The mean number of 3-torsion elements in ray class groups of quadratic fields

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Abstract

We determine the average number of 3-torsion elements in the ray class groups of fixed (integral) conductor c of quadratic fields ordered by absolute discriminant, generalizing Davenport and Heilbronn's theorem on class groups. A consequence of this result is that a positive proportion of such ray class groups of quadratic fields have trivial 3-torsion subgroup whenever the conductor c is taken to be a squarefree integer having very few prime factors none of which are congruent to 1 mod 3. Additionally, we compute the second main term for the number of 3-torsion elements in ray class groups with fixed conductor of quadratic fields ordered by discriminant.

1 Introduction

In 1971, Davenport-Heilbronn [8] determined that the mean number of 3-torsion elements in the class groups of real (respectively, imaginary) quadratic fields ordered by absolute discriminant is $\frac{4}{3}$ (resp., 2). In this paper, we determine the average number of 3-torsion elements in *ray class groups* of fixed integral conductor of quadratic fields ordered by their discriminant. More precisely, we prove the following theorem.

Theorem 1. *Fix a positive integer c, and let m denote the number of primes $p \mid c$ such that $p \equiv 1 \bmod 3$.*

(a) *The average size of the 3-torsion subgroups in the ray class groups of conductor c of real quadratic fields K ordered by discriminant is:*

$$\lim_{X\to\infty} \frac{\displaystyle\sum_{\substack{[K:\mathbb{Q}]=2\\ 0<\mathrm{Disc}(K)<X}} \#\mathrm{Cl}_3(K,c)}{\displaystyle\sum_{\substack{[K:\mathbb{Q}]=2\\ 0<\mathrm{Disc}(K)<X}} 1} = \begin{cases} 3^m \cdot \left(1 + \dfrac{1}{3}\cdot \displaystyle\prod_{p\mid c} 1 + \dfrac{p}{p+1}\right) & \text{if } 3 \nmid c, \\ 3^m \cdot \left(1 + \dfrac{2}{7}\cdot \displaystyle\prod_{p\mid c} 1 + \dfrac{p}{p+1}\right) & \text{if } 3 \mid\mid c, \text{ and} \\ 3^{m+1} \cdot \left(1 + \dfrac{5}{7}\cdot \displaystyle\prod_{p\mid c} 1 + \dfrac{p}{p+1}\right) & \text{if } 9 \mid c. \end{cases}$$

(b) *The average size of the 3-torsion subgroups in the ray class groups of conductor c of imaginary quadratic fields K ordered by discriminant is:*

$$\lim_{X\to\infty} \frac{\displaystyle\sum_{\substack{[K:\mathbb{Q}]=2\\ -X<\mathrm{Disc}(K)<0}} \#\mathrm{Cl}_3(K,c)}{\displaystyle\sum_{\substack{[K:\mathbb{Q}]=2\\ -X<\mathrm{Disc}(K)<0}} 1} = \begin{cases} 3^m \cdot \left(1 + \displaystyle\prod_{p\mid c} 1 + \dfrac{p}{p+1}\right) & \text{if } 3 \nmid c, \\ 3^m \cdot \left(1 + \dfrac{6}{7}\cdot \displaystyle\prod_{p\mid c} 1 + \dfrac{p}{p+1}\right) & \text{if } 3 \mid\mid c, \text{ and} \\ 3^{m+1} \cdot \left(1 + \dfrac{15}{7}\cdot \displaystyle\prod_{p\mid c} 1 + \dfrac{p}{p+1}\right) & \text{if } 9 \mid c. \end{cases}$$

Above, we denote by $\mathrm{Cl}_3(K,c)$ the 3-torsion subgroup of the ray class group of conductor c of a quadratic field K. Thus, the case $c=1$ in Theorem 1 recovers Davenport-Heilbronn's theorem on the average number of 3-torsion elements in the class groups of real and imaginary quadratic fields [8, Theorem 3]. For $c>1$, this is the first result of its kind.

The Cohen-Lenstra heuristics [6], which conjecture asymptotics for the distribution of torsion in class groups of quadratic fields ordered by discriminant, were inspired by the constants appearing in [8] as well as computations. The analogous distributions for p-torsion subgroups in ray class groups of quadratic fields can be predicted; for example, we expect that the average size of p-torsion subgroups in the ray class groups of conductor c coprime to p of real quadratic fields ordered by discriminant is

$$p^{\#\{\ell\mid c:\ \ell\equiv 1 \bmod p\}}\cdot\left(1+\frac{1}{p}\cdot\prod_{\substack{\ell\mid c\\ \ell\equiv\pm1(p)}}\left(1+\frac{p-1}{2}\cdot\frac{\ell}{\ell+1}\right)\right),\tag{1}$$

where ℓ runs over primes dividing c. Such a generalization of the Cohen-Lenstra heuristics predicting the full distribution for ray class groups of families of fixed degree fields would give a more explicit and tangible description of the maximal abelian extension over number fields other than $\mathbb{Q}$ and imaginary quadratic fields (see Pagano-Sofos [14].)

While the mean values in Theorem 1 do depend on the conductor c, if we instead average over quadratic fields with discriminant coprime to the conductor, we obtain different constants that only depend on the number of primes dividing c. Note that when averaging over a family of quadratic fields defined by prescribed splitting conditions at a finite set of primes, the average size of the 3-torsion subgroups in ray class groups only changes when the set of primes includes prime factors of c. This gives the expected generalization of the case when $c=1$, in which the mean values do not depend on the family one averages over (see Corollary 4 in [4]).

Theorem 2. *Fix a positive integer c with n distinct prime factors, and let m denote the number of distinct primes $p\mid c$ that are congruent to* 1 mod 3. *When quadratic fields are ordered by their absolute discriminant:*

(a) *The average number of* 3*-torsion elements in the ray class groups of conductor c of real quadratic fields with discriminant coprime to c is:*

$$\lim_{X\to\infty}\frac{\displaystyle\sum_{\substack{[K:\mathbb{Q}]=2\\ (\mathrm{Disc}(K),c)=1\\ 0<\mathrm{Disc}(K)<X}}\#\mathrm{Cl}_3(K,c)}{\displaystyle\sum_{\substack{[K:\mathbb{Q}]=2\\ (\mathrm{Disc}(K),c)=1\\ 0<\mathrm{Disc}(K)<X}}1}=\begin{cases}3^m\cdot\left(1+\dfrac{2^n}{3}\right) & \text{if } 3\nmid c,\\[2ex] 3^m\cdot\left(1+\dfrac{2^{n-1}}{3}\right) & \text{if } 3\,\|\,c, \text{ and}\\[2ex] 3^{m+1}\cdot\left(1+2^{n-1}\right) & \text{if } 9\mid c.\end{cases}$$

(b) *The average number of* 3*-torsion elements in the ray class groups of conductor c of imaginary quadratic fields with discriminant coprime to c is:*

$$\lim_{X\to\infty}\frac{\displaystyle\sum_{\substack{[K:\mathbb{Q}]=2\\ (\mathrm{Disc}(K),c)=1\\ -X<\mathrm{Disc}(K)<0}}\#\mathrm{Cl}_3(K,c)}{\displaystyle\sum_{\substack{[K:\mathbb{Q}]=2\\ (\mathrm{Disc}(K),c)=1\\ -X<\mathrm{Disc}(K)<0}}1}=\begin{cases}3^m\cdot\left(1+2^n\right) & \text{if } 3\nmid c,\\[1ex] 3^m\cdot\left(1+2^{n-1}\right) & \text{if } 3\,\|\,c, \text{ and}\\[1ex] 3^{m+1}\cdot\left(1+3\cdot 2^{n-1}\right) & \text{if } 9\mid c.\end{cases}$$

More generally, we find that the mean size of the 3-torsion subgroup in ray class groups of conductor c of quadratic fields defined by prescribing splitting conditions at a finite set of primes only depends on the specific primes p dividing c that are allowed to ramify in the family of quadratic fields and the number of primes $p \mid c$ that are required to remain unramified in the family (see Theorem 5.1). Such generalizations often shed light on the mass formulas that dictate class group asymptotics (see e.g. Theorem 3 of [4]. We further expect Theorem 5.1 to have concrete applications to questions surrounding the arithmetic properties of fundamental units (see e.g., [9, 16])

The averages obtained in [8] implied that a positive proportion of real (respectively, imaginary) quadratic fields have class number indivisible by 3 when ordered by discriminant, and this result was slightly refined by Nakagawa-Horie [13] in order to prove that there are infinitely many hyperelliptic curves over $\mathbb{Q}$ of a given genus with no integral points. In joint work with Bhargava, we prove the following (as a consequence of Corollary 4 in [4]):

Theorem 3 (Bhargava-Varma [4])**.** *Let $S_+ \cup S_0 \cup S_-$ be a disjoint union of finite sets of primes. There are infinitely many real (respectively, imaginary) quadratic fields K with class number indivisible by 3 such that K is ramified at each prime of S_0, inert at each prime of S_-, and split at each prime of S_+.*

This result and its generalizations (see Wiles [19] in conjunction with Beckwith [1]) have been utilized to imply unconditional versions of modularity lifting theorems in the residually reducible case as in Skinner-Wiles [15]. Furthermore, they are required in proving the nonvanishing of critical values of L-functions for positive proportions of quadratic twist families of elliptic curves with rational p-torsion points (see Vatsal [18]) and have applications to proving cases of the weak Goldfeld conjecture (see [10]).

