identical, but we shall sketch the details for the convenience of the reader. As in the case k even, it suffices to construct an isotrivial specialization $C \rightsquigarrow C'$ where C' satisfies (1) and (2). Suppose that $(C, \{p_i\}_{i=1}^n)$ is an A_k -stable curve, and let $q_1, \dots, q_r \in C$ be the A_k -singularities of C which satisfy one of the following two conditions: q_i is not contained in an $H_{m,2}$ -chain, or q_i is contained in an $H_{m,2}$ -chain as the A_k -singularity connecting two $H_{m,2}$ -curves. We define a curve C' as follows: First, normalize C at q_1, \dots, q_r and attach a monomial $H_{m,2}$ -bridge at each of the r pairs of points lying above q_1, \dots, q_r . Note that this has the effect of replacing every $H_{m,2}$ -chain of length r by a link of $2r - 1$ $H_{m,2}$ -curves. We then define C' by replacing each $H_{m,2}$ -curve by a monomial $H_{m,2}$ -curve. Note that since C' is naturally isomorphic to C in a neighborhood of the marked points, we may consider $(C', \{p_i\}_{i=1}^n)$ as an n -pointed curve. Evidently, $(C', \{p_i\}_{i=1}^n)$ is A_k -stable, and we claim that there exists an isotrivial specialization $(C, \{p_i\}_{i=1}^n) \rightsquigarrow (C', \{p_i\}_{i=1}^n)$.

We will construct the desired isotrivial specialization in two steps. First, let q_1, \dots, q_r be the set of A_k -singularities defined above, and consider the trivial family $C \times \Delta$ and let $\mathcal{C} \rightarrow C \times \Delta$ denote the normalization of $C \times \Delta$ along $\cup_{i=1}^r (q_i \times \Delta)$ and let $\{\sigma_i, \sigma'_i\}_{i=1}^r$ denote the sections of $\mathcal{C} \rightarrow \Delta$ lying above the A_k -singular locus. Now let $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ denote the blow-up of \mathcal{C} at the smooth points $\{\sigma_i(0), \sigma'_i(0)\}_{i=1}^r$, and let $\tilde{\sigma}_i, \tilde{\sigma}'_i$ denote the strict transforms of the sections. Note that the special fiber $\tilde{\mathcal{C}}$ now decomposes as $C \cup Z_1 \cup Z'_1 \cup \dots \cup Z_r \cup Z'_r$, where each Z_i and Z'_i is a smooth \mathbb{P}^1 meeting C in a single node, with $\tilde{\sigma}_i(0) \in Z_i, \tilde{\sigma}'_i(0) \in Z'_i$. Now let $\tilde{\mathcal{C}} \rightarrow \mathcal{C}^\dagger$ denote the map obtained by crimping each pair of sections $\tilde{\sigma}_i \cup \tilde{\sigma}'_i$ back to an A_k -singularities. (For the fact that the limit singularity is again an A_k -singularity, cf. the proof of Proposition 9.6.)

Now the family \mathcal{C}^\dagger is an isotrivial specialization $C \rightsquigarrow C^\dagger$ in which the curve C^\dagger has sprouted $H_{m,2}$ -bridges at q_1, \dots, q_r . It only remains to construct an isotrivial specialization $C^\dagger \rightsquigarrow C'$. This follows exactly as in the case k even, using Proposition 3.11. \square

Before we can show the converse direction, namely that any maximally degenerate A_k -stable curve is closed, we will need several lemmas. The first of these should be an explicit recipe for doing A_k^- -stable reduction on a family of curves acquiring an

A_k -singularity. It shows how to replace an A_k -singularity by an $H_{m,1}/H_{m,2}$ -curve using a single weighted blow-up.

Lemma 5.11. ([Fed10, Proposition 7.2]) *For any miniversal family $\mathcal{X} \rightarrow T$ of an A_k -singularity there is an alteration $f : T' \rightarrow T$ and a weighted blow-up $\mathcal{Y} \rightarrow \mathcal{X} \times_T T'$ of the A_k -locus such that $\mathcal{Y} \rightarrow T'$ is a flat and proper family of curves with at-worst A_{k-1} -singularities and such that $\mathcal{Y}|_{f^{-1}(0)} \rightarrow f^{-1}(0)$ is a family of $H_{m,1}$ -tails if $k = 2m$ is even (resp., $H_{m,2}$ -bridges if $k = 2m + 1$ is odd).*

Lemma 5.12. *Let $(C, \{p_i\}_{i=1}^n)$ be an A_k^- -stable curve, and let $(K, \{p_i\}_{i=1}^n, \{q_i\}_{i=1}^m)$ be the core of C , which is defined by considering C as an A_k -stable curve. Then $(C, \{p_i\}_{i=1}^n)$ is a maximally degenerate A_k^- -stable curve iff $(K, \{p_i\}_{i=1}^n, \{q_i\}_{i=1}^m)$ is a maximally degenerate A_k^- -stable curve.*

Proof. We shall prove the lemma in the case $k = 2m$ is even (the proof in the k odd case is similar). Let $(\mathcal{C} \rightarrow \Delta, \{\sigma_i\}_{i=1}^n)$ be any isotrivial specialization in $\overline{\mathcal{M}}_g(A_k^-)$ and apply Lemma 5.6 to decompose the normalization of \mathcal{C} along the attaching nodes of the canonical decomposition of the general fiber:

$$\tilde{\mathcal{C}} = \mathcal{K} \cup \mathcal{E}_1 \cup \dots \cup \mathcal{E}_r,$$

where $(\mathcal{K}, \{\alpha_i\}_{i=1}^r)$ is the connected component of $\tilde{\mathcal{C}}$ containing the core of the generic fiber and $(\mathcal{E}_1, \beta_1), \dots, (\mathcal{E}_r, \beta_r)$ comprise the $H_{m,1}$ -tails of the general fiber. To prove the lemma, we simply need to show that each of the isotrivial specializations (\mathcal{Z}_i, β_i) is trivial. Equivalently, that any $H_{m,1}$ -curve is a closed point of $\overline{\mathcal{M}}_{m,1}(A_k^-)$.

By Lemma 4.14, $\overline{\mathcal{H}}_{m,1} \subset \overline{\mathcal{M}}_{m,1}(A_k)$ is closed, so $\overline{\mathcal{H}}_{m,1} \cap \overline{\mathcal{M}}_{m,1}(A_k^-) \subset \overline{\mathcal{M}}_{m,1}(A_k^-)$ is closed. Evidently, $\overline{\mathcal{H}}_{m,1} \cap \overline{\mathcal{M}}_{m,1}(A_k^-)$ is simply $\overline{\mathcal{H}}_{m,1}$ minus the unique point corresponding to a monomial $\mathcal{H}_{m,1}$ -curve. By the explicit description of $\mathcal{H}_{m,1}$ (Proposition 3.11), we see that all points of $\overline{\mathcal{H}}_{m,1} \cap \overline{\mathcal{M}}_{m,1}(A_k^-)$ are closed in $\overline{\mathcal{H}}_{m,1} \cap \overline{\mathcal{M}}_{m,1}(A_k^-)$, hence in $\overline{\mathcal{M}}_{m,1}(A_k^-)$. \square

Lemma 5.13. *Suppose $(C, \{p_i\}_{i=1}^n)$ is a closed point of $\overline{\mathcal{M}}_{g,n}(A_k^-)$ curve with no nodally-attached $H_{m,1}$ -tail/ $H_{m,2}$ -bridge ($k = 2m/2m + 1$). Then $(C, \{p_i\}_{i=1}^n)$ is a closed point of $\overline{\mathcal{M}}_{g,n}(A_k)$.*

Proof. Let $(\mathcal{C} \rightarrow \Delta, \{\sigma_i\}_{i=1}^n)$ be any isotrivial specialization in $\overline{\mathcal{M}}_{g,n}(A_k)$ with geometric generic fiber isomorphic to $(C, \{p_i\}_{i=1}^n)$ and special fiber closed in $\overline{\mathcal{M}}_{g,n}(A_k)$. We must show that this specialization is trivial. Since we are assuming that $(C, \{p_i\}_{i=1}^n)$ is closed in $\overline{\mathcal{M}}_{g,n}(A_k^-)$, it is sufficient to prove that the special fiber lies in $\overline{\mathcal{M}}_{g,n}(A_k^-)$, i.e. has no A_k -singularities. Suppose, on the contrary, that the special fiber contains A_k -singularities; we will obtain a contradiction by showing that the generic fiber would then necessarily contain a nodally-attached $H_{m,1}$ -tail/ $H_{m,2}$ -bridge ($k = 2m/2m + 1$).

By induction and Theorem 9.1, we may assume that $\overline{\mathcal{M}}_{g,n}(A_k^-)$ is weakly proper, so there exists a family $(\mathcal{C}' \rightarrow \Delta, \{\sigma_i\}_{i=1}^n)$ such that $(\mathcal{C}')^* \simeq \mathcal{C}^*$ and with central fiber \mathcal{C}'

a closed point of $\overline{\mathcal{M}}_{g,n}(A_k^-)$. By assumption, the generic fiber of \mathcal{C}' is a closed point of $\overline{\mathcal{M}}_{g,n}(A_k^-)$, so the isotrivial specialization \mathcal{C}' must be trivial, i.e. the generic fiber of \mathcal{C} is isomorphic to the special fiber \mathcal{C}' . We will compute the A_k^- -stable limit \mathcal{C}' explicitly, and show that if \mathcal{C} contains A_k^- -singularities, then \mathcal{C}' necessarily contains $H_{m,1}$ -tails/ $H_{m,2}$ -bridges ($k = 2m/2m + 1$).

By Lemma 5.10, the closed points of $\overline{\mathcal{M}}_{g,n}(A_k)$ are maximally degenerate. Thus, all A_k^- -singularities of \mathcal{C} , say q_1, \dots, q_r , lie on monomial $H_{m,1}$ -curves (resp. $H_{m,2}$ -curves), there are no $H_{m,2}$ -chains, and the core of \mathcal{C} is maximally degenerate as an A_k^- -stable curve. We claim that the A_k^- -stable limit is then obtained in two steps as follows. By lemma 5.11, there exists (after a suitable base-change) a single weighted blow-up $\phi : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ centered over q_1, \dots, q_r such that $\phi^{-1}(q_i)$ is an $H_{m,1}$ -curve. Note that $\phi^{-1}(q_i)$ meets the special fiber $\tilde{\mathcal{C}}$ in one (resp. two) semistable \mathbb{P}^1 's in the case k even (resp. odd). Let $\tilde{\mathcal{C}} \rightarrow \mathcal{C}'$ be the blow-down of these semistable \mathbb{P}^1 's. By Lemma 5.12, \mathcal{C}' is maximally degenerate A_k^- -stable curve. Since \mathcal{C}' was obtained by replacing A_k^- -singularities by nodally attached $H_{m,1}/H_{m,2}$ -curves, we are done. \square

Theorem 5.14 (Closed Points of $\overline{\mathcal{M}}_{g,n}(A_k)$). *An A_k^- -stable curve $(\mathcal{C}, \{p_i\}_{i=1}^n)$ is maximally degenerate if and only if $(\mathcal{C}, \{p_i\}_{i=1}^n)$ is maximally degenerate.*

Proof of Theorem 5.14. The fact that a closed A_k^- -stable curve is maximally degenerate follows immediately from Lemma 5.10. It remains to show that a maximally-degenerate curve $(\mathcal{C}, \{p_i\}_{i=1}^n)$ is closed. As in Lemma 5.6, decompose the normalization $\tilde{\mathcal{C}}$ of \mathcal{C} along the locus of attaching nodes of the canonical decomposition of the general fiber to obtain:

$$\tilde{\mathcal{C}} = \mathcal{K} \cup \mathcal{E}_1 \cup \dots \cup \mathcal{E}_r,$$

where $(\mathcal{K}, \{\alpha_i\}_{i=1}^r)$ is the connected component of $\tilde{\mathcal{C}}$ containing the core of the generic fiber and $(\mathcal{E}_1, \beta_1), \dots, (\mathcal{E}_r, \beta_r)$ comprise the $H_{m,1}$ -tails ($H_{m,2}$ -bridges) of the general fiber. Since the monomial $H_{m,1}$ -tail is the unique closed point of $\overline{\mathcal{H}}_{m,1}$ (Proposition 3.11) and $\overline{\mathcal{H}}_{m,1} \subset \overline{\mathcal{M}}_{m,1}(A_k)$ is closed by Lemma 4.14, the isotrivial specializations (\mathcal{Z}_i, β_i) must be trivial. Thus, to show that the entire specialization $\mathcal{C} \rightarrow \Delta$ is trivial, it only remains to show that the isotrivial specialization $(\mathcal{K}, \{\alpha_i\}_{i=1}^r)$ is trivial. By hypothesis, the generic fiber is a maximally-degenerate A_k^- -stable curve with no nodally attached $H_{m,1}$ -tail (resp. $H_{m,2}$ -bridge). By Lemma 5.13, it is actually a maximally-degenerate A_k^- -stable curve, so the specialization $(\mathcal{K}, \{\alpha_i\}_{i=1}^r)$ is trivial, as desired. \square

A corollary of this characterization, which will be used in the proof of weak properness of $\overline{\mathcal{M}}_{g,n}(A_k)$, is that the isomorphism class of a closed A_k^- -stable curve is uniquely determined by its core.

Corollary 5.15 (Isomorphism class determined by core). *When k is even, the isomorphism class of maximally degenerate A_k^- -stable curve $(\mathcal{C}, \{p_i\}_{i=1}^n)$ is determined by the isomorphism class of its core $(\mathcal{K}, \{q_i\}_{i=1}^r)$. When k is odd, the isomorphism*

class of a maximally degenerate A_k -stable curve $(C, \{p_i\}_{i=1}^n)$ is determined by its core $(K, \{p_i\}_{i=1}^{n-s}, \{q_i\}_{i=1}^{2r+s})$, along with the sequence of integer lengths l_1, \dots, l_r of the links of monomial $H_{m,2}$ curves connecting q_{2i-1} to q_{2i} , and the lengths l_1, \dots, l_s of the link of monomial $H_{m,2}$ curves attached at q_{2r+i} .

6. DEFORMATION THEORY

Let $(C, \{p_i\}_{i=1}^n)$ be an A_k -stable curve. We denote by $T^1(C, \{p_i\}_{i=1}^n)$ the vector space of first-order deformations of $(C, \{p_i\}_{i=1}^n)$. The *deformation space* of $(C, \{p_i\}_{i=1}^n)$ is

$$\mathrm{Def}(C, \{p_i\}_{i=1}^n) := \mathrm{Spec} \mathrm{Sym} T^1(C, \{p_i\}_{i=1}^n)^\vee;$$

the *formal deformation space* of $(C, \{p_i\}_{i=1}^n)$ is

$$\widehat{\mathrm{Def}}(C, \{p_i\}_{i=1}^n) := \mathrm{Spf} \widehat{\mathcal{O}}_{\mathrm{Def}(C, \{p_i\}_{i=1}^n), 0}.$$

Denote by $\mathrm{Aut}(C, \{p_i\}_{i=1}^n)$ the automorphism group scheme of $(C, \{p_i\}_{i=1}^n)$ and also, abusing notation, just the automorphism group. Denote by $\mathrm{Aut}(C, \{p_i\}_{i=1}^n)^\circ$ the subgroup (scheme) of $\mathrm{Aut}(C, \{p_i\}_{i=1}^n)^\circ$ of automorphisms which restrict to the identity automorphism on the core of C (Definition 5.2 and Definition 5.5).

