# SINGULARITIES WITH $\mathbb{G}_m$-ACTION AND THE LOG MINIMAL MODEL PROGRAM FOR $\overline{M}_g$

JAROD ALPER, MAKSYM FEDORCHUK, AND DAVID SMYTH

ABSTRACT. We give a precise formulation of the modularity principle for the log canonical models $\overline{M}_g(\alpha) := \operatorname{Proj} \bigoplus_{d \geq 0} \Gamma(\overline{\mathscr{M}}_g, d(K_{\overline{\mathscr{M}}_g} + \alpha\delta))$. Assuming the modularity principle holds, we develop and compare two methods for determining the critical $\alpha$-values at which a singularity or complete curve with $\mathbb{G}_m$-action arises in the modular interpretation of $\overline{M}_g(\alpha)$. The first method involves a new invariant of curve singularities with $\mathbb{G}_m$-action, constructed via the characters of the induced $\mathbb{G}_m$-action on spaces of pluricanonical forms. The second method involves intersection theory on the variety of stable limits of a singular curve. We compute the expected $\alpha$-values for large classes of singular curves, including curves with ADE, toric and monomial unibranch Gorenstein singularities as well as for ribbons, and show that the two methods yield identical predictions. We use these results to give a conjectural outline of the log MMP for $\overline{M}_g$.

## 1. INTRODUCTION

In an effort to understand the canonical model of $\overline{M}_g$, Hassett and Keel initiated a program to give modular interpretations to the log canonical models

$$\overline{M}_g(\alpha) := \operatorname{Proj} \bigoplus_{d \geq 0} \mathrm{H}^0(\overline{\mathscr{M}}_g, \lfloor d(K_{\overline{\mathscr{M}}_g} + \alpha\delta) \rfloor),$$

for all $\alpha \in [0,1] \cap \mathbb{Q}$ such that $K_{\overline{\mathscr{M}}_g} + \alpha\delta$ is effective. The assertion that these log canonical models should admit modular interpretations, which is implicit in the work of Hassett, Hyeon, and Keel, can be formulated precisely as follows:

**Modularity principle for the log MMP for $\overline{M}_g$.** *For every $\alpha \in [0,1] \cap \mathbb{Q}$ such that $K_{\overline{\mathscr{M}}_g} + \alpha\delta$ is effective, there exists an open substack $\overline{\mathscr{M}}_g(\alpha) \subseteq \mathscr{U}_g$ of the stack of all complete connected Gorenstein curves of arithmetic genus g and a good moduli space $\phi \colon \overline{\mathscr{M}}_g(\alpha) \to \overline{M}_g(\alpha)$ (i.e., $\phi$ is cohomologically affine and $\phi_* \mathscr{O}_{\overline{\mathscr{M}}_g(\alpha)} = \mathscr{O}_{\overline{M}_g(\alpha)}$).*

We refer to [Alp08] for a discussion of the essential properties of good moduli spaces. They may be thought of as best-possible approximations to a coarse moduli space, in cases where the existence of a coarse moduli space is precluded by non-separatedness of the moduli stack. Hassett, Hyeon, and Lee have verified the modularity principle for the log MMP for $\overline{M}_g$ for $g = 2,3$ and for $\alpha > \frac{7}{10} - \varepsilon$ in arbitrary genus [HH09, HH08, HL10]. In exploring possible extensions of their work, it is natural to consider the following question: Assuming the modularity principle holds, what curves must appear in the stacks $\overline{\mathscr{M}}_g(\alpha)$? In this paper,

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we develop two different methods for answering this question, at least for curves with a $\mathbb{G}_m$-action.

Consider a complete curve $C$ with a $\mathbb{G}_m$-action $\eta \colon \mathbb{G}_m \to \operatorname{Aut}(C)$ and an isolated singularity at a fixed point $p \in C$. If $\mathscr{L}$ be a line bundle on $\overline{\mathscr{M}}_g$ which extends to a neighborhood of $[C]$ in the stack of all curves, there is an induced action of $\mathbb{G}_m$ on the fiber of $\mathscr{L}$ over $[C]$ and a corresponding character $\chi_{\mathscr{L}}(C,\eta)$. Moreover, if $T$ is a complete family of stable limits of $C$, one can consider the intersection number $T \cdot \mathscr{L}$. We prove in fact that there is a precise relationship between $\chi_{\mathscr{L}}(C,\eta)$ and $T \cdot \mathscr{L}$; see Theorem 4.2.

The key observation connecting characters with the modularity principle is that if a curve $C$ is to appear in $\overline{\mathscr{M}}_g(\alpha)$ for some $\alpha$, then the character of $K_{\overline{\mathscr{M}}_g(\alpha)} + \alpha\delta = 13\lambda + (\alpha-2)\delta$ is necessarily trivial as the line bundle descends to $\overline{M}_g(\alpha)$; see Corollary 2.2. In this paper, we compute the characters of line bundles which generate $\operatorname{Pic}_{\mathbb{Q}}(\overline{\mathscr{M}}_g)$ for a large class of singular curves with $\mathbb{G}_m$-action. In particular, we calculate the characters for curves with ADE singularities, toric singularities, and monomial unibranch Gorenstein singularities as well as for ribbons; see Tables 1 and 2. As a consequence we get predictions for which singular curves arise in the modular interpretations $\overline{\mathscr{M}}_g(\alpha)$ (see Table 3).

On the other hand, if a locus $\mathscr{T} \subseteq \overline{\mathscr{M}}_g$ is covered by $(K_{\overline{\mathscr{M}}_g} + \alpha\delta)$-negative curves, then $\mathscr{T}$ is contracted under $\overline{\mathscr{M}}_g \dashrightarrow \overline{M}_g(\alpha)$. When $\mathscr{T} = \mathscr{T}_C$ is the variety of stable limits of a singular curve $C$, we compute the value $\alpha$ at which $\mathscr{T}_C$ is covered by $(K_{\overline{\mathscr{M}}_g} + \alpha\delta)$-negative curves in Section 6. This provides a more refined description of the log minimal model program as it predicts the critical values at which certain curves are removed from $\overline{\mathscr{M}}_g(\alpha)$. The critical values for $\alpha$ obtained by character theory and intersection theory are the same which is unsurprising considering Theorem 4.2.

**Notation.** We work over an algebraically closed field $k$ of characteristic 0. Let $\mathscr{U}_g$ be the stack of all complete connected Gorenstein curves $C$ with $\omega_C$ ample. Let $\pi \colon \mathscr{C}_g \to \mathscr{U}_g$ be the universal curve. Denote by $\omega_{\mathscr{C}_g/\mathscr{U}_g}$ the relative dualizing sheaf on $\mathscr{C}_g$. The following line bundles are defined on all of $\mathscr{U}_g$:

$$\begin{aligned} \lambda = \lambda_1 &= \det \pi_*(\omega_{\mathscr{C}_g/\mathscr{U}_g}) \\ \lambda_n &= \det \pi_*(\omega_{\mathscr{C}_g/\mathscr{U}_g}^{\otimes n}) \end{aligned}$$

The following divisor classes are defined on any open substack of $\mathscr{U}_g$ that satisfies Serre's condition $S_2$ and whose locus of non-nodal curves is of codimension at least 2:

$$\begin{aligned} \kappa &= \pi_*(c_1^2(\omega_{\mathscr{C}_g/\mathscr{U}_g})) \\ K &= \text{canonical divisor} \\ \delta_0 = \delta_{\mathrm{irr}} &= \text{divisor of irreducible singular curves} \\ \delta = \delta_0 + \delta_{\mathrm{red}} &= \delta_0 + \delta_1 + \cdots + \delta_{\lfloor g/2 \rfloor} \end{aligned}$$

where we have the relations

$$\begin{aligned} K &= 13\lambda - 2\delta = -13\lambda + 2\lambda_2 \\ \lambda_2 &= 13\lambda - \delta = K + \delta \\ \kappa &= 12\lambda - \delta = -\lambda + \lambda_2 = \frac{12}{13}(K + \frac{11}{12}\delta) \end{aligned} \tag{1.1}$$

We define the *slope* of a divisor $s\lambda - \delta$ to be $s$ and the *$\alpha$-value* of a divisor $K + \alpha\delta$ to be $\alpha$. In particular, the slope of $K + \alpha\delta$ is $\dfrac{13}{2-\alpha}$ and the $\alpha$-value of $s\lambda - \delta$ is $2 - 13/s$.

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## 2. CHARACTER THEORY

To begin, recall that if $\mathscr{L}$ is a line bundle on $\overline{\mathscr{M}}_g(\alpha)$ and $[C] \in \overline{\mathscr{M}}_g(\alpha)$ is any point, the natural action of $\mathrm{Aut}(C)$ on the fiber $\mathscr{L}|_{[C]}$ induces a character $\mathrm{Aut}(C) \to \mathbb{G}_m$. If $\eta\colon \mathbb{G}_m \to \mathrm{Aut}(C)$ is any one-parameter subgroup, then there is an induced character $\mathbb{G}_m \to \mathbb{G}_m$ which is necessarily of the form $z \to z^n$ for some integer $n \in \mathbb{Z}$. Given a curve $[C]$, a 1-parameter subgroup $\eta\colon \mathbb{G}_m \to \mathrm{Aut}(C)$, and a line bundle $\mathscr{L} \in \mathrm{Pic}(\overline{\mathscr{M}}_g(\alpha))$, we denote this integer by $\chi_{\mathscr{L}}(C,\eta)$. If $\mathscr{L} = \lambda_m$, we write simply $\chi_m(C,\eta)$. Furthermore, if $\mathrm{Aut}(C) \simeq \mathbb{G}_m$, then we write $\chi_{\mathscr{L}}(C)$ or $\chi_m(C)$, where the 1-parameter subgroup $\eta\colon \mathbb{G}_m \to \mathrm{Aut}(C)$ is understood to be the identity.[1]

**Lemma 2.1.** *Suppose the modularity principle for the log MMP for $\overline{\mathscr{M}}_g(\alpha)$ holds such that $\overline{\mathscr{M}}_g(\alpha)$ is $S_2$ and the locus of non-nodal curves in $\overline{\mathscr{M}}_g(\alpha)$ has codimension at least* 2. *Then some multiple of the $\mathbb{Q}$-line bundle $c(\alpha)\lambda_1 + \lambda_2$ descends to a $\mathbb{Q}$-line bundle on $\overline{M}_g(\alpha)$ where $c(\alpha) = \frac{13\alpha-13}{2-\alpha}$.*

*Proof.* Since $\lambda_2 = 13\lambda_1 - \delta$ on $\overline{\mathscr{M}}_g$, we have $K_{\overline{\mathscr{M}}_g} + \alpha\delta \sim c(\alpha)\lambda_1 + \lambda_2$. Now consider the commutative diagram

$$\begin{tikzcd} \overline{\mathscr{M}}_g \arrow[r, dashed] \arrow[d] & \overline{\mathscr{M}}_g(\alpha) \arrow[d, "\phi"] \\ \overline{M}_g \arrow[r, dashed, "f"] & \overline{M}_g(\alpha) \end{tikzcd}$$

By the definition of $\overline{M}_g(\alpha)$, the $\mathbb{Q}$-divisor class $K_{\overline{\mathscr{M}}_g} + \alpha\delta$ pushes forward to a Cartier divisor class on $\overline{M}_g(\alpha)$. We claim that $\phi^* f_*(K_{\overline{\mathscr{M}}_g} + \alpha\delta) = c(\alpha)\lambda_1 + \lambda_2$ on $\overline{\mathscr{M}}_g(\alpha)$. Evidently, this equality holds on $\overline{\mathscr{M}}_g(\alpha) \cap \overline{\mathscr{M}}_g$. Since the complement of $\overline{\mathscr{M}}_g(\alpha) \cap \overline{\mathscr{M}}_g$ has codimension 2 in $\overline{\mathscr{M}}_g(\alpha)$ and $\overline{\mathscr{M}}_g(\alpha)$ is $S_2$, equality holds over all of $\overline{\mathscr{M}}_g(\alpha)$.

