

INVARIANT RINGS THROUGH CATEGORIES

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ABSTRACT. We formulate a notion of “geometric reductivity” in an abstract categorical setting which we refer to as adequacy. The main theorem states that the adequacy condition implies that the ring of invariants is finitely generated. This result applies to the category of modules over a bialgebra, the category of comodules over a bialgebra, and the category of quasi-coherent sheaves on a finite type algebraic stack over an affine base.

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1. INTRODUCTION

A fundamental theorem in invariant theory states that if a reductive group G over a field k acts on a finitely generated k -algebra A , then the ring of invariants A^G is finitely generated over k (see [MFK94, Appendix 1.C]). Mumford’s conjecture, proven by Haboush in [Hab75], states that reductive groups are geometrically reductive; therefore this theorem is reduced to showing that the ring of invariants under an action by a geometrically reductive group is finitely generated, which was originally proved by Nagata in [Nag64].

Nagata’s theorem has been generalized to various settings. Seshadri showed an analogous result for an action of a “geometrically reductive” group scheme over a universally Japanese base scheme (see [Ses77]). In [BFS92], the result is generalized to an action of a “geometrically reductive” commutative Hopf algebra over a field on a coalgebra. In [KT08], an analogous result is proven for an action of a “geometrically reductive” (non-commutative) Hopf algebra over a field on an algebra. In [Alp08] and [Alp10], analogous results are shown for the invariants of certain pre-equivalence relations; moreover, [Alp10] systematically develops the theory of adequacy for algebraic stacks.

These settings share a central underlying “adequacy” property which we formulate in an abstract categorical setting. Namely, consider a homomorphism of commutative rings $R \rightarrow A$. Consider an R -linear \otimes -category \mathcal{C} with a faithful exact R -linear \otimes -functor

$$F : \mathcal{C} \longrightarrow \text{Mod}_A$$

such that \mathcal{C} is endowed with a ring object $\mathcal{O} \in \text{Ob}(\mathcal{C})$ which is a unit for \otimes . For precise definitions, please see Situation 2.1. One can then define

$$\Gamma : \mathcal{C} \longrightarrow \text{Mod}_R, \quad \mathcal{F} \longmapsto \text{Mor}_{\mathcal{C}}(\mathcal{O}, \mathcal{F}).$$

Adequacy means (roughly) in this setting that Γ satisfies: if $\mathcal{A} \rightarrow \mathcal{B}$ is a surjection of commutative ring objects and if $f \in \Gamma(\mathcal{B})$, then there exists $g \in \Gamma(\mathcal{A})$ with $g \mapsto f^n$ for some $n > 0$. The main theorem of this paper is Theorem 13.5 which states (roughly) that if Γ is adequate, then

- (1) $\Gamma(\mathcal{A})$ is of finite type over R if \mathcal{A} is of finite type, and
- (2) $\Gamma(\mathcal{F})$ is a finite type $\Gamma(\mathcal{A})$ -module if \mathcal{F} is of finite type.

Note that additional assumptions have to be imposed on the categorical setting in order to even formulate the result.

In the final sections of this paper, we show how the abstract categorical setting applies to (a) the category of modules over a bialgebra, (b) the category of comodules over a bialgebra, and (c) the category of quasi-coherent sheaves on a finite type algebraic stack over an affine base. Thus the main theorem above unifies and generalizes the results mentioned above, which was the original motivation for this research.

What is lacking in this theory is a practical criterion for adequacy. Thus we would like to ask the following questions: Is there a notion of reductivity in the categorical setting? Is there an abstract analogue of Haboush’s theorem? We hope to return to these questions in future research.

Conventions. Rings are associative with 1. Abelian categories are additive categories with kernels and cokernels such that $\text{Im} \cong \text{Coim}$ for any morphism.

2. SETUP

In this section, we introduce the types of structure we are going to work with. We keep the list of basic properties to an absolute minimum, and later we introduce additional axioms to impose.

Situation 2.1. We consider the following systems of data:

- (1) $R \rightarrow A$ is a map of commutative rings,

- (2) \mathcal{C} is an R -linear abelian category,
- (3) $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is an R -bilinear functor,
- (4) $F : \mathcal{C} \rightarrow \text{Mod}_A$ is a faithful exact R -linear functor,
- (5) there is a given bifunctorial isomorphism

$$\gamma_{\mathcal{F}, \mathcal{G}} : F(\mathcal{F}) \otimes_A F(\mathcal{G}) \longrightarrow F(\mathcal{F} \otimes \mathcal{G}),$$

- (6) there exist functorial isomorphisms

$$\tau_{\mathcal{F}, \mathcal{G}, \mathcal{H}} : (\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H} \longrightarrow \mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{H})$$

which are compatible with the usual associativity of tensor products of A -modules via γ , and

- (7) there is an object \mathcal{O} of \mathcal{C} endowed with functorial isomorphisms $\mu : \mathcal{O} \otimes \mathcal{F} \rightarrow \mathcal{F}$, and $\nu : \mathcal{F} \otimes \mathcal{O} \rightarrow \mathcal{F}$ such that $F(\mathcal{O}) = A$ and the isomorphisms correspond to the usual isomorphisms $A \otimes_A M = M \otimes_A A = M$ (via γ above).

If an associativity constraint τ as above exists, then it is uniquely determined by the condition that it agrees with the usual associativity constraint for A -modules (as F is faithful). Hence we often do not list it as part of the data, and we say ‘‘Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1’’.

Note that in particular $\mathcal{O} \otimes \mathcal{O} = \mathcal{O}$, and hence that \mathcal{O} is a ring object of \mathcal{C} (see Section 8), and for this ring structure every object of \mathcal{C} is in a canonical way an \mathcal{O} -module.

Definition 2.2. In the situation above we define the *global sections functor* to be the functor

$$\Gamma : \mathcal{C} \longrightarrow \text{Mod}_R, \quad \mathcal{F} \longmapsto \Gamma(\mathcal{F}) = \text{Mor}_{\mathcal{C}}(\mathcal{O}, \mathcal{F}).$$

Note that $\Gamma(\mathcal{F}) \subset F(\mathcal{F})$ since the functor F is faithful. There are canonical maps $\Gamma(\mathcal{F}) \otimes_R \Gamma(\mathcal{G}) \longrightarrow \Gamma(\mathcal{F} \otimes \mathcal{G})$ defined by mapping the pure tensor $f \otimes g$ to the map

$$\mathcal{O} = \mathcal{O} \otimes \mathcal{O} \xrightarrow{f \otimes g} \mathcal{F} \otimes \mathcal{G}$$

For any pair of objects \mathcal{F}, \mathcal{G} of \mathcal{C} there is a commutative diagram

$$\begin{array}{ccc} \Gamma(\mathcal{F}) \otimes_R \Gamma(\mathcal{G}) & \longrightarrow & \Gamma(\mathcal{F} \otimes \mathcal{G}) \\ \downarrow & & \downarrow \\ F(\mathcal{F}) \otimes_A F(\mathcal{G}) & \xlongequal{\quad} & F(\mathcal{F} \otimes \mathcal{G}) \end{array}$$

In particular, there is a natural $\Gamma(\mathcal{O})$ -module structure on $\Gamma(\mathcal{F})$ for every object \mathcal{F} of \mathcal{C} .

3. AXIOMS

The following axioms will be introduced throughout the text. For the convenience of the reader, we list them here.

Definition 3.1. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. We introduce the following axioms:

- (D) The category \mathcal{C} has arbitrary direct summands, and \otimes , F , and Γ commute with these.

- (C) There exist functorial isomorphisms $\sigma_{\mathcal{F}, \mathcal{G}} : \mathcal{F} \otimes \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{F}$ such that $\sigma_{\mathcal{F}, \mathcal{G}}$ is via F and γ compatible with the usual commutativity constraint $M \otimes_A N \cong N \otimes_A M$ on A -modules.
- (I) The category \mathcal{C} has arbitrary direct products, and F commutes with them.
- (S) For every object \mathcal{F} of \mathcal{C} and any $n \geq 1$ there exists a quotient

$$\mathcal{F}^{\otimes n} \longrightarrow \mathrm{Sym}_{\mathcal{C}}^n(\mathcal{F})$$

such that the map of A -modules $F(\mathcal{F}^{\otimes n}) \longrightarrow F(\mathrm{Sym}_{\mathcal{C}}^n(\mathcal{F}))$ factors through the natural surjection $F(\mathcal{F})^{\otimes n} \rightarrow \mathrm{Sym}_A^n(F(\mathcal{F}))$, and such that $\mathrm{Sym}_{\mathcal{C}}^n(\mathcal{F})$ is universal with this property.

- (L) Every object \mathcal{F} of \mathcal{C} is a filtered colimit $\mathcal{F} = \mathrm{colim} \mathcal{F}_i$ of finite type objects \mathcal{F}_i such that $F(\mathcal{F}) = \mathrm{colim} F(\mathcal{F}_i)$.
- (N) The ring A is Noetherian.
- (G) The functor Γ is exact.
- (A) For every surjection of weakly commutative ring objects $\mathcal{A} \rightarrow \mathcal{B}$ in \mathcal{C} with \mathcal{A} locally finite, and any $f \in \Gamma(\mathcal{B})$, there exists an $n > 0$ and an element $g \in \Gamma(\mathcal{A})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{B})$.

Terminology used above: An object \mathcal{F} of \mathcal{C} is *finite type* if $F(\mathcal{F})$ is finite type, see Definition 10.1. A ring object \mathcal{A} , see Definition 8.1, is *weakly commutative* if $F(\mathcal{A})$ is commutative, see Definition 9.1. An object \mathcal{F} of \mathcal{C} is *locally finite* if it is a filtered colimit $\mathcal{F} = \mathrm{colim} \mathcal{F}_i$ of finite type objects \mathcal{F}_i such that also $F(\mathcal{F}) = \mathrm{colim} F(\mathcal{F}_i)$, see Definition 11.2.

4. DIRECT SUMMANDS

We cannot prove much without the following axiom.

Definition 4.1. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. We introduce the following axiom:

- (D) The category \mathcal{C} has arbitrary direct summands, and \otimes , F , and Γ commute with these.

This implies that \mathcal{C} has colimits and that \otimes , F and Γ commute with these.