In this section, we describe the action of the group $\mathrm{Aut}(C, \{p_i\}_{i=1}^n)^\circ$ on $T^1(C, \{p_i\}_{i=1}^n)$ and more generally the action of the group scheme $\mathrm{Aut}(C, \{p_i\}_{i=1}^n)^\circ$ on $\widehat{\mathrm{Def}}(C, \{p_i\}_{i=1}^n)$. This prepares the way for the proof of Theorem 7.9 in the next section, which says that, formally locally around closed points of $\overline{\mathcal{M}}_{g,n}(A_k)$, the reduced closed substacks

$$\overline{\mathcal{S}}_{g,n}(A_k) := \overline{\mathcal{M}}_{g,n}(A_k) \setminus \overline{\mathcal{M}}_{g,n}(A_k^-)$$

and

$$\overline{\mathcal{H}}_{g,n}(A_k) := \overline{\mathcal{M}}_{g,n}(A_k) \setminus \overline{\mathcal{M}}_{g,n}(A_k^+)$$

of $\overline{\mathcal{M}}_{g,n}(A_k)$ correspond to the VGIT minus and plus chambers associated to the natural action of $\mathrm{Aut}(C, \{p_i\}_{i=1}^n)^\circ$ on $\widehat{\mathrm{Def}}(C, \{p_i\}_{i=1}^n)$.

Note that

$$\overline{\mathcal{S}}_{m,1}(A_{2m}) = \overline{\mathcal{S}}_{m,1} \quad \text{and} \quad \overline{\mathcal{S}}_{m,2}(A_{2m+1}) = \overline{\mathcal{S}}_{m,2}$$

and

$$\overline{\mathcal{H}}_{m,1}(A_{2m}) = \overline{\mathcal{H}}_{m,1} \quad \text{and} \quad \overline{\mathcal{H}}_{m,2}(A_{2m+1}) = \overline{\mathcal{H}}_{m,2}.$$

6.1. First-order deformations. Let (E, q) (resp., (E, q_1, q_2)) be a monomial $H_{m,1}$ -curve (resp., monomial $H_{m,2}$ -curve). Recall that $\xi \in E$ denotes the singular point. Denote by

$$\mathrm{LT}^1(E, q_1) \quad (\text{resp., } \mathrm{LT}^1(E, q_1, q_2))$$

the vector space of first-order deformations of (E, q_1) (resp., (E, q_1, q_2)) which induce a trivial deformation of $\widehat{\mathcal{O}}_{E,\xi}$, so that there is a short exact sequence

$$(6.1) \quad \begin{aligned} 0 \rightarrow \mathrm{LT}^1(E, q_1) \rightarrow \mathrm{T}^1(E, q_1) \rightarrow \mathrm{T}^1(\widehat{\mathcal{O}}_{E,\xi}) \rightarrow 0 \\ (\text{resp., } 0 \rightarrow \mathrm{LT}^1(E, q_1, q_2) \rightarrow \mathrm{T}^1(E, q_1, q_2) \rightarrow \mathrm{T}^1(\widehat{\mathcal{O}}_{E,\xi}) \rightarrow 0). \end{aligned}$$

(E, q_1) (resp., (E, q_1, q_2)) defines a point of $\overline{\mathcal{M}}_{m,1}(A_{2m})$ (resp., $\overline{\mathcal{M}}_{m,2}(A_{2m+1})$) contained in $\overline{\mathcal{S}}_{m,1} \cap \overline{\mathcal{H}}_{m,1}$ (resp., $\overline{\mathcal{S}}_{m,2} \cap \overline{\mathcal{H}}_{m,2}$) which we denote by 0.

Lemma 6.1. *With notation as above, the map of tangent spaces*

$$\mathrm{T}_{\overline{\mathcal{S}}_{m,1},0} \rightarrow \mathrm{T}_{\overline{\mathcal{M}}_{m,1}(A_{2m}),0} \quad (\text{resp., } \mathrm{T}_{\overline{\mathcal{S}}_{m,2},0} \rightarrow \mathrm{T}_{\overline{\mathcal{M}}_{m,2}(A_{2m+1}),0})$$

corresponds to the injective map in the short exact sequence (6.1). The map of tangent spaces

$$\mathrm{T}_{\overline{\mathcal{H}}_{m,1},0} \rightarrow \mathrm{T}_{\overline{\mathcal{M}}_{m,1}(A_{2m}),0} \quad (\text{resp., } \mathrm{T}_{\overline{\mathcal{H}}_{m,2},0} \rightarrow \mathrm{T}_{\overline{\mathcal{M}}_{m,2}(A_{2m+1}),0})$$

induces a splitting of (6.1) which is equivariant for the natural linear action of $\mathrm{Aut}(E, q_1)$ (resp., $\mathrm{Aut}(E, q_1, q_2)$).

Proof. The first statement is obvious from the definition. We prove the second. Let $\nu \in \{1, 2\}$. Let $G := \mathrm{Aut}(E, \{q_i\}_{i=1}^\nu) = \mathbb{G}_m$. It suffices to show that the composition

$$\alpha : \mathrm{T}_{\overline{\mathcal{H}}_{m,\nu},0} \rightarrow \mathrm{T}_{\overline{\mathcal{M}}_{m,\nu}(A_{2m+\nu-1}),0} = \mathrm{T}^1(E, \{q_i\}_{i=1}^\nu) \rightarrow \mathrm{T}^1(\widehat{\mathcal{O}}_{E,\xi})$$

is a G -equivariant isomorphism. The isomorphism $\overline{\mathcal{H}}_{m,\nu} \cong [\mathbb{A}^{2m+\nu-1}/\mathbb{G}_m]$ of Proposition 3.9 identifies $\mathrm{T}_{\overline{\mathcal{H}}_{m,\nu},0}$ with $\mathrm{T}_{\mathbb{A}^{2m+\nu-1},0}$, i.e.

$$\mathrm{T}_{\overline{\mathcal{H}}_{m,\nu},0} = \mathbb{C}\langle a_0, \dots, a_{2m+\nu-2} \rangle.$$

On the other hand, noting that $\mathrm{char}(\mathbb{C}) = 0$, an easy calculation shows that

$$\begin{aligned} \mathrm{T}^1(\widehat{\mathcal{O}}_{E,\xi}) = \\ \{ \mathrm{Spec} \mathbb{C}[[x, y, \varepsilon]] / (y^2 - x^{2m+\nu} - a_{2m+\nu-2}\varepsilon x^{2m+\nu-2} - a_{2m+\nu-3}\varepsilon x^{2m+\nu-3} - \dots - a_0\varepsilon, \varepsilon^2) \\ : a_{2m+\nu-2}, \dots, a_0 \in \mathbb{C} \}. \end{aligned}$$

We can describe explicitly the universal family of curves $\mathcal{C} \rightarrow \mathbb{A}^{2m+\nu-1}$ induced by the identification $\overline{\mathcal{H}}_{m,\nu} \cong [\mathbb{A}^{2m+\nu-1}/\mathbb{G}_m]$. It comes with an action of \mathbb{G}_m on \mathcal{C} compatible with the action of \mathbb{G}_m on $\mathbb{A}^{2m+\nu-1}$. Choose coordinates $a_0, \dots, a_{2m+\nu-2}$ on $\mathbb{A}^{2m+\nu-1}$. Let S_{m+1} be the surface obtained by identifying $\mathrm{Spec} \mathbb{C}[x, y]$ and $\mathrm{Spec} \mathbb{C}[u, v]$ along $D(x)$ and $D(u)$ via $u = x^{-1}$ and $v = x^{-m-1}y$. (S_{m+1} is the complement in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m-1))$ of the section corresponding to $\mathcal{O}_{\mathbb{P}^1} \subseteq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m-1)$.)

If $\nu = 1$, \mathcal{C} is the subscheme of $S_{m+1} \times \mathbb{A}_m^2$ defined by

$$y^2 - x^{2m+1} - a_{2m-1}x^{2m-1} - a_{2m-2}x^{2m-2} - \dots - a_0$$

and

$$v^2 - u - a_{2m-1}u^3 - a_{2m-2}u^4 - \dots - a_0u^{2m+2}.$$

The section of $\mathcal{C} \rightarrow \mathbb{A}^{2m}$ is

$$u \mapsto 0, v \mapsto 0.$$

The action of $\mathbb{G}_m = \text{Spec } \mathbb{C}[t, t^{-1}]$ on $S_{m+1} \times \mathbb{A}^k$ given by

$$x \mapsto t^{-2}x, y \mapsto t^{-2m-1}y, u \mapsto t^2u, v \mapsto t^v.$$

induces the action of \mathbb{G}_m on \mathcal{C} .

If $\nu = 2$, \mathcal{C} is the subscheme of $S_{m+1} \times \mathbb{A}^{2m+1}$ defined by

$$y^2 - x^{2m+2} - a_{2m}x^{2m} - a_{2m-1}x^{2m-1} - \dots - a_0$$

and

$$v^2 - 1 - a_{2m}u^2 - a_{2m-1}u^3 - \dots - a_0u^{2m+2}.$$

The two sections of $\mathcal{C} \rightarrow \mathbb{A}^k$ are

$$u \mapsto 0, v \mapsto \pm 1.$$

The action of $\mathbb{G}_m = \text{Spec } \mathbb{C}[t, t^{-1}]$ on $S_{m+1} \times \mathbb{A}^k$ given by

$$x \mapsto t^{-1}x, y \mapsto t^{-m-1}y, u \mapsto tu, v \mapsto v,$$

induces an action of \mathbb{G}_m on \mathcal{C} .

It follows from the explicit description of the universal curve over $\mathbb{A}^{2m+\nu-1}$ that α is the identity map in the coordinates described above, in particular an isomorphism. Moreover, the actions of G in these coordinates are the same (cf. Lemma 6.2 and Lemma 6.3 below), so α is a G -equivariant isomorphism as required. \square

We start with the case that $k = 2m$ is even. From Definition 5.2, it follows that there are three possible cases for the canonical decomposition of a maximally degenerate A_{2m} -stable curve $(C, \{p_i\}_{i=1}^n)$:

Case I: $(C, \{p_i\}_{i=1}^n) = (K \cup E_1 \cup \dots \cup E_r, \{p_i\}_{i=1}^n)$, where for $1 \leq i \leq r$, E_i meets the core K at a single node q_i and (E_i, q_i) is a monomial $H_{m,1}$ -curve, and $\{p_i\}_{i=1}^n \subseteq K$;

Case I': $C = E_1 \cup E_2$ where (E_1, q_1) and (E_2, q_2) are monomial $H_{m,1}$ -curves attached nodally via $q_1 \sim q_2$;

Case I'': $(C, p_1) = (E, q_1)$ is a monomial $H_{m,1}$ -curve.

Proposition 6.2. *Let $(C, \{p_i\}_{i=1}^n)$ be a maximally degenerate A_k -stable curve with $k = 2m$.*

Case I: *There are decompositions*

$$\text{Aut}(C, \{p_i\}_{i=1}^n)^\circ = \text{Aut}(K, \{p_i\}_{i=1}^n, \{q_i\}_{i=1}^r)^\circ \times \prod_{i=1}^r \text{Aut}(E_i, q_i)$$

and

$$T^1(C) = T^1(K, \{p_i\}_{i=1}^n, \{q_i\}_{i=1}^r) \oplus \bigoplus_{i=1}^r \left(\text{LT}^1(E_i, q_i) \oplus T^1(\widehat{\mathcal{O}}_{E_i, \xi_i}) \right) \oplus \bigoplus_{i=1}^r T^1(\widehat{\mathcal{O}}_{C, q_i}).$$

For $1 \leq i \leq r$, let t_i be the coordinate on $\text{Aut}(E_i, q_i) \cong \mathbb{G}_m$ given by the isomorphism following Definition 3.10. Set $\bar{g} := g - rm$. There are coordinates

“kore”	$\mathbf{k} = (k_1, \dots, k_{3\bar{g}-3+n+r})$	on	$T^1(K, \{p_i\}_{i=1}^n, \{q_i\}_{i=1}^r)$	
“crimping”	$\mathbf{c}_i = (c_{i,1}, \dots, c_{i,m-1})$	on	$\text{LT}^1(E_i, q_i)$	for $1 \leq i \leq r$
“singularity”	$\mathbf{s}_i = (s_{i,0}, \dots, s_{i,2m-1})$	on	$T^1(\widehat{\mathcal{O}}_{E_i, \xi_i})$	for $1 \leq i \leq r$
“node”	n_i	on	$T^1(\widehat{\mathcal{O}}_{C, q_i})$	for $1 \leq i \leq r$

such that the action of $\prod_{i=1}^r \text{Aut}(E_i, q_i)$ on $T^1(C, \{p_i\}_{i=1}^n)$ is given by

$$\begin{aligned} k_l &\mapsto k_l \\ c_{i,l} &\mapsto t_i^{2l-1} c_{i,l} \\ s_{i,l} &\mapsto t_i^{2l-4m-2} s_{i,l} \\ n_i &\mapsto t_i n_i \end{aligned}$$

Case I': There are decompositions

$$\text{Aut}(C)^\circ = \prod_{i=1}^2 \text{Aut}(E_i, q)$$

and

$$T^1(C) = \bigoplus_{i=1}^2 \left(\text{LT}^1(E_i, q) \oplus T^1(\widehat{\mathcal{O}}_{E_i, \xi_i}) \right) \oplus T^1(\widehat{\mathcal{O}}_{C, q}).$$

For $1 \leq i \leq 2$, let t_i be the coordinate on $\text{Aut}(E_i, q) \cong \mathbb{G}_m$ given by the isomorphism following Definition 3.10. There are coordinates

$\mathbf{c}_i = (c_{i,1}, \dots, c_{i,m-1})$	on	$\text{LT}^1(E_i, q)$	for $1 \leq i \leq 2$
$\mathbf{s}_i = (s_{i,0}, \dots, s_{i,2m-1})$	on	$T^1(\widehat{\mathcal{O}}_{E_i, \xi_i})$	for $1 \leq i \leq 2$
n	on	$T^1(\widehat{\mathcal{O}}_{C, q})$	

such that the action of $\prod_{i=1}^2 \text{Aut}(E_i, q)$ on $T^1(C)$ is given by

$$\begin{aligned} c_{i,l} &\mapsto t_i^{2l-1} c_{i,l} \\ s_{i,l} &\mapsto t_i^{2l-4m-2} s_{i,l} \\ n &\mapsto t_1 t_2 n. \end{aligned}$$

Case I'': In this case $(C, p) = (E, q)$ so

$$\text{Aut}(C, p)^\circ = \text{Aut}(E, q)$$

and there is a decomposition

$$T^1(C, p) = \text{LT}^1(E, q) \oplus T^1(\widehat{\mathcal{O}}_{E, \xi}) \oplus T^1(\widehat{\mathcal{O}}_{C, q}).$$

Let t be the coordinate on $\text{Aut}(E, q) \cong \mathbb{G}_m$ given by the isomorphism following Definition 3.10. There are coordinates

$$\begin{aligned} \mathbf{c} &= (c_1, \dots, c_{m-1}) \quad \text{on } \text{LT}^1(E, q) \\ \mathbf{s} &= (s_0, \dots, s_{2m-1}) \quad \text{on } \text{T}^1(\widehat{\mathcal{O}}_{E, \xi}) \end{aligned}$$

such that the action of $\text{Aut}(E, q)$ on $\text{T}^1(C, p)$ is given by

$$\begin{aligned} c_l &\mapsto t^{2l-1}c_l \\ s_l &\mapsto t^{2l-4m-2}s_l \end{aligned}$$

Proof. We give the proof for case I under the assumption that $r = 1$. The proof for arbitrary r is not more difficult and the proofs for cases I' and I'' are similar. Write (E, q) for (E_1, q_1) and set $G := \text{Aut}(E, q)$.