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[1] Note that, in general, the integers $\chi_m(C)$ are only defined up to sign since the choice of isomorphism $\mathrm{Aut}(C) \simeq \mathbb{G}_m$ depends on a sign. The ratios $\chi_l(C)/\chi_m(C)$ however are well-defined, and this is all we will need.

As a consequence, we obtain:

**Corollary 2.2.** *Suppose the modularity principle for the log MMP holds for* $\overline{\mathcal{M}}_g(\alpha)$ *such that* $\overline{\mathcal{M}}_g(\alpha)$ *is* $S_2$ *and the locus of non-nodal curves in* $\overline{\mathcal{M}}_g(\alpha)$ *has codimension at least* 2*. Let* $C$ *be a curve in* $\overline{\mathcal{M}}_g(\alpha)$*. Let* $\eta\colon \mathbb{G}_m \to \operatorname{Aut}(C)$ *be any* 1*-parameter subgroup. Then either* $\chi_m(C,\eta)=0$ *for all* $m$ *or*

$$\alpha = \frac{13-2\left(\frac{\chi_2(C,\eta)}{\chi_1(C,\eta)}\right)}{13-\left(\frac{\chi_2(C,\eta)}{\chi_1(C,\eta)}\right)}\,.$$

*In other words, either* $\operatorname{Aut}(C)$ *acts trivially on all the vector spaces* $\bigwedge H^0(C,\omega_C^m)$*, or else* $\alpha$ *is uniquely determined by the characters* $\chi_1(C,\eta)$ *and* $\chi_2(C,\eta)$*.*

*Proof.* By Lemma 2.1, the line-bundle $c(\alpha)\lambda_1+\lambda_2$ must descend from $\overline{\mathcal{M}}_g(\alpha)$ to $\overline{M}_g(\alpha)$. In particular, the 1-parameter subgroup $\eta\colon \mathbb{G}_m \to \operatorname{Aut}(C)$ must act trivially on the fiber $(c(\alpha)\lambda_1+\lambda_2)|_{[C]}$. Hence, the character for this action, given by $c(\alpha)\chi_1(C,\eta)+\chi_2(C,\eta)$, must be 0. We conclude that either $\chi_1(C,\eta)=\chi_2(C,\eta)=0$, or

$$c(\alpha) = -\chi_2(C,\eta)/\chi_1(C,\eta),$$

as desired. ☐

Evidently, Corollary 2.2 says nothing whatsoever concerning curves with no automorphisms. At critical values of $\alpha$ however, the stacks $\overline{\mathcal{M}}_g(\alpha)$ typically contain curves $[C]\in\overline{\mathcal{M}}_g(\alpha)$ with non-trivial characters $\chi_m(C)$. We define the $\alpha$*-value* of a complete curve $C$ with a $\mathbb{G}_m$-action $\eta\colon \mathbb{G}_m \to \operatorname{Aut}(C)$ as

$$\alpha(C,\eta) := \frac{13-2\left(\frac{\chi_2(C,\eta)}{\chi_1(C,\eta)}\right)}{13-\left(\frac{\chi_2(C,\eta)}{\chi_1(C,\eta)}\right)}\,.$$

We note that $\alpha(C,\eta)=2-13\chi_\lambda(C,\eta)/\chi_\delta(C,\eta)$ as long as the deformation space of $C$ is $S_2$ and the locus of non-nodal curves has codimension at least 2.

The Corollary 2.2 implies that the $\alpha$-value of any complete curve is the only $\alpha$ at which it can show up. We also note that when $\overline{M}_g(\alpha)$ is constructed as a GIT quotient, the curve $C$ can lie in the semistable locus only when the character of $K_{\overline{M}_g(\alpha)}+\alpha\delta$ is 0, as this character computes the Hilbert-Mumford index of $[C]$ with respect to $\eta$. We discuss the connection of the character theory with GIT in further detail in Section 7.

**The $\alpha$-value of a singularity.** Next, we explain how to define and extract critical $\alpha$-values for an arbitrary curve singularity with $\mathbb{G}_m$-action. Given a curve singularity $\widehat{\mathcal{O}}_{C,p}$ with $n$ branches and $\delta$-invariant $\delta(p)$, we may consider a curve of the form

$$C = E_1\cup\cdots\cup E_n\cup C_0,$$

where $C_0$ is any smooth curve of genus $g-\delta(p)-n+1$ and $E_1,\dots,E_n$ are rational curves attached to $C_0$ nodally and meeting in a singularity of type $\widehat{\mathcal{O}}_{C,p}$ (see Figure 1).

If $\mathbb{G}_m$ acts algebraically on $\widehat{\mathcal{O}}_{C,p}$ via $\eta$, then this action extends canonically to $C$, which induces a one-parameter subgroup $\widetilde{\eta}\colon \mathbb{G}_m \to \operatorname{Aut}(C)$. The characters $\chi_1(C,\widetilde{\eta})$ and $\chi_2(C,\widetilde{\eta})$

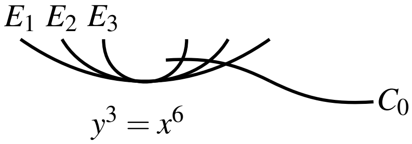

FIGURE 1. Three rational curves $E_1, E_2$ and $E_3$ meet at a monomial $y^3 = x^6$ singularity.

depend only on the singularity $\widehat{\mathscr{O}}_{C,p}$ and the $\mathbb{G}_m$-action. We define the *$\alpha$-value of $\widehat{\mathscr{O}}_{C,p}$ with respect to $\eta$*, denoted by $\alpha(\widehat{\mathscr{O}}_{C,p}, \eta)$, as the corresponding $\alpha$-value, $\alpha(C, \widetilde{\eta})$, of the complete curve $C$.

| Singularity type | $\lambda$ | $\lambda_2$ | $\delta$ | $\alpha$-value | slope |
|---|---|---|---|---|---|
| $A_{2k} : y^2 - x^{2k+1}$ | $k^2$ | $5k^2 - 4k + 1$ | $8k^2 + 4k - 1$ | $\frac{3k^2+8k-2}{8k^2+4k-1}$ | $\frac{8k^2+4k-1}{k^2}$ |
| $A_{2k+1} : y^2 - x^{2k+2}$ | $\frac{k^2+k}{2}$ | $\frac{5k^2+k}{2}$ | $4k^2 + 6k$ | $\frac{3k+11}{8k+12}$ | $\frac{8k+12}{k+1}$ |
| $D_{2k+1} : x(y^2 - x^{2k-1})$ | $k^2$ | $5k^2 - 2k$ | $8k^2 + 2k$ | $\frac{3k+4}{8k+2}$ | $\frac{8k+2}{k}$ |
| $D_{2k+2} : x(y^2 - x^{2k})$ | $\frac{k^2+k}{2}$ | $\frac{5k^2+3k}{2}$ | $4k^2 + 5k$ | $\frac{3k+7}{8k+10}$ | $\frac{8k+10}{k+1}$ |
| $E_6 : y^3 - x^4$ | 8 | 33 | 71 | $38/71$ | $71/8$ |
| $E_7 : y(y^2 - x^3)$ | 7 | 31 | 60 | $29/60$ | $60/7$ |
| $E_8 : y^3 - x^5$ | 14 | 63 | 119 | $8/17$ | $17/2$ |
| $y^3 - x^6$ | 7 | 34 | 57 | $23/57$ | $57/7$ |
| $y^3 - x^7$ | 31 | 152 | 251 | $99/251$ | $251/31$ |
| $y^3 - x^8$ | 42 | 211 | 335 | $124/335$ | $335/42$ |
| $T_{p,q} : y^p - x^q$ | See Proposition 6.6 for character values | | | | |
| monomial unibranch with gaps $\{b_1, \cdots, b_k\}$ | $\sum b_i$ | $(2k-1)^2 - \sum b_i$ | $14\sum_i b_i - (2k-1)^2$ | $\frac{15\sum_i b_i - 2(2k-1)^2}{14\sum_i b_i - (2k-1)^2}$ | $14 - \frac{(2k-1)^2}{\sum_i b_i}$ |
| Ribbon $C_\ell$ | $g\left(\ell - \frac{g-1}{2}\right)$ | $(5g-4)(\ell - \frac{g-1}{2})$ | $(8g+4)(\ell - \frac{g-1}{2})$ | $\frac{3g+8}{8g+4}$ | $8 + \frac{4}{g}$ |

TABLE 1. Character values of Gorenstein singularities

We compute the character values of all ADE singularities, toric, and monomial unibranch Gorenstein singularities, as well as for ribbons, in Section 5. The results are displayed in Table 1. We expect that the $\alpha$-values displayed in the table are the exactly the $\alpha$-values at which the given singularity type first appears on any curve in $\overline{\mathscr{M}}_g(\alpha)$, although there is no guarantee that curves with such singularities do appear. In addition, we note that these heuristic predictions for when $A_k$ singularities arise agree with the computations of Hyeon, who uses different heuristics [Hye10].

**Dangling singularities.** We explain a variant of the above ideas applied to curves with "dangling" singularities; such singularities have not yet appeared in the work of Hassett and Hyeon but we expect it to play an important roll in future stages of the program. We will define a collection of modified $\alpha$-values associated to a multi-branch singularity. If $\widehat{\mathscr{O}}_{C,p}$ is any curve singularity with $n \geq 2$ branches , we may enumerate the branches, and for any non-empty subset $S \subset \{1,\dots,n\}$, we may consider the curve

$$C^S = E_1 \cup \ldots \cup E_n \cup C_0,$$

where each $E_i \simeq \mathbb{P}^1$, each $E_i$ with $i \in S$ meets $C_0$ nodally at $\infty$, and all the $E_i$ are glued along singularity of type $\widehat{\mathscr{O}}_{C,p}$ at zero (see Figure 2).

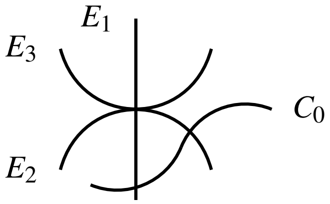

FIGURE 2. Dangling $D^{\{1,2\}}$-singularity.