In this article, we show that the mean values in Theorem 1 also imply that a positive proportion of quadratic fields have trivial 3-torsion subgroups in their ray class groups for certain conductors c, generalizing Theorem 3.

Corollary 4. *Assume c is equal to an odd prime number not congruent to $1 \bmod 3$ (in the real quadratic case, also consider those conductors c that are a product of two distinct primes that are not congruent to $1 \bmod 3$). Additionally, let $S_+ \cup S_- \cup S_0$ be a disjoint union of finite sets of prime numbers, none of which contain the primes dividing c. There are infinitely many real (respectively, imaginary) quadratic fields K that are split at each prime in S_+, inert at each prime in S_-, ramified at each prime in S_0, and have trivial 3-torsion subgroups in their ray class groups of conductor c.*

Finally, we may apply the methods of Taniguchi and Thorne [17] to compute the second main term for the mean number of 3-torsion elements in ray class groups of quadratic fields ordered by absolute discriminant. More precisely, we prove the following refinement of Theorem 1. Let $\mathrm{Cl}_3(K_2, c)$ denote the 3-torsion subgroup of the ray class group of conductor c for the quadratic field K_2.

Theorem 5. *For any positive integer c coprime to 3, let m denote the number of distinct primes dividing c that are congruent to $1 \bmod 3$. When quadratic fields are ordered by absolute discriminant:*

$$\begin{aligned}
\sum_{0<\mathrm{Disc}(K_2)<X} \#\mathrm{Cl}_3(K_2,c) &= 3^m \cdot \Bigg(\Big(1 + \frac{1}{3}\cdot\prod_{p\mid c}\Big(1+\frac{p}{p+1}\Big)\Big) \cdot \sum_{0<\mathrm{Disc}(K_2)<X} 1 \\
&+ \frac{\sqrt{3}\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{15\Gamma(2/3)\zeta(2)} \cdot \prod_p \left(1 - \frac{p^{1/3}+1}{p(p+1)}\right) \cdot \prod_{p\mid c}\left(1 + \frac{p(1-p^{1/3})}{1-\frac{p(p+1)}{p^{1/3}+1}}\right) \cdot X^{5/6} \Bigg) \\
&+ O_{\epsilon,c}(X^{5/6-7/138+\epsilon}), \text{ and} \\
\sum_{-X<\mathrm{Disc}(K_2)<0} \#\mathrm{Cl}_3(K_2,c) &= 3^m \cdot \Bigg(\Big(1 + \prod_{p\mid c}\Big(1+\frac{p}{p+1}\Big)\Big) \cdot \sum_{-X<\mathrm{Disc}(K_2)<0} 1 \\
&+ \frac{\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{5\Gamma(2/3)\zeta(2)} \cdot \prod_p \left(1 - \frac{p^{1/3}+1}{p(p+1)}\right) \cdot \prod_{p\mid c}\left(1 + \frac{p(1-p^{1/3})}{1-\frac{p(p+1)}{p^{1/3}+1}}\right) \cdot X^{5/6} \Bigg) \\
&+ O_{\epsilon,c}(X^{5/6-7/138+\epsilon}).
\end{aligned}$$

To derive the result for 3-torsion ideal classes in class groups of quadratic fields, Davenport and Heilbronn first provide asymptotic formulae for the number of cubic fields having bounded discriminant and sieve to count the *nowhere totally ramified* cubic fields. These are degree 3 extensions K_3 of $\mathbb{Q}$ in which any rational prime p that ramifies is of the form $(p) = \mathfrak{p}_1^2\mathfrak{p}_2$ where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two distinct primes of K_3. They prove that the number of nowhere totally ramified cubic fields having bounded discriminant determines the number of 3-torsion ideal classes in quadratic fields with the same bound on their discriminant, and so they deduce the above theorem.

To prove Theorems 1, 2 and 5, we first prove a new parametrization theorem that determines the number of 3-torsion elements of ray class groups of quadratic fields with bounded discriminant in terms of the number of appropriate (pairs of) cubic fields with related bounds on their discriminants. We then employ a generalization of Davenport and Heilbronn's asymptotics for cubic fields given in Bhargava-Shankar-Tsimerman [3] by simplifying the asymptotic count of the relevant pairs of cubic fields. It is important to note that Theorem 1 would not follow from the original asymptotics given in [8].

We begin this article by fixing an conductor c and a quadratic field K_2 in Section 2 in order to compare the number of 3-torsion ideal classes in the ray class group of K_2 of conductor c to the number of pairs of cubic fields whose discriminants satisfy certain c^2-divisibility conditions (see Theorem 2.5). We additionally study the action of $\mathrm{Gal}(K_2/\mathbb{Q})$ on this 3-torsion subgroup in order to relate the number of 3-torsion ideal classes with a fixed action of $\mathrm{Gal}(K_2/\mathbb{Q})$ to certain singleton cubic fields whose discriminants satisfy similar c^2-divisibility conditions. In Section 3, we recall and employ results of [3] that compute the density of discriminants of cubic fields satisfying certain *acceptable* local specifications. This allows us to determine in Section 4 the mean size of the 3-torsion subgroups in an eigenspace of the ray class groups of quadratic fields K_2 for the nontrivial action of $\mathrm{Gal}(K_2/\mathbb{Q})$. We are then able to conclude Theorems 1 and 2 as well as Corollary 4 in Section 5 by studying the 3-torsion elements in ray class groups of quadratic fields K_2 that are fixed by $\mathrm{Gal}(K_2/\mathbb{Q})$. Finally, in Section 6 we prove Theorem 5 by computing the second main term for the average number of 3-torsion elements in ray class groups of fixed conductor of quadratic fields with bounded discriminant, building on work of [17].

2 Parametrization of 3-torsion elements in ray class groups of quadratic fields

We begin by describing a bijection between index-3 subgroups of ray class groups of quadratic fields and certain pairs of cubic fields. This will allow us to determine the number of 3-torsion elements in ray class groups of fixed conductor of quadratic fields using a generalization given in [3] of Davenport-Heilbronn's asymptotic formulae on the density of discriminants of cubic fields.

2.1 Ray class groups and fields

First, we recall the definition of the ray class group of a number field K. Because we will eventually range over all quadratic fields, we only consider ray class groups whose finite part of the modulus is integral (so that it can be fixed independently of the quadratic field). Additionally, because ramification at infinity only affects the size of the 2-torsion subgroup in the (narrow) ray class groups, we work with ray class groups with trivial infinite part of the modulus. Under these restrictions, we refer to the rational positive generator of the modulus as the *conductor*.

Fix $c \in \mathbb{Z}$, and let $\mathcal{I}_c(\mathcal{O}_K)$ denote the subgroup of fractional ideals of $\mathcal{O}_K$ generated by prime ideals coprime to $c\mathcal{O}_K$. Additionally, let $\mathcal{P}_{1,c}(\mathcal{O}_K)$ denote the subgroup of principal ideals (α) such that $\alpha \equiv 1 \bmod c\mathcal{O}_K$. We then define the *ray class group of conductor* c as the quotient

$$\mathrm{Cl}(K, c) := \mathcal{I}_c(\mathcal{O}_K)/\mathcal{P}_{1,c}(\mathcal{O}_K). \tag{2}$$

In this notation, the ideal class group of a field K is denoted $\mathrm{Cl}(K,1)$. Additionally, let $\mathrm{Cl}_p(K,c)$ denote the p-torsion subgroup of $\mathrm{Cl}(K,c)$ for any prime p.