The action of G on $\text{T}^1(K, \{p_i\}_{i=1}^n, q)$ is trivial, since C is nodal at q .

The coordinates specified in the definition of the monomial $H_{m,1}$ -curve induce an isomorphism $\widehat{\mathcal{O}}_{E, \xi} \xrightarrow{\sim} \mathbb{C}[[x, y]]/(y^2 - x^{2m+1})$. Fix the following isomorphism $\text{T}^1(\widehat{\mathcal{O}}_{E, \xi}) \xrightarrow{\sim} \mathbb{C}^{\oplus 2m}$:

$$\text{Spec } \mathbb{C}[[x, y, \varepsilon]]/(y^2 - x^{2m+1} - s_{2m-1}\varepsilon x^{2m-1} - s_{2m-2}\varepsilon x^{2m-2} - \dots - s_0\varepsilon, \varepsilon^2) \mapsto (s_0, \dots, s_{2m-1}).$$

Then one computes from the definition that the action of G is:

$$s_l \mapsto t^{2l-4m-2}s_l \quad \text{for } 0 \leq l \leq 2m-1$$

Choose an isomorphism $\widehat{\mathcal{O}}_{C, q} \xrightarrow{\sim} \mathbb{C}[[s, u]]$ such that there is a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{C, q} & \longrightarrow & \mathbb{C}[[s, u]]/(su) \\ \downarrow & & \downarrow \\ \widehat{\mathcal{O}}_{E, q} & \longrightarrow & \mathbb{C}[[s]] \end{array}$$

where the lower horizontal map is the isomorphism induced by the coordinates specified in the definition of the monomial $H_{m,1}$ -curve. Fix the following isomorphism $\text{T}^1(\widehat{\mathcal{O}}_{C, q}) \xrightarrow{\sim} \mathbb{C}$:

$$\text{Spec } \mathbb{C}[[s, u, \varepsilon]]/(su - c\varepsilon, \varepsilon^2) \mapsto c.$$

Then one computes from the definition that the action of G is:

$$c \mapsto tc.$$

From Example 1.111 of [vdW10], one sees that there is the following isomorphism $\text{LT}^1(E, q) \xrightarrow{\sim} \mathbb{C}^{\oplus (m-1)}$:

$$\text{Spec } \mathbb{C}[(s + c_1\varepsilon s^2 + c_2\varepsilon s^4 + \dots + c_{m-1}\varepsilon s^{2m-2})^2, s^{2m}, \dots, s^{4m-1}, \varepsilon]/(\varepsilon)^2 \mapsto (c_1, \dots, c_{m-1}).$$

Then one computes from the definition that the action of G is:

$$c_l \mapsto t^{2l-1}c_l \quad \text{for } 1 \leq l \leq m-1.$$

□

Next we give the analogous proposition for the case that $k = 2m + 1$ is odd. Again, there are three possible cases for the canonical decomposition of a maximally degenerate A_{2m+1} -stable curve $(C, \{p_i\}_{i=1}^n)$: see Definition 5.5. Since the general case II is notationally complicated, we consider the following special case of II, case II*. We give the statement of Propositions 6.3 and 6.7 and the proof of Theorem 7.9 only in case II* but the same argument works for the general case.

Case II*: $(C, \{p_i\}_{i=1}^n) = (K \cup E_1 \cup \dots \cup E_r, \{p_i\}_{i=1}^n)$ where for $1 \leq i \leq r - 1$, E_i meets E_{i+1} in a node $q_{2i} = q_{2i+1}$, E_1 meets K in a node q_1 , E_r meets K in a node q_{2r} , and (E_i, q_{2i-1}, q_{2i}) is a monomial $H_{m,2}$ -curve. Set $\eta_i := q_{2i}$ for $1 \leq i \leq r - 1$ and $\eta_0 := q_1, \eta_r := q_{2r}$;

Case II': $(C, p_1, p_2) = (E_1 \cup \dots \cup E_r, q_1, q_{2r})$, where for $1 \leq i \leq r - 1$, E_i meets E_{i+1} in a node $q_{2i} = q_{2i+1}$ and for $1 \leq i \leq r$, (E_i, q_{2i-1}, q_{2i}) is a monomial $H_{m,2}$ -curve;

Case II'': $C = (E_1 \cup \dots \cup E_r)$ where for $1 \leq i \leq r - 1$, E_i meets E_{i+1} in a node $q_{2i} = q_{2i+1}$, and E_r meets E_1 in a node $q_{2r} = q_1$, and for $1 \leq i \leq r$, (E_i, q_{2i-1}, q_{2i}) is a monomial $H_{m,2}$ -curve.

Proposition 6.3. *Let $(C, \{p_i\}_{i=1}^n)$ be a maximally degenerate A_k -stable curve with $k = 2m + 1$.*

Case II*: *There are decompositions*

$$\mathrm{Aut}(C, \{p_i\}_{i=1}^n)^\circ = \mathrm{Aut}(K, \{p_i\}_{i=1}^n, q_1, q_{2r}) \times \prod_{i=1}^r \mathrm{Aut}(E_i, q_{2i-1}, q_{2i})$$

and

$$\mathrm{T}^1(C, \{p_i\}_{i=1}^n) = \mathrm{T}^1(K, \{p_i\}_{i=1}^n, q_1, q_{2r}) \oplus \bigoplus_{i=1}^r \left(\mathrm{LT}^1(E_i, q_{2i-1}, q_{2i}) \oplus \mathrm{T}^1(\widehat{\mathcal{O}}_{E_i, \xi_i}) \right) \oplus \bigoplus_{i=0}^r \mathrm{T}^1(\widehat{\mathcal{O}}_{C, \eta_i})$$

For $1 \leq i \leq r$ let t_i be the coordinate on $\mathrm{Aut}(E_i, q_{2i-1}, q_{2i}) \cong \mathbb{G}_m$ given by the isomorphism following Definition 3.10. There are coordinates

“kore”	\mathbf{k}	on	$\mathrm{T}^1(K, \{p_i\}_{i=1}^n, q_1, q_{2r})$	
“crimping”	\mathbf{c}_i	on	$\mathrm{LT}^1(E_i, q_{2i-1}, q_{2i})$	for $1 \leq i \leq r$
“singularity”	\mathbf{s}_i	on	$\mathrm{T}^1(\widehat{\mathcal{O}}_{E_i, \xi_i})$	for $1 \leq i \leq r$
“node”	\mathbf{n}_i	on	$\mathrm{T}^1(\widehat{\mathcal{O}}_{C, \eta_i})$	for $0 \leq i \leq r$

such that the action of $\prod_{i=1}^r \text{Aut}(E_i, q_{2i-1}, q_{2i})$ on $T^1(C, \{p_i\}_{i=1}^n)$ is given by

$$\begin{aligned} k_l &\mapsto k_l \\ c_{i,l} &\mapsto t_i^l c_{i,l} \\ s_{i,l} &\mapsto t_i^{l-2m-2} s_{i,l} \\ n_i &\mapsto t_i t_{i+1} n_i. \end{aligned}$$

Case II*: *There are decompositions*

$$\text{Aut}(C, \{p_i\}_{i=1}^n)^\circ = \text{Aut}(K, \{p_i\}_{i=1}^n, q_1, q_{2r}) \times \prod_{i=1}^r \text{Aut}(E_i, q_{2i-1}, q_{2i})$$

and

$$T^1(C, \{p_i\}_{i=1}^n) = T^1(K, \{p_i\}_{i=1}^n, q_1, q_{2r}) \oplus \bigoplus_{i=1}^r \left(\text{LT}^1(E_i, q_{2i-1}, q_{2i}) \oplus T^1(\widehat{\mathcal{O}}_{E_i, \xi_i}) \right) \oplus \bigoplus_{i=0}^r T^1(\widehat{\mathcal{O}}_{C, \eta_i})$$

For $1 \leq i \leq r$ let t_i be the coordinate on $\text{Aut}(E_i, q_{2i-1}, q_{2i}) \cong \mathbb{G}_m$ given by the isomorphism following Definition 3.10. There are coordinates

“kore”	k	on	$T^1(K, \{p_i\}_{i=1}^n, q_1, q_{2r})$	
“crimping”	c_i	on	$\text{LT}^1(E_i, q_{2i-1}, q_{2i})$	for $1 \leq i \leq r$
“singularity”	s_i	on	$T^1(\widehat{\mathcal{O}}_{E_i, \xi_i})$	for $1 \leq i \leq r$
“node”	n_i	on	$T^1(\widehat{\mathcal{O}}_{C, \eta_i})$	for $0 \leq i \leq r$

such that the action of $\prod_{i=1}^r \text{Aut}(E_i, q_{2i-1}, q_{2i})$ on $T^1(C, \{p_i\}_{i=1}^n)$ is given by

$$\begin{aligned} k_l &\mapsto k_l \\ c_{i,l} &\mapsto t_i^l c_{i,l} \\ s_{i,l} &\mapsto t_i^{l-2m-2} s_{i,l} \\ n_i &\mapsto t_i t_{i+1} n_i. \end{aligned}$$

Case II': *There are decompositions*

$$\text{Aut}(C, p_1, p_2)^\circ = \prod_{i=1}^r \text{Aut}(E_i, q_{2i-1}, q_{2i})$$

and

$$T^1(C, p_1, p_2) = \bigoplus_{i=1}^r \left(\text{LT}^1(E_i, q_{2i-1}, q_{2i}) \oplus T^1(\widehat{\mathcal{O}}_{E_i, \xi_i}) \right) \oplus \bigoplus_{i=1}^{r-1} T^1(\widehat{\mathcal{O}}_{C, q_{2i}})$$

For $1 \leq i \leq r$, let t_i be the coordinate on $\text{Aut}(E_i, q_{2i-1}, q_{2i}) \cong \mathbb{G}_m$ given by the isomorphism following Definition 3.10. There are coordinates

c_i = $(c_{i,1}, \dots, c_{i,m-1})$	on	$\text{LT}^1(E_i, q_{2i-1}, q_{2i})$	for $1 \leq i \leq r$
s_i = $(a_{i,0}, \dots, a_{i,2m})$	on	$T^1(\widehat{\mathcal{O}}_{E_i, \xi_i})$	for $1 \leq i \leq r$
n_i	on	$T^1(\widehat{\mathcal{O}}_{q_{2i}})$	for $1 \leq i \leq r-1$

such that the action of $\prod_{i=1}^r \text{Aut}(E_i, q_{2i-1}, q_{2i})$ on $T^1(C, p_1, p_2)$ is given by

$$\begin{aligned} c_{i,l} &\mapsto t_i^l c_{i,l} \\ s_{i,l} &\mapsto t_i^{l-2m-2} s_{i,l} \\ n_i &\mapsto t_i t_{i+1} n_i. \end{aligned}$$

Case II'': *There are decompositions*

$$\text{Aut}(C)^\circ = \prod_{i=1}^r \text{Aut}(E_i, q_{2i-1}, q_{2i})$$

and

$$T^1(C) = \bigoplus_{i=1}^r \left(\text{LT}^1(E_i, q_{2i-1}, q_{2i}) \oplus T^1(\widehat{\mathcal{O}}_{E_i, \xi_i}) \right) \oplus \bigoplus_{i=1}^r T^1(\widehat{\mathcal{O}}_{C, q_{2i}})$$

For $1 \leq i \leq r$, let t_i be the coordinate on $\text{Aut}(E_i, q_{2i-1}, q_{2i}) \cong \mathbb{G}_m$ given by the isomorphism in Definition 3.10. There are coordinates

$$\begin{array}{lll} \mathbf{c}_i = (c_{i,1}, \dots, c_{i,m-1}) & \text{on } \text{LT}^1(E_i, q_{2i-1}, q_{2i}) & \text{for } 1 \leq i \leq r \\ \mathbf{s}_i = (s_{i,0}, \dots, s_{i,2m}) & \text{on } T^1(\widehat{\mathcal{O}}_{E_i, \xi_i}) & \text{for } 1 \leq i \leq r \\ n_i & \text{on } T^1(\widehat{\mathcal{O}}_{q_{2i}}) & \text{for } 1 \leq i \leq r \end{array}$$

such that

$$\begin{aligned} c_{i,l} &\mapsto t_i^l c_{i,l} \\ s_{i,l} &\mapsto t_i^{l-2m-2} s_{i,l} \\ n_i &\mapsto t_i t_{i+1} n_i \end{aligned}$$

where $t_{r+1} := t_1$.

Proof. We give the proof for case II* under the assumption that $r = 1$. The proof for arbitrary r is not more difficult and the proofs for cases II' and II'' are similar. Write E for E_1 and set $G := \text{Aut}(E, q_1, q_2)$.

The coordinates specified in the definition of the monomial $H_{m,2}$ -curve induce an isomorphism $\widehat{\mathcal{O}}_{E, \xi} \xrightarrow{\sim} \mathbb{C}[[x, y]]/(y^2 - x^{2m+2})$. Fix the following isomorphism $T^1(\widehat{\mathcal{O}}_{E, \xi}) \xrightarrow{\sim} \mathbb{C}^{\oplus 2m+1}$:

$$\text{Spec } \mathbb{C}[[x, y, \varepsilon]]/(y^2 - x^{2m+2} - s_{2m}\varepsilon x^{2m} - s_{2m-1}\varepsilon x^{2m-1} - \dots - s_0\varepsilon, \varepsilon^2) \mapsto (s_0, \dots, s_{2m}).$$

Then one computes from the definition that the action of G is:

$$s_l \mapsto t^{l-2m-2} s_l \quad \text{for } 0 \leq l \leq 2m$$

For $i = 1, 2$, choose an isomorphism $\widehat{\mathcal{O}}_{C,q_i} \cong \mathbb{C}[[s_i, u]]$ such that there is a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{C,q_i} & \longrightarrow & \mathbb{C}[[s_i, u]]/(s_i u) \\ \downarrow & & \downarrow \\ \widehat{\mathcal{O}}_{E,q_i} & \longrightarrow & \mathbb{C}[[s_i]] \end{array}$$

where the lower horizontal map is the isomorphism induced by the coordinates specified in the definition of the monomial $H_{m,2}$ -curve. Fix the following isomorphism $T^1(\widehat{\mathcal{O}}_{C,q_i}) \xrightarrow{\sim} \mathbb{C}$:

$$\mathrm{Spec} \mathbb{C}[[s_i, u, \varepsilon]]/(s_i u - n_i \varepsilon, \varepsilon^2) \mapsto n_i.$$

Then one computes from the definition that the action of G is:

$$n_i \mapsto t n_i \quad \text{for } 1 \leq i \leq 2$$

From Example 1.112 of [vdW10], one sees that there is the the following isomorphism $\mathrm{LT}^1(E, q_1, q_2) \xrightarrow{\sim} \mathbb{C}^{\oplus(m-1)}$:

$$\begin{aligned} \mathrm{Spec} \mathbb{C}[s_1 + c_1 \varepsilon s_1^2 + \dots + c_{m-1} \varepsilon s_1^m \oplus s_2, s_1^{m+1} \oplus 0, \dots, s_1^{2m+1} \oplus 0, 0 \oplus s_2^{m+1}, \dots, 0 \oplus s_2^{2m+1}, \varepsilon]/(\varepsilon)^2 \\ \mapsto (c_1, \dots, c_{m-1}) \end{aligned}$$

Then one computes from the definition that the action of G is:

$$c_l \mapsto t^l c_l \quad \text{for } 1 \leq l \leq m-1.$$

□

6.2. Formal deformations.

Lemma 6.4. *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a map of algebraic stacks, x a \mathbb{C} -point of \mathcal{X} , y its image in \mathcal{Y} , and suppose that f is stabilizer-preserving at x . Set $G := G_x = G_y$. Given a miniversal deformation $\psi : \mathrm{Spf} B \rightarrow \mathcal{Y}$ of y , define $\mathrm{Spf} A$ by the 2-Cartesian square*

$$\begin{array}{ccc} \mathrm{Spf} A & \longrightarrow & \mathrm{Spf} B \\ \downarrow \theta & & \downarrow \psi \\ \mathcal{X} & \longrightarrow & \mathcal{Y}. \end{array}$$

Then $\theta : \mathrm{Spf} A \rightarrow \mathcal{X}$ is a miniversal deformation of x and the map $\mathrm{Spf} A \rightarrow \mathrm{Spf} B$ is equivariant for the actions of G on $\mathrm{Spf} A$ and $\mathrm{Spf} B$ induced by θ and ψ .