As before, if $\mathbb{G}_m$ acts algebraically on $\widehat{\mathscr{O}}_{C,p}$ via $\eta$, there is an induced one-parameter subgroup $\widetilde{\eta}\colon \mathbb{G}_m \to \mathrm{Aut}(C)$. We define the *$\alpha$-value of $\widehat{\mathscr{O}}_{C,p}$ with respect to $S$ and $\eta$*, denoted by $\alpha^S(\widehat{\mathscr{O}}_{C,p},\eta)$, as the corresponding $\alpha$-value, $\alpha(C^S,\widetilde{\eta})$, of the complete curve $C^S$. In this notation, $\alpha^{[n]}(\widehat{\mathscr{O}}_{C,p})$ is the standard $\alpha$-invariant defined above. In general, the invariants $\alpha^S(\widehat{\mathscr{O}}_{C,p})$ will depend on the subset $S$, which reflects the fact that curves $C^S$ may appear in the moduli stack $\overline{\mathscr{M}}_g(\alpha)$ at different values of $\alpha$.

The first example of this phenomenon should occur with the oscnode $(y^2 = x^6)$. As seen in Table 2, the $\alpha$-invariant of the oscnode is $17/28$ reflecting the fact that we expect oscnodes to replace genus-two bridges attached by conjugate points. By contrast, the $\alpha$-invariant of the dangling oscnode $A_5^{\{1\}}$ is $19/29$. The key point is that this is the precisely the threshold $\alpha$-value at which $\Delta_2$ is covered by $K+\alpha\delta$ negative curves, and indeed one can replace arbitrary genus 2 tails by a dangling oscnode, using the blow-up/blow-down procedure pictured in Figure 3.

While it would be too laborious to compute the associated $\alpha^S$-values for all possible quasitoric singularities, we will do a sample in order to given an indication. In Table 2, we list all the modified $\alpha$-invariants for ADE singularities. Note that since the branches of any $A_k$ or toric singularity are isomorphic the only relevant feature of the subset $S \subset \{1,\dots,n\}$ is the size. For $D_{2k+1}$ singularities, we use the labeling "1" for the smooth branch and "2" for the singular branch in the even case, and for $D_{2k+2}$-singularities, we use "1" for the smooth branch with unique tangent direction and "2,3" for the tangent branches in the odd case. Similarly for the $E_7$ singularity, we use the labeling "1" for the smooth branch and "2" the singular branch.

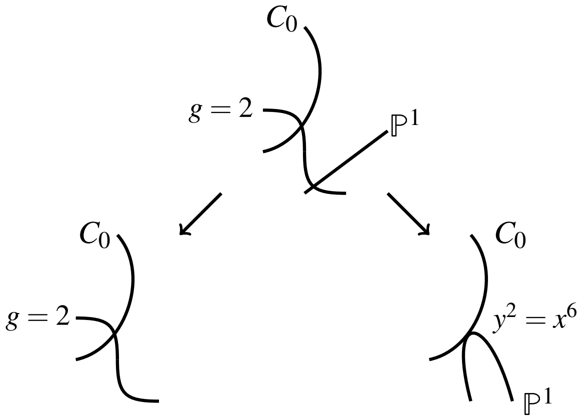

FIGURE 3. Given a smoothing of a curve with a genus 2 tail attached at an arbitrary point $p$, after blowing up the conjugate point of $p$ and contracting the genus 2 curve, one obtains a dangling $\mathbb{P}^1$ attached at an oscnode.

| Dangling type | $\lambda$ | $\lambda_2$ | $\delta$ | $\alpha$-value | slope |
|---|---|---|---|---|---|
| $A_{2k}^{\{\}}: y^2-x^{2k+1}$ | $k^2$ | $5k^2-4k$ | $8k^2+4k$ | $\frac{3k^2+8k}{8k^2+4k}$ | $\frac{8k+4}{k}$ |
| $A_{2k+1}^{\{\}}: y^2-x^{2k+2}$ | $\frac{k^2+k}{2}$ | $\frac{5k^2+3k^2-4}{2}$ | $4k^2+6k+2$ | $\frac{3k^2+11k+8}{8k^2+12k+4}$ | $\frac{8k^2+12k+4}{k^2+k}$ |
| $A_{2k+1}^{\{1\}}: y^2-x^{2k+2}$ | $\frac{k^2+k}{2}$ | $\frac{5k^2+3k^2-2}{2}$ | $4k^2+6k+1$ | $\frac{3k^2+11k+4}{8k^2+12k+2}$ | $\frac{8k^2+12k+2}{k^2+k}$ |
| $D_{2k+1}^{\{1\}}: x(y^2-x^{2k-1})$ | $k^2$ | $5k^2-2k-1$ | $8k^2+2k+1$ | $\frac{3k^2+4k+2}{8k^2+2k+1}$ | $\frac{8k^2+2k+1}{k^2}$ |
| $D_{2k+2}^{\{1\}}: x(y^2-x^{2k})$ | $\frac{k^2+k}{2}$ | $5k^2+3k-4$ | $4k^2+5k+2$ | $\frac{3k^2+7k+8}{8k^2+10k+4}$ | $\frac{8k^2+10k+4}{k^2+k}$ |
| $D_{2k+2}^{\{1,2\}}: x(y^2-x^{2k})$ | $\frac{k^2+k}{2}$ | $5k^2+3k-2$ | $4k^2+5k+1$ | $\frac{3k^2+7k+4}{8k^2+10k+2}$ | $\frac{8k^2+10k+2}{k^2+k}$ |
| $E_6^{\{\}}: y^3-x^4$ | 8 | 32 | 72 | $5/9$ | 9 |
| $E_7^{\{1\}}: y(y^2-x^3)$ | 7 | 30 | 61 | $31/61$ | $61/7$ |
| $E_8^{\{\}}: y^3-x^5$ | 14 | 62 | 120 | $29/60$ | $60/7$ |

TABLE 2. Character values for dangling $ADE$ singularities with ample $\omega_C$

## 3. PREDICTIONS FOR THE LOG MMP

Using heuristics provided by both intersection calculations and character computations, we offer predictions in Table 3 for moduli interpretations for $\overline{M}_g(\alpha)$ for $\alpha \geq 5/9$.

Giving a complete description of $\overline{\mathscr{M}}_g(\alpha)$ is much more subtle than generally describing the singularities added and the loci removed. For instance, after removing elliptic tails (connected genus 1 subcurves which are attached nodally) at $\alpha = 9/11-\varepsilon$, in each subsequent moduli stack parameterizing curves with additional singularities, one needs to redefine what is meant by an elliptic tail by specifying the allowed attaching singularities.

| $\alpha$-value | Singularity added at $\alpha$ | Locus removed at $\alpha-\varepsilon$ |
|---|---|---|
| $9/11$ | $A_2$ | elliptic tails attached nodally |
| $7/10$ | $A_3$ | elliptic bridges/chains attached nodally |
| $2/3$ | $A_4$ | $g=2$ tails attached nodally at a Weierstrass point |
| $19/29$ | $A_5^{\{1\}}$ | $g=2$ tails attached nodally/tacnodally |
| $17/28$ | $A_5$ | $g=2$ bridges/chains attached nodally at conjugate points |
| $49/83$ | $A_6$ | hyperelliptic $g=3$ tails attached nodally at a Weierstrass point |
| $32/55$ | $A_7^{\{1\}}$ | hyperelliptic $g=3$ tails attached nodally/tacnodally/oscnodally |
| $5/9$ | $A_7$ | hyperelliptic $g=3$ bridges/chains attached nodally at conjugate points |
| | $D_4$ | elliptic triboroughs attached nodally |
| | $D_5$ | $g=2$ bridges/chains attached nodally at a Weierstrass and free point |
| | $D_6^{\{1,2\}}$ | $g=2$ bridges/chains attached nodally at two free points |

TABLE 3. Predictions for the log MMP for $\alpha \geq 5/9$

## 4. CHARACTER THEORY VS INTERSECTION THEORY

Let $C$ be a complete curve of arithmetic genus $g$ with a $\mathbb{G}_m$-action $\eta\colon\ \mathbb{G}_m \to \operatorname{Aut}(C)$. A *versal deformation space with* $\mathbb{G}_m$*-action* is an affine scheme $\operatorname{Def}(C)$ with a $\mathbb{G}_m$-action together with a smooth morphism $[\operatorname{Def}(C)/\mathbb{G}_m] \to \mathcal{V}_g$ to the *stack of all complete curves of arithmetic genus* $g$ and a fixed point $0 \in \operatorname{Def}(C)$ over $[C]$. In this case, we define:

$$\operatorname{Def}^-(C) := \{x \in \operatorname{Def}(C) : \lim_{t\to\infty} t\cdot x = 0\},$$

$$\operatorname{Def}^0(C) := \{x \in \operatorname{Def}(C) : \forall t,\ t\cdot x = x\},$$

$$\operatorname{Def}^+(C) := \{x \in \operatorname{Def}(C) : \lim_{t\to 0} t\cdot x = 0\}.$$

*Remark* 4.1. If $\omega_C$ is ample, the versal deformation space of $C$ is normal and $\operatorname{Aut}(C)$ is linearly reductive, then it follows from [Alp10, Theorem 3] that there exists a miniversal deformation space with $\mathbb{G}_m$-action.

If $\mathscr{L}$ is a line bundle defined on $\mathcal{V}_g$ in a neighborhood of $C$ and $\operatorname{Def}(C)$ is a versal deformation space with $\mathbb{G}_m$-action, then after shrinking $\operatorname{Def}(C)$, we may assume that $\mathscr{L}$ is defined on $[\operatorname{Def}(C)/\mathbb{G}_m]$.

The main relationship between intersection numbers and characters is:

**Theorem 4.2.** *Let $C$ be a complete curve of arithmetic genus $g$ and* $0 \in \operatorname{Def}(C)$ *be a versal deformation space with* $\mathbb{G}_m$*-action. Let* $B \to \operatorname{Def}^-(C)$ *be a complete family of stable curves and* $\mathscr{L}$ *be a line bundle on* $[\operatorname{Def}(C)/\mathbb{G}_m]$. *Then*

$$\chi_{\mathscr{L}}(C,\eta) = \frac{B\cdot\mathscr{L}}{\deg(B)}$$

*where* $\deg(B)$ *is the degree with respect to the natural* $\mathscr{O}(1)$ *on* $[\operatorname{Def}^-(C)/\mathbb{G}_m]$*. In particular, if* $C$ *is Gorenstein (resp. the discriminant locus in* $\operatorname{Def}(C)$ *is Cartier), then*

$$\chi_{\lambda_i}(C,\eta)=\frac{B\cdot\lambda_i}{\deg(B)}\qquad\left(\textit{resp. }\chi_{\delta_i}(C,\eta)=\frac{B\cdot\delta_i}{\deg(B)}\right).$$

*Proof.* We can write $\operatorname{Def}^-(C)=\operatorname{Spec}A$ with $A$ a positively graded $k$-algebra. The line bundle $\mathscr{L}$ corresponds to a graded projective $A$-module which is free of rank 1 by [Eis95, Theorem 19.2]. It follows that $\mathscr{L}=\widetilde{A(d)}=\mathscr{O}(d)$ for some $d$. Therefore $\chi_{\mathscr{L}}(C,\eta)=-d$ and $B\cdot\mathscr{L}=\deg_B(\mathscr{L})=-d\deg_B(\mathscr{O}(1))$. □

**Question 4.3.** *When does the locus* $\mathscr{U}\subseteq[(\operatorname{Def}^-(C)\smallsetminus 0)/\mathbb{G}_m]$ *of stable curves contain complete curves? When is the codimension of its complement is at least* $2$*?*

*Remark* 4.4. Theorem 4.2 suggests the possibility of computing slopes of various loci inside $\overline{\mathscr{M}}_g$. For example, since $\operatorname{Def}^-(\widehat{\mathscr{O}}_{C,p})=\operatorname{Def}(\widehat{\mathscr{O}}_{C,p})$ for every $\widehat{\mathscr{O}}_{C,p}$ of type ADE, we obtain slopes of stable limits of all ADE singularities. Furthermore, by considering the singularity $y^3=x^{3k+1}$ (resp. $y^3=x^{3k+2}$), we obtain a family of trigonal curves of genus $g=3k$ (resp. $g=3k+1$) attaining the maximal possible slope $\dfrac{36(g+1)}{5g+1}$ among families of trigonal curves (cf. Remark 5.6). This bound on the slope of trigonal families is due to [SF00] and is obtained there by a laborious construction. It is an interesting question whether the slopes obtained by our methods are maximal among those of $q$-gonal curves for $q\geq 4$.