Lemma 4.2. *Assume that we are in Situation 2.1 and that the axiom (D) holds. Then Γ has a left adjoint*

$$\mathcal{O} \otimes_R - : \mathrm{Mod}_R \longrightarrow \mathcal{C}$$

with $\mathcal{O} \otimes_R R \cong \mathcal{O}$, and $F(\mathcal{O} \otimes_R M) = A \otimes_R M$. Moreover, for any object \mathcal{F} of \mathcal{C} there is a canonical isomorphism $\mathcal{F} \otimes (\mathcal{O} \otimes_R M) = (\mathcal{O} \otimes_R M) \otimes \mathcal{F}$ which reduces to the obvious isomorphism on applying F .

Proof. For any R -module M choose a presentation $\bigoplus_{j \in J} R \rightarrow \bigoplus_{i \in I} R \rightarrow M \rightarrow 0$ and define

$$\mathcal{O} \otimes_R M = \mathrm{Coker}\left(\bigoplus_{j \in J} \mathcal{O} \longrightarrow \bigoplus_{i \in I} \mathcal{O}\right)$$

where the arrow is given by the same matrix as the matrix used in the presentation for M . With this definition it is clear that $F(\mathcal{O} \otimes_R M) = A \otimes_R M$. Moreover, since there is an exact sequence

$$\bigoplus_{j \in J} \mathcal{O} \longrightarrow \bigoplus_{i \in I} \mathcal{O} \longrightarrow \mathcal{O} \otimes_R M \longrightarrow 0$$

it is straightforward to verify that $\mathrm{Mor}_{\mathcal{C}}(\mathcal{O} \otimes_R M, \mathcal{F}) = \mathrm{Mor}_R(M, \Gamma(\mathcal{F}))$. We leave the proof of the last statement to the reader. \square

In the situation of the lemma we will write $M \otimes_R \mathcal{F}$ instead of the more clumsy notation $M \otimes_R \mathcal{O} \otimes \mathcal{F}$.

Remark 4.3. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1, and further assume (D) holds. By Lemma 4.2 above, we have a diagram of functors

$$\begin{array}{ccc} \text{Mod}_R & \begin{array}{c} \xrightarrow{\mathcal{O} \otimes_R -} \\ \xleftarrow{\Gamma} \\ \xrightarrow{A \otimes_R -} \end{array} & \mathcal{C} \\ & & \downarrow F \\ & & \text{Mod}_A \end{array}$$

where $F \circ (\mathcal{O} \otimes_R -) = (A \otimes_R -)$, and $\mathcal{O} \otimes_R -$ is a left adjoint to Γ .

5. COMMUTATIVITY

Definition 5.1. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. We introduce the following axiom:

- (C) There exist functorial isomorphisms $\sigma_{\mathcal{F}, \mathcal{G}} : \mathcal{F} \otimes \mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{F}$ such that $\sigma_{\mathcal{F}, \mathcal{G}}$ is via F and γ compatible with the usual commutativity constraint $M \otimes_A N \cong N \otimes_A M$ on A -modules.

As in the case of the associativity constraint, if such maps $\sigma_{\mathcal{F}, \mathcal{G}}$ exist, then they are unique.

6. DIRECT PRODUCTS

Definition 6.1. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. We introduce the following axiom:

- (I) The category \mathcal{C} has arbitrary direct products, and F commutes with them.

If this is the case, then the category \mathcal{C} has inverse limits and the functor F commutes with them, which is why we use the letter (I) to indicate this axiom.

In the following lemma and its proof we will use the following abuse of notation. Suppose that \mathcal{F}, \mathcal{G} are two objects of \mathcal{C} , and that $\alpha : F(\mathcal{F}) \rightarrow F(\mathcal{G})$ is an A -module map. We say that α is a *morphism of \mathcal{C}* if there exists a morphism $a : \mathcal{F} \rightarrow \mathcal{G}$ in \mathcal{C} such that $F(a) = \alpha$. Note that if a exists it is unique.

Lemma 6.2. *Assume we are in Situation 2.1 and that (I) holds. Let \mathcal{F}, \mathcal{G} be two objects of \mathcal{C} . Let $\alpha : F(\mathcal{F}) \rightarrow F(\mathcal{G})$ be an A -module map. The functor*

$$\mathcal{C} \longrightarrow \text{Sets}, \quad \mathcal{H} \longmapsto \{\varphi \in \text{Mor}_{\mathcal{C}}(\mathcal{G}, \mathcal{H}) \mid F(\varphi) \circ \alpha \text{ is a morphism of } \mathcal{C}\}$$

is representable. The universal object $\mathcal{G} \rightarrow \mathcal{G}'$ is a surjection.

Proof. Since \mathcal{C} is abelian, any morphism $\pi : \mathcal{G} \rightarrow \mathcal{H}$ factors uniquely as $\mathcal{G} \rightarrow \mathcal{H}' \rightarrow \mathcal{H}$ where the first map π' is a surjection and the second is an injection. If $F(\pi) \circ \alpha = F(a)$ is a morphism of \mathcal{C} , then a factors through \mathcal{H}' and we see that $F(\pi') \circ \alpha$ is a morphism of \mathcal{C} . Hence it suffices to consider surjections. Consider the set $T = \{\pi : \mathcal{G} \rightarrow \mathcal{H}_\pi\}$ of surjections π such that $F(\pi) \circ \alpha$ is a morphism of \mathcal{C} . Set

$$\mathcal{G}' = \text{Im}(\mathcal{G} \longrightarrow \prod_{\pi \in T} \mathcal{H}_\pi).$$

The rest is clear. □

7. SYMMETRIC PRODUCTS

We introduce the axiom (S) and show that either axiom (I) or (C) implies (S).

Definition 7.1. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. We introduce the following axiom:

(S) For every object \mathcal{F} of \mathcal{C} and any $n \geq 1$ there exists a quotient

$$\mathcal{F}^{\otimes n} \longrightarrow \mathrm{Sym}_{\mathcal{C}}^n(\mathcal{F})$$

such that the map of A -modules $F(\mathcal{F}^{\otimes n}) \longrightarrow F(\mathrm{Sym}_{\mathcal{C}}^n(\mathcal{F}))$ factors through the natural surjection $F(\mathcal{F})^{\otimes n} \rightarrow \mathrm{Sym}_A^n(F(\mathcal{F}))$, and such that $\mathrm{Sym}_{\mathcal{C}}^n(\mathcal{F})$ is universal with this property.

Note that if axiom (S) holds, then the universality implies the rule $\mathcal{F} \rightsquigarrow \mathrm{Sym}_{\mathcal{C}}^n(\mathcal{F})$ is a functor. Moreover, for every $n, m \geq 0$ there are canonical maps

$$\mathrm{Sym}_{\mathcal{C}}^n(\mathcal{F}) \otimes \mathrm{Sym}_{\mathcal{C}}^m(\mathcal{F}) \longrightarrow \mathrm{Sym}_{\mathcal{C}}^{n+m}(\mathcal{F}).$$

If axiom (D) holds as well, then this will turn $\bigoplus_{n \geq 0} \mathrm{Sym}_{\mathcal{C}}^n(\mathcal{F})$ into a weakly commutative ring object of \mathcal{C} (see Definitions 8.1 and 9.1 below).

Lemma 7.2. *In Situation 2.1, if either axiom (C) or (I) holds, then axiom (S) holds.*

Proof. Suppose (C) holds. If \mathcal{F} is an object of \mathcal{C} , using the maps $\sigma_{\mathcal{F}, \mathcal{F}}$ we get an action of the symmetric group S_n on n letters on $\mathcal{F}^{\otimes n}$ (to see that it is an action of S_n apply the faithful functor F). Thus, $\mathrm{Sym}_{\mathcal{C}}^n(\mathcal{F})$ can be defined as the cokernel of a map

$$\bigoplus_{\tau \in S_n} \mathcal{F}^{\otimes n} \longrightarrow \mathcal{F}^{\otimes n}$$

where in the summand corresponding to τ we use the difference of the identity and the map corresponding to τ .

Suppose (I) holds. Let \mathcal{F} be an object of \mathcal{C} . The quotient $\mathcal{F}^{\otimes n} \rightarrow \mathrm{Sym}_{\mathcal{C}}^n(\mathcal{F})$ is characterized by the property that if $a : \mathcal{F}^{\otimes n} \rightarrow \mathcal{G}$ is a map such that $F(a)$ factors through $F(\mathcal{F})^{\otimes n} \rightarrow \mathrm{Sym}_A^n(F(\mathcal{F}))$ then a factors in \mathcal{C} through the map to $\mathrm{Sym}_{\mathcal{C}}^n(\mathcal{F})$. To prove such a quotient exists apply Lemma 6.2 to the map

$$\bigoplus_{\tau \in S_n} F(\mathcal{F})^{\otimes n} \longrightarrow F(\mathcal{F})^{\otimes n}$$

mentioned above. □

8. RING OBJECTS

Definition 8.1. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1.

- (1) A *ring object* \mathcal{A} in \mathcal{C} consists of an object \mathcal{A} of \mathcal{C} endowed with maps $\mathcal{O} \rightarrow \mathcal{A}$ and $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ which on applying F induce an A -algebra structure on $F(\mathcal{A})$.
- (2) If \mathcal{A} is a ring object of \mathcal{C} , then a (*left*) *module object* over \mathcal{A} is an object \mathcal{F} endowed with a morphism $\mu_{\mathcal{F}} : \mathcal{A} \otimes \mathcal{F} \rightarrow \mathcal{F}$ such that $F(\mathcal{A}) \otimes_A F(\mathcal{F}) \rightarrow F(\mathcal{F})$ induces an $F(\mathcal{A})$ -module structure on $F(\mathcal{F})$.

If \mathcal{A} is a ring object of \mathcal{C} , then $\Gamma(\mathcal{A})$ inherits an R -algebra structure in a natural manner. In other words, we have the following diagram of rings

$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow & & \downarrow \\ \Gamma(\mathcal{O}) & \longrightarrow & \Gamma(\mathcal{A}) \longrightarrow F(\mathcal{A}) \end{array}$$

In the same vein, given a \mathcal{A} -module \mathcal{F} the global sections $\Gamma(\mathcal{F})$ are a $\Gamma(\mathcal{A})$ -module in a natural way. Let $\text{Mod}_{\mathcal{A}}$ denote the category of \mathcal{A} -modules.