Proof. The proof is straightforward: cf. [Alp10]. □

Proposition 6.5 and Proposition 6.7 below will be used in the proof of Theorem 7.9 in the next section.

Proposition 6.5. *Let $(C, \{p_i\}_{i=1}^n)$ be a maximally-degenerate A_{2m} -stable curve in case I (see Proposition 6.2). Set*

$$A := \mathbb{C}[[\mathbf{k}, \mathbf{c}_1, \dots, \mathbf{c}_r, \mathbf{s}_1, \dots, \mathbf{s}_r, n_1, \dots, n_r]].$$

There is a miniversal formal deformation $\psi : \mathrm{Spf} A \rightarrow \overline{\mathcal{M}}_{g,n}(A_k)$ of $(C, \{p_i\}_{i=1}^n)$ such that there is a 2-cartesian diagram

$$\begin{array}{ccccc} \mathrm{Spf} A / \cap_{i=1}^r (\mathbf{s}_i) & \longrightarrow & \mathrm{Spf} A & \longleftarrow & \mathrm{Spf} A / \cap_{i=1}^r (\mathbf{c}_i, n_i) \\ \downarrow \theta & & \downarrow \psi & & \downarrow \phi \\ \overline{\mathcal{S}}_{g,n}(A_{2m}) & \xrightarrow{i} & \overline{\mathcal{M}}_{g,n}(A_{2m}) & \xleftarrow{j} & \overline{\mathcal{H}}_{g,n}(A_{2m}) \end{array}$$

and the action of the group scheme $\mathrm{Aut}(C, \{p_i\}_{i=1}^n)^\circ$ on $\widehat{\mathrm{Def}}(C, \{p_i\}_{i=1}^n)$ induced by ψ coincides with the action induced by the action of the group $\mathrm{Aut}(C, \{p_i\}_{i=1}^n)^\circ$ on $T^1(C, \{p_i\}_{i=1}^n)$ described in Proposition 6.2.

Proof. Let $G := \mathrm{Aut}(C, \{p_i\}_{i=1}^n)^\circ = \prod_{i=1}^r \mathrm{Aut}(E_i, q_i)$. Choose a miniversal formal deformation $\psi' : \mathrm{Spf} A' \rightarrow \overline{\mathcal{M}}_{g,n}(A_{2m})$ of $(C, \{p_i\}_{i=1}^n)$. Let $\mathcal{C}' \rightarrow \mathrm{Spf} A'$ be the corresponding family of curves. Let $\mathrm{Spf} S'_i$ be the formal closed subscheme of $\mathrm{Spf} A'$ where the A_{2m} -singularity ξ_i is preserved in \mathcal{C}' ($1 \leq i \leq r$). Let $\mathrm{Spf} H'_i$ be the formal closed subscheme of $\mathrm{Spf} A'$ where the node q_i is preserved in \mathcal{C}' and the induced deformation of (E_i, q_i) is a family of $H_{m,1}$ -curves ($1 \leq i \leq r$). Clearly there is a 2-cartesian diagram

$$\begin{array}{ccccc} \bigcup_{i=1}^r \mathrm{Spf} S'_i & \longrightarrow & \mathrm{Spf} A' & \longleftarrow & \bigcup_{i=1}^r \mathrm{Spf} H'_i \\ \downarrow \cup_{i=1}^r \theta'_i & & \downarrow \psi' & & \downarrow \cup_{i=1}^r \phi'_i \\ \overline{\mathcal{S}}_{g,n}(A_{2m}) & \longrightarrow & \overline{\mathcal{M}}_{g,n}(A_{2m}) & \longleftarrow & \overline{\mathcal{H}}_{g,n}(A_{2m}). \end{array}$$

We claim that we can choose an isomorphism $\mathrm{Spf} A \rightarrow \mathrm{Spf} A'$ such that for $1 \leq i \leq r$, the base change of $\mathrm{Spf} S'_i$ is $\mathrm{Spf} A/(\mathbf{s}_i)$ and the base change of $\mathrm{Spf} H'_i$ is $\mathrm{Spf} A/(\mathbf{c}_i, n_i)$ and in addition the action of G on $\mathrm{Spf} A$ is as claimed.

Let $S'_i = A'/I'_i$ and $H'_i = A'/J'_i$. The inclusions $\mathrm{Spf} S'_i \rightarrow \mathrm{Spf} A'$ and $\mathrm{Spf} H'_i \rightarrow \mathrm{Spf} A'$ are equivariant with respect to the actions of G induced by θ'_i , ψ' and ϕ'_i by Lemma 6.4.

There is a commutative diagram of equivariant exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & I'_i & \longrightarrow & m_{A'} & \longrightarrow & m_{S'_i} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I'_i/m_{A'} I'_i & \longrightarrow & m_{A'}/m_{A'}^2 & \longrightarrow & m_{S'_i}/m_{S'_i}^2 \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow = \\ 0 & \longrightarrow & T^1(\widehat{\mathcal{O}}_{E_i, \xi_i})^\vee & \longrightarrow & T^1(C, \{p_i\}_{i=1}^n)^\vee & \longrightarrow & T_{S'_i}^\vee \longrightarrow 0. \end{array}$$

The second row is exact on the left because A' and S'_i are both formally smooth. All the maps are equivariant.

By Proposition 6.2, we can choose, for $1 \leq i \leq r$, a basis $s'_i = (s'_{i,0}, \dots, s'_{i,2m-1})$ for $T^1(\widehat{\mathcal{O}}_{E_i, \xi_i})^\vee$ such that the action of G is

$$s'_{i,l} \rightarrow t_i^{2l-4m-2} s'_{i,l}.$$

Choose an equivariant section $\sigma_i : T^1(\widehat{\mathcal{O}}_{E_i, \xi_i})^\vee \rightarrow I'_i$ and, abusing notation, write $s'_{i,l}$ also for the image of $s'_{i,l}$ under σ_i . By Nakayama's Lemma, $s'_{i,0}, \dots, s'_{i,2m-1}$ generate I'_i .

Recall from Lemma 6.1 that for $1 \leq i \leq r$ there is an equivariant inclusion

$$\mathrm{LT}^1(E_i, q_i)^\vee \rightarrow T^1(C, \{p_i\}_{i=1}^n)^\vee$$

There is a commutative diagram of equivariant exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J'_i & \longrightarrow & m_{A'} & \longrightarrow & m_{S'_i} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J'_i/m_{A'}J'_i & \longrightarrow & m_{A'}/m_{A'}^2 & \longrightarrow & m_{S'_i}/m_{S'_i}^2 & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow = & & \\ 0 & \longrightarrow & \mathrm{LT}^1(E_i, q_i)^\vee \oplus T^1(\widehat{\mathcal{O}}_{C, q_i})^\vee & \longrightarrow & T^1(C, \{p_i\}_{i=1}^n)^\vee & \longrightarrow & T_{H_i}^\vee & \longrightarrow & 0. \end{array}$$

Therefore, using Proposition 6.2 and using an equivariant section as above, we can choose, for $1 \leq i \leq r$, $\mathbf{c}'_i = (c'_{i,1}, \dots, c'_{i,m-1})$ and n'_i such that $J'_i = (\mathbf{c}'_i, n'_i)$ and the action of G is

$$c'_{i,l} \rightarrow t_i^{2l-1} c'_{i,l} \quad n'_i \rightarrow t_i n'_i.$$

The subset $S := \{\mathbf{c}'_1, \dots, \mathbf{c}'_r, n'_1, \dots, n'_r, \mathbf{s}'_1, \dots, \mathbf{s}'_r\}$ of $T^1(C, \{p_i\}_{i=1}^n)^\vee$ is linearly independent. Choose $k'_1, \dots, k'_{\bar{g}}$, where $\bar{g} := g - rm$ such that $S \cup \{k'_1, \dots, k'_{\bar{g}}\}$ is a basis and the action of G is

$$k'_l \mapsto k'_l.$$

Choose an equivariant section $\sigma : T^1(C, \{p_i\}_{i=1}^n)^\vee \rightarrow m_{A'}$ and, abusing notation, write k'_l also for the image of k'_l under σ . Define a map $A \rightarrow A'$ by

$$n_i \rightarrow n'_i \quad c_{i,l} \rightarrow c'_{i,l} \quad n_i \rightarrow n'_i \quad s_{i,l} \rightarrow s'_{i,l}.$$

This is a local homomorphism inducing an isomorphism $A/m_A^2 \rightarrow A'/m_{A'}^2$ and therefore an isomorphism. \square

Remark 6.6. Similar descriptions can be given for cases I' and I''.

Proposition 6.7. *Set $m = 1$. Let $(C, \{p_i\}_{i=1}^n)$ be a maximally degenerate A_{2m+1} -stable curve in case II* (see Proposition 6.3). Set*

$$A := \mathbb{C}[[\mathbf{k}, \mathbf{c}_1, \dots, \mathbf{c}_r, \mathbf{s}_1, \dots, \mathbf{s}_r, n_0, \dots, n_r]].$$

There is a miniversal formal deformation $\psi : \mathrm{Spf} A \rightarrow \overline{\mathcal{M}}_{g,n}(A_k)$ of $(C, \{p_i\}_{i=1}^n)$ such that there is a 2-cartesian diagram

$$\begin{array}{ccccc} \mathrm{Spf} A / \cap_{i=1}^r (s_i) & \longrightarrow & \mathrm{Spf} A & \longleftarrow & \mathrm{Spf} A / \cap_{\mu,\nu \in S} J_{\mu,\nu} \\ \downarrow \theta & & \downarrow \psi & & \downarrow \phi \\ \overline{\mathcal{S}}_{g,n}(A_{2m+1}) & \xrightarrow{i} & \overline{\mathcal{M}}_{g,n}(A_{2m+1}) & \xleftarrow{j} & \overline{\mathcal{H}}_{g,n}(A_{2m+1}), \end{array}$$

where $S := \{\mu, \nu : 1 \leq \mu \leq \lceil \frac{r}{2} \rceil, 0 \leq \nu \leq r - 2\mu + 1\}$ and

$$J_{\mu,\nu} = (n_\nu, \mathbf{c}_{\nu+1}, \mathbf{s}_{\nu+2}, \mathbf{c}_{\nu+3}, \dots, \mathbf{s}_{\nu+2\mu-2}, \mathbf{c}_{\nu+2\mu-1}, n_{\nu+2\mu-1}),$$

and the action of the group scheme $\mathrm{Aut}(C, \{p_i\}_{i=1}^n)^\circ$ on $\widehat{\mathrm{Def}}(C, \{p_i\}_{i=1}^n)$ induced by ψ coincides with the action induced by the action of the group $\mathrm{Aut}(C, \{p_i\}_{i=1}^n)^\circ$ on $\mathbb{T}^1(C, \{p_i\}_{i=1}^n)$ described in Proposition 6.2.

Proof. The proof is similar to the proof of Proposition 6.5. Let $G := \mathrm{Aut}(C, \{p_i\}_{i=1}^n)^\circ = \prod_{i=1}^r \mathrm{Aut}(E_i, q_i)$. Choose a miniversal formal deformation $\psi' : \mathrm{Spf} A' \rightarrow \overline{\mathcal{M}}_{g,n}(A_{2m+1})$ of $(C, \{p_i\}_{i=1}^n)$. Let $C' \rightarrow \mathrm{Spf} A'$ be the corresponding family of curves. Let $\mathrm{Spf} S'_i$ be the formal closed subscheme of $\mathrm{Spf} A'$ where the A_{2m+1} -singularity ξ_i is preserved in C' ($1 \leq i \leq r$). Let $\mathrm{Spf} N'_i$ be the formal closed subscheme of $\mathrm{Spf} A'$ where the node η_i is preserved in C' ($0 \leq i \leq r$). Clearly there is a 2-cartesian diagram

$$\begin{array}{ccccc} \bigcup_{i=1}^r \mathrm{Spf} S'_i & \longrightarrow & \mathrm{Spf} A' & \longleftarrow & \bigcup_{\mu,\nu \in S} \mathrm{Spf} H'_{\mu,\nu} \\ \downarrow \bigcup_{i=1}^r \theta'_i & & \downarrow \psi' & & \downarrow \bigcup_{\mu,\nu \in S} \phi'_{\mu,\nu} \\ \overline{\mathcal{S}}_{g,n}(A_{2m}) & \longrightarrow & \overline{\mathcal{M}}_{g,n}(A_{2m}) & \longleftarrow & \overline{\mathcal{H}}_{g,n}(A_{2m}), \end{array}$$

where

$$\mathrm{Spf} H'_{\mu,\nu} = \mathrm{Spf} N'_\nu \cap \mathrm{Spf} N'_{\nu+2\mu-1} \cap \bigcap_{i=1}^{m-1} \mathrm{Spf} S'_{\nu+2i}.$$

Now argue as in the proof of Proposition 6.5, this time using Proposition 6.3, that we can choose an isomorphism $\mathrm{Spf} A \rightarrow \mathrm{Spf} A'$ such that for $1 \leq i \leq r$, the base change of $\mathrm{Spf} S'_i$ is $\mathrm{Spf} A/(s_i)$ and the base change of $\mathrm{Spf} N'_i$ is $\mathrm{Spf} A/(n_i)$ and in addition the action of G on $\mathrm{Spf} A$ is as claimed. \square

We believe that Proposition 6.7 is valid also for $m > 1$ but there are some further details to verify in this case.

Remark 6.8. Similar descriptions can be given for cases II' and II''.

7. LOCAL VARIATION OF GIT

In this section, we calculate the plus and minus variation of GIT chambers for the action of the automorphism group on the deformation space of a maximally degenerate curve in $\overline{\mathcal{M}}_{g,n}(A_k)$. The main result is Theorem 7.9, which states vaguely that the minus and plus GIT chambers cut out formally locally the closed loci $\overline{\mathcal{S}}_{g,n}(A_k)$ and $\overline{\mathcal{H}}_{g,n}(A_k)$.

7.1. Variation of GIT. Let G be a linearly reductive group scheme over \mathbb{C} acting on an affine $X = \text{Spec } A$ finite type over \mathbb{C} . Let $\sigma : A \rightarrow \Gamma(G) \otimes A$ denote the dual action. Let $\chi : G \rightarrow \mathbb{G}_m = \text{Spec } \mathbb{C}[t]_t$ be a character. For an integer n , define

$$A_n = \{f \in A \mid \sigma(f) = (\chi^*t)^n f\}.$$

Note that $A_0 = A^G$. Define V^- and V^+ to be the reduced G -invariant closed subschemes of X defined by the ideals

$$\begin{aligned} I^- &= (f \in A \mid f \in A_n \text{ for } n < 0), \\ I^+ &= (f \in A \mid f \in A_n \text{ for } n > 0). \end{aligned}$$

We define $X^- = X \setminus V^-$ and $X^+ = X \setminus V^+$ be the G -invariant open subschemes; of course, these depend on the character χ . Then it is easy to see (see [DH98] or [Tha96]) that that there is a commutative diagram

$$\begin{array}{ccccc} X^- & \hookrightarrow & X & \longleftarrow & X^+ \\ \downarrow & & \downarrow & & \downarrow \\ \text{Proj } \bigoplus_{n \geq 0} A_{(-n)} = X^-//G & \longrightarrow & X//G = \text{Spec } A^G & \longleftarrow & X^+//G = \text{Proj } \bigoplus_{n \geq 0} A_n \end{array}$$

where the vertical arrows are good GIT quotients. Note that the induced morphisms $X^- \hookrightarrow X$ and $X^+ \hookrightarrow X$ are open immersions while $X^-//G \rightarrow X//G$ and $X^+//G \rightarrow X//G$ are projective.