The space $\operatorname{Def}^-(C)$ of negative deformations corresponds to non-equisingular deformations which give rise to stable limits of $C$. On the other hand, $\operatorname{Def}^0(C)$ and $\operatorname{Def}^+(C)$ correspond to equisingular deformations. To make this precise, consider the rational morphism $\operatorname{Def}(C)\dashrightarrow\overline{\mathscr{M}}_g$ and its graph

$$\begin{tikzcd} & \mathscr{Z} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \operatorname{Def}(C) \arrow[rr, dashed] & & \overline{\mathscr{M}}_g \end{tikzcd}$$

We define $\mathscr{T}_C:=q(p^{-1}(0))\subset\overline{\mathscr{M}}_g$ to be the *stack of stable limits* of $C$ which parameterizes stable limits of all possible smoothings of $C$.

**Proposition 4.5.** *Let* $C$ *be a complete curve with* $\omega_C$ *ample. Suppose there is an isomorphism* $\eta\colon\mathbb{G}_m\to\operatorname{Aut}^\circ(C)$ *fixing a singularity* $p\in C$ *such that* $\widehat{\mathscr{O}}_{C,p}$ *is positively graded under* $\eta$*. Let* $\operatorname{Def}(C)$ *be a versal deformation space with* $\mathbb{G}_m$*-action. Then the rational morphism* $f\colon[\operatorname{Def}^-(C)/\mathbb{G}_m]\dashrightarrow\overline{\mathscr{M}}_g$ *is dominant onto* $\mathscr{T}_C$*.*

*Proof.* Let $\mathscr{U}\subseteq[\operatorname{Def}^-(C)/\mathbb{G}_m]$ be the locus of stable curves. For a point $u_1\in\mathscr{U}(k)$, the isotrivial specialization $u_1\rightsquigarrow 0$ lifts to a specialization $z_1\rightsquigarrow z$ in $\mathscr{Z}$ which maps to $f(u_1)\rightsquigarrow[C_0]$ in $\overline{\mathscr{M}}_g$ with $[C_0]\in\mathscr{T}_C$. It follows that $f(u_1)\in\mathscr{T}_C$, so that $f\colon[\operatorname{Def}^-(C)/\mathbb{G}_m]\dashrightarrow\overline{\mathscr{M}}_g$ factors through $\mathscr{T}_C$.

For a general curve $[C_1] \in \mathscr{T}_C$, there is a smoothing $\mathscr{C} \to \Delta$ with central fiber $C$ such that the stable limit $\mathscr{C}_1 \to \Delta$ has central fiber $C_1$. This gives us two families of curves in $[\mathrm{Def}(C)/\mathbb{G}_m]$:

$$\begin{tikzcd}
\mathscr{C} \arrow[dr] & & \mathscr{C}_1 \arrow[dl] \\
& \Delta &
\end{tikzcd}$$

Since $0 = [C] \in [\mathrm{Def}(C)/\mathbb{G}_m]$ is a closed point, there is an isotrivial specialization $C_1 \rightsquigarrow C$ which is induced by one-parameter subgroup. In particular, we have either $\lim\limits_{t\to 0} t\cdot[C_1] = [C]$, or $\lim\limits_{t\to\infty} t\cdot[C_1] = [C]$. This isotrivial specialization can be realized by a $\mathbb{G}_m$-equivariant family $\mathscr{D} \to \mathbb{A}^1$, trivial over $\mathbb{A}^1 \setminus 0$, with central fiber $C$. The closed points of the total family $\mathscr{D}$ consist of $p \in C$ together with components of $C$ not containing $p$. By the assumptions on $C$, for a general point $x$ on a component containing $p$, we have $\lim\limits_{t\to 0} t\cdot x = p$ which gives a contradiction if $\lim\limits_{t\to 0} t\cdot[C_1] = [C]$. It follows that $[C_1] \in \mathrm{Def}^-(C)$. □

## 5. CHARACTER THEORY COMPUTATIONS

**Computing the characters of $\lambda$ and $\lambda_2$.** Suppose we are given a curve $C$ and a 1-parameter subgroup $\eta : \mathbb{G}_m \to \mathrm{Aut}^0(C)$. Then there is an induced action on the sequence of one dimensional vector spaces

$$\lambda_m|_{[C]} := \bigwedge H^0(C, \omega_C^m).$$

For many classes of singularities, the induced character $\chi_m(C,\eta) \in \mathbb{Z}$ of this action can be explicitly be computed. We provide the calculations for only $A_{2k}$, $D_{2k+2}$, elliptic $m$-fold points, monomial unibranch Gorenstein singularities, and ribbons. Using the same procedure, one can establish the characters for ADE and toric singularities as listed in Table 1.

**Example 5.1** ($A_{2k} : y^2 = x^{2k+1}$)**.** Let $C = C_1 \cup E_1$, where $E_1$ is a smooth rational curve with a higher cusp $y^2 = x^{2k+1}$ at zero, and a nodal attachment to $C_1$ at $\infty$. If $t$ is a uniformizer at zero, then we may write down a basis for $H^0(C, \omega_C)$ as follows:

$$\left(0, \frac{dt}{t^{2k}}\right), \left(0, \frac{dt}{t^{2k-2}}\right), \dots, \left(0, \frac{dt}{t^2}\right), (\omega_1, 0), \dots, \left(\omega_{g-k}, 0\right),$$

where $\omega_1, \dots, \omega_{g-k}$ is a basis for $\omega_{C_1}$. This basis diagonalizes the action $t \to \lambda^{-1}t$ with weights $(2k-1), (2k-3), \dots, 1$. Thus, the character $\chi_1(C,\eta)$ is

$$\chi_1 = \sum_{i=1}^{k} (2i-1) = k^2.$$

Similarly, we may write down a basis for $H^0(C, \omega_C^2)$ as

$$\left(0, \frac{(dt)^2}{t^{4k}}\right), \left(0, \frac{(dt)^2}{t^{4k-2}}\right), \dots, \left(0, \frac{(dt)^2}{t^{2k}}\right), \left(0, \frac{(dt)^2}{t^{2k-1}}\right), \dots, \left(\omega_0, \frac{(dt)^2}{t^2}\right),$$
$$(\omega_1, 0), \dots, \left(\omega_{3g-3k-2}, 0\right),$$

where $\omega_1, \dots, \omega_{3g-3k-2}$ is basis for $H^0(C_1, \omega_{C_1}^2(p))$, and $\omega_0$ is an appropriately chosen element of $H^0(C_1, \omega_{C_1}^2(2p)) \setminus H^0(C_1, \omega_{C_1}^2(p))$.

Thus the character

$$\chi_2 = \sum_{i=0}^{k-1}(2k+2i) + \sum_{i=0}^{2k-2} i = 5k^2 - 4k + 1.$$

**Example 5.2** ($D_{2k+2} : x(y^2 - x^{2k})$)**.** The $\delta$-invariant of this singularity is $k+2$. The map from the normalization is given by:

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & t_1 \\ t_2 & t_2^k \\ t_3 & -t_3^k \end{pmatrix}, \qquad k \geq 1.$$

A generator for $\omega_C$ is $\left(\frac{2dt_1}{t_1^2}, \frac{dt_2}{t_2^{k+1}}, \frac{-dt_3}{t_3^{k+1}}\right)$. From this, one computes:

$$\chi_1 = 1 + 2 + \ldots + k = \frac{k(k+1)}{2}.$$

A generator for $\omega_C^2$ is $\left(\frac{4(dt_1)^2}{t_1^4}, \frac{(dt_2)^2}{t_2^{2k+2}}, \frac{(dt_3)^2}{t_3^{2k+2}}\right)$, so one may write out an array of $(3k+3)$ quadratic differentials ($2k+1$ in the first column, $k+1$ in the second column, 1 in the third):

$$\begin{array}{ccc} \left(\frac{4(dt_1)^2}{t_1^4}, \frac{(dt_2)^2}{t_2^{2k+2}}, \frac{(dt_3)^2}{t_3^{2k+2}}\right) & \left(\frac{4(dt_1)^2}{t_1^3}, \frac{(dt_2)^2}{t_2^{k+2}}, \frac{-(dt_3)^2}{t_3^{k+2}}\right) & \left(\frac{4(dt_1)^2}{t_1^2}, \frac{(dt_2)^2}{t_2^2}, \frac{(dt_3)^2}{t_3^2}\right) \\ \left(0, \frac{(dt_2)^2}{t_2^{2k+1}}, \frac{(dt_3)^2}{t_3^{2k+1}}\right) & \left(0, \frac{(dt_2)^2}{t_2^{k+1}}, \frac{-(dt_3)^2}{t_3^{k+1}}\right) & \\ \vdots & \vdots & \end{array}$$

By summing the weights, we conclude:

$$\chi_2 = \sum_{i=0}^{2k} i + \sum_{i=0}^{k} i = (2k+1)(2k)/2 + (k+1)k/2 = \frac{5k^2+3k}{2}.$$

**Example 5.3** (Elliptic $m$-fold points)**.** Let $m \geq 3$. An elliptic $m$-fold point is the Gorenstein union of $m$ general lines through a point in $\mathbb{P}^{m-1}$ [Smy]. Every such singularity is isomorphic to the cone over points $p_1 = (1,0,\dots,0)$, $p_2 = (0,1,\dots,0)$, $\dots$, $p_{m-1} = (0,0,\dots,1)$, and $p_m = (1,\dots,1)$ in $\mathbb{A}^{m-1}$. If $(x_1, \dots, x_{m-1})$ are coordinates centered at the vertex then the normalization map from $m-1$ copies of $\mathbb{P}^1$ to $E$ is given by

$$\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_{m-1} \end{pmatrix} \rightarrow \begin{pmatrix} t_1 & 0 & \dots & 0 & t_m \\ 0 & t_2 & \ddots & \vdots & t_m \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & t_{m-1} & t_m \end{pmatrix}.$$

We let $C$ be the singular curve obtained by attaching $E$ to a smooth curve $C_0$ nodally at points $p_1, \dots, p_m$. A generator for $\omega_C$ in the neighborhood of the $m$-fold point is

$$\omega_0 = \left(\frac{dt_1}{t_1^2}, \dots, \frac{dt_{m-1}}{t_{m-1}^2}, -\frac{dt_m}{t_m^2}\right).$$