There is another (equivalent) definition of $\mathrm{Cl}(K,c)$ as a quotient of the ideles. More precisely, let $\mathbb{A}_K^\times$ denote the ideles of K, and for any $\mathcal{O}_K$-prime $\mathfrak{p} \mid c$, define

$$W_c(\mathfrak{p}) = 1 + \mathfrak{m}_{\mathfrak{p}}^{c(\mathfrak{p})},$$

where $\mathfrak{m}_{\mathfrak{p}}$ is the maximal ideal of $\mathcal{O}_{K_{\mathfrak{p}}}$ and $c(\mathfrak{p})$ denotes the largest power of $\mathfrak{p}$ which contains $c\mathcal{O}_K$. Let

$$W_c = \prod_{\mathfrak{p}\mid c} W_c(\mathfrak{p}) \times \prod_{\mathfrak{p}\nmid c} \mathcal{O}_{K_{\mathfrak{p}}}^\times .$$

We can then define

$$\mathrm{Cl}(K,c) := \mathbb{A}_K^\times / K^\times \cdot W_c. \tag{3}$$

The fact that these two definitions are equivalent can be found, e.g. in Milne [11].

Let $K(c)$ denote the *ray class field of conductor* c of K, which is characterized as the unique abelian extension of K such that the Artin map provides an isomorphism between $\mathrm{Cl}(K,c)$ and $\mathrm{Gal}(K(c)/K)$. It is well-known that every finite abelian extension is contained in some ray class field. The *conductor* of a finite abelian extension L/K is defined to be the conductor of the smallest ray class field of K that L lies in (note that if $c \mid c'$, then $K(c) \subset K(c')$). Additionally, it is true that any prime $\mathfrak{p}$ of $\mathcal{O}_K$ that ramifies in L must divide c.

The importance of the conductor of a finite abelian extension is that it determines exactly which primes ramify. We next show that conductors of cubic cyclic extensions over a quadratic field are squarefree away from 3 and never divisible by 27.

Lemma 2.1. *Fix a integer $c \in \mathbb{Z}$ with prime factorization $c = 3^k \cdot \prod_{j=1}^n p_j^{k_j}$.*

(a) *If $k = 0$, any cubic cyclic extension of a quadratic field K that is unramified away from primes dividing c is contained in the ray class field $K\big(\prod_{j=1}^n p_j\big)$.*

(b) *If $k > 0$, any cubic cyclic extension of a quadratic field K that is unramified away from the primes dividing c contained in the ray class field $K\big(9 \cdot \prod_{j=1}^n p_j\big)$.*

Proof. Part (6) of Theorem 9.2.6 in [5] implies that the conductor f of a cubic extension L over K is squarefree away from 3. We deduce part (a) by noting that $f \mid \prod_{j=1}^n p_j$ since L cannot ramify at any prime which is coprime to c.

If $3 \mid f$ and $\mathfrak{p}$ is a prime ideal of K above 3, then $(\mathcal{O}_{K_{\mathfrak{p}}}^\times)^3$ contains $1 + 9\mathcal{O}_{K_{\mathfrak{p}}} = (1 + 3\mathcal{O}_{K_{\mathfrak{p}}})^3$. Using the definition $\mathrm{Cl}(K,f) = \mathbb{A}_K^\times / K^\times W_f$, there is an index-3 subgroup of $\mathbb{A}_K^\times$ corresponding to the extension L. Since any such index-3 subgroup of $\mathbb{A}_K^\times$ will contain the cubes $(\mathcal{O}_{K_{\mathfrak{p}}}^\times)^3$, we deduce that $9 \mid f$, but $27 \nmid f$, and we can then combine this fact with part (a) to deduce part (b). ☐

Lemma 2.1 implies that the minimality restriction on conductors of cubic extensions of quadratic fields requires that such conductors are integers c which are squarefree away from 3 and additionally, $27 \nmid c$. We next study the relationship between conductors and discriminants of cubic extensions.

Lemma 2.2. (a) *Let K be a quadratic field. If L is a non-Galois cubic field such that the compositum LK is Galois over $\mathbb{Q}$, then $\mathrm{Disc}(L) = \mathrm{Disc}(K) f^2$, where f is equal to the conductor of LK over K.*

(b) *If L is a Galois cubic field and $\mathrm{Disc}(L) = f_0^2$, then $f_0 = 3^e \cdot p_1 \cdot \ldots \cdot p_m$ where $e = 0$ or 2 and each p_i denotes a distinct prime satisfying $p_i \equiv 1 \bmod 3$ for all i. Additionally, $L \subset \mathbb{Q}(f_0)$.*

Proof. Part (a) follows from Theorem 9.2.6(4) in [5]. Part (b) follows from class field theory (see [7]). ☐

Next, we explicitly determine cubic fields that lie inside the normal closure (over $\mathbb{Q}$) of a cubic cyclic extension of a quadratic field K. We show that the quantity of such cubic fields can be used to compute the number of the 3-torsion elements in the ray class groups of K.

2.2 Index-3 subgroups of ray class groups of quadratic fields

For the remainder of the section, fix a conductor c as described by Lemma 2.1, i.e. let c be a positive integer which is squarefree away from 3, and $27 \nmid c$. We describe the relationship between index-3 subgroups of $\mathrm{Cl}(K_2, c)$ for a quadratic field K_2 and certain pairs of cubic field. To do so, we must first introduce some notation. Call an integer $d \in \mathbb{Z}$ *fundamental* if it is the discriminant of some quadratic field.

Definition 2.3. *We say that a pair of fields (K^+, K^-) is* **c-valid** *if:*

- *$K^+ = \mathbb{Q}$ or a Galois cubic field with $\mathrm{Disc}(K^+) \mid c^2$, and*
- *$K^- = \mathbb{Q}$ or a non-Galois cubic field with $\mathrm{Disc}(K^-) = df^2$ where $d \in \mathbb{Z}$ is fundamental and $f \mid c$.*

Two c-valid pairs (K^+, K^-) and (M^+, M^-) are **isomorphic** *if both $K^+ \cong M^+$ and $K^- \cong M^-$.*

To see an explicit example, let $c = 7$, and take $K^+ = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, where ζ_7 denotes a 7th root of unity. If θ denotes a root of $f(x) = x^3 - x^2 + 5x + 1$, then $K^- = \mathbb{Q}(\theta)$ totally ramifies at 7. Since $\mathrm{Disc}(K^+) = 49$ and $\mathrm{Disc}(K^-) = -3 \cdot 2^2 \cdot 7^2$, it follows that (K^+, K^-) is a 7-valid pair.

The only 1-valid pairs have $K^+ = \mathbb{Q}$, and $K^- = \mathbb{Q}$ or has discriminant equal to the discriminant of a quadratic field. It is straightforward to check that such cubic fields K^- cannot be totally ramified at any prime.

Below, we give names to certain special classes of c-valid pairs.

Definition 2.4. *Let c be a positive integer which is squarefree away from 3, and $27 \nmid c$.*

(a) *The pair $(\mathbb{Q}, \mathbb{Q})$ is the* **trivial c-valid pair** *for any c.*

(b) *For any cyclic cubic field K_3 satisfying $\mathrm{Disc}(K_3) \mid c^2$, $(K_3, \mathbb{Q})$ is a c-valid pair. We refer to K_3 as a* **(cyclic) c-valid cubic field***.*

(c) *For any non-cyclic cubic field K_3 whose discriminant can be written as df^2 where $f \mid c$ and d is a fundamental, $(\mathbb{Q}, K_3)$ is a c-valid pair. We refer to K_3 as a* **(non-cyclic) c-valid cubic field***.*

We now state the main result of this section, which describes a correspondence between c-valid pairs and index-3 subgroups of $\mathrm{Cl}(K_2, c)$. This will allow us to later determine the size of $\mathrm{Cl}_3(K_2, c)$ in terms of c-valid cubic fields.

Theorem 2.5. *Let c be a positive integer which is squarefree away from 3, and additionally, $27 \nmid c$. There is a natural bijection between pairs (K_2, G) consisting of a quadratic field K_2 along with an index-3 subgroup G of $\mathrm{Cl}(K_2, c)$ and isomorphism classes of non-trivial c-valid pairs.*

When $c = 1$ and $\mathrm{Cl}_3(K_2) = \mathrm{Cl}_3(K_2, 1)$, Theorem 2.5 is simply the bijection used in [8] between nowhere totally ramified cubic fields and index-3 subgroups of the class groups of quadratic fields (see also [3]). We prove this generalization by studying prime ramification in (cubic) subfields contained within the Galois closure of an arbitrary cubic cyclic extension K_6 over K_2. These cubic subfields turn out to be c-valid iff K_6 is unramified away from c.

The goal for the remainder of this section is to prove Theorem 2.5. We first discuss the Galois theory of an arbitrary cubic cyclic extension of a quadratic number field.