We will also use the following stack-theoretic language: set $\mathcal{X} = [X/G]$, $\mathcal{X}^- = [X^-/G]$ and $\mathcal{X}^+ = [X^+/G]$. We have a commutative diagram

$$\begin{array}{ccccc} \mathcal{X}^- & \hookrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}^+ \\ \downarrow & & \downarrow & & \downarrow \\ X^-//G & \longrightarrow & X//G & \longleftarrow & X^+//G \end{array}$$

where the vertical arrows are good moduli spaces.

Remark 7.1. The character χ induces a G -linearization \mathcal{L}_χ of the structure sheaf \mathcal{O}_X . The semi-stable locus $X_{\mathcal{L}_\chi}^{\text{ss}}$ (resp., $X_{\mathcal{L}_\chi^\vee}^{\text{ss}}$) is identified with X^+ (resp., X^-).

Proposition 7.2 (Affine Hilbert-Mumford criterion). *Suppose G is a linearly reductive group scheme over \mathbb{C} acting on an affine scheme $X = \text{Spec } A$ finite type over \mathbb{C} . Let $\chi : G \rightarrow \mathbb{G}_m$ be a character. Let $x \in X(\mathbb{C})$. Then $x \in V^-$ (resp., $x \in V^+$) if and only if there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow G$ with $\chi \circ \lambda > 0$ (resp., $\chi \circ \lambda < 0$) such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists.*

Proof. Suppose there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow G$ with $\chi \circ \lambda > 0$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists. Let $f \in A$ satisfy $\sigma(f) = \chi^*(t)^n f$ for $n < 0$. Then under $\mathbb{G}_m \rightarrow X, t \mapsto \lambda(t) \cdot x$, the function f pulls back to $t^{n(\chi \circ \lambda)} f(x)$. Since the limit exists and $n(\chi \circ \lambda) < 0$, $f(x) = 0$. Therefore $x \in V^-$.

Conversely, let $x \in V^-$. Consider the induced action of G on $\text{Spec } A[y]$ via $y \mapsto (\chi^* t)y$. (This is precisely the G -line bundle over X corresponding to χ .) Then $\overline{O(x, 1)} \cap \{y = 0\} \neq \emptyset$. Otherwise, there would exist a function $f \in A[y]^G$ with $f(x, 1) \neq 0$ and $f(x, 0) = 0$; by writing $f = \sum_n f_n y^n$, we see that for some $n > 0$, $f_n \mapsto (\chi^* t)^{-n} f_n$ and $f_n(x) \neq 0$, which contradicts $x \in V^-$. Therefore, in the closure of the G° -orbit of $(x, 1)$ there is a point $(x_0, 0)$ with closed orbit with $x_0 \neq 0$. By the Hilbert-Mumford criterion ([Mum65, Theorem 2.1]), there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow G$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot (x, 1) = (x_0, 0)$. This gives the desired one-parameter subgroup. The V^+ case is similar. \square

Lemma 7.3. *Let G_i be linearly reductive groups acting on affine schemes X_i and $\chi_i : G_i \rightarrow \mathbb{G}_m$ be characters for $i = 1, \dots, n$. Consider the diagonal action of $G = \prod_i G_i$ on $X = \prod_i X_i$ and the character $\prod_i \chi_i : G \rightarrow \mathbb{G}_m$. Then*

$$V^- = \bigcup_{i=1}^n X_1 \times \cdots \times V_i^- \times \cdots \times X_n$$

$$V^+ = \bigcup_{i=1}^n X_1 \times \cdots \times V_i^+ \times \cdots \times X_n$$

Proof. This follows from Proposition 7.2. \square

Lemma 7.4. *Let G be a linearly reductive group scheme over \mathbb{C} acting on an affine $X = \text{Spec } A$ finite type over \mathbb{C} . Let $\chi : G \rightarrow \mathbb{G}_m$ be a character. Let $Z \subseteq X$ be a G -invariant closed subscheme. Then with respect to the character χ , we have $Z^- = X^- \cap Z$ and $Z^+ = X^+ \cap Z$.*

Proof. Clearly $Z^- \subseteq X^- \cap Z$. Let $I \subseteq A$ be the invariant ideal defining Z . If we consider the induced action of G on $\text{Spec } A[y]$ where $y \mapsto \chi^* t y$, then since G is linearly reductive $A[y]^G \rightarrow (A[y]/I)^G \cong (A/I[y])^G$ is surjective. Let $z \in Z^-$ and $f \in A/I$ with $f \mapsto \chi^* t^d f$ for $d < 0$ with $f(z) \neq 0$. It follows that there exists a lift $\tilde{f} \in A$ with $\tilde{f} \mapsto \chi^* t^d \tilde{f}$ for $d < 0$ with $\tilde{f}(z) \neq 0$ so $z \in X^-$. The Z^+ case is similar. \square

Now we will analyze the V^- and V^+ chambers for the natural action of $\text{Aut}(C, \{p_i\}_{i=1}^n)^\circ$ on the first order deformation space $\text{Def}(C, \{p_i\}_{i=1}^n)$ in a sequence of successively more general cases.

7.2. The case of a monomial $H_{m,1}$ -curve/ $H_{m,2}$ -curve. If $k = 2m$ (resp., $k = 2m + 1$), let (E, q) (resp., (E, q_1, q_2)) be the monomial $H_{m,1}$ -curve (resp., $H_{m,2}$ -curve) as specified in Definition 3.10. By Section 6, we can write

$$\text{Def}(E, q) = \text{Spec } \mathbb{C}[s_0, \dots, s_{k-1}, c_1, \dots, c_{m-1}]$$

where $\mathbf{s} = (s_1, \dots, s_{k-1})$ are the ‘‘singularity’’ coordinates and $\mathbf{c} = (c_1, \dots, c_{m-1})$ are the ‘‘crimping’’ coordinates. The action by $\text{Aut}(E, q) = \mathbb{G}_m$ (resp., $\text{Aut}(E, q_1, q_2) = \mathbb{G}_m$) is given by:

$$s_i \mapsto t^{2(i-k-1)} s_i \quad c_i \mapsto t^{2i-1} c_i$$

Lemma 7.5. *For the monomial $H_{m,1}$ -curve (E, q) or monomial $H_{m,2}$ -curve (E, q_1, q_2) and with notation as above, we have*

$$\begin{aligned} V^- &= V(\mathbf{s}) = V(s_0, \dots, s_{k-1}) \\ V^+ &= V(\mathbf{c}) = V(c_1, \dots, c_{m-1}) \end{aligned}$$

Proof. This is immediate from the definitions of V^- and V^+ . □

7.3. The case of an $H_{m,2}$ -link. In this section, we handle the special case of an $H_{m,2}$ -link when $k = 2m + 1$ is odd, $n = 0$ and $C = K \cup E_1 \cup \dots \cup E_r$ as in case II* in Proposition 6.3. Recall that we can write

$$\text{Def}(C) = \text{Spec } \mathbb{C}[\mathbf{s}_1, \dots, \mathbf{s}_r, \mathbf{c}_1, \dots, \mathbf{c}_r, n_0, \dots, n_r]$$

where the s_i are the ‘‘singularity’’ coordinates, the c_i are the ‘‘crimping’’ coordinates and n_i are the ‘‘node’’ coordinates. The action by $\text{Aut}(C)^\circ = \mathbb{G}_m^r$ is given by:

$$\begin{aligned} \mathbf{s}_{j,i} &\mapsto t_i^{2(i-k-1)} \mathbf{s}_{j,i} & \mathbf{c}_{j,i} &\mapsto t_i^{2i-1} \mathbf{c}_{j,i} & n_0 &\mapsto t_1 n_0 \\ & & & & n_i &\mapsto t_i t_{i+1} n_i, & k_i &\mapsto k_i \\ & & & & n_r &\mapsto t_r n_r \end{aligned}$$

Lemma 7.6. *With the above notation,*

$$V^- = \bigcup_{j=1}^r V(\mathbf{s}_j) \quad V^+ = \bigcup_{\mu \geq 1} \bigcup_{\nu=0}^{r-2\mu+1} V_{\mu,\nu}$$

where $V_{\mu,\nu} = V(n_\nu, \mathbf{c}_{\nu+1}, \mathbf{s}_{\nu+2}, \mathbf{c}_{\nu+3}, \dots, \mathbf{s}_{\nu+2\mu-2}, \mathbf{c}_{\nu+2\mu-1}, n_{\nu+2\mu-1})$.

Remark 7.7. For instance, $V_{1,\nu} = V(n_\nu, \mathbf{c}_{\nu+1}, n_{\nu+1})$ and $V_{2,\nu} = V(n_\nu, \mathbf{c}_{\nu+1}, \mathbf{s}_{\nu+2}, \mathbf{c}_{\nu+3}, n_{\nu+3})$.

Proof. We will use the Hilbert-Mumford criterion of Proposition 7.2. For the V^- case, suppose for $x \in \text{Def}(C)$ that for some j , $s_j(x) = 0$. Set $\lambda = (\delta_s^j) : \mathbb{G}_m \rightarrow \mathbb{G}_m^r \cong \prod_{i=1}^r \text{Aut}(E_i, q_{2i-1}, q_{2i})$. Then $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists so $x \in V^-$. Conversely, let $\lambda = (\lambda_s)$ be a one-parameter subgroup with $\sum_s \lambda_s > 0$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists. Then for some j , $\lambda_j > 0$ which implies that $s_j(x) = 0$.

For the V^+ case, the inclusion \supseteq is easy: suppose that $x \in V_{\mu, \nu}$ for $\mu \geq 1$ and $\nu = 0, \dots, r - 2\mu + 1$. Set

$$\lambda = \left(\underbrace{0, \dots, 0}_{\nu}, \underbrace{-1, 1, -1, \dots, 1, -1}_{2\mu-1}, \underbrace{0, \dots, 0}_{r-2\mu-\nu+1} \right)$$

Then $\sum_s \lambda_s = -1$ and $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists so $x \in V^+$. For the \subseteq inclusion, we will use induction on r . If $r = 1$, then clearly $V^+ = V(n_0, \mathbf{c}_1, n_1)$. For $r > 1$, suppose $x \in V^+$ and $\lambda = (\lambda_s) : \mathbb{G}_m \rightarrow \mathbb{G}_m^r$ is a one-parameter subgroup with $\sum_{s=1}^r \lambda_s < 0$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists. If $\lambda_r \geq 0$, then $\sum_{s=1}^{r-1} \lambda_s < 0$ so by the induction hypothesis $x \in V_{\mu, \nu}$ for some $\mu \geq 1$ and $\nu = 0, \dots, r - 2\mu$. If $\lambda_r < 0$, then we immediately conclude that $n_r(x) = \mathbf{c}_r(x) = 0$. If $\lambda_{r-1} + \lambda_r < 0$, then $n_{r-1}(x) = 0$ so $x \in V_{1, r-1}$. If $\lambda_{r-1} + \lambda_r \geq 0$, then $\lambda_{r-1} \geq 0$ so $s_{r-1}(x) = 0$. Furthermore, $\sum_{s=1}^{r-2} \lambda_s < 0$ so by applying the induction hypothesis and restricting to the locus $V(n_{r-2}, \mathbf{s}_{r-1}, \mathbf{c}_{r-1}, n_{r-1}, \mathbf{s}_r, \mathbf{c}_r, n_r)$, we can conclude either: (1) $x \in V_{\mu, \nu}$ for $\mu \geq 1$ and $\nu = 0, \dots, r - 2\mu - 1$, or (2) $x \in V(n_{r-\mu-4}, \mathbf{c}_{r-\mu-3}, \mathbf{s}_{r-\mu-2}, \mathbf{c}_{r-\mu-1}, \dots, \mathbf{s}_{r-3}, \mathbf{c}_{r-2})$ for some $\mu \geq 1$. In case (2), since $s_{r-1}(x) = \mathbf{c}_r(x) = n_r(x) = 0$, we have $x \in V_{\mu+1, r-\mu-4}$. \square

Remark 7.8. The chamber V^- is the closed locus in the deformation space where an A_k -singularity is preserved. The chamber V^+ is the closed locus of curves containing an $H_{m,2}$ -chain.

7.4. The general case. Let $(C, \{p_i\}_{i=1}^n)$ be a maximally degenerate A_k -stable curve. Consider the action of $\text{Aut}(C, \{p_i\}_{i=1}^n)$ on $\text{Def}(C, \{p_i\}_{i=1}^n)$ described in Section 6.2. Let

$$\chi^\circ : \text{Aut}(C, \{p_i\}_{i=1}^n)^\circ \rightarrow \mathbb{G}_m$$

be the character which is the product of the natural characters on the monomial $H_{m,1}$ and $H_{m,2}$ -subcurves and trivial on the core (see Definition 5.2).

Let V^- and V^+ be the reduced closed subschemes of $\text{Def}(C, \{p_i\}_{i=1}^n)$ defined by the character χ° .

Let $\text{Spf } \widehat{A} \rightarrow \overline{\mathcal{M}}_{g,n}(A_k)$ be a miniversal deformation space of $(C, \{p_i\}_{i=1}^n)$. We can identify \widehat{A} with the completion of the origin in the 1st order deformation space $\text{Def}(C, \{p_i\}_{i=1}^n)$. Define the closed formal subschemes \mathfrak{z}^- and \mathfrak{z}^+ of $\text{Spf } \widehat{A}$ as the cartesian

products

(7.1)

$$\begin{array}{ccccc} \mathfrak{Z}^- & \hookrightarrow & \mathrm{Spf} \widehat{A} & \longleftarrow & \mathfrak{Z}^+ \\ \downarrow & & \downarrow & & \downarrow \\ V^- & \hookrightarrow & \mathrm{Def} & \longleftarrow & V^+ \end{array}$$

Recall that $\overline{\mathcal{S}}_{g,n}(A_k) = \overline{\mathcal{M}}_{g,n}(A_k) \setminus \overline{\mathcal{M}}_{g,n}(A_k^-)$ is the locus of curves with an A_k -singularity and $\overline{\mathcal{H}}_{g,n}(A_k) = \overline{\mathcal{M}}_{g,n}(A_k) \setminus \overline{\mathcal{M}}_{g,n}(A_k^+)$ is the locus of curves containing an $H_{m,1}$ -tail (resp., $H_{m,2}$ -chain) if $k = 2m$ or $H_{m,2}$ -chain if $k = 2m + 1$.

Theorem 7.9. *Let $(C, \{p_i\}_{i=1}^n)$ be a maximally degenerate A_k -stable curve with $k \leq 4$. With the notation above, there is a cartesian diagram*

$$\begin{array}{ccccc} \mathfrak{Z}^- & \hookrightarrow & \mathrm{Spf} \widehat{A} & \longleftarrow & \mathfrak{Z}^+ \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\mathcal{S}}_{g,n}(A_k) & \hookrightarrow & \overline{\mathcal{M}}_{g,n}(A_k) & \longleftarrow & \overline{\mathcal{H}}_{g,n}(A_k) \end{array}$$

Proof. We split the proof into the cases according to the canonical decomposition of Definition 5.2.