In fact $(\omega_0, 0)$ spans the only[2] non-trivial eigenspace of $\mathrm{H}^0(C, \omega_C)$. Thus, $\chi_1(C) := 1$. A generator for $\omega_C^2$ in the neighborhood of the $m$-fold point is

$$\omega_0^2 = \left( \frac{(dt_1)^2}{t_1^4}, \dots, \frac{(dt_{m-1})^2}{t_{m-1}^4}, \frac{(dt_m)^2}{t_m^4} \right),$$

and the only non-trivial eigenspaces of $\mathrm{H}^0(C, \omega_C^2)$ are spanned by

$$(\omega_0^2, 0), (x_1\omega_0^2, 0), \dots, (x_{m-1}\omega_0^2, 0).$$

It follows that $\chi_2(C) = 2 + (m-1) = m+1$. Thus, the $\alpha$-invariant of the elliptic $m$-fold point is

$$\alpha := \frac{11-2m}{12-m}.$$

**Example 5.4** (Monomial unibranch singularities)**.** Let $C$ be a rational curve of arithmetic genus $k$ with an isolated monomial Gorenstein singularity $k[[t^{a_1}, \dots, t^{a_n}]] \subseteq k[[t]]$ at $p \in C$. Let $\{b_1 = 1, \cdots, b_k\}$ be the corresponding gap sequence. Since $C$ is Gorenstein, the gap sequence is symmetric; that is, for each $i$, $b_k - b_i$ is not a gap. In particular, $b_k + 1 = 2k$. A generator for $\omega_C$ in a neighborhood of $p$ is given by $\frac{dt}{t^{b_k+1}}$. Therefore, we can write down bases

$$\begin{aligned}
\mathrm{H}^0(C, \omega_C) &= \left\langle \frac{dt}{t^{b_1+1}}, \frac{dt}{t^{b_2+1}}, \dots, \frac{dt}{t^{b_k+1}} \right\rangle \\
\mathrm{H}^0(C, \omega_C^2) &= \left\langle \frac{(dt)^2}{t^{2b_k+2-j}} \quad : \quad j \in \{0, \dots, 2b_k - 2\} \text{ is not a gap} \right\rangle
\end{aligned}$$

and we compute

$$\begin{aligned}
\chi_1 &= \sum_{i=1}^{k} b_i \\
\chi_2 &= \sum_{j=0}^{2b_k-2} (2b_k - j) - \sum_{i=1}^{k} (2b_k - b_i) = b_k(2b_k + 1 - 2k) + \sum_{i=1}^{k} b_i - 1 \\
&= (2k-1)^2 - \sum_{i=1}^{k} b_i - 1.
\end{aligned}$$

If $C' = C \cup E$ is the nodal union of $C$ and a genus $g-k$ curve, the corresponding characters are $\chi_1 = \sum_{i=1}^{k} b_i$ and $\chi_2 = (2k-1)^2 - \sum_{i=1}^{k} b_i$.

In particular, for a toric singularity $x^p = y^q$ with $p$ and $q$ coprime, the genus is $k = pq - 1$ and the gap sequence is the set of integers $i = 1, \cdots, pg - 1$ relatively prime to $p$ and $q$. It is easy to compute that $\chi_1 = pq(pq-1)(q-2)/2$ and $\chi_2 = (2(pq-1)-1)^2 - pq(pq-1)(q-2)/2 - 1$ which agrees with the intersection numbers in Proposition 6.6.

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[2] The remaining eigenspaces $\mathrm{H}^0(C, \omega_C)$ are spanned by $(x_1\omega_0, \omega_0'), \dots, (x_{m-1}\omega_0, \omega_{m-1}'), (0, \omega_1''), \dots, (0, \omega_{g-m}'')$, where $\omega_i' \in \mathrm{H}^0(C_0, \omega_{C_0}(p_i))$ satisfies $\mathrm{Res}_{p_i}\omega_i' = 1$, and where $\omega_j'' \in \mathrm{H}(C_0, \omega_{C_0})$.

**Example 5.5** (Non-reduced curves: A case study of ribbons)**.** The character theory is particularly suited to the study of GIT stability of non-reduced Gorenstein schemes. Here, we treat the case of *ribbons*, certain flat limits of canonically embedded curves under degenerations of their moduli points to the hyperelliptic locus in $\overline{M}_g$ [Fon93]. Our exposition is self-contained but we refer the reader to [BE95] for a more systematic study of ribbons.

A ribbon of genus $g$ is a scheme obtained by gluing together two copies of $\mathbb{A}^1[\varepsilon] := \operatorname{Spec} k[x,\varepsilon]/(\varepsilon^2)$. Precisely, let $U_1 = \operatorname{Spec} k[x,\varepsilon]/(\varepsilon^2)$ and $U_2 = \operatorname{Spec} k[y,\eta]/(\eta^2)$, and let $(U_1)_x$ and $(U_2)_y$ be the corresponding open subschemes. Then a ribbon over $\mathbb{P}^1$ is given by a gluing isomorphism $\varphi\colon (U_1)_y \to (U_2)_x$ defined by

$$\begin{aligned} x &\mapsto y^{-1} - y^{-2} f(y)\eta, \\ \varepsilon &\mapsto y^{-g-1}\eta, \end{aligned}$$

where $f(y) \in \dfrac{k[y,y^{-1}]}{k[y]+t^{-g+1}k[y^{-1}]}$ [BE95, p. 733].

We focus here on non-split ribbons that admit a $\mathbb{G}_m$-action. There are $g-2$ such ribbons, each given by $f(y) = y^{-\ell}$, for $\ell \in \{1,\dots,g-2\}$. Denote the ribbon corresponding to $f(y) = y^{-\ell}$ by $C_\ell$. Then the $\mathbb{G}_m$-action on $C_\ell$ is given by $t\cdot(x,y,\varepsilon,\eta) = (tx, t^{-1}y, t^{g-\ell}\varepsilon, t^{-\ell-1}\eta)$.

By adjunction, the sections of $\omega_{C_\ell}$ over $U_1$ are identified with restrictions to $U_1$ of 2-forms $f(x,\varepsilon)\frac{dx\wedge d\varepsilon}{\varepsilon^2}$ on $\operatorname{Spec} k[x,\varepsilon]$, and the sections of $\omega_{C_\ell}$ over $U_2$ are identified with restrictions to $U_2$ of 2-forms $f(y,\eta)\frac{dy\wedge d\eta}{\eta^2}$ on $\operatorname{Spec} k[y,\eta]$. With this in mind, we can write down $g$ linearly independent global sections of $\omega_{C_\ell}$:

For $k = 0,\dots,g-\ell-2$,

$$\omega_k = x^k \frac{dx\wedge d\varepsilon}{\varepsilon^2} = -(y^{g-1-k} + (g-\ell-k-1)y^{g-\ell-k-2}\eta)\frac{dy\wedge d\eta}{\eta^2},$$

for $k = g-\ell-1,\dots,g-1$,

$$\omega_k = (x^k + (\ell+k+1-g)x^{\ell+k-1}\varepsilon)\frac{dx\wedge d\varepsilon}{\varepsilon^2} = -y^{g-1-k}\frac{dy\wedge d\eta}{\eta^2}.$$

It follows that $\{\omega_i\}_{i=0}^{g-1}$ form the basis of $\mathrm{H}^0(C_\ell,\omega_{C_\ell})$. Note that we recover the second part of [BE95, Theorem 5.1], namely the identification of the sections of $\mathrm{H}^0(C_\ell,\omega_{C_\ell})$ with functions

$$1, y, y^2, \dots, y^\ell, y^{\ell+1}+\eta, y^{\ell+2}+2y\eta, \dots, y^{g-1}+(g-\ell-1)y^{g-\ell-2}\eta,$$

under a trivialization of $\omega_{C_\ell}$ on $U_2$.

We now proceed with character computations. Under the $\mathbb{G}_m$-action above, we have that

$$t\cdot\omega_k = t^{k-g+\ell+1}\omega_k.$$

Summing up the weights of the $\mathbb{G}_m$-action on the basis $\{\omega_i\}_{i=0}^{g-1}$, we obtain the character of $\lambda$:

$$\chi_1(C_\ell) = \sum_{k=0}^{g-1}(k-g+\ell+1) = g(\ell+1-g) + g(g-1)/2 = g\left(\ell - \frac{g-1}{2}\right).$$

It remains to compute the weights of the $\mathbb{G}_m$-action on a basis of $\mathrm{H}^0(C_\ell, \omega_{C_\ell}^2)$ and the corresponding character $\chi_2(C_\ell)$. Since $h^0(C_\ell, \omega_{C_\ell}^2) = 3g-3$, it suffices to exhibit $3g-3$ linearly independent sections. One such choice is presented by

$$\begin{aligned}&1, y, y^2, \dots, y^{2\ell}, y^{2\ell+1} + y^\ell \eta, \dots, y^{\ell+g-1} + (g-\ell-1)y^{g-2}\eta,\\ &\qquad y^{\ell+g} + (g-\ell)y^{g-1}, \dots, y^{2g-2} + (2g-2\ell-2)y^{2g-\ell-3}, \eta, y\eta, \dots, y^{g-3}\eta.\end{aligned}$$

In particular, taking into account that the weight of $\frac{dy\wedge d\eta}{\eta^2}$ is $2\ell$, we see that the weights of the $\mathbb{G}_m$-action on $\mathrm{H}^0(C_\ell, \omega_{C_\ell}^2)$ are

$$2\ell, 2\ell-1, \dots, 2\ell-(2g-2),\ \ell-1, \dots, \ell-g+2.$$

Summing up these weights, we obtain the character of $\lambda_2$:

$$\begin{aligned}\chi_2(C_\ell) &= 2(2g-1)\ell - (g-1)(2g-1) + (g-2)\ell - (g-2)(g-1)/2\\ &= (5g-4)(\ell - \frac{g-1}{2}) = (5g-4)\chi_1(C_\ell).\end{aligned}$$

By Equation (1.1), the character of $\delta$ is $\chi_\delta(C_\ell) = (8g+4)(\ell - \frac{g-1}{2})$. In particular, if $\ell \neq (g-1)/2$, then all three characters $\chi_1(C_\ell), \chi_2(C_\ell), \chi_\delta(C_\ell)$ are non-zero, and we have

$$\frac{\chi_\delta(C_\ell)}{\chi_\lambda(C_\ell)} = \frac{8g+4}{g}.$$

*Remark* 5.6. Generalizing the computations of Example 5.5 above, Anand Deopurkar recently computed characters of Gorenstein $n$-ribbons[3] with $\mathbb{G}_m$-action and verified that always

$$\chi_\delta = \frac{12(2g+n-1)n}{n^2+(4g-3)n+2-2g}\chi_\lambda.$$

This recovers the ratio $\frac{8g+4}{g}$ for 2-ribbons, gives the ratio $\frac{36(g+1)}{5g+1}$ for 3-ribbons (see Remark 4.4 for the significance of this slope), and more generally gives the same ratio $\chi_\delta/\chi_\lambda$ as that of the toric singularity $y^n = x^{2g/(n-1)+1}$ computed in Corollary 6.7 (the arithmetic genus of an $n$-ribbon always satisfies $n-1 \mid 2g$).