Lemma 8.2. *Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. If \mathcal{A} is a ring object in \mathcal{C} , then the category $\text{Mod}_{\mathcal{A}}$ is abelian.*

Proof. Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a map of \mathcal{A} -modules. Set $\mathcal{K} = \text{Ker}(\varphi)$ and $\mathcal{Q} = \text{Coker}(\varphi)$ in \mathcal{C} . We claim that both \mathcal{K} and \mathcal{Q} have natural \mathcal{A} -module structure that turn them into the kernel and cokernel of φ in $\text{Mod}_{\mathcal{A}}$. To see this for \mathcal{K} consider the map

$$\mathcal{A} \otimes \mathcal{K} \rightarrow \mathcal{A} \otimes \mathcal{F} \rightarrow \mathcal{F}$$

Its composition with the map to \mathcal{G} is zero as φ is a map of \mathcal{A} -modules. Hence we see that it factors into a map $\mathcal{A} \otimes \mathcal{K} \rightarrow \mathcal{K}$. To get the module structure for \mathcal{Q} , note that the sequence

$$\mathcal{A} \otimes \mathcal{F} \rightarrow \mathcal{A} \otimes \mathcal{G} \rightarrow \mathcal{A} \otimes \mathcal{Q} \rightarrow 0$$

is exact, because it is exact on applying F . Hence the module structure on \mathcal{G} induces one on \mathcal{Q} . We omit checking that these structures do indeed give the kernel and cokernel of φ in $\text{Mod}_{\mathcal{A}}$. \square

Let us use $\text{Hom}_{\mathcal{A}}(-, -)$ for the morphisms in the category $\text{Mod}_{\mathcal{A}}$. Note that

$$\Gamma(\mathcal{F}) = \text{Mor}_{\mathcal{C}}(\mathcal{O}, \mathcal{F}) = \text{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{F})$$

for $\mathcal{F} \in \text{Mod}_{\mathcal{A}}$. The map from the left to the right associates to $f : \mathcal{O} \rightarrow \mathcal{F}$ the map

$$\mathcal{A} = \mathcal{A} \otimes \mathcal{O} \xrightarrow{1 \otimes f} \mathcal{A} \otimes \mathcal{F} \xrightarrow{\mu_{\mathcal{F}}} \mathcal{F}.$$

Lemma 8.3. *In Situation 2.1 assume axiom (D) and let \mathcal{A} be a ring object in \mathcal{C} . Then the functor*

$$\Gamma : \text{Mod}_{\mathcal{A}} \longrightarrow \text{Mod}_{\Gamma(\mathcal{A})}$$

has a right adjoint

$$\mathcal{A} \otimes_{\Gamma(\mathcal{A})} - : \text{Mod}_{\Gamma(\mathcal{A})} \longrightarrow \text{Mod}_{\mathcal{A}}.$$

We have $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{A}) = \mathcal{A}$ and $F(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} M) = F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} M$.

Proof. The proof is identical to the argument of Lemma 4.2 using that $\Gamma(\mathcal{F}) = \text{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{F})$ for any \mathcal{A} -module \mathcal{F} . \square

Remark 8.4. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. Assume axiom (D). Let \mathcal{A} be a ring object, and let S be a set. We can define the *polynomial algebra over \mathcal{A}* as the ring object

$$\mathcal{A}[x_s; s \in S] = \mathcal{A} \otimes_{\Gamma(\mathcal{A})} (\Gamma(\mathcal{A})[x_s; s \in S])$$

Explicitly $\mathcal{A}[x_s; s \in S] = \bigoplus_I \mathcal{A}x^I$ where I runs over all functions $I : S \rightarrow \mathbf{Z}_{\geq 0}$ with finite support. The symbol $x^I = \prod x_s^{I(s)}$ indicates the corresponding monomial.

The multiplication on $\mathcal{A}[x_s; s \in S]$ is defined by requiring the “elements” of \mathcal{A} to commute with the variables x_s .

A homomorphism $\mathcal{A}[x_s; s \in S] \rightarrow \mathcal{B}$ of ring objects is given by a homomorphism $\mathcal{A} \rightarrow \mathcal{B}$ of ring objects together with some elements $y_s \in \Gamma(\mathcal{B})$ which commute with all elements in the image of $F(\mathcal{A}) \rightarrow F(\mathcal{B})$.

9. COMMUTATIVE RING OBJECTS AND MODULES

Definition 9.1. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. A ring object \mathcal{A} is called *weakly commutative* if $F(\mathcal{A})$ is commutative.

Lemma 9.2. *In Situation 2.1. If \mathcal{A} is a weakly commutative ring and $\mathcal{I} \subset \mathcal{A}$ is a left ideal, then \mathcal{I} is a two-sided ideal and \mathcal{A}/\mathcal{I} is a weakly commutative ring.*

Proof. Consider the image \mathcal{I}' of the multiplication $\mathcal{A} \otimes \mathcal{I} \rightarrow \mathcal{A}$. By assumption $F(\mathcal{I}') = F(\mathcal{I})$, hence we have equality. The final assertion is clear. \square

In order to define the tensor product of two modules over a ring object we use the notion of commutative modules.

Definition 9.3. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1.

- (1) A ring object \mathcal{A} is called *commutative* if there exists an isomorphism $\sigma : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ which under F gives the usual flip isomorphism and which is compatible with the multiplication (so in particular \mathcal{A} is weakly commutative).
- (2) A module object \mathcal{F} over a ring object \mathcal{A} is said to be *commutative* if there exists an isomorphism $\sigma : \mathcal{F} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{F}$ which on applying F gives the usual flip isomorphism.

It is clear that if axiom (C) holds, then any weakly commutative ring object is commutative and all module objects are automatically commutative. Let us denote $\text{Mod}_{\mathcal{A}}^c$ the category of all commutative \mathcal{A} -modules. This category always has cokernels, but not necessarily kernels.

Lemma 9.4. *Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. Let \mathcal{A} be a commutative ring object of \mathcal{C} . The category $\text{Mod}_{\mathcal{A}}^c$ is abelian in each of the following cases:*

- (1) *axiom (C) holds, or*
- (2) *the ring map $F(\mathcal{A}) \rightarrow F(\mathcal{A}) \otimes_A F(\mathcal{A})$ is flat.*

The second condition holds for example if $A \rightarrow F(\mathcal{A})$ is either flat or surjective.

Proof. In case (1) we have $\text{Mod}_{\mathcal{A}} = \text{Mod}_{\mathcal{A}}^c$ so the statement follows from Lemma 8.2. For case (2), let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a map of commutative \mathcal{A} -modules. We set $\mathcal{K} = \text{Ker}(\varphi)$ and $\mathcal{Q} = \text{Coker}(\varphi)$ in \mathcal{C} , and we know that these are kernels and cokernels in $\text{Mod}_{\mathcal{A}}$. The diagram with exact rows

$$\begin{array}{ccccccc}
 \mathcal{F} \otimes \mathcal{A} & \longrightarrow & \mathcal{G} \otimes \mathcal{A} & \longrightarrow & \mathcal{Q} \otimes \mathcal{A} & \longrightarrow & 0 \\
 \downarrow \sigma & & \downarrow \sigma & & \downarrow \text{dotted} & & \\
 \mathcal{A} \otimes \mathcal{F} & \longrightarrow & \mathcal{A} \otimes \mathcal{G} & \longrightarrow & \mathcal{A} \otimes \mathcal{Q} & \longrightarrow & 0
 \end{array}$$

defines the commutativity map σ for \mathcal{Q} . But in general we do not know that the map $\mathcal{K} \otimes \mathcal{A} \rightarrow \mathcal{F} \otimes \mathcal{A}$ is injective. After applying F this becomes the map

$$F(\mathcal{K}) \otimes_A F(\mathcal{A}) \rightarrow F(\mathcal{F}) \otimes_A F(\mathcal{A})$$

By our discussion in Section 8 we know that $B = F(\mathcal{A})$ is a commutative A -algebra, and $F(\mathcal{K}) \subset F(\mathcal{F})$ is an inclusion of B -modules. Note that for a B -module M we have $M \otimes_A B = M \otimes_B (B \otimes_A B)$. Hence the injectivity of the last displayed map is clear if property (2) holds, and in this case we get the commutativity restraint for \mathcal{K} also. \square

If \mathcal{A} is a commutative ring object of \mathcal{C} and \mathcal{F}, \mathcal{G} are module objects over \mathcal{A} , and \mathcal{F} is commutative then we define

$$\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G} := \text{Coequalizer of } \begin{array}{ccc} \mathcal{A} \otimes \mathcal{F} \otimes \mathcal{G} & \xrightarrow{\sigma \otimes 1} & \mathcal{F} \otimes \mathcal{A} \otimes \mathcal{G} \\ \downarrow \mu \otimes 1 & & \downarrow 1 \otimes \mu \\ \mathcal{F} \otimes \mathcal{G} & \xrightarrow{1} & \mathcal{F} \otimes \mathcal{G} \end{array}$$

going around both ways

Then it is clear that there is a canonical isomorphism

$$\gamma_{\mathcal{A}} : F(\mathcal{F}) \otimes_{F(\mathcal{A})} F(\mathcal{G}) \longrightarrow F(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G})$$

which is functorial in the pair $(\mathcal{F}, \mathcal{G})$. In particular, it is clear that there are functorial isomorphisms

$$\mu_{\mathcal{A}} : \mathcal{A} \otimes_{\mathcal{A}} \mathcal{F} \longrightarrow \mathcal{F}, \quad \mu_{\mathcal{A}} : \mathcal{F} \otimes_{\mathcal{A}} \mathcal{A} \longrightarrow \mathcal{F}$$

for any commutative \mathcal{A} -module \mathcal{F} (via σ and the multiplication map for \mathcal{F}).