2.3 Cubic cyclic extensions of quadratic fields

In order to prove Theorem 2.5, we first show that for a fixed quadratic field, any cubic cyclic extension of conductor c is determined by a (unique up to isomorphism) non-trivial c-valid pair. To find a candidate for this c-valid pair, we look within the normal closure (over $\mathbb{Q}$) of such sextic fields. For any number field K, let $\widetilde{K}$ denote its normal closure over $\mathbb{Q}$.

Fix a quadratic extension $K_2/\mathbb{Q}$. If K_6/K_2 is a cyclic cubic extension, then the Galois group $\mathrm{Gal}(\widetilde{K}_6/\mathbb{Q})$ is equal to S_3, C_6, or $S_3 \times C_3$, which are the transitive subgroups with order at least 6 in the wreath product

$$\mathrm{Gal}(K_6/K_2) \wr \mathrm{Gal}(K_2/\mathbb{Q}) \cong (C_3 \times C_3) \rtimes C_2 \cong S_3 \times C_3. \tag{4}$$

Note that in the first two cases, K_6 is already Galois. We have the following field diagram when $K_6 \neq \widetilde{K_6}$.

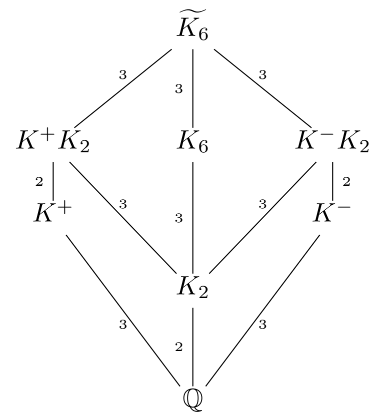

Figure 1: Some subfields of $\widetilde{K_6}$ when $\mathrm{Gal}(K_6/\mathbb{Q}) \cong S_3 \times C_3$

When K_6 is not Galois, denote the subfield of $\widetilde{K}_6$ fixed by $C_2 \times C_3 \subset S_3 \times C_3$ as K^-, and the subfield fixed by S_3 as K^+. We then have that $\mathrm{Gal}(K^+/\mathbb{Q}) = C_3$, and $\mathrm{Gal}(\widetilde{K}^-/\mathbb{Q}) = S_3$ with $\widetilde{K}^- = K^-K_2$. It follows that $\widetilde{K_6} = \widetilde{K^+K^-}$. We have thus proven the following lemma stating that K^+ and K^- determine K_6 and vice versa.

Lemma 2.6. *Let K_6 denote a cubic cyclic extension over a quadratic field K_2. If K_6 is not Galois over $\mathbb{Q}$, then*

$$\mathrm{Gal}(\widetilde{K_6}/\mathbb{Q}) \cong S_3 \times C_3.$$

Additionally, there exists a unique pair of (isomorphism classes of) cubic subfields K^+ and K^-, where K^+ is cyclic and K^- is not Galois such that the normal closure of K^+K^- is equal to $\widetilde{K_6}$. In particular, we can explicitly write

$$\widetilde{K_6} = K^+K_2K^-.$$

This lemma implies that any degree-18 field with Galois group over $\mathbb{Q}$ equal to $S_3 \times C_3$ is either determined by a degree 6 non-Galois subfield that is cyclic over a quadratic subextension or equivalently, by the fixed field of $C_2 \times C_3$ and the fixed field of S_3. We will use this to prove that the pair of fields denoted in Figure 1 by K^+ and K^- make a c-valid pair whenever K_6 is unramified away from c.

2.4 Ramification in fields with Galois group $S_3 \times C_3$

We next show that a pair of cubic fields associated to a cubic cyclic extension of conductor c over a fixed quadratic field by Lemma 2.6 is indeed c-valid. We do so by understanding how ramification in the sextic field determines ramification in the pair and vice versa. We begin by reviewing properties of the discriminants of subfields within a Galois sextic field.

Lemma 2.7. *Fix a quadratic field K_2. Any c-valid cubic field K such that K_2K is Galois over $\mathbb{Q}$ satisfies:*

(a) $\mathrm{Disc}(K)^2 \mid \mathrm{Disc}(K_2K)$;

(b) $\mathrm{Nm}_K(\mathrm{Disc}(K_2K/K)) \mid \mathrm{Disc}(K_2)^3$*;*

(c) $\mathrm{Disc}(K_2K) \mid c^4 \cdot \mathrm{Disc}(K_2)^3$.

Proof. There are two different ways we can calculate the discriminant of K_2K using the field towers $K_2K/K_2/\mathbb{Q}$ and $K_2K/K/\mathbb{Q}$:

$$\mathrm{Nm}_{K_2}(\mathrm{Disc}(K_2K/K_2)) \cdot \mathrm{Disc}(K_2)^3 = \mathrm{Disc}(K_2K) = \mathrm{Nm}_K(\mathrm{Disc}(K_2K/K)) \cdot \mathrm{Disc}(K)^2. \tag{5}$$

(a) By the second equality in (5), we conclude that $\mathrm{Disc}(K)^2 \mid \mathrm{Disc}(K_2K)$.

(b) Let $[\beta_1, \beta_2]$ denote an integral basis of $\mathcal{O}_{K_2}$, and define the 2×2 matrix $M = [\sigma_i(\beta_j)]_{i,j}$ where σ_i run through elements of $\mathrm{Gal}(K_2K/K)$. First, we know that $\det(M)^2 = \mathrm{Disc}(K_2)$, and second, we have that $[\beta_1, \beta_2]$ is a K-basis for K_2K. This implies that $\det(M)^2 \in \mathrm{Disc}(K_2K/K)$, and hence

$$\mathrm{Disc}(K_2K/K) \mid \mathrm{Disc}(K_2)$$

as $\mathcal{O}_K$-ideals. Part (b) then follows by taking norms.

(c) The extension K_2K/K_2 is abelian, and by Theorem 9.2.6 of [5], it has an integral conductor f. It is related to the relative discriminant by $\mathrm{Disc}(K_2K/K_2) = (f\mathcal{O}_{K_2})^2$. If K is non-Galois, then $\mathrm{Disc}(K) = df^2$ by Lemma 2.2(a), and so $f \mid c$. If K is Galois, then $\mathrm{Disc}(K) = f^2$ where $f \mid c$.

From (5), we have that

$$\mathrm{Disc}(K_2K) = \mathrm{Nm}_{K_2}(f^2) \cdot \mathrm{Disc}(K_2)^3.$$

Since $f \mid c$, we obtain $\mathrm{Disc}(K_2K) \mid \mathrm{Nm}_{K_2}(c^2) \cdot \mathrm{Disc}(K_2)^3$, which implies that

$$\mathrm{Disc}(K_2K) \mid c^4 \cdot \mathrm{Disc}(K_2)^3.$$

□

Proposition 2.8. *Fix a quadratic field K_2. Any cubic cyclic extension K_6 over K_2 of conductor c that is not Galois over $\mathbb{Q}$ has a c-valid pair (K^+, K^-) contained within the normal closure $\widetilde{K_6}$ satisfying $\widetilde{K^+K^-} = \widetilde{K_6}$. It is unique up to isomorphism.*

Proof. Given a cubic cyclic extension K_6 over K_2 of conductor c, the candidate c-valid pair (K^+, K^-) associated to K_6 comes from Lemma 2.6 and Figure 1. It remains to show that $\mathrm{Disc}(K^+) \mid c^2$ and $\frac{\mathrm{Disc}(K^-)}{\mathrm{Disc}(K_2)} \mid c^2$. To do so, we study the relationship between total ramification in K^+ or K^- and ramification in K_6/K_2.

We begin with K^-. By Lemma 2.2(a), the conductor of K^-K_2/K_2 is equal to f where $\mathrm{Disc}(K^-) = \mathrm{Disc}(K_2)f^2$. Combined with Lemma 2.7, we obtain $\mathrm{Disc}(K_2)^2 f^4 \mid c^4\,\mathrm{Disc}(K_2)^3$, and so

$$f^4 \mid c^4 \cdot \mathrm{Disc}(K_2).$$

Recall that $\mathrm{Disc}(K_2)$ is squarefree away from 2, and $2^4 \nmid \mathrm{Disc}(K_2)$ for any quadratic field, so we conclude that $f \mid c$.

We now turn to K^+. Lemma 2.7 implies that

$$\mathrm{Disc}(K^+)^2 \mid c^4\,\mathrm{Disc}(K_2)^3.$$

In this case, $\mathrm{Disc}(K^+) = f_0^2$ for some integer f_0, and so we obtain

$$f_0^4 \mid c^4 \cdot \mathrm{Disc}(K_2)^3.$$

If $\mathrm{Disc}(K_2)$ is odd, then it is squarefree by Lemma 2.2(b). This implies $f_0 \mid c$.