**Computing the characters of $\delta_i$.** In this section, we illustrate how the characters of $\delta_i$ can be computed for a curve with a $\mathbb{G}_m$-action. If $C$ is a curve with a $\mathbb{G}_m$-action such that the discriminant locus is Cartier, then line bundles $\delta_i$ can be defined in a neighborhood of $[C]$ in the stack of all curves. The following proposition shows that the character of $\delta_i$ is precisely the weighted degree of the discriminant.

---

[3] An $n$-ribbon is a non-reduced scheme supported on $\mathbb{P}^1$ and locally isomorphic to $U \times \operatorname{Spec} k[\varepsilon]/(\varepsilon^n)$, where $U \subset \mathbb{P}^1$ is affine.

**Proposition 5.7.** *Let $C$ be a complete curve with miniversal deformation space* $\operatorname{Spf} A$ *and a one-parameter subgroup $\eta\colon\ \mathbb{G}_m \to \operatorname{Aut}(C)$. Let $D$ be a Cartier divisor defined on a neighborhood $\mathscr{V}$ in the stack of all complete curves. Suppose that there is a cartesian diagram*

$$\begin{tikzcd}
V(f) \arrow[r] \arrow[d] & \operatorname{Spf} A \arrow[d] \\
D \arrow[r] & \mathscr{V}
\end{tikzcd}$$

*such that $f \mapsto t^d f$ under the induced action of $\mathbb{G}_m = \operatorname{Spec} k[t]_t$ on* $\operatorname{Spf} A$. *Then $\chi_{\mathscr{L}(D)}(C,\eta) = -d$.*

*Proof.* Denote by $\sigma : A \to k[t]_t \widehat{\otimes} A$ the dual action of $\mathbb{G}_m$ on $\operatorname{Spf} A$. The exact sequence $0 \to \mathscr{L}(-D) \to \mathscr{O}_{\mathscr{U}_g} \to \mathscr{O}_D \to 0$ restricted to $\operatorname{Spf} A$ corresponds to the exact sequence

$$0 \to A_\eta \xrightarrow{f} A \to A/f \to 0$$

where $A_\eta$ is the $\mathbb{G}_m$-$A$-module corresponding to the character $\mathbb{G}_m \xrightarrow{d} \mathbb{G}_m$; that is, $A_\eta$ is $A$ as an $A$-module with coaction $a \mapsto t^d \sigma(a)$. Therefore $\mathscr{L}(-D)|_{B\mathbb{G}_m}$ corresponds to the character $\mathbb{G}_m \xrightarrow{d} \mathbb{G}_m$ and $\chi_{\mathscr{L}(-D)}(C,\eta) = d$. □

We include the computation of the character of $\delta_i$ for certain curves with $A_{2k}$ and $D_{2k-2}$ but the same approach can be applied to a large class of curves (see Table 1). The characters of $\delta_i$ will depend on the global geometry of the curve. For instance, if an $A_{2k+1}$-singularity lies on a component intersecting the rest of the curve at two nodes, the character of $\delta_0$ will depend on whether the component is separating or not. Furthermore, if an $A_{2k+1}$-singularity lies on a rational curve attached to the rest of the curve at one point which we refer to as a "dangling" singularity, the value of $\delta_0$ will be different from the non-dangling case. Due to a lack of space, we only include the calculations in a sampling of cases.

**Example 5.8** ($A_{2k+1}$-singularity: non-separating case)**.** Let $C = C_0 \cup E$ be a curve where $C_0$ is a smooth curve of genus $g-k$, $E = E_1 \cup E_2$ is the union of two rational curves at a monomial $A_{2k+1}$-singularity at $p$, and $E_i$ intersects $C_0$ at a node $q_i$. The first order deformation space can be written as

$$\operatorname{Def}(C) = \operatorname{Def}(C_0,q_1,q_2) \times \operatorname{Cr}(\widehat{\mathscr{O}}_{C,p}) \times \operatorname{Def}(\widehat{\mathscr{O}}_{C,p}) \times \operatorname{Def}(\widehat{\mathscr{O}}_{C,q_1}) \times \operatorname{Def}(\widehat{\mathscr{O}}_{C,q_2})$$

where $\operatorname{Cr}(\widehat{\mathscr{O}}_{C,p})$ denotes the "crimping" deformations (see [AvdWS10] for more details); the $\mathbb{G}_m$-action on $\operatorname{Cr}(\widehat{\mathscr{O}}_{C,p})$ can be explicitly determined but doesn't affect this calculation. We can choose coordinates

$$\operatorname{Def}(\widehat{\mathscr{O}}_{C,p}) = \{y^2 - x^{2k+2} + a_{2k}x^{2k} + \cdots + a_1 x + a_0 = 0\}$$
$$\operatorname{Def}(\widehat{\mathscr{O}}_{C,q_i}) = \{xy + n_i = 0\}$$

and a one-parameter subgroup $\eta\colon\ \mathbb{G}_m \to \operatorname{Aut}(C)$ such that $\mathbb{G}_m$ acts via $a_i \mapsto t^{2i-2k-2}a_i$ and $n_i \mapsto t n_i$. The discriminant $\Delta$ of the $A_{2k+1}$-singularity has weighted degree $-(2k+1)(2k+2)$ and each node has weighted degree 1. Since $\delta_0 = V(\Delta) \cup V(n_1) \cup V(n_2)$, we conclude:

$$\chi_{\delta_0} = (2k+1)(2k+2) - 2, \quad \chi_{\delta_i} = 0 \text{ for } i \neq 0.$$

**Example 5.9** ($A_{2k+1}$-singularity: separating case)**.** Let $C = C_1 \cup C_2 \cup E$ where $C_i$ is a smooth curve of genus $h_i$ where $k+h_1+h_2 = g$, $E = E_1 \cup E_2$ is the union of two rational curves at a monomial $A_{2k+1}$-singularity, and $E_i$ intersects $C_i$ at a node. Using the above calculation, since $\delta_0 = V(\Delta)$ and $\delta_{h_i} = V(n_i)$, we conclude:

$$\chi_{\delta_0} = (2k+1)(2k+2), \quad \chi_{\delta_{h_1}} = \chi_{\delta_{h_2}} = -1, \quad \chi_{\delta_i} = 0 \text{ for } i \neq 0, h_1, h_2.$$

**Example 5.10** ($A^{\{1\}}_{2k+1}$-singularity: dangling case)**.** Let $C = C \cup E$ where $C$ is a smooth curve of genus $g-k$ , $E = E_1 \cup E_2$ is the union of two rational curves at a monomial $A_{2k+1}$-singularity, and $E_1$ intersects $C_1$ at a node. Then

$$\chi_{\delta_0} = (2k+1)(2k+2), \quad \chi_{\delta_k} = -1, \quad \chi_{\delta_i} = 0 \text{ for } i \neq 0, k.$$

**Example 5.11** ($A^{\{\}}_{2k+1}$-singularity: isolated case)**.** Let $C = E_1 \cup E_2$ is the union of two rational curves at a monomial $A_{2k+1}$-singularity. Then

$$\chi_{\delta_0} = (2k+1)(2k+2), \quad \chi_{\delta_i} = 0 \text{ for } i \neq 0.$$

For instance, if $k = 2$, the $\alpha$-value is $7/10$ and if $k = 3$, the $\alpha$-value is $17/28$.

**Example 5.12** ($D_{2k+2}$-singularity: non-separating case)**.** Let $C = C_0 \cup E$ where $C_0$ is a genus $g-k$ curve, $E = E_1 \cup E_2 \cup E_3$ is the union of three rational curves at a monomial $D_{2k+2}$-singularity at $p \in E$, and each $E_i$ intersects $C_0$ at a node $q_i$. We write

$$\operatorname{Def}(C) = \operatorname{Def}(C_0, \{q_i\}) \times \operatorname{Cr}(\widehat{\mathscr{O}}_{C,p}) \times \operatorname{Def}(\widehat{\mathscr{O}}_{C,p}) \times \prod_{i=1}^{3} \operatorname{Def}(\widehat{\mathscr{O}}_{C,q_i})$$

We can choose coordinates

$$\operatorname{Def}(\widehat{\mathscr{O}}_{C,p}) = \{xy^2 - x^{2k+1} + a_{2k-1}x^{2k-1} + \cdots + a_1x + a_0 + by = 0\}$$
$$\operatorname{Def}(\widehat{\mathscr{O}}_{C,q_i}) = \{yz + n_i = 0\}$$

and an isomorphism $\eta\colon \mathbb{G}_m \to \operatorname{Aut}(E, \{q_i\})$ such that $\mathbb{G}_m$ acts via

$$a_i \mapsto t^{i-2k-1} \qquad b \mapsto t^{-k-1}b \qquad n_1 \mapsto t^k n_1 \qquad n_2 \mapsto tn_2 \qquad n_3 \mapsto tn_3$$

The discriminant has weight $-(2k+1)(2k+2)$ so we conclude that

$$\chi_{\delta_0} = (2k+1)(2k+2) - (k+2), \quad \chi_{\delta_i} = 0 \text{ for } i \neq 0.$$

**Example 5.13** ($D^{\{1\}}_{2k+2}$-singularity)**.** Let $C = C_0 \cup E$ where $C_0$ is a genus $g-k$ curve, $E = E_1 \cup E_2 \cup E_3$ is the union of three rational curves at a monomial $D_{2k+2}$-singularity with $E_1$ being the smooth branch, and $E_1$ and $E_2$ intersect $C_0$ at nodes. Using the calculation above, we conclude that:

$$\chi_{\delta_0} = (2k+1)(2k+2) - (k+1), \quad \chi_{\delta_i} = 0 \text{ for } i \neq 0.$$

**Example 5.14** ($D^{\{1,2\}}_{2k+2}$-singularity)**.** Let $C$ be a curve as in the previous example except that only the smooth branch $E_1$ intersects $C_0$. Then

$$\chi_{\delta_0} = (2k+1)(2k+2) - k, \quad \chi_{\delta_i} = 0 \text{ for } i \neq 0.$$

## 6. INTERSECTION THEORY

The deformation theory of *ADE* curve singularities enjoys two key properties that enable hands-on analysis of their varieties of stable limits. They are:

- The versal deformation space $\mathrm{Def}(C)$ can be taken to be an affine space.
- $\mathrm{Def}(C)$ is (strictly) negatively graded by a natural $\mathbb{G}_m$-action.

In particular, let $C$ be a complete curve with a unique *ADE* singularity $p \in C$ and a $\mathbb{G}_m$-action such that $\widehat{\mathscr{O}}_{C,p}$ is positively graded. Then there exists a complete family $B \to \mathrm{Def}^-(C)$ of Deligne-Mumford stable curves. By Theorem 4.2, the intersection numbers of $B$ are computed in terms of characters of the $\mathbb{G}_m$-action, and by Proposition 4.5 we can think of $B$ as a family of stable limits of $C$. In this section, we study $B$ from the point of view of intersection theory. In Examples 6.1– 6.4 below, we write down explicit families of stable limits of $A$ and $D$ singularities. Our computations agree with the results obtained in Section 5. As singularities become more complicated, the task of writing down a complete family of stable limits becomes nearly impossible. In the case of toric singularities $x^p = y^q$, we can circumvent some of the difficulties by settling for an implicitly defined family of stable limits.