Lemma 9.5. *Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. Let \mathcal{A} be a commutative ring object of \mathcal{C} . Assume the category $\text{Mod}_{\mathcal{A}}^{\mathcal{C}}$ is abelian. Then*

$$(R \rightarrow F(\mathcal{A}), \text{Mod}_{\mathcal{A}}^{\mathcal{C}}, \otimes_{\mathcal{A}}, F, \gamma_{\mathcal{A}}, \mathcal{A}, \mu_{\mathcal{A}})$$

is another set of data as in Situation 2.1. Furthermore, if axiom (D) is satisfied for $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$, then it is also satisfied for $(R \rightarrow F(\mathcal{A}), \text{Mod}_{\mathcal{A}}^{\mathcal{C}}, \otimes_{\mathcal{A}}, F, \gamma_{\mathcal{A}}, \mathcal{A}, \mu_{\mathcal{A}})$.

Proof. This is clear from the discussion above. \square

In the situation of the lemma we have the global sections functor

$$\Gamma_{\mathcal{A}} : \text{Mod}_{\mathcal{A}} \longrightarrow \text{Mod}_R, \quad \mathcal{F} \longmapsto \text{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{F}).$$

We have seen in Section 8 that for an object $\mathcal{F} \in \text{Mod}_{\mathcal{A}}$ we have $\Gamma_{\mathcal{A}}(\mathcal{F}) = \Gamma(\mathcal{F})$ as R -modules. We will often abuse notation by writing $\Gamma = \Gamma_{\mathcal{A}}$.

10. FINITENESS CONDITIONS

Here are some finiteness conditions we can impose.

Definition 10.1. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1.

- (1) An object \mathcal{F} of \mathcal{C} is said to be *of finite type* if $F(\mathcal{F})$ is a finitely generated A -module.
- (2) A ring object \mathcal{A} of \mathcal{C} is said to be *of finite type* if $F(\mathcal{A})$ is a finitely generated A -algebra.
- (3) A module object \mathcal{F} over a ring object \mathcal{A} of \mathcal{C} is said to be *of finite type* if $F(\mathcal{F})$ is of finite type over $F(\mathcal{A})$.

Note that the ring objects in this definition need not be commutative. A noncommutative algebra S over A is finitely generated if it is isomorphic to a quotient of the free algebra $A\langle x_1, \dots, x_n \rangle$ for some n .

11. ADEQUACY

The notion of *adequacy*, which is our analogue of geometric reductivity, can be formulated in a variety of different ways.

Definition 11.1. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. We introduce the following axiom:

(N) The ring A is Noetherian.

Definition 11.2. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. An object \mathcal{F} of \mathcal{C} is called *locally finite* if it is a filtered colimit $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ of finite type objects \mathcal{F}_i such that also $F(\mathcal{F}) = \operatorname{colim} F(\mathcal{F}_i)$.

Definition 11.3. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. We introduce the axiom:

(L) Every object \mathcal{F} of \mathcal{C} is locally finite.

Lemma 11.4. *Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. A quotient of a locally finite object of \mathcal{C} is locally finite. If axioms (N) and (D) hold, then a subobject of a locally finite object is locally finite and the subcategory of locally finite objects is abelian.*

Proof. Suppose that $\mathcal{F} \rightarrow \mathcal{Q}$ is surjective and that \mathcal{F} is locally finite. Write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ of finite type objects \mathcal{F}_i such that also $F(\mathcal{F}) = \operatorname{colim} F(\mathcal{F}_i)$. Set $\mathcal{Q}_i = \operatorname{Im}(\mathcal{F}_i \rightarrow \mathcal{Q})$. We claim that $\mathcal{Q} = \operatorname{colim}_i \mathcal{Q}_i$ and that $F(\mathcal{Q}) = \operatorname{colim} F(\mathcal{Q}_i)$. The last statement follows from exactness of F and the fact that colimits commute with images in Mod_A . If $\beta_i : \mathcal{Q}_i \rightarrow \mathcal{G}$ is a compatible system of maps to an object of \mathcal{C} , then composing with the surjections $\mathcal{F}_i \rightarrow \mathcal{Q}_i$ gives a compatible system of maps also, whence a morphism $\beta : \mathcal{F} \rightarrow \mathcal{G}$. But $F(\beta)$ factors through $F(\mathcal{F}) \rightarrow F(\mathcal{Q})$ and hence is zero on $F(\operatorname{Ker}(\mathcal{F} \rightarrow \mathcal{Q}))$. Because F is faithful and exact we see that β factors as $\mathcal{Q} \rightarrow \mathcal{G}$ as desired.

Suppose that $\mathcal{J} \rightarrow \mathcal{F}$ is injective, that \mathcal{F} is locally finite and that (N) and (D) hold. Write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ of finite type objects \mathcal{F}_i such that also $F(\mathcal{F}) = \operatorname{colim} F(\mathcal{F}_i)$. By the argument of the preceding paragraph applied to $\operatorname{id}_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{F}$ we may assume $\mathcal{F}_i \subset \mathcal{F}_i$ for each i . Set $\mathcal{J}_i = \mathcal{F}_i \cap \mathcal{J}$. Since axiom (N) holds we see that each \mathcal{J}_i is of finite type. As F is exact we see that $\operatorname{colim} F(\mathcal{J}_i) = F(\mathcal{J})$. As axiom (D) holds we know that $\mathcal{J}' = \operatorname{colim} \mathcal{J}_i$ exists and $\operatorname{colim} F(\mathcal{J}_i) = F(\mathcal{J}')$. Hence we get a canonical map $\mathcal{J}' \rightarrow \mathcal{J}$ which has to be an isomorphism as F is exact and faithful. This proves that \mathcal{J} is locally finite.

Assume (N) and (D). Let $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of locally finite objects. We have to show that the kernel and cokernel of α are locally finite. This is clear by the results of the preceding two paragraphs. \square

Lemma 11.5. *Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. Assume axiom (D) holds. The tensor product of locally finite objects is locally finite. For any R -module M the object $M \otimes_R \mathcal{O}$ is locally finite. If \mathcal{A} is a locally finite ring object, then $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} M$ is locally finite for any $\Gamma(\mathcal{A})$ -module M .*

Proof. This is clear since in the presence of (D), the tensor product commutes with colimits. \square

Lemma 11.6. *Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. Assume axiom (S) holds. Consider the following conditions*

- (1) *For every surjection of finite type objects $\mathcal{G} \rightarrow \mathcal{F}$ and $f \in \Gamma(\mathcal{F})$ there exists an $n > 0$ and a $g \in \Gamma(\text{Sym}_{\mathcal{C}}^n(\mathcal{G}))$ which maps to f^n in $\Gamma(\text{Sym}_{\mathcal{C}}^n(\mathcal{F}))$.*
- (2) *For every surjection $\mathcal{G} \rightarrow \mathcal{O}$ with \mathcal{G} of finite type and $f \in \Gamma(\mathcal{O})$ there exists an $n > 0$ and a $g \in \Gamma(\text{Sym}_{\mathcal{C}}^n(\mathcal{G}))$ which maps to f^n in $\Gamma(\mathcal{O})$.*
- (3) *For every surjection of weakly commutative ring objects $\mathcal{A} \rightarrow \mathcal{B}$ in \mathcal{C} with \mathcal{A} locally finite, and any $f \in \Gamma(\mathcal{B})$, there exists an $n > 0$ and an element $g \in \Gamma(\mathcal{A})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{B})$.*

We always have (1) \Rightarrow (2) and (1) \Rightarrow (3). If axiom (N) holds, then (2) \Rightarrow (1). If axiom (D) holds, then (3) \Rightarrow (1). Furthermore, consider the following variations

- (1') *For every surjection of objects $\mathcal{G} \rightarrow \mathcal{F}$ and $f \in \Gamma(\mathcal{F})$ there exists an $n > 0$ and a $g \in \Gamma(\text{Sym}_{\mathcal{C}}^n(\mathcal{G}))$ which maps to f^n in $\Gamma(\text{Sym}_{\mathcal{C}}^n(\mathcal{F}))$.*
- (2') *For every surjection $\mathcal{G} \rightarrow \mathcal{O}$ and $f \in \Gamma(\mathcal{O})$ there exists an $n > 0$ and a $g \in \Gamma(\text{Sym}_{\mathcal{C}}^n(\mathcal{G}))$ which maps to f^n in $\Gamma(\mathcal{O})$.*
- (3') *For every surjection of weakly commutative ring objects $\mathcal{A} \rightarrow \mathcal{B}$ in \mathcal{C} , and any $f \in \Gamma(\mathcal{B})$, there exists an $n > 0$ and an element $g \in \Gamma(\mathcal{A})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{B})$.*

If axiom (L) holds, then (1) \Leftrightarrow (1'), (2) \Leftrightarrow (2'), and (3) \Leftrightarrow (3').

Proof. It is clear that (1) implies (2). Assume (N) + (2) and let us prove (1). Consider $\mathcal{G} \rightarrow \mathcal{F}$ and f as in (1). Let $\mathcal{H} = \mathcal{G} \times_{\mathcal{F}} \mathcal{O}$. Then $\mathcal{H} \rightarrow \mathcal{O}$ is surjective, and $F(\mathcal{H}) = F(\mathcal{G}) \times_{F(\mathcal{F})} A$. By assumption (N) this implies that $F(\mathcal{H})$ is a finite A -module.

Let us prove that (1) implies (3). Let $\mathcal{A} \rightarrow \mathcal{B}$ and f be as in (3). Write $\mathcal{A} = \text{colim}_i \mathcal{G}_i$ as a directed colimit such that $F(\mathcal{A}) = \text{colim}_i F(\mathcal{G}_i)$ and such that each \mathcal{G}_i is of finite type. Think of $f \in \Gamma(\mathcal{B}) \subset F(\mathcal{B})$. Then for some i there exists a $\tilde{f} \in F(\mathcal{G}_i)$ which maps to f . Set $\mathcal{G} = \mathcal{G}_i$, set $\mathcal{F} = \text{Im}(\mathcal{G}_i \rightarrow \mathcal{B})$. The map $\mathcal{G} \rightarrow \mathcal{F}$ is surjective. Since F is exact we see that $f \in F(\mathcal{F}) \subset F(\mathcal{B})$. Hence, as Γ is left exact we conclude that $f \in \Gamma(\mathcal{F})$ as well. Thus property (1) applies and we find an $n > 0$ and a $g \in \Gamma(\text{Sym}_{\mathcal{C}}^n(\mathcal{G}))$ which maps to f^n in $\Gamma(\text{Sym}_{\mathcal{C}}^n(\mathcal{F}))$. Since \mathcal{A} and \mathcal{B} are ring objects we obtain a canonical diagram

$$\begin{array}{ccc} \mathcal{G}^{\otimes n} & \longrightarrow & \mathcal{F}^{\otimes n} \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{B} \end{array}$$

Since \mathcal{A} and \mathcal{B} are weakly commutative this produces a commutative diagram

$$\begin{array}{ccc} \text{Sym}_{\mathcal{C}}^n(\mathcal{G}) & \longrightarrow & \text{Sym}_{\mathcal{C}}^n(\mathcal{F}) \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{B} \end{array}$$

Hence the element $g \in \Gamma(\text{Sym}_{\mathcal{C}}^n(\mathcal{G}))$ maps to the desired element of $\Gamma(\mathcal{A})$.