Case I: $C = K \cup E_1 \cup \dots \cup E_r$ where (E_i, q_i) is an $H_{m,1}$ -tail. By Lemma 7.3, we may assume $r = 1$. By Proposition 6.5, the miniversal deformation space is $\widehat{A} \cong \mathbb{C}[[\mathbf{k}, \mathbf{c}, s, n]]$ with the action of $\mathrm{Aut}^*(C) = \mathbb{G}_m$ given by

$$s_i \mapsto t^{2(i-k-1)} s_i \quad c_i \mapsto t^{2i-1} c_i \quad n \mapsto tn, \quad k_i \mapsto k_i$$

where the $\mathbf{s} = (s_1, \dots, s_{k-1})$ are the ‘‘singularity’’ coordinates, the $\mathbf{c} = (c_1, \dots, c_{m-1})$ are the ‘‘crimping’’ coordinates, the n variable is the ‘‘node’’ coordinate, and the $\mathbf{k} = (k_i)$ are ‘‘kore’’ coordinates.

We see that $V^- = V(\{s_i\}_{i=0}^{k-1})$ which defines the closed locus in the deformation space where the A_k -singularity is preserved. On the other hand, $V^+ = V(n, \{c_i\}_{i=1}^{m-1})$ defines the closed locus in the deformation space where an $H_{m,1}$ -tail is attached at a node.

Case I': $C = E_1 \cup E_2$ where (E_1, q_1) and (E_2, q_2) are $H_{m,1}$ -tails. We have $\widehat{A} \cong \mathbb{C}[[s_1, s_2, \mathbf{c}_1, \mathbf{c}_2, n]]$ with $\mathbf{s}_j = (s_{j,0}, \dots, s_{j,k-1})$ and $\mathbf{c}_j = (c_{j,1}, \dots, c_{j,m-1})$ for $j = 1, 2$. The action of $\mathrm{Aut}(C)^\circ \cong \mathbb{G}_m^2$ is given by

$$s_{j,i} \mapsto t_i^{2(i-k-1)} s_{j,i} \quad c_{j,i} \mapsto t_i^{2i-1} c_{j,i} \quad n \mapsto t_1 t_2 n$$

Let $x \in \mathrm{Def}(C)$. If $\lambda = (l_1, l_2) : \mathbb{G}_m \rightarrow \mathbb{G}_m^2$ is a one-parameter subgroup with $l_j > 0$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists then $s_{j,0}(x) = \dots = s_{j,k-1}(x) = 0$. Conversely, if $s_{j,0}(x) = \dots = s_{j,k-1}(x) = 0$ for $j = 1, 2$, then $\lambda = (1, 0)$ if $j = 1$ or $\lambda = (0, 1)$ if $j = 2$ is a one-parameter subgroup such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists. By Proposition 7.2,

$V^- = V(\{s_{1,i}\}_{i=0}^{k-1}) \cup V(\{s_{2,i}\}_{i=0}^{k-1})$ which corresponds in the deformation space to where one of the two A_k -singularities is preserved.

Let $x \in \text{Def}(C)$. If $\lambda = (l_1, l_2) \rightarrow \mathbb{G}_m^2$ is a one-parameter subgroup with $l_1 + l_2 < 0$ and $l_j < 0$ such that $\lim_{t \rightarrow 0}$ exists, then $n(x) = c_{j,1}(x) = \cdots = c_{j,m-1}(x) = 0$. Conversely, if $n = c_{j,1} = \cdots = c_{j,m-1} = 0$, then $\lambda = (-1, 0)$ if $j = 1$ or $\lambda = (0, -1)$ if $j = 2$ is a one-parameter subgroup such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists. By Proposition 7.2, $V^+ = V(n, \{c_{1,i}\}_{i=1}^{m-1}) \cup V(n, \{c_{2,i}\}_{i=1}^{m-1})$ which corresponds in the deformation space to where the node is preserved and one of the components is an $H_{m,1}$ -tail.

Case I'': This is Lemma 7.5.

Case II*: $C = K \cup E_1 \cup \cdots \cup E_r \cup F_1 \cup \cdots \cup F_s$ where (E_i, q_{2i-1}, q_{2i}) is an $H_{m,2}$ -link intersecting K at two nodes $q_{2i-1}, q_{2i} \in C$ and (F_i, r_{2i-1}, r_{2i}) is an $H_{m,2}$ -link intersection K at a node $r_{2i-1} \in C$ and r_{2i} is a marked point.

By Lemma 7.3, it is enough to consider the case when either $r = 1, s = 0$ or $r = 0, s = 1$. The case of $r = 1$ and $s = 0$ is the example worked out in Lemma 7.6; the addition of marked points in the present situation doesn't affect the calculation of Lemma 7.6. If $r = 1, s = 0$, the action of $\text{Aut}(C, \{p_i\}_{i=1}^n)^*$ on $\text{Def}(C, \{p_i\}_{i=1}^n)$ is precisely the action given in Section 7.3 restricted to the closed subscheme $V(n_{r+1}) = 0$. This case therefore follows from Lemmas 7.4 and 7.6.

Case II': $(C, p_1, p_2) = (E_1 \cup \cdots \cup E_r, q_1, q_{2r})$, where for $1 \leq i \leq r-1$, E_i meets E_{i+1} in a node $q_{2i} = q_{2i+1}$ and for $1 \leq i \leq r$, (E_i, q_{2i-1}, q_{2i}) is a monomial $H_{m,2}$ -curve.

The action of $\text{Aut}(C, \{p_i\}_{i=1}^n)^\circ$ on $\text{Def}(C, \{p_i\}_{i=1}^n)$ is the action given in Section 7.3 restricted to the closed subscheme $V(n_0, n_{r+1}) = 0$ so this case follows from Lemmas 7.4 and 7.6.

Case II'': $C = (E_1 \cup \cdots \cup E_r)$ where for $1 \leq i \leq r-1$, E_i meets E_{i+1} in a node $q_{2i} = q_{2i+1}$, and E_r meets E_1 in a node $q_{2r} = q_1$, and for $1 \leq i \leq r$, (E_i, q_{2i-1}, q_{2i}) is a monomial $H_{m,2}$ -curve. We have $A \cong \mathbb{C}[[s_0, \dots, s_{r-1}, c_0, \dots, c_{r-1}, n_0, \dots, n_{r-1}]]$ with the action of $\text{Aut}(C)^\circ \cong \mathbb{G}_m^r$ given by:

$$s_{j,i} \mapsto t_i^{2(i-k-1)} s_{j,i} \quad c_{j,i} \mapsto t_i^{2i-1} c_{j,i} \quad n_i \mapsto t_i t_{i+1} n_i$$

where subscripts are taken modular the integer r . We claim that

$$V^- = \bigcup_{j=0}^{r-1} V(s_j) \quad V^+ = \bigcup_{\mu \geq 1} \bigcup_{\nu=0}^{\lceil \frac{r}{2} \rceil - 1} V_{\mu,\nu}$$

where $V_{\mu,\nu} = V(n_\nu, c_{\nu+1}, s_{\nu+2}, c_{\nu+3}, \dots, s_{\nu+2\mu-2}, c_{\nu+2\mu-1}, n_{\nu+2\mu-1})$ (where the subscripts are taken modular r). (We note that if r is odd and $\mu = (r+1)/2$, then $\nu + 2\mu - 1 = \nu$ so that the coordinate $n_{\nu+2\mu-1}$ in $V_{\mu,\nu}$ is redundant.)

The V^- case is straightforward and the containment \supseteq for the V^+ case follows as in the proof of Lemma 7.6 from choosing an appropriate one parameter

subgroup. For the inclusion \subseteq , let $x \in \text{Def}(C, \{p_i\}_{i=1}^n)$ and suppose $\lambda = (\lambda_s) : \mathbb{G}_m \rightarrow \mathbb{G}_m^r$ is a one-parameter subgroup with $\sum_{s=0}^{r-1} \lambda_s < 0$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists. After possibly reordering, we may assume that $\lambda_0 + \lambda_1 < 0$ and $\lambda_0 < 0$. In particular $n_0(x) = c_0(x) = 0$. If $\lambda_0 + \lambda_{r-1} < 0$, then $x \in V_{1,r-1}$. Otherwise, $\lambda_1 + \cdots + \lambda_{r-2} < 0$. If $\lambda_{r-1} < 0$, then $\lambda_1 + \cdots + \lambda_{r-1} < 0$ so by restricting the locus $n_{r-1} = s_0 = c_0 = n_0 = 0$ and by applying Lemma 7.6, we see that $x \in V_{\mu,\nu}$ for some μ, ν . Otherwise, $\lambda_{r-1} > 0$ so $s_{r-1}(x) = 0$.

We have reduced to the case where $\lambda_0 + \lambda_1 < 0$, $\lambda_0 < 0$ and $\lambda_{r-1} > 0$ so that $n_0(x) = c_0(x) = s_{r-1}(x) = 0$. By restricting to the locus $n_{r-2} = s_{r-1} = c_{r-1} = s_0 = c_0 = n_0 = 0$ and applying Lemma 7.6, we have four cases:

- (a) $x \in V_{\mu,\nu}$ for $\mu \geq 1$ and $1 \leq \nu \leq (r-2) - 2\mu$.
- (b) $x \in V(\mathbf{c}_1, \mathbf{s}_2, \dots, \mathbf{s}_{2\mu-2}, \mathbf{c}_{2\mu-1}, n_{2\mu-1})$ for $\mu \geq 1$ with $2\mu - 1 < r - 2$.
- (c) $x \in V(n_{r-2\mu-1}, \mathbf{c}_{r-2\mu}, \mathbf{s}_2, \dots, \mathbf{s}_{r-3}, \mathbf{c}_{r-2})$ for $1 \leq \nu \leq r - 3$ for $\mu \geq 1$ with $2\mu - 1 < r - 2$. $2\mu - 1$
- (d) r is even and $x \in V(\mathbf{c}_1, \mathbf{s}_2, \dots, \mathbf{s}_{r-3}, \mathbf{c}_{r-2})$.

Case (a) is ok. In case (b), $x \in V_{\mu,0}$ since $n_0(x) = 0$. In case (c), $x \in V_{\mu+1,r-2\mu-1}$ since $s_{r-1}(x) = c_0(x) = n_0(x) = 0$. In case (d), $x \in V_{r/2,0}$ since $s_{r-1}(x) = c_0(x) = n_0(x) = 0$.

□

8. ÉTALE LOCAL PRESENTATIONS BY GIT CHAMBERS

The goal of this section to prove Theorem 8.3 which states that the inclusions

$$\overline{\mathcal{M}}_{g,n}(A_k^-) \subseteq \overline{\mathcal{M}}_{g,n}(A_k) \supseteq \overline{\mathcal{M}}_{g,n}(A_k^+)$$

correspond étale locally around closed points in $\overline{\mathcal{M}}_{g,n}(A_k)$ to variation of GIT chambers on the deformation space. The theorem will follow from the algebraization result Proposition 8.1 and the formally local statement Theorem 7.9.

Proposition 8.1. ([Alp10, Theorem 3]) *Let \mathcal{X} be an algebraic stack of finite type over \mathbb{C} . Suppose \mathcal{X} is a quotient stack $[X/G]$, where G is a connected algebraic group acting on a smooth and separated scheme X . If $x \in X(\mathbb{C})$ has linearly reductive stabilizer, there exists a locally closed G_x -invariant affine $W \hookrightarrow X$ with $w \in W$ such that*

$$[W/G_x] \rightarrow [X/G]$$

is affine and étale.

Corollary 8.2. *Let \mathcal{X} be an algebraic stack finite type over \mathbb{C} . Suppose \mathcal{X} is a quotient stack $[X/G]$, where G is a connected algebraic group acting on a smooth and separated scheme X . If $x \in X(\mathbb{C})$ has linearly reductive stabilizer, there is an affine scheme*

$W = \text{Spec } A$ with an action by the stabilizer G_x , a closed G_x -invariant point $w \in W$ and a commutative diagram

$$(8.1) \quad \begin{array}{ccccc} \widehat{\text{Def}}(x) & \xrightarrow{j} & \mathcal{W} = [W/G_x] & & \\ & \searrow f & \downarrow & \searrow g & \\ & \mathcal{X} & W//G_x & & [\text{Def}(x)/G_x] \\ & & \searrow \bar{g} & & \downarrow \\ & & & & \text{Def}(x)//G_x \end{array}$$

such that

- (1) There is an isomorphism $\widehat{\text{Def}}(x) \rightarrow \text{Spf } \widehat{\mathcal{O}}_{W,w}$ inducing j and $f \circ j : \widehat{\text{Def}}(x) \rightarrow \mathcal{X}$ is a miniversal deformation space of x ,
- (2) f is étale, affine, and stabilizer preserving at x with $f(w) = x$,
- (3) $[W/G_x] \rightarrow W//G_x$ and $[\text{Def}(x)/G_x] \rightarrow \text{Def}(x)//G_x$ are good moduli spaces,
- (4) g is affine, étale, stabilizer preserving and saturated (in particular, g maps closed points to closed points), and
- (5) the right parallelogram is cartesian.

Proof. Proposition 8.1 gives the existence of an affine scheme $W = \text{Spec } A$ with a G_x -action, a closed G_x -invariant point $w \in W$ and a morphism

$$f : [\text{Spec } A/G_x] \rightarrow \mathcal{X}$$

which is étale and affine. Furthermore, $f(w) = x$ and f is stabilizer preserving at w . The maximal ideal $\mathfrak{m} \subseteq A$ corresponding to w is G_x -invariant which induces a G_x -representation $\mathfrak{m}/\mathfrak{m}^2$. Since G_x is linearly reductive, there exists a splitting $\mathfrak{m}/\mathfrak{m}^2 \hookrightarrow \mathfrak{m}$ of the surjection $\mathfrak{m} \rightarrow \mathfrak{m}/\mathfrak{m}^2$. The inclusion $\mathfrak{m}/\mathfrak{m}^2 \hookrightarrow \mathfrak{m} \subseteq A$ induces a morphism on algebras $\text{Sym}^* \mathfrak{m}/\mathfrak{m}^2 \rightarrow A$ which is G_x -equivariant which in turns gives a morphism

$$g : [\text{Spec } A/G_x] \rightarrow [\text{Def}(x)/G_x]$$

such that $g(x)$ is the origin, g is étale at x and stabilizer preserving at x . Therefore we have a commutative diagram as in (8.1) where (1) and (2) satisfied. By [Alp10, Theorem 6.10], we may shrink $\text{Spec } A$ by choosing an affine saturated open of $\text{Spec } A$ containing w such that the parallelogram is cartesian and \bar{g} is étale which establishes (3) – (5). \square

Theorem 8.3. *Let $x \in \overline{\mathcal{M}}_{g,n}(A_k)$ be a closed point for $k \leq 4$. There exists a morphism $f : \mathcal{W} = [\text{Spec } A/G_x] \rightarrow \overline{\mathcal{M}}_{g,n}(A_k)$ where G_x acts on an affine scheme $\text{Spec } A$ fixing a*

point w with $f(w) = x$, a morphism $g : \mathcal{W} \rightarrow [\mathrm{Def}(x)/G_x]$ and a commutative diagram

$$(8.2) \quad \begin{array}{ccccc} \mathcal{W}^- & \hookrightarrow & \mathcal{W} = [\mathrm{Spec} A/G_x] & \hookleftarrow & \mathcal{W}^+ \\ \downarrow & & \downarrow f & & \downarrow \\ \overline{\mathcal{M}}_{g,n}(A_k^-) & \hookrightarrow & \overline{\mathcal{M}}_{g,n}(A_k) & \hookleftarrow & \overline{\mathcal{M}}_{g,n}(A_k^+) \end{array}$$

such that

- (1) f is étale, affine and stabilizer preserving at w ,
- (2) the induced map $\widehat{\mathrm{Def}}(x) \rightarrow \overline{\mathcal{M}}_{g,n}(A_k)$ is a miniversal deformation space,
- (3) the squares are cartesian,
- (4) there exist good moduli spaces $\mathcal{W} \rightarrow Y = \mathrm{Spec} A^{G_x}$, $\mathcal{W}^- \rightarrow Y^-$ and $\mathcal{W}^+ \rightarrow Y^+$ and a commutative diagram

$$\begin{array}{ccccc} \mathcal{W}^- & \hookrightarrow & \mathcal{W} = [\mathrm{Spec} A/G_x] & \hookleftarrow & \mathcal{W}^+ \\ \downarrow & & \downarrow & & \downarrow \\ Y^- & \longrightarrow & Y & \longleftarrow & Y^+ \end{array}$$

with $Y^- \rightarrow Y$ and $Y^+ \rightarrow Y$ projective,

- (5) the morphism g is affine, étale, stabilizer preserving and saturated, and
- (6) $g^{-1}([V^+/G_x]) = \mathcal{W}^+$ and $g^{-1}([V^-/G_x]) = \mathcal{W}^-$ where V^- and V^+ are the open GIT chambers of the deformation space $\mathrm{Def}(x)$ given by the character $\chi : G_x^\circ \rightarrow \mathbb{G}_m$ defined in Section 7.4.