If $\mathrm{Disc}(K_2)$ is even, but $2 \nmid f_0$, then we also conclude $f_0 \mid c$. If $2 \mid f_0$, assume for the sake of contradiction that $2 \nmid c$. Then $\widetilde{K_6}$ is unramified over K_2 at the primes above 2. This implies that the ramification degree of $\widetilde{K_6}/\mathbb{Q}$ is at most 2 for $p = 2$, and thus the ramification degree is at most 2 in K^+. Since K^+ is cyclic of degree 3, 2 is thus unramified in K^+, which contradicts $2 \mid f_0$ since $\mathrm{Disc}(K^+) = f_0^2$. Thus, $2 \mid c$, and since $4 \nmid f_0$ by Lemma 2.2(b), we obtain $f_0 \mid c$. □

2.5 Proof of Theorem 2.5

We finally return to Theorem 2.5 and give a proof. We begin by giving an explicit description of the map. Let c be as in the statement of Theorem 2.5. If (K_2, G) is a quadratic field along with an index-3 subgroup G of $\mathrm{Cl}(K_2, c)$, then let K_6 denote the fixed field in $K_2(c)$ for the subgroup G so that $\mathrm{Gal}(K_6/K_2) = \mathrm{Cl}(K_2, c)/G$. We then have:

- If K_6 is Galois over $\mathbb{Q}$ and $\mathrm{Gal}(K_6/\mathbb{Q}) = C_6$, then (K_2, G) corresponds to $(K_3, \mathbb{Q})$, where K_3 is the cubic subfield of K_6;
- If K_6 is Galois over $\mathbb{Q}$ and $\mathrm{Gal}(K_6/\mathbb{Q}) = S_3$, then (K_2, G) corresponds to $(\mathbb{Q}, K_3)$, where K_3 is the cubic subfield of K_6;
- If K_6 is not Galois over $\mathbb{Q}$, then (K_2, G) corresponds to (K^+, K^-) as constructed in Lemma 2.6.

If K_6 is Galois, then its cubic subfield K_3 can only totally ramify at primes dividing c. Indeed, this is clear when $\mathrm{Gal}(K_6/\mathbb{Q}) = C_6$. When $\mathrm{Gal}(K_6/\mathbb{Q}) = S_3$, a prime p ramifies in the extension K_6/K_2 if and only if $p^2 \mid \frac{\mathrm{Disc}(K_3)}{\mathrm{Disc}(K_2)}$. Thus, K_3 is a c-valid cubic field in either case.

When K_6 is not Galois, Proposition 2.8 implies that the above map sends (K_2, G) to a c-valid pair (K^+, K^-). To prove the other direction, we begin with a c-valid pair (K^+, K^-) and construct a cubic cyclic extension over K_2 of conductor dividing c. Then, the Galois group of $K_2(c)$ over this cubic cyclic extension will be equal to G.

Recall that the compositum $K^+K_2K^-$ is Galois over $\mathbb{Q}$ of degree 6 or 18. If it is degree 6, then (K^+, K^-) is in fact a non-trivial c-valid *cubic field*, i.e., exactly one of $K^\pm$ is equal to $\mathbb{Q}$. Thus, we take K_6 to be the compositum $K^+K_2K^-$. In this case, it remains to show that K_6 has conductor dividing c over K_2, i.e. $K_6 \subset K_2(c)$. If $K^- = \mathbb{Q}$, then by assumption $\mathrm{Disc}(K^+) = f_0^2$ where $f_0 \mid c$; thus, $K^+ \subset \mathbb{Q}(f_0) \subset \mathbb{Q}(c)$, so $K_6 = K^+K_2 \subset K_2(c)$. If $K^+ = \mathbb{Q}$, then $\mathrm{Disc}(K^-) = \mathrm{Disc}(K_2)f^2$ where $f \mid c$, and so $K^-K_2 = \widetilde{K^-}$ is a cubic extension of K_2 contained in $K_2(f) \subset K_2(c)$ by Lemma 2.2(a).

If $K^+K_2K^-$ is degree 18, it has Galois group equal to $S_3 \times C_3$, and we define K_6 to be the fixed field of any non-normal $C_3 \subset S_3 \times C_3$. It remains to show that K_6 has conductor dividing c as an extension over K_2. We do so by proving that K_2K^- and K_2K^+ have conductor dividing c over K_2.

Lemma 2.2(a) implies that K_2K^- has conductor f where $\mathrm{Disc}(K^-) = \mathrm{Disc}(K_2) \cdot f^2$, and $f \mid c$. Additionally, if $\mathrm{Disc}(K^+) = f_0^2$, suppose p is a prime such that $p \nmid f_0$. Then p cannot ramify in K^+, which implies that K^+K_2/K_2 is unramified above p. By Lemma 2.1, we conclude that K_2K^+/K_2 has conductor dividing

$$\begin{cases} \displaystyle\prod_{p \mid f_0} p & \text{if } 3 \nmid f_0, \text{ or} \\ \displaystyle 9 \cdot \prod_{3 \neq p \mid f_0} p & \text{if } 3 \mid f_0. \end{cases}$$

If $3 \nmid f_0$, then f_0 is squarefree by Lemma 2.2(b), and so the conductor of K_2K^+ over K_2 divides c since $f_0 \mid c$. If $3 \mid f_0$, note that $9 \mid\mid f_0$; thus, we altogether obtain that K_2K^+ and K_2K^- both have conductor dividing c. Since $\widetilde{K_6} = K^+K_2K^-$, it must have conductor dividing c as an extension over K_2, and so K_6 as a subextension must also have conductor dividing c.

It is easy to check that two non-isomorphic c-valid pairs (K^+, K^-) correspond to distinct non-isomorphic cubic cyclic extensions of K_2 of conductor c, and thus, they correspond to distinct index 3-subgroups of $\mathrm{Cl}(K_2, c)$. □

Using the fact that the number of order-3 subgroups is equal to the number of index-3 subgroups in a finite abelian group, we directly relate the number of non-trivial c-valid pairs to the number of 3-torsion elements in ray class groups of conductor c.

Corollary 2.9. *If c is a positive integer which is squarefree away from 3 and $27 \nmid c$, then*

$$\#\mathrm{Cl}_3(K_2, c) = 2 \cdot \# \left\{ \begin{array}{c} \text{non-trivial } c\text{-valid pairs of fields } (K^+, K^-) \\ \text{s.t. } K^- = \mathbb{Q} \text{ or has quadratic resolvent } K_2 \end{array} \right\} + 1.$$

Recall that the *quadratic resolvent* of a non-Galois cubic field K_3 is the quadratic subfield of the normal closure $\widetilde{K_3}$. If K_3 has quadratic resolvent K_2, then $\mathrm{Disc}(K_2) \mid \mathrm{Disc}(K_3)$ and $\frac{\mathrm{Disc}(K_3)}{\mathrm{Disc}(K_2)}$ is equal to the square of the conductor of $\mathrm{Gal}(\widetilde{K_3}/K_2)$ by Lemma 2.2(b).

2.6 The action of $\mathrm{Gal}(K_2/\mathbb{Q})$ on $\mathrm{Cl}_3(K_2, c)$

We next consider the action of $\mathrm{Gal}(K_2/\mathbb{Q})$ on $\mathrm{Cl}_3(K_2, c)$. The number of c-valid *cubic fields* on the right-hand side of the equality in Corollary 2.9 is related to the sizes of eigenspaces for the action of $\mathrm{Gal}(K_2/\mathbb{Q})$. Note that $\mathrm{Cl}_3(K_2, c)$ is a $\mathrm{Gal}(K_2/\mathbb{Q})$-module of odd order, and thus we have two well-defined submodules of $\mathrm{Cl}_3(K_2, c)$:

$$\begin{aligned} \mathrm{Cl}_3^+(K_2, c) &:= \{[I] \in \mathrm{Cl}_3(K_2, c) : \sigma(I) = I\}, \text{ and} \\ \mathrm{Cl}_3^-(K_2, c) &:= \{[I] \in \mathrm{Cl}_3(K_2, c) : \sigma(I) = J \text{ where } [I]^{-1} = [J]\}. \end{aligned}$$

We then have $\mathrm{Cl}_3(K_2, c) = \mathrm{Cl}_3^+(K_2, c) \oplus \mathrm{Cl}_3^-(K_2, c)$, and thus

$$\#\mathrm{Cl}_3(K_2, c) = \#\mathrm{Cl}_3^+(K_2, c) \cdot \#\mathrm{Cl}_3^-(K_2, c). \tag{6}$$