If (D) holds, then given $\mathcal{G} \rightarrow \mathcal{F}$ as in (1) we can form the map of “symmetric” algebras

$$\mathrm{Sym}_{\mathcal{C}}^*(\mathcal{G}) \longrightarrow \mathrm{Sym}_{\mathcal{C}}^*(\mathcal{F})$$

and we see that (3) implies (1).

The final statement is clear. \square

We do not know of an example of Situation 2.1 where axiom (D) does not hold. On the other hand, we do know cases where (S) does not hold, namely, the category of comodules over a general bialgebra. Hence we take property (3) of the lemma above as the defining property, since it also make sense in those situations.

Definition 11.7. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. We introduce the following axiom:

- (A) For every surjection of weakly commutative rings $\mathcal{A} \rightarrow \mathcal{B}$ in \mathcal{C} with \mathcal{A} locally finite, and any $f \in \Gamma(\mathcal{B})$, there exists an $n > 0$ and an element $g \in \Gamma(\mathcal{A})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{B})$.

A much stronger condition is the notion of *goodness*, which is our analogue of linear reductivity. It can hold even in geometrically interesting situations.

Definition 11.8. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. We introduce the following axiom:

- (G) The functor Γ is exact.

12. PRELIMINARY RESULTS

Let \mathcal{A} be a weakly commutative ring object of \mathcal{C} . This implies that $\Gamma(\mathcal{A}) \subset F(\mathcal{A})$ is a commutative ring. Let $I \subset \Gamma(\mathcal{A})$ be an ideal. Assuming the axiom (D) we have the object $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} I$ (see Lemma 8.3) and a canonical map

$$(12.1) \quad \mathcal{A} \otimes_{\Gamma(\mathcal{A})} I \longrightarrow \mathcal{A}.$$

Namely, this is the adjoint to the map $I \rightarrow \Gamma(\mathcal{A})$. Applying F to the the map (12.1) gives the obvious map $F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} I \rightarrow F(\mathcal{A})$. The image of (12.1) will be denoted $\mathcal{A}I$ in the sequel. We have $F(\mathcal{A}I) = F(\mathcal{A})I$ by exactness of the functor F .

For an ideal I of a commutative ring B we set

$$I^* = \{f \in B \mid \exists n > 0, f^n \in I^n\}.$$

Note that it is not clear (or even true) in general that I^* is an ideal. (Our notation is not compatible with notation concerning integral closure of ideals in algebra texts. We will only use this notation in this section.)

Lemma 12.1. *Assume that we are in Situation 2.1 and that axiom (D) holds. Let \mathcal{A} be a locally finite, weakly commutative ring object of \mathcal{C} . Let $I \subset \Gamma(\mathcal{A})$ be an ideal. Consider the ring map*

$$\varphi : \Gamma(\mathcal{A})/I \longrightarrow \Gamma(\mathcal{A}/\mathcal{A}I).$$

- (1) *If the axiom (G) holds, φ is an isomorphism.*
- (2) *If the axiom (A) holds, then*
 - (a) *the kernel of φ is contained in $I^*\Gamma(\mathcal{A})/I$; in particular it is locally nilpotent, and*
 - (b) *for every element $f \in \Gamma(\mathcal{A}/\mathcal{A}I)$ there exists an integer $n > 0$ and an element $g \in \Gamma(\mathcal{A})/I$ which maps to f^n via φ .*

Proof. The surjectivity of φ in (1) is immediate from axiom (G). The ring object $\mathcal{A}/\mathcal{A}I$ is weakly commutative (by Lemma 9.2). Hence (2b) is implied by axiom (A).

Suppose that $f \in \Gamma(\mathcal{A})$ maps to zero in $\Gamma(\mathcal{A}/\mathcal{A}I)$. This means that $f \in \Gamma(\mathcal{A}I)$. Choose generators $f_s \in I$, $s \in S$ for I . Consider the ring map

$$\mathcal{A}[x_s; s \in S] \longrightarrow \mathcal{B} = \bigoplus I^n \mathcal{A}$$

which maps x_s to $f_s \in \Gamma(I\mathcal{A})$, see Remark 8.4. This is a surjection of ring objects of \mathcal{C} . Hence if (G) holds, then we see that f is in the image of $\bigoplus_{s \in S} \Gamma(\mathcal{A}) \rightarrow \Gamma(\mathcal{A}I)$, i.e., f is in $\Gamma(\mathcal{A})I$ and injectivity in (1) holds. For the rest of the proof assume (A). Clearly the polynomial algebra $\mathcal{A}[x_s; s \in S]$ is weakly commutative and locally finite. Hence (A) implies there exists an $n > 0$ and an element

$$g \in \Gamma(\mathcal{A}[x_s; s \in S])$$

which maps to f^n in the summand $\Gamma(\mathcal{A}I^n)$ of $\Gamma(\mathcal{B})$. Hence we may also assume that g is in the degree n summand

$$\Gamma\left(\bigoplus_{|J|=n} \mathcal{A}x^J\right)$$

of $\Gamma(\mathcal{A}[x_s; s \in S])$. Now, note that there is a ring map $\mathcal{B} \rightarrow \mathcal{A}$ and that the composition

$$\mathcal{A}[x_s; s \in S] \longrightarrow \mathcal{B} \longrightarrow \mathcal{A}$$

in degree n maps $\Gamma(\bigoplus_{|J|=n} \mathcal{A}x^J)$ into $\Gamma(\mathcal{A})I^n$, because x_s maps to f_s . Hence $f^n \in I^n$. This finishes the proof. \square

Let \mathcal{A} be a weakly commutative ring object of \mathcal{C} . Let $\Gamma(\mathcal{A}) \rightarrow \Gamma'$ be a homomorphism of commutative rings. Write $\Gamma' = \Gamma(\mathcal{A})[x_s; s \in S]/I$. Assume axiom (D) holds. Then we see that we have the equality

$$\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma' = \mathcal{A}[x_s; s \in S]/(\mathcal{A}[x_s; s \in S])I$$

where the polynomial algebra is as in Remark 8.4 and the tensor product as in Lemma 8.3. The reason is that there is an obvious map (from right to left) and that we have

$$F(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma') = F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} \Gamma' = F(\mathcal{A})[x_s; s \in S]/(F(\mathcal{A})[x_s; s \in S])I$$

by the properties of the functor F and the results mentioned above. Hence $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma'$ is a weakly commutative ring object (see Lemma 9.2). Note that if \mathcal{A} is locally finite, then so is $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma'$, see Lemma 11.5.

Lemma 12.2. *Assume that we are in Situation 2.1 and that axiom (D) holds. Let \mathcal{A} be a ring object.*

- (1) *Assume that also axiom (G) holds. If M is a left $\Gamma(\mathcal{A})$ -module, then the adjunction map*

$$\varphi : M \longrightarrow \Gamma(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} M)$$

is an isomorphism.

- (2) *Assume the axiom (A) holds, and that \mathcal{A} is locally finite and weakly commutative. Let $\Gamma(\mathcal{A}) \rightarrow \Gamma'$ be a commutative ring map. Consider the adjunction map*

$$\varphi : \Gamma' \longrightarrow \Gamma(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma')$$

- (a) *the kernel of φ is locally nilpotent, and*

- (b) for every element $f \in \Gamma(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} \Gamma')$ there exists an integer $n > 0$ and an element $g \in \Gamma'$ which maps to f^n via φ .

Proof. For (1), since both functors $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} -$ and Γ commute with arbitrary direct sums, the map φ is an isomorphism when M is free. Furthermore, since $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} -$ is right exact and Γ is exact, the general case follows. For (2), the map is an isomorphism when Γ' is a polynomial algebra (since we are assuming all functors commute with direct sums). And the general case follows from this, the discussion above the lemma and Lemma 12.1. \square

Lemma 12.3. *Assume that we are in Situation 2.1 and that axioms (D) and (A) hold. Then for every locally finite, weakly commutative ring object \mathcal{A} of \mathcal{C} the map*

$$\mathrm{Spec}(F(\mathcal{A})) \longrightarrow \mathrm{Spec}(\Gamma(\mathcal{A}))$$

is surjective.

Proof. Let $\Gamma(\mathcal{A}) \rightarrow K$ be a ring map to a field. We have to show that the ring

$$F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} K = F(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} K)$$

is not zero. This follows from Lemma 12.2 and the fact that K is not the zero ring. \square

In the following lemma we use the notion of a *universally subtrusive* morphism of schemes $f : X \rightarrow Y$. This means that f satisfies the following valuation lifting property: for every valuation ring V and every morphism $\mathrm{Spec}(V) \rightarrow Y$ there exists a local map of valuation rings $V \rightarrow V'$ and a morphism $\mathrm{Spec}(V') \rightarrow X$ such that

$$\begin{array}{ccc} X & \longleftarrow & \mathrm{Spec}(V') \\ \downarrow & & \downarrow \\ Y & \longleftarrow & \mathrm{Spec}(V) \end{array}$$

is commutative. It turns out that if $f : X \rightarrow Y$ is of finite type, and Y is Noetherian, then this notion is equivalent to f being *universally submersive*.

Lemma 12.4. *Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. Let \mathcal{A} be a ring object. Assume that*

- (1) axioms (D) and (A) hold, and
- (2) \mathcal{A} is locally finite and weakly commutative.