Proof. Since the stack $\overline{\mathcal{M}}_{g,n}(A_k)$ is smooth and parameterizes canonically polarized curves, we may apply Corollary 8.2 to find morphisms $f : [\mathrm{Spec} A/G_x] \rightarrow \overline{\mathcal{M}}_{g,n}(A_k)$ and $g : [\mathrm{Spec} A/G_x] \rightarrow [\mathrm{Def}(x)/G_x]$ giving a diagram as in (8.1) such that 8.2(1)–8.2(5) are satisfied. In particular, (1), (2) and (5) in this theorem are satisfied.

Let $V^-, V^+ \hookrightarrow \mathrm{Def}(x)$ and $\mathfrak{Z}^-, \mathfrak{Z}^+ \hookrightarrow \mathrm{Spf} \hat{A}$ be as in Diagram (7.1). By Theorem 7.9, $f^{-1}(\overline{\mathcal{S}}_{g,n}(A_k))$ and $g^{-1}([V^-/G_x])$ (resp., $f^{-1}(\overline{\mathcal{H}}_{g,n}(A_k))$ and $g^{-1}([V^+/G_x])$) are closed substacks that agree in a formal neighborhood of w . Therefore, they agree in a Zariski-open neighborhood. We may restrict to a saturated G_x -invariant open affine neighborhood of w in $\mathrm{Spec} A$ giving a diagram as in (8.1) still satisfying 8.2(1)–8.2(5) and such that

$$\begin{aligned} f^{-1}(\overline{\mathcal{S}}_{g,n}(A_k)) &= g^{-1}([V^-/G_x]), \\ f^{-1}(\overline{\mathcal{H}}_{g,n}(A_k)) &= g^{-1}([V^+/G_x]). \end{aligned}$$

This establishes that there exists the desired diagram (8.2) satisfying properties (1)–(3). Furthermore, by variation of GIT on $\mathrm{Def}(x)$ with respect to χ (see Section 7.1), we

have a commutative diagram

$$\begin{array}{ccccc} [V^-/G_x] & \hookrightarrow & [\text{Def}(x)/G_x] & \longleftarrow & [V^+/G_x] \\ \downarrow & & \downarrow & & \downarrow \\ V^-//G_x & \longrightarrow & \text{Def}(x)//G_x & \longleftarrow & V^+//G_x \end{array}$$

where the vertical arrows are good moduli spaces and the morphisms $\text{Def}^-//G_x \rightarrow \text{Def}//G_x$ and $\text{Def}^+//G_x \rightarrow \text{Def}//G_x$ are projective. Base changing this diagram by $\text{Spec } A^{G_x} \rightarrow \text{Def}//G_x$ gives properties (4) and (6). \square

9. WEAK PROPERNESS OF $\overline{\mathcal{M}}_{g,n}(A_k^-)$, $\overline{\mathcal{M}}_{g,n}(A_k)$, $\overline{\mathcal{M}}_{g,n}(A_k^+)$

In this section, we prove our main theorem.

Theorem 9.1. *For $k = 2, 3, 4$, the stacks $\overline{\mathcal{M}}_{g,n}(A_k^-)$, $\overline{\mathcal{M}}_{g,n}(A_k)$ and $\overline{\mathcal{M}}_{g,n}(A_k^+)$ are weakly proper.*

The proof is by induction. Since

$$\overline{\mathcal{M}}_{g,n}(A_2^-) = \overline{\mathcal{M}}_{g,n}$$

is weakly proper, it suffices to show that

$$\overline{\mathcal{M}}_{g,n}(A_k^-) \text{ weakly proper} \implies \overline{\mathcal{M}}_{g,n}(A_k) \text{ weakly proper}$$

and

$$\overline{\mathcal{M}}_{g,n}(A_k) \text{ weakly proper} \implies \overline{\mathcal{M}}_{g,n}(A_k^+) \text{ weakly proper.}$$

These implications are proved in the following two sections as Theorem 9.5 and Theorem 9.7 respectively.

9.1. $\overline{\mathcal{M}}_{g,n}(A_k^-)$ weakly proper $\implies \overline{\mathcal{M}}_{g,n}(A_k)$ weakly proper.

Lemma 9.2. *Suppose $\mathcal{C} \rightarrow \Delta$, $\{\sigma_i\}_{i=1}^n$ is an isotrivial specialization in $\overline{\mathcal{M}}_{g,n}(A_k)$ satisfying*

- (1) *the generic fiber is a closed point of $\overline{\mathcal{M}}_{g,n}(A_k^-)$,*
- (2) *the special fiber is a closed point of $\overline{\mathcal{M}}_{g,n}(A_k)$.*

Then the core of $C_{\overline{\eta}}$ is isomorphic to the core of C_0 .

Proof. We will prove the lemma in the case $k = 2m$ is even (the case when k odd is essentially identical.) Let $C_{\overline{\eta}} = K \cup E_1 \cup \dots \cup E_r$ be the canonical decomposition of the geometric generic fiber. By Lemma 4.15, the limits of the attaching nodes of the canonical decomposition are again nodes. Let $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ denotes the normalization of \mathcal{C} along this locus of nodes, so we have

$$\tilde{\mathcal{C}} = \mathcal{K} \cup \mathcal{E}_1 \cup \dots \cup \mathcal{E}_r,$$

where \mathcal{K} and \mathcal{E}_i are isotrivial specializations with generic fiber K and E_i respectively. Since K is a maximally-degenerate A_k^- -stable curve with no nodally attached $H_{m,1}$ -tails ($k = 2m$) or $H_{m,2}$ -chains ($k = 2m + 1$), K is a maximally-degenerate A_k -stable curve by Lemma 5.13. It follows that the isotrivial specialization \mathcal{K} is trivial, i.e. $(\mathcal{K})_0 \simeq K$. To complete the proof of the lemma, we only need to show that K_0 is the core of C_0 .

On the one hand, since $\overline{\mathcal{H}}_{m,1} \subset \overline{\mathcal{M}}_{m,1}(A_k)$ is closed by Proposition 4.16, it is clear that the limits $(\mathcal{E}_i)_0$ are all $H_{m,1}$ -curves. Thus, we only need to see that K_0 contains no nodally attached $H_{m,1}$ -curves. By our characterization of closed points of $\overline{\mathcal{M}}_g(A_k)$ (Theorem 5.14), any nodally attached $H_{m,1}$ -curve in K_0 must be monomial and hence contains an A_k -singularity. But since $K_0 \simeq K$ is A_k^- -stable, this is clearly impossible. \square

Lemma 9.3. *Suppose $(C, \{p_i\}_{i=1}^n) \rightsquigarrow (C_0, \{p_i\}_{i=1}^n)$ is an isotrivial specialization in $\overline{\mathcal{M}}_{g,n}(A_k)$ satisfying:*

- (1) $(C, \{p_i\}_{i=1}^n)$ is an A_k^- -stable curve,
- (2) $(C_0, \{p_i\}_{i=1}^n)$ is a closed point in $\overline{\mathcal{M}}_{g,n}(A_k)$.

Then there exists an isotrivial specialization $(C, \{p_i\}_{i=1}^n) \rightsquigarrow (C^-, \{p_i\}_{i=1}^n)$ in $\overline{\mathcal{M}}_{g,n}(A_k^-)$ satisfying:

- (1) $(C^-, \{p_i\}_{i=1}^n)$ is a closed point in $\overline{\mathcal{M}}_{g,n}(A_k^-)$.
- (2) *There exists an isotrivial specialization $(C^-, \{p_i\}_{i=1}^n) \rightsquigarrow (C_0, \{p_i\}_{i=1}^n)$.*

Proof. We will prove the case when $k = 2m$ is even (the case k odd is essentially identical). Let $(\mathcal{C} \rightarrow \Delta, \{\sigma_i\}_{i=1}^n)$ be the family of A_k -stable curves witnessing the isotrivial specialization $(C, \{p_i\}_{i=1}^n) \rightsquigarrow (C_0, \{p_i\}_{i=1}^n)$. Let $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ be the normalization of \mathcal{C} along the locus of attaching nodes of the canonical decomposition of the generic fiber, so that

$$\tilde{\mathcal{C}} = \mathcal{K} \cup \mathcal{E}_1 \cup \dots \cup \mathcal{E}_r$$

where $\mathcal{K} \rightarrow \Delta$ picks out the core of the special fiber, while each of the others, say $\mathcal{E}_1, \dots, \mathcal{E}_r$, picks out the $H_{m,1}$ -tails. Note that, by Theorem 5.14, K_0 is a maximally-degenerate A_k^- -stable curve.

We construct a new isotrivial specialization $(C, \{p_i\}_{i=1}^n) \rightsquigarrow (C^-, \{p_i\}_{i=1}^n)$ simply by gluing the family \mathcal{K} with trivial families $\mathcal{E}'_1, \dots, \mathcal{E}'_r$ whose fibers are all isomorphic to the geometric general fiber of $\mathcal{E}_1, \dots, \mathcal{E}_r$ respectively. The special fiber of this new family is a maximally-degenerate A_k^- -stable curve by Lemma 4.16 and it is obvious from the construction that it specializes to $(C_0, \{p_i\}_{i=1}^n)$. \square

Lemma 9.4. *Let $j : \Delta \rightarrow \overline{\mathcal{M}}_{g,n}(A_k)$ be any map such that $j(\eta) \in \overline{\mathcal{M}}_{g,n}(A_k^-)$ and $j(0)$ is closed. Then there exists a lift $j^- : \Delta \rightarrow \overline{\mathcal{M}}_{g,n}(A_k^-)$ such that*

- $j|_\eta = j^-|_\eta$,

- $j^-(0)$ is closed,
- $j(0) \in \overline{j^-(0)}$.

Proof. To construct the lift j^- , apply Theorem 8.3 with $x = j(0)$. We obtain an affine, étale morphism $f : \mathcal{W} \rightarrow \overline{\mathcal{M}}_{g,n}(A_k)$ inducing the following Cartesian diagram:

$$\begin{array}{ccccc}
 & & \mathcal{W}^- & \hookrightarrow & \mathcal{W} \\
 & & \downarrow f^- & & \downarrow f \\
 \Delta^* & \longrightarrow & \overline{\mathcal{M}}_{g,n}(A_k^-) & \hookrightarrow & \overline{\mathcal{M}}_{g,n}(A_k) \\
 \downarrow & & & \nearrow j & \\
 \Delta & & & &
 \end{array}$$

Since f is étale, we may lift j to a morphism $\Delta \rightarrow \mathcal{W}$. Since the square is Cartesian, we may then lift the map $\Delta^* \rightarrow \overline{\mathcal{M}}_{g,n}(A_k^-)$ to a map $\Delta^* \rightarrow \mathcal{W}^-$. Now we have a commutative diagram:

$$\begin{array}{ccccc}
 \Delta^* & \longrightarrow & \mathcal{W}^- & \hookrightarrow & \mathcal{W} \\
 \downarrow & & \downarrow & \nearrow & \downarrow \\
 \Delta & & \mathcal{W}^- & \longrightarrow & \mathcal{W}
 \end{array}$$

Now, since the morphism $\mathcal{W}^- \rightarrow \mathcal{W}$ is projective, the composition $\Delta \rightarrow \mathcal{W}$ may be lifted to a morphism $\Delta \rightarrow \mathcal{W}^-$. Then, since $\mathcal{W}^- \rightarrow \mathcal{W}^-$ is universally closed, we may lift to a morphism $\Delta \rightarrow \mathcal{W}^-$. Finally, composing with f^- , we obtain a map $j' : \Delta \rightarrow \overline{\mathcal{M}}_{g,n}(A_k^-)$. The point $j'(0)$ necessarily admits an isotrivial specialization to $j(0)$, the only problem is that $j'(0)$ may not be closed. However, by applying lemma 9.3 we see that $j'(0)$ admits an isotrivial specialization to a closed point in $\overline{\mathcal{M}}_{g,n}(A_k^-)$ which still specializes isotrivially to $j(0)$. Thus, using the valuative criterion for algebraic stacks, there exists a map $j^- : \Delta \rightarrow \overline{\mathcal{M}}_{g,n}(A_k^-)$ with the desired properties. \square

Theorem 9.5. *If $\overline{\mathcal{M}}_{g,n}(A_k^-)$ is weakly proper over \mathbb{C} , then $\overline{\mathcal{M}}_{g,n}(A_k)$ is weakly proper over \mathbb{C} .*

Proof. Existence of A_k -stable limits: Let $\mathcal{C}^* \rightarrow \Delta^*$ be a family of A_k -stable curves. If $\mathcal{C}^* \rightarrow \Delta^*$ is a family of A_k^- -stable curves, then since $\overline{\mathcal{M}}_{g,n}(A_k^-)$ is universally closed by hypothesis, there exists a limit (after a base change). Otherwise $\mathcal{C}^* \rightarrow \Delta^*$ is a family of curves in $\overline{\mathcal{S}}_{g,n}(A_k)$, which is universally closed by Proposition 9.6, so again there exists a limit after a base change.

Uniqueness of closed A_k -stable limits: Suppose we have a diagram

$$\begin{array}{ccc} \Delta^* & \longrightarrow & \overline{\mathcal{M}}_{g,n}(A_k) \\ \downarrow & \nearrow^{h_1} & \downarrow \\ \Delta & \longrightarrow & \mathrm{Spec} \mathbb{C} \end{array}$$

with two lifts $h_1, h_2 : \Delta \rightarrow \overline{\mathcal{M}}_{g,n}(A_k)$ such that $h_1(0), h_2(0) \in |\overline{\mathcal{M}}_{g,n}(A_k) \times_{\mathbb{C}} \Delta|$ are closed points. If $h_1(\eta) = h_2(\eta)$ lies in $\overline{\mathcal{S}}_{g,n}(A_k)$, then since $\overline{\mathcal{S}}_{g,n}(A_k)$ is weakly proper by Proposition 9.6, $h_1(0) = h_2(0)$. Otherwise $h_1(\eta) = h_2(\eta)$ lies in $\overline{\mathcal{M}}_{g,n}(A_k^-)$. We claim that there exist lifts $\tilde{h}_1, \tilde{h}_2 : \Delta \rightarrow \overline{\mathcal{M}}_{g,n}(A_k^-)$ satisfying:

- (1) $\tilde{h}_1(0), \tilde{h}_2(0) \in \overline{\mathcal{M}}_{g,n}(A_k^-)$ are closed points;
- (2) $\tilde{h}_1(0) \rightsquigarrow h_1(0)$ and $\tilde{h}_2(0) \rightsquigarrow h_2(0)$.