### Weierstrass tails, bridges and triboroughs.

**Example 6.1** (Hyperelliptic bridges). Let $k \geq 1$ be an integer. Then there exists a complete one-parameter family $B_k$ of 2-pointed curves of genus $k$ such that the generic fiber is a smooth hyperelliptic curve with marked points conjugate under the hyperelliptic involution. Furthermore,

$$\begin{aligned}
\lambda \cdot B_k &= (k^2+k)/2,\\
\delta_0 \cdot B_k &= (2k+1)(2k+2),\\
\psi_1 \cdot B_k &= \psi_2 \cdot B_k = 1,\\
\delta_1 \cdot B_k &= \cdots = \delta_{\lfloor k/2 \rfloor} \cdot B_k = 0.
\end{aligned}$$

To construct the said family, take a Hirzebruch surface $\mathbb{F}_1$ realized as a $\mathbb{P}^1$-bundle $\mathbb{F}_1 \to B$ over $B \simeq \mathbb{P}^1$. Denote by $E$ the unique $(-1)$-section and by $F$ the fiber. Next, choose $2k+2$ general divisors $S_1, \dots, S_{2k+2}$ in the linear system $|E+F|$ (these are sections of $\mathbb{F}_1 \to B$ of self-intersection 1). The divisor $\sum_{i=1}^{2k+2} S_i$ is divisible by 2 in $\mathrm{Pic}(\mathbb{F}_1)$ and so there is a cyclic degree 2 cover $\pi\colon X \to \mathbb{F}_1$ branched over $\sum_{i=1}^{2k+2} S_i$. It follows easily from the Riemann-Hurwitz formula that $\pi^{-1}(B) = \Sigma_1 + \Sigma_2$, the disjoint union of two $\mathbb{P}^1$'s. Thus, $\pi\colon (X;\Sigma_1,\Sigma_2) \to B$ is a family of at worst nodal hyperelliptic curves of genus $k$ with two conjugate sections.

From $K_{X/B} = \pi^*(K_{\mathbb{F}_1/B} + \frac{1}{2}\sum_{i=1}^{2k+2} S_i)$ we deduce that

$$12\lambda_{X/B} - \delta_{X/B} = K_{X/B}^2 = 2(K_{\mathbb{F}_1} + 2F + (2k+2)(E+F))^2 = 2(k^2-1).$$

From the construction, there are $\binom{2k+2}{2}$ nodes in the fibers of $\pi$. Because $X$ has $A_1$ singularity at each of these nodes, we have

$$\delta_{X/B} = 2\binom{2k+2}{2} = (2k+1)(2k+2).$$

Finally, the self-intersection of each $\Sigma_i$ is $-1$. It follows that $\pi\colon X \to B$ is the requisite family.

**Example 6.2** (Hyperelliptic tails attached at arbitrary points)**.** Taking family $B_k$ constructed in Example 6.1 above and forgetting one marked section, we arrive at a family $H_k$ of hyperelliptic curves with a single marked section whose intersection numbers are

$$\begin{aligned}\lambda \cdot H_k &= (k^2+k)/2,\\ \delta_0 \cdot H_k &= (2k+1)(2k+2),\\ \psi \cdot H_k &= 1,\\ \delta_1 \cdot H_k &= \cdots = \delta_{\lfloor g/2 \rfloor} \cdot H_k = 0.\end{aligned}$$

In particular, the locus of curves with a hyperelliptic tail of genus $k$ attached at an arbitrary point falls in the base locus of $K_{\overline{\mathscr{M}}_g} + \alpha\delta$ for $\alpha = \dfrac{3k^2+11k+4}{8k^2+12k+2}$. For example, when $k=2$, this shows that $\Delta_2 \subset \overline{\mathscr{M}}_g$ is covered by curves orthogonal to $K_{\overline{\mathscr{M}}_g} + \frac{19}{29}\delta$.

**Example 6.3** (Hyperelliptic triboroughs)**.** There exists a complete one-parameter family $Tri_k$ of 3-pointed curves of genus $k$ such that the generic fiber is a smooth hyperelliptic curve and two of its marked points are conjugate. Furthermore,

$$\begin{aligned}\lambda \cdot Tri_k &= k^2+k,\\ \delta_0 \cdot Tri_k &= 2(2k+1)(2k+2),\\ \psi_1 \cdot Tri_k &= \psi_2 \cdot Tri_k = 2,\\ \psi_3 \cdot Tri_k &= 2k,\\ \delta_1 \cdot Tri_k &= \cdots = \delta_{\lfloor k/2 \rfloor} \cdot Tri_k = 0.\end{aligned}$$

The construction of $Tri_k$ parallels that of the family $B_k$ in above. Namely, in the notation of Example 6.1, consider an additional section $S_0$ of $\mathbb{F}_1$ of self-intersection 1 such that $S_0$ is transverse to $\sum_{i=1}^{2k+2} S_i$. Set $C := \pi^{-1}(S_0)$. Then $C$ is a degree 2 cover of $B$ (of genus $k$). Note that $C^2 = 2$ on $X$. Consider the base extension $\pi' \colon Y := X \times_B C \to C$. The preimage of $C$ on $Y$ is the union of two sections $C_1$ and $C_2$, intersecting transversally in $2k+2$ points. By construction, $(C_1+C_2)^2 = 2C^2 = 4$, and so $C_1^2 = C_2^2 = -2k$. Setting $\Sigma_3 := C_1$, we obtain the requisite family $\pi' \colon Y \to C$ of hyperelliptic triboroughs with two conjugate sections (the preimages of $\Sigma_1$ and $\Sigma_2$) of self-intersection $-2$ and the third section $\Sigma_3$ of self-intersection $-2k$.

**Example 6.4** (Hyperelliptic bridges attached at arbitrary points)**.** Taking family $Tri_k$ constructed in Example 6.3 above and forgetting one conjugate section, we arrive at a family $BH_k$ of hyperelliptic curves with two marked points. The intersection numbers of $BH_k$ are

$$\begin{aligned}\lambda \cdot BH_k &= k^2+k,\\ \delta_0 \cdot BH_k &= 2(2k+1)(2k+2),\\ \psi_1 \cdot BH_k &= 2, \quad \psi_2 \cdot BH_k = 2k,\\ \delta_1 \cdot BH_k &= \cdots = \delta_{\lfloor k/2 \rfloor} \cdot BH_k = 0.\end{aligned}$$

In particular, the locus of curves with a hyperelliptic bridge of genus $k$ attached at arbitrary points falls in the base locus of $K_{\overline{\mathscr{M}}_g} + \alpha\delta$ for $\alpha = \dfrac{3k^2+7k+4}{8k^2+10k+2}$. For example, when $k =$

2, this shows that the locus of curves with genus 2 bridges in $\overline{\mathcal{M}}_g$ is covered by curves orthogonal to $K_{\overline{\mathcal{M}}_g}+\frac{5}{9}\delta$.

**Toric singularities.** How can we write down a complete 1-parameter family of stable limits of a singularity, equivalently, a complete family of tails, in such a way that its intersection numbers with the divisors on $\overline{\mathcal{M}}_g$ can be computed? We give a complete answer only in the case of planar toric singularities $x^p=y^q$, even though our method applies more generally to any planar singularity.

Our approach is via degenerations: Begin with a complete family $F_1$ of at-worst nodal curves – a generic pencil of plane curves of degree $d\gg 0$ will do. Now vary $F_1$ in a 1-parameter family $F_t$ in such a way that among curves in $F_0$ exactly one curve $C$ has singularity $f(x,y)=0$ while the rest are at worst nodal. Since the generic points of $F_0$ and $F_1$ are smooth curves of genus $g=\binom{d-1}{2}$, we obtain 1-cycles $F_0,F_1\in \mathrm{N}_1(\overline{\mathcal{M}}_g)$. For a line bundle $\mathscr{L}\in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ the numbers $\mathscr{L}\cdot F_0$ and $\mathscr{L}\cdot F_1$ will differ. If we denote by $\mathscr{F}$ the total space of $\{F_t\}$, then the discrepancy between $\mathscr{L}\cdot F_0$ and $\mathscr{L}\cdot F_1$ is accounted for by indeterminacy of a rational map $\mathscr{F}\dashrightarrow\overline{\mathcal{M}}_g$ at the point $[C]$. In fact, if

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathscr{F} \arrow[rr, dashed] & & \overline{\mathcal{M}}_g \end{tikzcd}$$

is a resolution, then in $\mathrm{N}_1(\overline{\mathcal{M}}_g)$ we have

$$F_1=F_0+q(p^{-1}([C])).$$

By construction, $q(p^{-1}([C]))$ is a 1-cycle inside $\mathscr{T}_{f(x,y)=0}$ and its slope equals to

$$\frac{\delta\cdot F_1-\delta\cdot F_0}{\lambda\cdot F_1-\lambda\cdot F_0}.$$

We now perform the necessary computation for toric planar singularities. To begin, let $C$ be a plane curve of degree $d\gg 0$ with a singularity $x^{pb}=y^{qb}$, where $p$ and $q$ are coprime. The space of stable limits of $C$ has the following description due to Hassett:

**Proposition 6.5** ([Has00, Theorem 6.5]). *The stable limits of $C$ are of the form $\widetilde{C}\cup T$, where the tail $(T,p_1,\dots,p_b)$ is a $b$-pointed curve of arithmetic genus $g=(pqb^2-pb-qb-b+2)/2$. Moreover,*

*(1) $K_T=(pqb-p-q-1)(p_1+\cdots+p_b)$.*
*(2) $T$ is $qb$-gonal with $g^1_{qb}$ given by $|q(p_1+\cdots+p_b)|$.*

**Proposition 6.6.** *Let $(p,q)=1$. Suppose $F_0$ is a pencil of plane curves of degree $d\gg 0$ containing a curve $C$ with a unique singularity $x^{pb}=y^{qb}$ and such that the total family over*

*$F_0$ is smooth. Consider a deformation $\mathscr{F}=\{F_t\}$ of $F_0$ such that $F_1$ is a general pencil. If*

$$\begin{tikzcd}
& W \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathscr{F} \arrow[rr, dashed] & & \overline{\mathscr{M}}_{\binom{d-1}{2}}
\end{tikzcd}$$

*is a resolution of the natural moduli map, then the $1$-cycle $Z:=q(p^{-1}([C]))$ inside $\mathscr{T}_C\subset\overline{\mathscr{M}}_{g,b}$ (where $g=(pqb^2-pb-qb-b+2)/2$) has the following intersection numbers:*

$$\begin{aligned}
\lambda\cdot Z &= \frac{b}{12}\left((pqb-p-q)^2+pq(pqb^2-pb-qb+1)-1\right),\\
\delta_{irr}\cdot Z &= pqb(pqb^2-pb-qb+1),\\
\psi\cdot Z &= b.
\end{aligned}$$

*Proof.* Let $\mathscr{X}_i$ be the total families of pencils $F_i$ $(i=0,1)$. Our first goal is to compare numerical invariants $\delta$ and $\kappa$ of $\mathscr{X}_0$ and $\mathscr{X}_1$. To begin, since $F_1$ is a general pencil, we have $\delta(\mathscr{X}_1)=\delta_{\mathrm{irr}}(\mathscr{X}_1)$.