Then $\mathrm{Spec}(F(\mathcal{A})) \rightarrow \mathrm{Spec}(\Gamma(\mathcal{A}))$ is universally subtrusive. If in addition,

- (3) $R \rightarrow A$ is finite type,
- (4) \mathcal{A} is of finite type, and
- (5) $\Gamma(\mathcal{A})$ is Noetherian.

Then $\mathrm{Spec}(F(\mathcal{A})) \rightarrow \mathrm{Spec}(\Gamma(\mathcal{A}))$ is universally submersive.

Proof. To show the first part, let $\mathrm{Spec}(V) \rightarrow \mathrm{Spec}(\Gamma(\mathcal{A}))$ be a morphism where V is a valuation ring with fraction field K . We must show that

$$f : \mathrm{Spec}(F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} V) \longrightarrow \mathrm{Spec}(V)$$

is subtrusive. Let $\eta \in \mathrm{Spec}(V)$ be the generic point. It suffices to show that the closure of $f^{-1}(\eta)$ in $\mathrm{Spec}(F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} V)$ surjects onto $\mathrm{Spec}(V)$. If we set

$$\mathcal{I} = \ker(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V \longrightarrow \mathcal{A} \otimes_{\Gamma(\mathcal{A})} K)$$

then $F(\mathcal{I})$ is the kernel of $F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} V \rightarrow F(\mathcal{A}) \otimes_{\Gamma(\mathcal{A})} K$ and defines the closure of $f^{-1}(\eta)$. The ring object $(\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V)/\mathcal{I}$ is weakly commutative and locally finite. By Lemma 12.3,

$$\mathrm{Spec}(F((\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V)/\mathcal{I})) \longrightarrow \mathrm{Spec}(\Gamma((\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V)/\mathcal{I}))$$

is surjective. Axiom (A) applied to the surjection $\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V \rightarrow (\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V)/\mathcal{I}$ implies that

$$\mathrm{Spec}(\Gamma((\mathcal{A} \otimes_{\Gamma(\mathcal{A})} V)/\mathcal{I})) \longrightarrow \mathrm{Spec}(V)$$

is integral. Therefore the composition of the two morphisms above is surjective so that the closure of $f^{-1}(\eta)$ surjects onto $\mathrm{Spec}(V)$.

The hypotheses in the second part imply that $\Gamma(\mathcal{A}) \rightarrow F(\mathcal{A})$ is of finite type and $\Gamma(\mathcal{A})$ is Noetherian, hence the remark preceding the lemma applies. \square

Below we will use the following algebraic result to get finite generation.

Theorem 12.5. *Consider ring maps $R \rightarrow B \rightarrow A$ such that*

- (1) *B and R are noetherian,*
- (2) *$R \rightarrow A$ is of finite type, and*
- (3) *$\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(B)$ is universally submersive.*

Then $R \rightarrow B$ is of finite type.

Proof. This is a special case of Theorem 6.2.1 of [Alp10]. It was first discovered while writing an earlier version of this paper. \square

13. THE MAIN RESULT

The main argument in the proof of Theorem 13.5 is an induction argument. In order to formulate it we use the following condition.

Definition 13.1. Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. Let \mathcal{A} be a weakly commutative ring object. Consider the following property of \mathcal{A}

- (\star) The ring $\Gamma(\mathcal{A})$ is a finite type R -algebra and for every finite type module \mathcal{F} over \mathcal{A} the $\Gamma(\mathcal{A})$ -module $\Gamma(\mathcal{F})$ is finite.

Lemma 13.2. *Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. Let $\mathcal{A} \rightarrow \mathcal{B}$ be a surjection of ring objects. Assume*

- (1) *R is Noetherian and axiom (A) holds,*
- (2) *\mathcal{A} is locally finite and weakly commutative, and*
- (3) *$\Gamma(\mathcal{B})$ is a finitely generated R -algebra.*

Then $\Gamma(\mathcal{B})$ is a finite $\Gamma(\mathcal{A})$ -module and there exists a finitely generated R -subalgebra $B \subset \Gamma(\mathcal{A})$ such that

$$\mathrm{Im}(\Gamma(\mathcal{A}) \longrightarrow \Gamma(\mathcal{B})) = \mathrm{Im}(B \longrightarrow \Gamma(\mathcal{B})).$$

Proof. Since \mathcal{A} is weakly commutative, so is \mathcal{B} . Hence $\Gamma(\mathcal{B})$ is a commutative R -algebra. Pick $f_1, \dots, f_n \in \Gamma(\mathcal{B})$ which generate as an R -algebra. By axiom (A) we can find $g_1, \dots, g_n \in \Gamma(\mathcal{A})$ which map to $f_1^{n_1}, \dots, f_n^{n_n}$ in $\Gamma(\mathcal{B})$ for some $n_i > 0$. Then we see that $\Gamma(\mathcal{B})$ is generated by the elements

$$f_1^{e_1} \dots f_n^{e_n}, \quad 0 \leq e_i \leq n_i - 1$$

and so $\Gamma(\mathcal{B})$ is finite over $\Gamma(\mathcal{A})$. As a first approximation, let $B = R[g_1, \dots, g_n] \subset \Gamma(\mathcal{A})$. Then the equality of the lemma may not hold, but in any case $\Gamma(\mathcal{A})$ is

finite over B . Since B is a Noetherian ring, $\text{Im}(\Gamma(\mathcal{A}) \rightarrow \Gamma(\mathcal{B}))$ is a finite B -module so we choose finitely many generators $g_{n+1}, \dots, g_{n+m} \in \Gamma(\mathcal{A})$. Hence by setting $B = R[g_1, \dots, g_{n+m}]$, the lemma is proved. \square

Lemma 13.3. *Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. Let \mathcal{A} be a ring object and let $\mathcal{I} \subset \mathcal{A}$ be a left ideal. Assume*

- (1) R is Noetherian and axiom (A) holds,
- (2) \mathcal{A} is locally finite and weakly commutative,
- (3) (\star) holds for \mathcal{A}/\mathcal{I} , and
- (4) there is a quotient $\mathcal{A} \rightarrow \mathcal{A}'$ such that (\star) holds for \mathcal{A}' and such that \mathcal{I} is a finite \mathcal{A}' -module.

Then (\star) holds for \mathcal{A} .

Proof. Since \mathcal{A} is weakly commutative and locally finite so are \mathcal{A}/\mathcal{I} and \mathcal{A}' . By Lemma 13.2 the rings $\Gamma(\mathcal{A}')$ and $\Gamma(\mathcal{A}/\mathcal{I})$ are finite $\Gamma(\mathcal{A})$ -algebras. Consider the exact sequence

$$0 \rightarrow \Gamma(\mathcal{I}) \rightarrow \Gamma(\mathcal{A}) \rightarrow \Gamma(\mathcal{A}/\mathcal{I}).$$

By (\star) for \mathcal{A}' we see that $\Gamma(\mathcal{I})$ is a finite $\Gamma(\mathcal{A}')$ -module, hence a finite $\Gamma(\mathcal{A})$ -module. Choose generators $x_1, \dots, x_s \in \Gamma(\mathcal{I})$ as a $\Gamma(\mathcal{A})$ -module. By Lemma 13.2 we can find a finite type R -subalgebra $B \subset \Gamma(\mathcal{A})$ such that the image of B in $\Gamma(\mathcal{A}')$ and the image of B in $\Gamma(\mathcal{A}/\mathcal{I})$ is the same as the image of $\Gamma(\mathcal{A})$ in those rings. We claim that

$$\Gamma(\mathcal{A}) = B[x_1, \dots, x_s]$$

as subrings of $\Gamma(\mathcal{A})$. Namely, if $h \in \Gamma(\mathcal{A})$ then we can find an element $b \in B$ which has the same image as h in $\Gamma(\mathcal{A}/\mathcal{I})$. Hence replacing h by $h - b$ we may assume $h \in \Gamma(\mathcal{I})$. By our choice of x_1, \dots, x_s we may write $h = \sum a_i x_i$ for some $a_i \in \Gamma(\mathcal{A})$. But since \mathcal{I} is a \mathcal{A}' -module, we can write this as $h = \sum a'_i x_i$ with $a'_i \in \Gamma(\mathcal{A}')$ the image of a_i . By choice of B we can find $b_i \in B$ mapping to a'_i . Hence we see that $h \in B[x_1, \dots, x_s]$ as desired. This proves that $\Gamma(\mathcal{A})$ is a finitely generated R -algebra.

Let \mathcal{F} be a finite type \mathcal{A} -module. Set \mathcal{IF} equal to the image of the map $\mathcal{I} \otimes \mathcal{F} \rightarrow \mathcal{F}$ which is the restriction of the multiplication map of \mathcal{F} . Consider the exact sequence

$$0 \rightarrow \mathcal{IF} \rightarrow \mathcal{F} \rightarrow \mathcal{F}/\mathcal{IF} \rightarrow 0$$

This gives rise to a similar short exact sequence on applying F , and a surjective map $F(\mathcal{I}) \otimes_A F(\mathcal{F}) \rightarrow F(\mathcal{IF})$ which factors through $F(\mathcal{I}) \otimes_{F(\mathcal{A})} F(\mathcal{F})$ as \mathcal{A} is weakly commutative. Since $F(\mathcal{F})$ is finite as a $F(\mathcal{A})$ -module, and $F(\mathcal{I})$ is finite as a $F(\mathcal{A}')$ -module, we conclude that $F(\mathcal{IF})$ is a finite $F(\mathcal{A}')$ -module, i.e., that \mathcal{IF} is a finite \mathcal{A}' -module. In the same way we see that \mathcal{F}/\mathcal{IF} is a finite \mathcal{A}/\mathcal{I} -module. Hence in the exact sequence

$$0 \rightarrow \Gamma(\mathcal{IF}) \rightarrow \Gamma(\mathcal{F}) \rightarrow \Gamma(\mathcal{F}/\mathcal{IF})$$

we see that the modules on the left and the right are finite $\Gamma(\mathcal{A})$ -modules. Since $\Gamma(\mathcal{A})$ is Noetherian by the result of the preceding paragraph we see that $\Gamma(\mathcal{F})$ is a finite $\Gamma(\mathcal{A})$ -module. This concludes the proof that property (\star) holds for \mathcal{A} . \square

Lemma 13.4. *Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. Let \mathcal{A} be a ring object, and let $\mathcal{I} \subset \mathcal{A}$ be a left ideal. Assume that*

- (1) axioms (N) and (A) hold and R is Noetherian,

- (2) \mathcal{A} is locally finite, weakly commutative and of finite type,
- (3) $\mathcal{I}^n = 0$ for some $n \geq 0$, and
- (4) \mathcal{A}/\mathcal{I} has property (\star) .