Proposition 2.10. *Fix a quadratic field K_2, and let c be a positive integer which is squarefree away from 3, and $27 \nmid c$. Then:*

$$\begin{aligned} &\text{(a)} \quad \#\mathrm{Cl}_3^+(\mathrm{K_2, c}) &= \quad &2 \cdot \# \{\text{Cyclic } c\text{-valid cubic fields } K^+\} + 1; \\ &\text{(b)} \quad \#\mathrm{Cl}_3^-(K_2, c) &= \quad &2 \cdot \# \{\text{Non-cyclic } c\text{-valid cubic fields } K^- \text{ with quadratic resolvent } K_2\} + 1. \end{aligned}$$

Proof. The second part follows from Lemma 1.10 of [12] and Proposition 35 of [4]. To prove the first part, consider some cyclic cubic field K^+ unramified away from c. By class field theory and the proof of Proposition 2.5, K^+K_2/K_2 corresponds to a index-3 subgroup H of $\mathrm{Cl}(K_2, c)^{(3)}$, the 3-Sylow subgroup of $\mathrm{Cl}(K_2, c)$. Since K^+K_2 is Galois over $\mathbb{Q}$, H has an action of $\mathrm{Gal}(K_2/\mathbb{Q})$. Artin reciprocity implies that

$$\sigma(K^+K_2) = K^+K_2 \quad \Rightarrow \quad \sigma(H) = H.$$

Thus, we write $H = H^+ \oplus H^-$, where $H^\pm := \{[I] \in H : \sigma([I]) = [I]^\pm\}$. Let $\mathrm{Cl}^\pm(K_2, c)^{(3)}$ be defined analogously. Since H is index 3, it is clear that $H^+ = \mathrm{Cl}^+(K_2, c)^{(3)}$ or $H^- = \mathrm{Cl}^-(K_2, c)^{(3)}$.

We now show that $H^- = \mathrm{Cl}^-(K_2, c)^{(3)}$, so that $\mathrm{Cl}(K_2, c)^{(3)}/H \cong \mathrm{Cl}^+(K_2, c)^{(3)}/H^+$. For any lift $\tilde{\sigma}$ of σ to $\mathrm{Gal}(K^+K_2/K_2)$, Artin reciprocity implies the action of conjugation on $\mathrm{Gal}(K^+K_2/K_2)$ by $\tilde{\sigma}$ corresponds to acting by σ on $\mathrm{Cl}(K_2,c)^{(3)}/H$. Since $\mathrm{Gal}(K^+K_2/K_2)$ is isomorphic to C_6, σ acts trivially on $\mathrm{Cl}(K_2,c)^{(3)}/H$ so $H^- = \mathrm{Cl}^-(K_2,c)^{(3)}$. We then have that the number of index-3 subgroups of $\mathrm{Cl}^+(K_2,c)^{(3)}$ is the same as the number of order-3 subgroups, which are generated by nontrivial elements of $\mathrm{Cl}_3^+(K_2,c)$. Since powers of an element generate the same subgroup we then deduce the first part. □

We additionally remark that for any quadratic field K_2, $\mathrm{Cl}_3^+(K_2,c) = \mathrm{Cl}_3(\mathbb{Q},c)$, independent of K_2. This is a crucial fact that greatly simplifies the computation for the average size of ray class groups of conductor c when K_2 is allowed to vary. We next compute asymptotics for both $\mathrm{Cl}_3^\pm(K_2,c)$ by counting the relevant c-valid cubic fields as given in Proposition 2.10.

3 Counting c-valid cubic fields

The results of the previous section allow us to determine the number of 3-torsion elements in ray class groups of quadratic fields simply in terms of c-valid cubic fields instead of c-valid pairs. We first compute the size of $\mathrm{Cl}_3^+(K_2,c)$ for any quadratic field K_2 by enumerating the number of cyclic c-valid cubic fields. In order to obtain asymptotics for the size of $\mathrm{Cl}_3^-(K_2,c)$, we then employ the results of [3] building on those of [8] for computing the number of cubic fields with bounded discriminant that satisfy certain ramification restrictions.

3.1 The size of the 3-torsion subgroup in ray class groups of $\mathbb{Q}$

As before, let K_2 be a quadratic field. In this section, we prove that the number of $\mathrm{Gal}(K_2/\mathbb{Q})$-stable elements in the 3-torsion subgroups of the ray class group of conductor c depends only on the number of distinct primes congruent to 1 mod 3 that divide c. More precisely,

Proposition 3.1. *Let K_2 be a quadratic field, and let c be a positive integer. Let the number of distinct prime factors $p_i \mid c$ such that $p_i \equiv 1 \bmod 3$ be denoted by m. Then*

$$\#\mathrm{Cl}_3^+(K_2,c) = \begin{cases} 3^m & \text{if } 9 \nmid c, \text{ and} \\ 3^{m+1} & \text{if } 9 \mid c, \end{cases}$$

independent of the quadratic field K_2.

Proof. By Proposition 2.10(a), we can enumerate elements of $\mathrm{Cl}_3^+(K_2,c)$ by counting cyclic c-valid cubic fields, i.e., normal cubic extensions of $\mathbb{Q}$ with discriminant dividing c^2. If K^+ is such a cyclic field of degree 3, by Lemma 2.2(b), the conductor of K^+ is equal to $c_0 = 3^e \cdot p_1 \cdot \ldots \cdot p_m$ where $e = 0$ or 2, and p_i denotes distinct primes satisfying $p_i \equiv 1 \bmod 3$ for all i. Furthermore, if $e = 0$, then there are 2^{m-1} cubic cyclic fields of conductor c_0, and if $e = 2$, then there are 2^m cubic cyclic fields of conductor c_0 (see [7]). We must therefore enumerate cyclic c-valid cubic fields with discriminant c_0^2 where c_0 is as above.

If $9 \nmid c$, let $c = 3^e \cdot p_1^{k_1} \cdot \ldots \cdot p_m^{k_m} \cdot q_{m+1}^{k_{m+1}} \cdot \ldots \cdot q_n^{k_n}$ where each p_i is a distinct prime congruent to 1 mod 3, q_j are distinct primes congruent to 2 mod 3, and $e = 0$ or 1. In conjunction with Proposition 2.10(a), we obtain

$$\#\mathrm{Cl}_3^+(K_2,c) = 1 + 2 \cdot \left(\sum_{j=1}^{m} \binom{m}{j} 2^{j-1} \right) = 3^m.$$

Similarly, if $e \geq 2$, we deduce

$$\#\mathrm{Cl}_3^+(K_2,c) = 1 + 2 \cdot \left(\sum_{j=1}^{m+1} \binom{m+1}{j} 2^{j} \right) = 3^{m+1}.$$

□

3.2 The asymptotic number of non-cyclic c-valid cubic fields

We want to next determine the asymptotics for the number of cubic fields that are totally ramified at a certain fixed set of primes. Let $\mathcal{K}^{\text{full}}$ denote the set of isomorphism classes of cubic fields, and for any subset $\mathcal{K} \subseteq \mathcal{K}^{\text{full}}$, define for $i = 0$ or 1:

$$N_3^{(i)}(\mathcal{K}, X) := \#\{K_3 \in \mathcal{K} \mid 0 < (-1)^i \operatorname{Disc}(K_3) < X\}.$$

Theorem 3.2. *Let S denote a set of primes, and let $n_i = \# \operatorname{Aut}(\mathbb{R}^{3-2i} \oplus \mathbb{C}^i)$ for $i = 0$ or 1.*

(a) *Let $\mathcal{K}_S$ denote the set of isomorphism classes of cubic fields that are totally ramified exactly at the primes $p \in S$.*

$$\lim_{X\to\infty} \frac{N_3^{(i)}(\mathcal{K}_S, X)}{X} = \frac{1}{n_i} \cdot \frac{1}{2 \cdot \zeta(2)} \cdot \prod_{p\in S} \frac{1}{p(p+1)}$$

(b) *If $3 \in S$, let $\mathcal{K}_S^{(3)}$ denote the set of isomorphism classes of cubic fields that are totally ramified exactly at $p \in S$ and have discriminant that is* not *divisible by 81.*

$$\lim_{X\to\infty} \frac{N_3^{(i)}(\mathcal{K}_S^{(3)}, X)}{X} = \frac{1}{n_i} \cdot \frac{1}{3 \cdot \zeta(2)} \cdot \prod_{p\in S} \frac{1}{p(p+1)}$$

(c) *If $3 \in S$, let $\mathcal{K}_S^{(9)}$ denote the set of isomorphism classes of cubic fields that are totally ramified exactly at $p \in S$ and have discriminant divisible by 81.*

$$\lim_{X\to\infty} \frac{N_3^{(i)}(\mathcal{K}_S^{(9)}, X)}{X} = \frac{1}{n_i} \cdot \frac{1}{6 \cdot \zeta(2)} \cdot \prod_{p\in S} \frac{1}{p(p+1)}.$$