Let $C_1 \rightarrow \Delta$ and $C_2 \rightarrow \Delta$ be the two families induced by h_1 and h_2 . We must show that $C_1 \simeq C_2$. Using Lemma 9.4, we can lift the induced maps $\Delta \rightarrow \overline{\mathcal{M}}_{g,n}(A_k)$ to $\overline{\mathcal{M}}_{g,n}(A_k^-)$ so we get corresponding families $C_1^- \rightarrow \Delta$ and $C_2^- \rightarrow \Delta$ whose special fibers C_1^- and C_2^- satisfy:

- (1) $[C_1] \in \overline{[C_1^-]}$, $[C_2] \in \overline{[C_2^-]}$;
- (2) $[C_1^-], [C_2^-]$ are closed points of $\overline{\mathcal{M}}_{g,n}(A_k^-)$.

Now by the weak properness of $\overline{\mathcal{M}}_{g,n}(A_k^-)$, we know that $C_1^- \simeq C_2^-$. In particular, their cores are isomorphic. By Lemma 9.2, the isotrivial specialization $C_i^- \rightsquigarrow C_i$ induces an isomorphism of cores, for $i = 1, 2$. Thus, the core of C_1 is isomorphic to the core of C_2 . By Corollary 5.15, it follows that C_1 is isomorphic to C_2 as desired. \square

Finally, it remains to prove the existence and uniqueness of limits in the case when the general fiber of $C^* \rightarrow \Delta^*$ has A_k -singularities, or, equivalently, the weak properness of $\overline{\mathcal{S}}_{g,n}(A_k)$. As in the case of nodal curves, the result is reduced to the case where the general fiber has no ramphoid cusps by considering the pointed normalization along the locus of A_k -singularities. However, there is one additional wrinkle to be considered: since the crimping data of an A_k -singularity can vary in families, we must take care to specify how to take the limits of such degenerations.

Proposition 9.6. $\overline{\mathcal{S}}_{g,n}(A_k)$ is weakly proper for $2 \leq k \leq 4$.

Proof. Existence of limits: Let us assume first that $k = 2m$ is even. Given a family $(C^* \rightarrow \Delta^*, \{\sigma_i^*\}_{i=1}^n)$ in $\overline{\mathcal{S}}_{g,n}(A_k)$, the limits of the attaching nodes of the core decomposition are again nodes, so we may decompose C^* as:

$$\mathcal{K}^* \cup \mathcal{E}_1^* \cup \dots \cup \mathcal{E}_r^*,$$

where \mathcal{K}^* picks out the core of the general fiber, and $\mathcal{E}_1^*, \dots, \mathcal{E}_r^*$ comprise the nodally attached $H_{m,1}$ -tails. Since $\overline{H}_{m,1}$ is weakly proper, we may complete the families \mathcal{E}_i^* to

families \mathcal{E}_i such that the special fiber is a monomial $H_{m,1}$ -tail. Thus, it suffices to complete the family $\mathcal{K}^* \subset \mathcal{K}$.

Possibly after a finite base change, we may assume there exist sections $\{\tau_i\}_{i=1}^m$ picking out the A_k -singularities of the general fiber, and let $\tilde{\mathcal{K}}^* \rightarrow \mathcal{K}$ be the pointed normalization of \mathcal{K} along these sections. Note that $(\tilde{\mathcal{K}}^*, \{\tilde{\tau}_i^*\}_{i=1}^m)$ is now a family of A_k -stable curves with no A_k -singularities in the generic fiber, i.e. the family is in fact A_k^- -stable. Thus, by Theorem 9.5 and induction on the genus, we may complete $(\tilde{\mathcal{K}}^*, \{\tilde{\tau}_i^*\}_{i=1}^m)$ to a family $(\tilde{\mathcal{K}}, \{\tilde{\tau}_i\}_{i=1}^m)$. Now we must “recrimp” the sections $\{\tilde{\tau}_i\}_{i=1}^m$ back to A_k -singularities. We will show that this is always possible, possibly after making some blow-ups in the special fiber which have the effect of introducing a single monomial $H_{m,1}$ -tail at $\tilde{\tau}_i(0)$. Since the special fiber of $(\tilde{\mathcal{K}}, \{\tilde{\tau}_i\}_{i=1}^m)$ was a pointed A_k -stable curve, the curve obtained by attaching monomial $H_{m,1}$ -tails at any subset of the marked points is still A_k -stable, and this completes the proof of existence of limits.

To prove the crimping statement, fix an isomorphism of the $(2m - 1)$ th-order neighborhood of $\tilde{\tau}_i$ in $\tilde{\mathcal{K}}$ with $\text{Spec } R[s]/(s)^{2m}$. Using the induced isomorphism of the $(2m - 1)$ th-order neighborhood of σ^* in $\tilde{\mathcal{K}}^*$ with $\text{Spec } K[s]/(s)^{2m}$, there corresponds to $\tilde{\mathcal{K}}^* \rightarrow C^* \rightarrow \Delta^*$ a map from Δ^* to the Grassmannian $\mathbb{G}(m - 1, 2m - 1)$. Fill this in to a map $\Delta \rightarrow \mathbb{G}(m - 1, 2m - 1)$ and correspondingly obtain a map $\tilde{\mathcal{K}} \rightarrow \mathcal{K}$ over Δ . The limit of the A_k -singularity of \mathcal{K}^* in C_0 may not be an A_k -singularity but we will show that this can be rectified by blowing up $\tilde{\mathcal{K}}$ at $\sigma(0)$ a suitable number of times.

The K -subalgebra of $K[s]/(s)^{2m}$ corresponding to the map $\Delta^* \rightarrow \mathbb{G}(m - 1, 2m - 1)$ can be written in the form

$$K[(s + c_1 s^2 + \dots + c_{m-1} s^{2m-2})^2, s^{2m}, \dots, s^{4m-1}],$$

where $c_1, \dots, c_{m-1} \in K$. (Cf. the proof of Proposition 6.2 and see [vdW10, Example 1.111] for details.) Let t be a uniformizing parameter of R . Let $\tilde{\mathcal{K}}'$ be obtained by blowing $\tilde{\mathcal{K}}$ up b times at $\sigma(0)$. This has the effect of making the coordinate change $s \mapsto t^b s$, so the K -algebra associated to the map $\Delta^* \rightarrow \mathbb{G}(m - 1, 2m - 1)$ corresponding to the map $(\tilde{\mathcal{K}}')^* \rightarrow \mathcal{K}^*$ over Δ^* is

$$K[(s + t^b c_1 s^2 + \dots + t^{(2m-3)b} c_{m-1} s^{2m-2})^2, s^{2m}, \dots, s^{4m-1}].$$

For $1 \leq i \leq m - 1$, write $c_i = u_i t^{b_i}$, where u_i is a unit. Choose $b \geq 0$ minimal such that $(2i - 1)b + b_i \geq 0$ for each i , so that $t^{(2i-1)b} c_i \in R$ for each i . The R -algebra corresponding to the filling-in $\Delta \rightarrow \mathbb{G}(m - 1, 2m - 1)$ is then

$$R[(s + t^b c_1 s^2 + \dots + t^{(2m-3)b} c_{m-1} s^{2m-2})^2, s^{2m}, \dots, s^{4m-1}].$$

Form the corresponding map $\tilde{\mathcal{K}}' \rightarrow \mathcal{K}$ over Δ . The limit in K_0 of the A_k -singularity of C^* is then an A_k -singularity, as required. Blow down the $b - 1$ nodally-attached \mathbb{P}^1 's of K_0 in \mathcal{K} to obtain the desired family.

The entire proof of existence of limits in the case that $k = 2m + 1$ is odd is similar. Here, we just discuss the analogue of the crimping statement. The relevant K -algebra is the K -subalgebra

$$K[s_1 + c_1 s_1^2 + \dots + c_{m-1} s_1^m \oplus s_2, s_1^{m+1} \oplus 0, \dots, s_1^{2m+1} \oplus 0, 0 \oplus s_2^{m+1}, \dots, 0 \oplus s_2^{2m+1}]$$

of $K[s_1]/(s_1)^{m+1} \oplus K[s_2]/(s_2)^{m+1}$. One can arrange that the limit R -subalgebra of $R[s_1]/(s_1)^{m+1} \oplus R[s_2]/(s_2)^{m+1}$ is of the required form by blowing $\tilde{\mathcal{K}}$ up b times at $\sigma_1(0)$, which has the effect of making the coordinate change $s_1 \rightarrow t^b s_1$: choose b minimal such that $ib + b_i \geq 0$ for $1 \leq i \leq m-1$, where $c_i = u_i t^{b_i}$ as before.

Uniqueness of closed limits: Let $(\mathcal{C} \rightarrow \Delta, \{\sigma_i\}_{i=1}^n)$ be any completion of a given family $(\mathcal{C}^* \rightarrow \Delta^*, \{\sigma_i\}_{i=1}^n)$ such that the central fiber is a closed point C_0 . We will show that the isomorphism class of the central fiber is uniquely determined. As usual, we consider the case $k = 2m$ is even (the case where k is odd is essentially identical).

The limits of the attaching nodes of the core decomposition of the generic fiber are again nodes, so we may decompose the family \mathcal{C}^* as:

$$\mathcal{K} \cup \mathcal{E}_1 \cup \dots \cup \mathcal{E}_r,$$

where \mathcal{K} picks out the core of the general fiber, and $\mathcal{E}_1, \dots, \mathcal{E}_r$ comprise the nodally attached $H_{m,1}$ -tails. Now, since $\overline{\mathcal{H}}_{m,1}$ is weakly proper, the limits of the families \mathcal{E}_i^* are certainly unique, i.e. they are the unique monomial $H_{m,1}$ -tail. It remains to show that the limit of the family \mathcal{K}^* is uniquely determined.

If \mathcal{K}^* has no A_k -singularities, then the family is A_k^- -stable and the limit is uniquely determined by Theorem 9.5. Thus, we may assume that \mathcal{K}^* contains at least one A_k -singularity, and we let $\{\tau_i\}_{i=1}^m$ be sections of \mathcal{K} picking out the A_k -singularities of the generic fiber. Note that the limits $\tau_i(0)$ are necessarily A_k -singularities, furthermore, by the characterization of closed points in Theorem 5.14, each of the limits $\tau_i(0)$ sits on a monomial $H_{m,1}$ -tail. Now let $(\tilde{\mathcal{K}}, \tilde{\tau}_i)$ be the pointed normalization of $\{\tau_i\}_{i=1}^m$. Evidently, the general fiber of $(\tilde{\mathcal{K}}, \tilde{\tau}_i)$ is A_k^- -stable and the limit is A_k^- -stable, save for the existence of m semistable \mathbb{P}^1 's containing each of the m points $\tau_i(0)$. Let $\tilde{\mathcal{K}} \rightarrow \mathcal{K}'$ be the blowdown of these m \mathbb{P}^1 's, so that the special fiber \mathcal{K}' is A_k^- -stable. By the weak properness of $\overline{\mathcal{M}}_{g,n}(A_k^-)$, this limit \mathcal{K}' is uniquely determined. Since the limit \mathcal{K} is obtained simply by adjoining m monomial $H_{m,1}$ -tails at the appropriate marked points of \mathcal{K}' , we conclude that \mathcal{K} is uniquely determined as desired. \square

9.2. $\overline{\mathcal{M}}_{g,n}(A_k)$ weakly proper $\implies \overline{\mathcal{M}}_{g,n}(A_k^+)$ weakly proper. The following theorem is a formal consequence of the étale local description of $\overline{\mathcal{M}}_{g,n}(A_k)$ at a closed point given in Theorem 8.3.

Theorem 9.7. *If $\overline{\mathcal{M}}_{g,n}(A_k)$ is weakly proper over \mathbb{C} , then $\overline{\mathcal{M}}_{g,n}(A_k^+)$ is weakly proper over \mathbb{C} .*

Uniqueness of closed A_k^+ -stable limits: Suppose we have a diagram

$$\begin{array}{ccc} \Delta^* & \longrightarrow & \overline{\mathcal{M}}_{g,n}(A_k^+) \\ \downarrow & \nearrow^{h_1} & \downarrow \\ \Delta & \longrightarrow & \mathrm{Spec} \mathbb{C} \end{array}$$

with two lifts $h_1, h_2 : \Delta \rightarrow \overline{\mathcal{M}}_{g,n}(A_k^+)$ such that $h_1(0), h_2(0) \in |\overline{\mathcal{M}}_{g,n}(A_k^+) \times_{\mathbb{C}} \Delta|$ are closed points. Since $\overline{\mathcal{M}}_{g,n}(A_k)$ is weakly separated by hypothesis, after possibly making a base change there is a lift $h_0 : \Delta \rightarrow \overline{\mathcal{M}}_{g,n}(A_k)$ such that $h_0(0) \in |\overline{\mathcal{M}}_{g,n}(A_k) \times_{\mathbb{C}} \Delta|$ is closed and there are specializations $h_1(0) \rightsquigarrow h_0(0)$ and $h_2(0) \rightsquigarrow h_0(0)$ in $|\overline{\mathcal{M}}_{g,n}(A_k) \times_{\mathbb{C}} \Delta|$. By Theorem 8.3, there exists a morphism $f : \mathcal{W} \rightarrow \overline{\mathcal{M}}_{g,n}(A_k)$ with $h_0(0) = f(w_0)$ for $w_0 \in |\mathcal{W}|$ which induces a cartesian diagram as in Diagram 9.2. We have a commutative diagram

$$\begin{array}{ccccc} & & \mathcal{W}^+ & \longrightarrow & \mathcal{W} \\ & & \downarrow & & \downarrow f \\ \Delta^* & \longrightarrow & \overline{\mathcal{M}}_{g,n}(A_k^+) & \xrightarrow{\subset} & \overline{\mathcal{M}}_{g,n}(A_k) \\ \downarrow & \nearrow^{h_1} & \nearrow^{h_2} & \searrow^{h_0} & \uparrow \\ \Delta & \longrightarrow & & & \end{array}$$

Since f is étale and $f^{-1}(\overline{\mathcal{M}}_{g,n}(A_k^+)) = \mathcal{W}^+$, there exist unique points $w_1, w_2 \in |\mathcal{W}^+|$ and specializations $w_i \rightsquigarrow w_0$ over $h_i(0) \rightsquigarrow h_0(0)$. Let $\xi = h_0(\eta) \in \overline{\mathcal{M}}_{g,n}(A_k)$ be the image of the generic point. There exist $\chi \in |\mathcal{W}|$ over ξ and specializations $\chi \rightsquigarrow w_i$ for $i = 0, 1, 2$. The specializations $\chi \rightsquigarrow w_i$ for $i = 1, 2$ can be realized, after a base change, by morphisms $\tilde{h}_i : \Delta \rightarrow \mathcal{W}^+$ which lift $h_i : \Delta \rightarrow \overline{\mathcal{M}}_{g,n}(A_k^+)$. Note that w_1 and w_2 are necessarily closed in $|\mathcal{W}^+ \times_{\mathbb{C}} \Delta|$ as $h_1(0)$ and $h_2(0)$ are closed in $|\overline{\mathcal{M}}_{g,n}(A_k^+) \times_{\mathbb{C}} \Delta|$. Since \mathcal{W}^+ is weakly separated, it follows that $w_1 = w_2$ so $h_1(0) = h_2(0)$ as required. \square

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