Then \mathcal{A} has property (\star) .

Proof. We argue by induction on n and hence we may assume that $\mathcal{I}^2 = 0$. Then we get an exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{A} \rightarrow \mathcal{A}/\mathcal{I} \rightarrow 0.$$

Because (N) holds and \mathcal{A} is of finite type we see that $F(\mathcal{A})$ is a finitely generated A -algebra hence Noetherian. Thus \mathcal{I} is a finite type \mathcal{A} -module, and hence also a finite type \mathcal{A}/\mathcal{I} -module. This means that Lemma 13.3 applies, and we win. \square

Theorem 13.5. *Let $(R \rightarrow A, \mathcal{C}, \otimes, F, \gamma, \mathcal{O}, \mu)$ be as in Situation 2.1. Assume*

- (1) R is Noetherian,
- (2) $R \rightarrow A$ is of finite type, and
- (3) the axioms (A) and (D) hold.

Then for every finite type, locally finite, weakly commutative ring object \mathcal{A} of \mathcal{C} property (\star) holds.

Proof. Let \mathcal{A} be a finite type, locally finite, weakly commutative ring object \mathcal{A} of \mathcal{C} . For every left ideal $\mathcal{I} \subset \mathcal{A}$ the quotient \mathcal{A}/\mathcal{I} is also a finite type, locally finite, weakly commutative ring object of \mathcal{C} . Consider the set

$$\{\mathcal{I} \subset \mathcal{A} \mid (\star) \text{ fails for } \mathcal{A}/\mathcal{I}\}.$$

To get a contradiction assume that this set is nonempty. By Noetherian induction on the ideal $F(\mathcal{I}) \subset F(\mathcal{A})$ we see there exists a maximal left ideal $\mathcal{I}_{max} \subset \mathcal{A}$ such that (\star) holds for any ideal strictly containing \mathcal{I}_{max} but (\star) does not hold for \mathcal{I}_{max} . Replacing \mathcal{A} by $\mathcal{A}/\mathcal{I}_{max}$ we may assume (in order to get a contradiction) that (\star) does not hold for \mathcal{A} but does hold for every proper quotient of \mathcal{A} .

Let $f \in \Gamma(\mathcal{A})$ be nonzero. If $\text{Ker}(f : \mathcal{A} \rightarrow \mathcal{A})$ is nonzero, then we see that we get an exact sequence

$$0 \rightarrow (f) \rightarrow \mathcal{A} \rightarrow \mathcal{A}/(f) \rightarrow 0$$

Since we are assuming (\star) holds for both $\mathcal{A}/\text{Ker}(f : \mathcal{A} \rightarrow \mathcal{A})$ and $\mathcal{A}/(f)$ and since $\text{Ker}(f)$ is a finite $\mathcal{A}/(f)$ -module, we can apply Lemma 13.3. Hence we see that we may assume that any nonzero element $f \in \Gamma(\mathcal{A})$ is a nonzero divisor on \mathcal{A} . In particular, $\Gamma(\mathcal{A})$ is a domain.

Again, assume that $f \in \Gamma(\mathcal{A})$ is nonzero. Consider the sequence

$$0 \rightarrow \mathcal{A} \xrightarrow{f} \mathcal{A} \rightarrow \mathcal{A}/f\mathcal{A} \rightarrow 0$$

which gives rise to the sequence

$$0 \rightarrow \Gamma(\mathcal{A}) \xrightarrow{f} \Gamma(\mathcal{A}) \rightarrow \text{Im}(\Gamma(\mathcal{A}) \rightarrow \Gamma(\mathcal{A}/f\mathcal{A})) \rightarrow 0$$

We know that the ring on the right is a finite type R -algebra which is finite over $\Gamma(\mathcal{A})$, see Lemma 13.2. Hence any ideal $I \subset \Gamma(\mathcal{A})$ containing f maps to a finitely generated ideal in it. This implies that $\Gamma(\mathcal{A})$ is Noetherian.

Next, we claim that for any finite type \mathcal{A} -module \mathcal{F} the module $\Gamma(\mathcal{F})$ is a finite $\Gamma(\mathcal{A})$ -module. Again we can do this by Noetherian induction applied to the set

$$\{\mathcal{G} \subset \mathcal{F} \text{ is an } \mathcal{A}\text{-submodule such that finite generation fails for } \Gamma(\mathcal{F}/\mathcal{G})\}.$$

In other words, we may assume that \mathcal{F} is a minimal counter example in the sense that any proper quotient of \mathcal{F} gives a finite $\Gamma(\mathcal{A})$ -module. Pick $s \in \Gamma(\mathcal{F})$ nonzero (if $\Gamma(\mathcal{F})$ is zero, we're done). Let $\mathcal{A} \cdot s \subset \mathcal{F}$ denote the image of $\mathcal{A} \rightarrow \mathcal{F}$ which is multiplying against s . Now we have

$$0 \rightarrow \mathcal{A} \cdot s \rightarrow \mathcal{F} \rightarrow \mathcal{F}/\mathcal{A} \cdot s \rightarrow 0$$

which gives the exact sequence

$$0 \rightarrow \Gamma(\mathcal{A} \cdot s) \rightarrow \Gamma(\mathcal{F}) \rightarrow \Gamma(\mathcal{F}/\mathcal{A} \cdot s)$$

By minimality we see that the module on the right is finite over the Noetherian ring $\Gamma(\mathcal{A})$. On the other hand, the module on the left is $\Gamma(\mathcal{A}/\mathcal{I})$ for the ideal $\mathcal{I} = \text{Ker}(s : \mathcal{A} \rightarrow \mathcal{F})$. If $\mathcal{I} = 0$ then this is $\Gamma(\mathcal{A})$ and therefore finite, and if $\mathcal{I} \neq 0$ then this is a finite $\Gamma(\mathcal{A})$ -module by Lemma 13.2 and minimality of \mathcal{A} . Hence we conclude that the middle module is finite over the Noetherian ring $\Gamma(\mathcal{A})$ which is the desired contradiction.

Finally, we show that $\Gamma(\mathcal{A})$ is of finite type over R which will finish the proof. Namely, by Lemma 12.4 the morphism of schemes

$$\text{Spec}(F(\mathcal{A})) \longrightarrow \text{Spec}(\Gamma(\mathcal{A}))$$

is universally submersive. We have already seen that $\Gamma(\mathcal{A})$ is a Noetherian ring. Thus Theorem 12.5 kicks in and we are done. \square

Remark 13.6. We note that the proof of Theorem 13.5 can be simplified if the axiom (G) is also satisfied. In fact, if axiom (G) holds in addition to the conditions (1) - (3) of Theorem 13.5, then for every finite type, weakly commutative (but not necessarily locally finite) ring object \mathcal{A} , property (\star) holds. Lemma 12.2 implies that for any ideal $I \subseteq \Gamma(\mathcal{A})$, $I = IF(\mathcal{A}) \cap \Gamma(\mathcal{A})$; therefore $\Gamma(\mathcal{A})$ is Noetherian. We can then apply Theorem 12.5 to conclude that $\Gamma(\mathcal{A})$ is a finite type R -algebra. Furthermore, a simple noetherian induction argument shows that for every finite type module \mathcal{F} over \mathcal{A} the $\Gamma(\mathcal{A})$ -module $\Gamma(\mathcal{F})$ is finite type.

14. QUASI-COHERENT SHEAVES ON ALGEBRAIC STACKS

Let $S = \text{Spec}(R)$ be an affine scheme. Let \mathcal{X} be a quasi-compact algebraic stack over S . Let $p : T \rightarrow \mathcal{X}$ be a smooth surjective morphism from an affine scheme $T = \text{Spec}(A)$.

Lemma 14.1. *In the situation above, the category $QCoh(\mathcal{O}_{\mathcal{X}})$ endowed with its natural tensor product, pullback functor $F : QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow QCoh(\mathcal{O}_T) = Mod_A$ and structure sheaf $\mathcal{O} = \mathcal{O}_{\mathcal{X}}$ is an example of Situation 2.1. The functor $\Gamma : QCoh(\mathcal{O}_{\mathcal{X}}) \rightarrow Mod_R$ is identified with the functor of global sections*

$$\mathcal{F} \longmapsto \Gamma(\mathcal{X}, \mathcal{F}).$$

Axioms (D), (C), and (S) hold. If \mathcal{X} is noetherian (eg. \mathcal{X} is quasi-separated and A is Noetherian), then axiom (L) holds.

Proof. The final statement is [LMB00, Prop 15.4]. The rest is clear. \square

The following definition reinterprets the adequacy axiom (A).

Definition 14.2. Let \mathcal{X} be an quasi-compact algebraic stack over $S = \text{Spec}(R)$. We say that \mathcal{X} is *adequate* if for every surjection $\mathcal{A} \rightarrow \mathcal{B}$ of quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -algebras with \mathcal{A} locally finite and $f \in \Gamma(\mathcal{X}, \mathcal{B})$, there exists an $n > 0$ and a $g \in \Gamma(\mathcal{X}, \mathcal{A})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{X}, \mathcal{B})$.

Lemma 14.3. *Let \mathcal{X} be an quasi-compact algebraic stack over $S = \text{Spec}(R)$. The following are equivalent:*

- (1) \mathcal{X} is adequate.
- (2) For every surjection of finite type $\mathcal{O}_{\mathcal{X}}$ -modules $\mathcal{G} \rightarrow \mathcal{F}$ and $f \in \Gamma(\mathcal{X}, \mathcal{F})$, there exists an $n > 0$ and a $g \in \Gamma(\mathcal{X}, \text{Sym}^n \mathcal{G})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{X}, \text{Sym}^n \mathcal{F})$.