(d) *Let S' be a set of primes containing S. Let $\mathcal{K}_{S,S'}$ denote the set of isomorphism classes of cubic fields that are totally ramified exactly at $p \in S$ and unramified at $p \in S' \smallsetminus S$.*

$$\lim_{X\to\infty} \frac{N_3^{(i)}(\mathcal{K}_{S,S'}, X)}{X} = \frac{1}{n_i} \cdot \frac{1}{2 \cdot \zeta(2)} \cdot \prod_{p\in S'\smallsetminus S} \frac{p}{p+1} \cdot \prod_{p\in S} \frac{1}{p(p+1)}.$$

(e) *Let S' be a set of primes containing S. If $3 \in S$, let $\mathcal{K}_{S,S'}^{(9)}$ denote the set of isomorphism classes of cubic fields K_3 that are totally ramified exactly at $p \in S$, unramified at $p \in S' \smallsetminus S$, and $81 \mid\mid \operatorname{Disc}(K_3)$.*

$$\lim_{X\to\infty} \frac{N_3^{(i)}(\mathcal{K}_{S,S'}^{(9)}, X)}{X} = \frac{1}{n_i} \cdot \frac{1}{9 \cdot \zeta(2)} \cdot \prod_{p\in S'\smallsetminus S} \frac{p}{p+1} \cdot \prod_{p\in S} \frac{1}{p(p+1)}.$$

Proof. For any prime p, let Σ_p^{tr} denote the set of all isomorphism classes of maximal cubic étale algebras over $\mathbb{Z}_p$ that are totally ramified. Let Σ_p^{ur} denote the set of all isomorphism classes of maximal cubic étale algebras over $\mathbb{Z}_p$ that are unramified. Additionally, let Σ_p^{ntr} denote the set of all isomorphism classes of maximal cubic étale algebras over $\mathbb{Z}_p$ that are not totally ramified. When $p = 3$, let $\Sigma_3^{(3)}$ denote the subset of Σ_3^{tr} whose discriminant over $\mathbb{Z}_3$ is not divisible by 81, and let $\Sigma_3^{(9)}$ denote the subset of Σ_3^{tr} whose discriminant is divisible by 81. Finally, let $\Sigma_3^{(81)}$ denote the subset of Σ_3^{tr} whose discriminant is equal to 81.

We now describe collections $\Sigma = (\Sigma_p)_p$ of local specifications at each prime p which exactly determine the (maximal orders of) cubic fields contained in $\mathcal{K}_S$, $\mathcal{K}_S^{(3)}$, $\mathcal{K}_S^{(9)}$, $\mathcal{K}_{S,S'}$, and $\mathcal{K}_{S,S'}^{(9)}$, respectively. In each case, Σ is an *acceptable* collection of local specifications as defined in [3]. Indeed, we can equivalently define the family of cubic fields (a) $\mathcal{K}_S$, (b) $\mathcal{K}_S^{(3)}$, (c) $\mathcal{K}_S^{(9)}$, (d) $\mathcal{K}_{S,S'}$, or (e) $\mathcal{K}_{S,S'}^{(9)}$ as containing exactly the fraction fields of all maximal orders R for which $R \otimes \mathbb{Z}_p \in \Sigma_p$ for all p where:

(a) $\Sigma_p = \begin{cases} \Sigma_p^{\text{ntr}} & \text{if } p \notin S, \\ \Sigma_p^{\text{tr}} & \text{if } p \in S; \end{cases}$

(b) $\Sigma_p = \begin{cases} \Sigma_p^{\text{ntr}} & \text{if } p \notin S, \\ \Sigma_p^{\text{tr}} & \text{if } p \in S \smallsetminus \{3\}, \\ \Sigma_3^{(3)} & \text{if } p = 3; \end{cases}$

(c) $\Sigma_p = \begin{cases} \Sigma_p^{\text{ntr}} & \text{if } p \notin S, \\ \Sigma_p^{\text{tr}} & \text{if } p \in S \smallsetminus \{3\}, \\ \Sigma_3^{(9)} & \text{if } p = 3; \end{cases}$

(d) $\Sigma_p = \begin{cases} \Sigma_p^{\text{ntr}} & \text{if } p \notin S_0, \\ \Sigma_p^{\text{tr}} & \text{if } p \in S, \\ \Sigma_p^{\text{ur}} & \text{if } p \in S' \smallsetminus S; \end{cases}$

(e) $\Sigma_p = \begin{cases} \Sigma_p^{\text{ntr}} & \text{if } p \notin S_0, \\ \Sigma_p^{\text{tr}} & \text{if } p \in S \smallsetminus \{3\}, \\ \Sigma_p^{\text{ur}} & \text{if } p \in S' \smallsetminus S, \\ \Sigma_3^{(81)} & \text{if } p = 3. \end{cases}$

Let $N_3^{(i)}(\Sigma, X)$, the number of (isomorphism classes) of maximal cubic rings R such that $R \otimes \mathbb{Z}_p \in \Sigma_p$ for all p with $0 < (-1)^i \operatorname{Disc}(R) < X$. We asymptotically compute $N_3^{(i)}(\Sigma, X)$ using Theorem 7 in [3], which determines the main term in terms of a mass formula whenever Σ is defined by an acceptable collection of local conditions. More precisely, they prove

$$\lim_{X\to\infty} \frac{N_3^{(i)}(\Sigma, X)}{X} = \frac{1}{2n_i} \cdot \prod_p \left(\frac{p-1}{p} \cdot \sum_{R \in \Sigma_p} \frac{1}{\operatorname{Disc}_p(R)} \cdot \frac{1}{\#\operatorname{Aut}(R)} \right), \tag{7}$$

where $\operatorname{Disc}_p(R)$ denotes the discriminant of R over $\mathbb{Z}_p$ as a power of p. We compute (or combine Lemmas 18, 19, and 32 in [3] to deduce):

$$\sum_{R\in\Sigma_p} \frac{1}{\operatorname{Disc}_p(R)} \cdot \frac{1}{\#\operatorname{Aut}(R)} = \begin{cases} \frac{p+1}{p} & \text{if } \Sigma_p = \Sigma_p^{\text{ntr}}, \\ \frac{1}{p^2} & \text{if } \Sigma_p = \Sigma_p^{\text{tr}}, \\ 1 & \text{if } \Sigma_p = \Sigma_p^{\text{ur}}; \end{cases} \qquad \sum_{R\in\Sigma_3} \frac{1}{\operatorname{Disc}_3(R)} \cdot \frac{1}{\#\operatorname{Aut}(R)} = \begin{cases} \frac{2}{27} & \text{if } \Sigma_3 = \Sigma_3^{(3)}, \\ \frac{1}{27} & \text{if } \Sigma_3 = \Sigma_3^{(9)}, \\ \frac{2}{81} & \text{if } \Sigma_3 = \Sigma_3^{(81)}. \end{cases}$$

In conjunction with (7), we thus obtain the desired asymptotes in Theorem 3.2. As an example, we give the calculation below in case (e):

$$\begin{aligned} \lim_{X\to\infty} \frac{N_3^{(i)}(\mathcal{K}_{S,S'}^{(9)}, X)}{X} &= \frac{1}{2n_i} \cdot \frac{4}{243} \cdot \prod_{p\notin S'} \frac{p^2-1}{p^2} \cdot \prod_{p\in S'\smallsetminus S} \frac{p-1}{p} \cdot \prod_{3\neq p\in S} \frac{p-1}{p^3} \\ &= \frac{1}{n_i} \cdot \frac{1}{9\cdot\zeta(2)} \cdot \prod_{p\in S'\smallsetminus S} \frac{p}{p+1} \cdot \prod_{p\in S} \frac{1}{p(p+1)}. \end{aligned}$$

□

3.3 Prescribing splitting conditions on the quadratic resolvents of c-valid cubic fields

Now, let $\underline{S} = (S_+, S_-, S_0)$ be three disjoint sets of primes. We will next consider families $\mathcal{K}_3(\underline{S})$ consisting of all cubic fields whose quadratic resolvent field is in $\mathcal{K}_2(\underline{S})$. (Recall that $\mathcal{K}_2(\underline{S})$ consists of all quadratic fields that split at the primes in S_+, remain inert at the primes in S_-, and ramify at the primes in S_0.)

Theorem 3.3. *Let S denote a set of primes not containing 3, and let $n_i = \#\operatorname{Aut}(\mathbb{R}^{3-2i} \oplus \mathbb{C}^i)$ for $i = 0$ or 1. Additionally, let $\underline{S} = (S_+, S_-, S_0)$ be three disjoint sets of primes such that:*

- *$S_+ \cap S$ only contains primes congruent to* $1 \bmod 3$.