To find the number of singular fibers in $\mathscr{X}_0\smallsetminus C$, we observe that the topological Euler characteristic of $C$ equals to

$$2-2g(C)-(b-1),$$

where $g(C)=\binom{d-1}{2}-g-b+1$ is the geometric genus of $C$. It follows that

$$\delta_{\mathrm{irr}}(\mathscr{X}_0\smallsetminus C)=\delta_{\mathrm{irr}}(\mathscr{X}_1)-(pqb^2-pb-qb+1).$$

Since $\mathscr{X}_0$ and $\mathscr{X}_1$ are total families of pencils of degree $d$ curves, we have

$$\kappa(\mathscr{X}_0)=\kappa(\mathscr{X}_1).$$

Next, to compare intersection numbers of $F_0$ and $F_1$ with $\lambda$ and $\delta_{\mathrm{irr}}$, we need to write down a family of stable curves over each $F_i$. There is nothing to do in the case of $F_1$, since it is a general pencil. In particular, we have $\lambda\cdot F_1=(\kappa(\mathscr{X}_1)+\delta_{\mathrm{irr}}(\mathscr{X}_1))/12$.

To write down a stable family over $F_0$, we perform stable reduction. It proceed in two steps. *Base change:* To begin, make a base change of order $bpq$ to obtain a family $\mathscr{Y}$ with local equation $x^{pb}-y^{qb}=t^{pqb}$.

The numerical invariants of $\mathscr{Y}$ are

$$\begin{aligned}
\kappa(\mathscr{Y}) &= pqb\kappa(\mathscr{X}_0)=pqb\kappa(\mathscr{X}_1),\\
\delta_{\mathrm{irr}}(\mathscr{Y}\smallsetminus C) &= pqb\delta_{\mathrm{irr}}(\mathscr{X}_0\smallsetminus C)=pqb(\delta_{\mathrm{irr}}(\mathscr{X}_1)-(pqb^2-pb-qb+1)).
\end{aligned}$$

*Weighted blow-up:* Assign weights $q,p,1$ to $x,y,t$. Denote by $\mathscr{Z}$ the weighted blow-up of $\mathscr{Y}$, centered at $x=y=t=0$. The central fiber of $\mathscr{Z}$ becomes $\widetilde{C}\cup T$ of the form described in Proposition 6.5. The self-intersection of the tail $T$ inside $\mathscr{Z}$ is $-b$. It follows from the formula

$$K_{\mathscr{Z}}=\pi^*K_{\mathscr{Y}}+aT$$

that $a=p+q-pqb$. We compute that

$$\kappa(\mathscr{Z})=\kappa(\mathscr{Y})-b(pqb-p-q)^2=pqb\kappa(\mathscr{X}_1)-b(pqb-p-q)^2.$$

The number $\delta_{\text{irr}}(\mathscr{Z}-C)$ of singular fibers in $\mathscr{Z}\smallsetminus C$ is the same as in $\mathscr{Y}\smallsetminus C$ and equals to

$$pqb\delta_{\text{irr}}(\mathscr{X}_1)-pqb(pqb^2-pb-qb+1).$$

Remembering that the central fiber of $\mathscr{Z}$ has $b$ nodes, we compute

$$\begin{aligned}pqb\delta(\mathscr{X}_1)-\delta(\mathscr{Z})&=pqb(pqb^2-pb-qb+1)-b,\\ pqb\kappa(\mathscr{X}_1)-\kappa(\mathscr{Z})&=b(pqb-p-q)^2.\end{aligned}$$

In particular,

$$\delta_{\text{irr}}\cdot Z=pqb(pqb^2-pb-qb+1).$$

Using the Mumford's formula $\lambda=(\kappa+\delta)/12$, we obtain

$$\lambda\cdot Z=\lambda\cdot(pqbF_2-Z_0)=\frac{b}{12}\left((pqb-p-q)^2+pq(pqb^2-pb-qb+1)-1\right).$$

We leave it as an exercise for the reader to verify that $\psi\cdot Z=b$.

□

**Corollary 6.7.** *Suppose $p$ and $q$ are coprime. Then for any family $Z$ of irreducible 1-pointed tails of stable limits of $x^p=y^q$, we have*

$$\frac{(\delta-\psi)\cdot Z}{\lambda\cdot Z}=12\frac{pq(p-1)(q-1)-1}{(p-1)(q-1)(2pq-p-q-1)}.$$

*Considered as a family of unpointed curves of genus $g=(p-1)(q-1)/2$, the family $Z$ has slope*

$$\frac{\delta\cdot Z}{\lambda\cdot Z}=\frac{12pq}{2pq-p-q-1}.$$

## 7. CONNECTIONS TO GIT

The $\alpha$-invariant can be reinterpreted in terms of the Hilbert-Mumford index in geometric invariant theory. Recall that for every $g$, $n$ and $m$, we have the Hilbert and Chow quotients

$$\overline{\text{Hilb}}^{m,\text{ss}}_{g,n}/\!/\,\text{SL}_{N+1}\quad\text{and}\quad\overline{\text{Chow}}^{\text{ss}}_{g,n}/\!/\,\text{SL}_{N+1}$$

parameterizing, respectively, semistable $m^{th}$ Hilbert points of $n$-canonically embedded curves of genus $g$, and semistable Chow points of $n$-canonically embedded curves of genus $g$ curves. Here, $N=g$ if $n=1$, and $N=(2n-1)(g-1)$ if $n>1$.

**Proposition 7.1.** *Let $T$ be a curve singularity with a $\mathbb{G}_m$-action. Let $C=C_0\cup E$ be a Gorenstein $n$-canonically embedded genus $g$ curve with an $S_2$ miniversal deformation space where $E$ is the union of rational curves with a singularity $p$ of type $T$ as in Section 2. Let $\eta\colon\mathbb{G}_m\to\text{Aut}(C)$ be a one-parameter subgroup which induces $\widetilde{\eta}\colon\mathbb{G}_m\to\text{Aut}(C)\to\text{PGL}_{N+1}$. Then the Hilbert-Mumford indices are*

$$\mu^{\overline{\text{Hilb}}^m_{g,n}}([C],\widetilde{\eta})=\begin{cases}\chi_\lambda+(m-1)\left[(4g+2)m-g+1)\chi_\lambda-\frac{gm}{2}\chi_\delta\right], & \text{if } n=1,\\ (m-1)(g-1)\left[(6mn^2-2mn-2n+1)\chi_\lambda-\frac{mn^2}{2}\chi_\delta\right], & \text{if } n>1,\end{cases}$$

*and*

$$\mu^{\overline{Chow}_{g,n}}([C],\widetilde{\eta}) = \begin{cases} (4g+2)\chi_\lambda - \frac{g}{2}\chi_\delta & \text{if } n=1, \\ (g-1)n[(6n-2)\chi_\lambda - \frac{n}{2}\chi_\delta] & \text{if } n>1. \end{cases}$$

*Proof.* This result follows directly by computing the divisor classes of the GIT polarizations as in [Mum77, Theorem 5.15] and [HH08, Section 5]. □

This proposition implies that if one can compute the characters of $\lambda$ and $\delta$ (or equivalently $\lambda$ and $\lambda_2$) with respect to one-parameter subgroups of the automorphism group, then one immediately knows the Hilbert-Mumford indices for all Hilbert and Chow quotients for such one-parameter subgroups; if the Hilbert (resp. Chow) index is non-zero, then the curve is Hilbert (resp. Chow) unstable. In particular, we recover the result of [Hye10, Propositions 2 and 3].

**GIT stability of ribbons.** Since the Hilbert-Mumford index can be interpreted in terms of the characters of $\lambda$ and $\delta$ (see Proposition 7.1), we obtain the following result as a corollary of the computations made in Example 5.5. We keep notation of Example 5.5. In particular,

**Theorem 7.2** (Hilbert stability of ribbons)**.** *Let $C_\ell$ be a ribbon defined by $f(y) = y^{-\ell}$ for some $\ell \in \{1,\dots,g-2\}$. Then $C_\ell$ admits a $\mathbb{G}_m$-action $\rho\colon \mathbb{G}_m \to \mathrm{Aut}(C)$ and*

*(1) If $\ell \neq (g-1)/2$, then the $m^{th}$ Hilbert point of the $n$-canonical embedding of the ribbon $C_\ell$ is unstable, for all $m \geq 2$ and $n \geq 1$.*

*(2) If $\ell = (g-1)/2$, then the $m^{th}$ Hilbert point of the $n$-canonical embedding of the ribbon $C_\ell$ is unstable for all $m \geq 2$ and $n \geq 2$.*

*(3) If $\ell = (g-1)/2$, then the $m^{th}$ Hilbert point of the canonical embedding of $C_\ell$ is $\rho$-strictly semistable for all $m \geq 2$.*

*Proof.* The ribbon $C_\ell$ is obtained by a gluing $\operatorname{Spec} k[x,x^{-1},\varepsilon]/(\varepsilon^2)$ and $\operatorname{Spec} k[y,y^{-1},\eta]/(\eta^2)$ via

$$x \mapsto y^{-1} - y^{-\ell-2}\eta,$$
$$\varepsilon \mapsto y^{-g-1}\eta,$$

and the $\mathbb{G}_m$-action on $C_\ell$ is given by $t\cdot(x,y,\varepsilon,\eta) = (tx, t^{-1}y, t^{g-\ell}\varepsilon, t^{-\ell-1}\eta)$. By Example 5.5,

$$\chi_\lambda(C_\ell) = g\left(\ell - \frac{g-1}{2}\right), \qquad \chi_\delta(C_\ell) = (5g-4)\left(\ell - \frac{g-1}{2}\right).$$

In follows by Proposition 7.1 that the Hilbert-Mumford index with respect to $\rho$ of the $m^{th}$ Hilbert point of the canonical embedding of $C_\ell$ is

$$\mu^{\overline{\mathrm{Hilb}}^{ss,m}_{g,1}}([C],\rho) = \chi_\lambda(C_\ell)(g+m-gm) = g(g+m-gm)\left(\ell - \frac{g-1}{2}\right).$$

In particular, it is 0 iff $\ell = (g-1)/2$. Similarly, we verify that the Hilbert-Mumford index $\mu^{\overline{\mathrm{Hilb}}^{ss,m}_{g,n}}([C],\rho)$ of the $m^{th}$ Hilbert point of the $n$-canonical embedding of $C_\ell$ is non-zero for every $n,m \geq 2$. This finishes the proof. □

**Corollary 7.3.** *If $g$ is even, then every ribbon of genus $g$ with a $\mathbb{G}_m$-action is unstable.*

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(Alper) DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, 2990 BROADWAY, NEW YORK, NY 10027

*E-mail address*: `jarod@math.columbia.edu`

(Fedorchuk) DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, 2990 BROADWAY, NEW YORK, NY 10027

*E-mail address*: `mfedorch@math.columbia.edu`

(Smyth) DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, 1 OXFORD STREET, CAMBRIDGE, MA 01238

*E-mail address*: `smyth@math.harvard.edu`