If \mathcal{X} is noetherian, then the above are also equivalent to:

- (3) For every surjection $\mathcal{G} \rightarrow \mathcal{O}$ with \mathcal{G} of finite type and $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, there exists an $n > 0$ and a $g \in \Gamma(\mathcal{X}, \text{Sym}^n \mathcal{G})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.
- (1') For every surjection $\mathcal{A} \rightarrow \mathcal{B}$ of quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -algebras and $f \in \Gamma(\mathcal{X}, \mathcal{B})$, there exists an $n > 0$ and a $g \in \Gamma(\mathcal{X}, \mathcal{A})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{X}, \mathcal{B})$.
- (2') For every surjection of $\mathcal{O}_{\mathcal{X}}$ -modules $\mathcal{G} \rightarrow \mathcal{F}$ and $f \in \Gamma(\mathcal{X}, \mathcal{F})$, there exists an $n > 0$ and a $g \in \Gamma(\mathcal{X}, \text{Sym}^n \mathcal{G})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{X}, \text{Sym}^n \mathcal{F})$.
- (3') For every surjection $\mathcal{G} \rightarrow \mathcal{O}$ and $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, there exists an $n > 0$ and a $g \in \Gamma(\mathcal{X}, \text{Sym}^n \mathcal{G})$ such that $g \mapsto f^n$ in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

Proof. This is Lemma 11.6. □

Corollary 14.4. *Let \mathcal{X} be an algebraic stack finite type over an affine noetherian scheme $\text{Spec}(R)$. Suppose \mathcal{X} is adequate. Let \mathcal{A} be a finite type $\mathcal{O}_{\mathcal{X}}$ -algebra. Then $\Gamma(\mathcal{X}, \mathcal{A})$ is finitely generated over R and for every finite type \mathcal{A} -module \mathcal{F} , the $\Gamma(\mathcal{X}, \mathcal{A})$ -module $\Gamma(\mathcal{X}, \mathcal{F})$ is finite.*

Proof. This is Theorem 13.5. □

15. BIALGEBRAS, MODULES AND COMODULES

In this section we discuss how modules and comodules over a bialgebra form an example of our abstract setup. If A is a commutative ring, recall that a *bialgebra* H over A is an A -module H endowed with maps $(A \rightarrow H, H \otimes_A H \rightarrow A, \epsilon : H \rightarrow A, \delta : H \rightarrow H \otimes_A H)$. Here $H \otimes_A H \rightarrow H$ and $A \rightarrow H$ define an unital A -algebra structure on H , the maps δ and ϵ are unital A -algebra maps. Moreover, the comultiplication μ is associative and ϵ is a counit.

Let H be a bialgebra over A . A *left H -module* is a left module over the R -algebra structure on H ; that is, there is a A -module homomorphism $H \otimes_A M \rightarrow M$ satisfying the two commutative diagrams for an action. A *left H -comodule* M is an R -module homomorphism $\sigma : M \rightarrow H \otimes_A M$ satisfying the two commutative diagram for a coaction. See [Kas95, Chapter 3] and [Mon93, Chapter 1] for the basic properties of H -modules and H -comodules.

Definition 15.1. Let A be a commutative ring. Let H be a bialgebra over A .

- (1) Let Mod_H be the category of left H -modules. It is endowed with the forgetful functor to A -modules, the tensor product

$$(M, N) \longmapsto M \otimes_A N$$

where H acts on $M \otimes_A N$ via the comultiplication, and the object \mathcal{O} given by the module A where H acts via the counit.

- (2) Let Comod_H be the category of left H -comodules. It is endowed with the forgetful functor to A -modules, the tensor product

$$(M, N) \longmapsto M \otimes_A N$$

where comodule structure on $M \otimes_A N$ comes from the multiplication in H , and the object \mathcal{O} given by the module A where H acts via the A -algebra structure H .

Lemma 15.2. *Let $R \rightarrow A$ be a map of commutative rings. Let H be a bialgebra over A .*

- (1) *The category Mod_H with its additional structure introduced in Definition 15.1 is an example of Situation 2.1. The functor $\Gamma : \text{Mod}_H \rightarrow \text{Mod}_R$ is identified with the functor of invariants*

$$M \longmapsto M^H = \{m \in M \mid h \cdot m = \epsilon(h)m\}.$$

Axioms (D), (I) and (S) hold. Axiom (C) holds if H is cocommutative.

- (2) *The category Comod_H with its additional structure introduced in Definition 15.1 is an example of Situation 2.1. The functor $\Gamma : \text{Comod}_H \rightarrow \text{Mod}_R$ is identified with the functor of coinvariants*

$$M \longmapsto M_H = \{m \in M \mid \sigma(m) = 1 \otimes m\}$$

where $\sigma : M \rightarrow H \otimes_A M$ indicates the coaction of M . Axiom (D) holds. Axiom (C) holds if H is commutative.

Proof. The first two statements in both part (1) and (2) are clear. It also clear that axiom (D) holds in both cases. Arbitrary direct products exist in the category Mod_H , which is axiom (I), and so by Lemma 7.2 axiom (S) holds. The final statement concerning axiom (C) is straightforward, see [Mon93, Section 1.8]. \square

16. ADEQUACY FOR A BIALGEBRA

Let $R \rightarrow A$ be map of commutative rings. Let H be a bialgebra over A . Let M be an H -module. We can identify $\text{Sym}_H^n M := \text{Sym}_{\text{Mod}_H}^n M$ of axiom (S) with the H -module

$$\underbrace{M \otimes_A \cdots \otimes_A M}_n / M'$$

where M' is the submodule generated by elements $h \cdot ((\cdots \otimes m_i \otimes \cdots \otimes m_j \otimes \cdots) - (\cdots \otimes m_j \otimes \cdots \otimes m_i \otimes \cdots))$ for $h \in H$ and $m_1, \dots, m_n \in M$. And $\text{Sym}_H M := \bigoplus_n \text{Sym}_H^n M$ is the largest H -module quotient of the tensor algebra on M which is commutative.

An H -algebra is an H -module C which is an algebra over the algebra structure on H such that $A \rightarrow C$ and $C \otimes_A C \rightarrow C$ are H -module homomorphisms. We say that C is *commutative* if C is commutative as an algebra. An H -module M is *locally finite* if it is the filtered colimit of finite type H -modules.

The following definition reinterprets adequacy axiom (A) for the category Mod_H .

Definition 16.1. Let $R \rightarrow A$ be map of commutative rings. Let H be a bialgebra over A . We say that H is *adequate* if for every surjection of commutative H -algebras $C \rightarrow D$ in Mod_H with C locally finite, and any $f \in D^H$, there exists an $n > 0$ and an element $g \in C^H$ such that $g \mapsto f^n$ in D^H .

Lemma 16.2. *Let $R \rightarrow A$ be map of commutative rings. Let H be a bialgebra over A . The following are equivalent:*

- (1) H is adequate.
- (2) For every surjection of finite type H -modules $N \rightarrow M$ and $f \in M^H$, there exists an $n > 0$ and a $g \in (\text{Sym}_H^n N)^H$ such that $g \mapsto f^n$ in $(\text{Sym}_H^n M)^H$.

If A is Noetherian, then the above are also equivalent to:

- (3) For every surjection of finite type H -modules $N \rightarrow A$ and $f \in A$, there exists an $n > 0$ and a $g \in (\text{Sym}_H^n N)^H$ such that $g \mapsto f^n$ in A .

Proof. This is Lemma 11.6. □

Corollary 16.3. *Let $R \rightarrow A$ be a finite type map of commutative rings where R is Noetherian. Let H be an adequate bialgebra over A . Let C be a finitely generated, locally finite, commutative H -algebra. Then C^H is a finitely generated R -algebra and for every finite type C -module M , the C^H -module M^H is finite.*

Proof. This is Theorem 13.5. □

Remark 16.4. If $R = A = k$ where k is a field, then [KT08] define a Hopf algebra H over k to be *geometrically reductive* if any finite dimensional H -module M and any non-zero homomorphism of H -modules $N \rightarrow k$ there exist $n > 0$ such that $\text{Sym}_H^n(N)^H \rightarrow k$ is non-zero. By Lemma 16.2, H is geometrically reductive if and only if H is adequate.

In [KT08, Theorem 3.1], Kalniuk and Tyc prove that with the hypotheses of the above corollary and with $R = A = k$ is a field, C^H is finitely generated over k .

17. COADEQUACY FOR A BIALGEBRA

Let $R \rightarrow A$ be map of commutative rings. Let H be a bialgebra over A . An H -coalgebra is an H -comodule C which is an algebra over the algebra structure on H such that $A \rightarrow C$ and $C \otimes_A C \rightarrow C$ are H -comodule homomorphisms; C is *commutative* if C is commutative as an algebra. An H -comodule M is *locally finite* if it is the filtered colimit of finite type H -comodules.

Here we reinterpret the adequacy axiom (A) for the category Comod_H .

Definition 17.1. Let $R \rightarrow A$ be map of commutative rings. Let H be a bialgebra over A . We say that H is *coadequate* if for every surjection of commutative H -coalgebras $C \rightarrow D$ with C locally finite, and any $f \in D_H$, there exists an $n > 0$ and an element $g \in C_H$ such that $g \mapsto f^n$ in D_H .

Recall that we only know that axiom (S) holds for Comod_H when H is commutative.

Lemma 17.2. *Let $R \rightarrow A$ be map of commutative rings. Let H be a commutative bialgebra over A . The following are equivalent:*

- (1) H is adequate.
- (2) For every surjection of finite type H -modules $N \rightarrow M$ and $f \in M^H$, there exists an $n > 0$ and a $g \in (\text{Sym}_H^n N)^H$ such that $g \mapsto f^n$ in $(\text{Sym}_H^n M)^H$.

If A is Noetherian, then the above are also equivalent to:

- (3) For every surjection of finite type H -modules $N \rightarrow A$ and $f \in A$, there exists an $n > 0$ and a $g \in (\text{Sym}_H^n N)^H$ such that $g \mapsto f^n$ in A .

Proof. This is Lemma 11.6. □

Corollary 17.3. *Let $R \rightarrow A$ be a finite type of commutative rings where R is Noetherian. Let H be an adequate bialgebra over A . Let C be a finitely generated, locally finite, commutative H -coalgebra. Then C_H is a finitely generated R -algebra and for every finite type C -module M , the C_H -module M_H is finite.*

Proof. This is Theorem 13.5. □

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