- *$S_- \cap S$ only contains primes congruent to* 2 mod 3.

As before, let $\mathcal{K}_S$ denote the set of isomorphism classes of cubic fields that are totally ramified exactly at the primes $p \in S$. If $\mathcal{K}_3^S(\underline{\mathcal{S}}) = \mathcal{K}_S \cap \mathcal{K}_3(\underline{\mathcal{S}})$, then we have:

$$\lim_{X\to\infty} \frac{N_3^{(i)}(\mathcal{K}_3^S(\underline{\mathcal{S}}), X)}{X} = \frac{1}{n_i} \cdot \frac{1}{2\cdot\zeta(2)} \cdot \prod_{p\in S} \frac{1}{p(p+1)} \cdot \prod_{p\in S_0} \frac{1}{p+1} \cdot \prod_{p\in S_\pm \smallsetminus (S\cap S_\pm)} \frac{p}{2(p+1)}$$

Proof. For any prime p, let:

1. Σ_p^{mr} denote the set of all (isomorphism classes of) maximal cubic étale algebras over $\mathbb{Z}_p$ that are minimally ramified, i.e., they decompose as $\mathbb{Z}_p \oplus Q$ where Q is a totally ramified quadratic étale algebra over $\mathbb{Z}_p$;
2. Σ_p^+ consist of the ring of integers of the unique unramified extension of degree 3 over $\mathbb{Q}_p$ as well as the algebra $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$;
3. $\Sigma_p^- = \{\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}\}$ where $\mathbb{Z}_{p^2}$ denotes the ring of integers of $\mathbb{Q}_{p^2}$, the unique unramified extension of degree 2 over $\mathbb{Q}_p$.
4. $\Sigma_p^{\mathrm{tr}+}$ consists of maximal cubic algebras over $\mathbb{Z}_p$ that are totally ramified and whose quadratic resolvent algebra is contained in $\mathbb{Q}_p \oplus \mathbb{Q}_p$.
5. $\Sigma_p^{\mathrm{tr}-}$ consists of maximal cubic algebras over $\mathbb{Z}_p$ that are totally ramified and whose quadratic resolvent is contained in $\mathbb{Q}_{p^2}$.

As before, denote the set of all (isomorphism classes of) maximal cubic étale algebras over $\mathbb{Z}_p$ that are totally ramified as $\Sigma_p^{\mathrm{tr}} = \Sigma_p^{\mathrm{tr}+} \cup \Sigma_p^{\mathrm{tr}-}$, denote the set of all unramified cubic étale algebras over $\mathbb{Z}_p$ as $\Sigma_p^{\mathrm{ur}} = \Sigma_p^+ \cup \Sigma_p^-$, and denote the set of all cubic étale algebras over $\mathbb{Z}_p$ that are not totally ramified as $\Sigma_p^{\mathrm{ntr}} = \Sigma_p^{\mathrm{ur}} \cup \Sigma_p^{\mathrm{mr}}$.

If $\Sigma = (\Sigma_p)_p$ denotes the acceptable collection of local specifications defining $\mathcal{K}_3^S(\underline{\mathcal{S}})$, then we have:

$$\Sigma_p = \begin{cases} \Sigma_p^{\mathrm{ntr}} & \text{if } p \notin S_+ \cup S_- \cup S_0 \cup S, \\ \Sigma_p^{\mathrm{tr}} & \text{if } p \in S \smallsetminus (S \cap (S_+ \cup S_-)), \\ \Sigma_p^{\mathrm{mr}} & \text{if } p \in S_0, \\ \Sigma_p^{\pm} & \text{if } p \in S_\pm \smallsetminus (S \cap S_\pm), \\ \Sigma_p^{\mathrm{tr}\pm} & \text{if } p \in S \cap S_\pm. \end{cases}$$

It is straightforward to determine that

$$\sum_{R\in\Sigma_p} \frac{1}{\mathrm{Disc}_p(R)} \cdot \frac{1}{|\operatorname{Aut}(R)|} = \begin{cases} \frac{1}{p} & \text{if } \Sigma_p = \Sigma_p^{\mathrm{mr}}, \\ \frac{1}{2} & \text{if } \Sigma_p = \Sigma_p^{\pm}, \\ \frac{1}{p^2} & \text{if } \Sigma_p = \Sigma_p^{\mathrm{tr}+} \text{ and } p \equiv 1 \bmod 3. \\ \frac{1}{p^2} & \text{if } \Sigma_p = \Sigma_p^{\mathrm{tr}-} \text{ and } p \equiv 2 \bmod 3. \end{cases}$$

Using (7) and the computations following it, we conclude the theorem:

$$\begin{aligned} \lim_{X\to\infty} \frac{N_3^{(i)}(\mathcal{K}_3^S(\underline{\mathcal{S}}), X)}{X} &= \frac{1}{2n_i} \cdot \prod_{p\notin S_\pm\cup S_0\cup S} \frac{p^2-1}{p^2} \cdot \prod_{p\in S} \frac{p-1}{p^3} \cdot \prod_{p\in S_0} \frac{p-1}{p^2} \cdot \prod_{p\in S_\pm\smallsetminus(S\cap S_\pm)} \frac{p-1}{2p} \\ &= \frac{1}{n_i} \cdot \frac{1}{2\cdot\zeta(2)} \cdot \prod_{p\in S} \frac{1}{p(p+1)} \cdot \prod_{p\in S_0} \frac{1}{p+1} \cdot \prod_{p\in S_\pm\smallsetminus(S\cap S_\pm)} \frac{p}{2(p+1)}. \end{aligned}$$

(Above, we have abused notation slightly by letting $S_\pm = S_+ \cup S_-$.) ☐

4 The mean size of $\mathrm{Cl}_3^-(K_2, c)$ over families of quadratic fields K_2

In this section, we begin by computing the average number of 3-torsion elements in the minus eigenspace of their ray class groups of fixed conductor c in families of quadratic fields ordered by discriminant. We then determine the mean size of $\mathrm{Cl}_3^-(K_2, c)$ over certain subfamilies of quadratic fields K_2, namely those defined by local specifications at a finite number of primes. We first vary over the quadratic fields whose discriminants are coprime to the choice of conductor c and obtain a different average that only depends on the number of primes dividing c. We then average over quadratic fields that have prescribed splitting conditions at a finite number of primes.

4.1 The average number of 3-torsion elements in the minus eigenspaces of the ray class groups of quadratic fields

For shorthand, let

$$\mathrm{Avg}^{(i)}(\mathrm{Cl}_3^-(c)) := \lim_{X\to\infty} \frac{\displaystyle\sum_{0<(-1)^i \operatorname{Disc}(K_2)<X} \#\mathrm{Cl}_{3,}^-(K_2, c)}{\displaystyle\sum_{0<(-1)^i \operatorname{Disc}(K_2)<X} 1}.$$

Proposition 4.1. *Fix a positive integer c, and recall that $n_0 = 6$ and $n_1 = 2$. Then*

(a) *If $3 \nmid c$, then* $\mathrm{Avg}^{(i)}(\mathrm{Cl}_3^-(c)) = 1 + \dfrac{2}{n_i} \cdot \displaystyle\prod_{p|c}\left(1 + \frac{p}{p+1}\right)$;

(b) *If $3 \,||\, c$, then* $\mathrm{Avg}^{(i)}(\mathrm{Cl}_3^-(c)) = 1 + \dfrac{12}{7n_i} \cdot \displaystyle\prod_{p|c}\left(1 + \frac{p}{p+1}\right)$;

(c) *If $9 \mid c$, then* $\mathrm{Avg}^{(i)}(\mathrm{Cl}_3^-(c)) = 1 + \dfrac{30}{7n_i} \cdot \displaystyle\prod_{p|c}\left(1 + \frac{p}{p+1}\right)$.

Proof. Let S_c denote the set of primes dividing c, and recall that the density of fundamental discriminants is:

$$\lim_{X\to\infty} \frac{\displaystyle\sum_{0<(-1)^i \operatorname{Disc}(K_2)<X} 1}{X} = \frac{1}{2\cdot\zeta(2)}. \tag{8}$$

(a) Assume $3 \nmid c$. Recall that a c-valid cubic field K_3 has discriminant $\operatorname{Disc}(K_3) = df^2$ where d is the discriminant of its quadratic resolvent field and $f \mid c$. Furthermore, it follows (for example, from Proposition 8.4.1(1) of [5]) that a prime p totally ramifies in K_3 if and only if $p \mid f$. Proposition 2.10 in conjunction with (8) therefore implies:

$$\mathrm{Avg}^{(i)}(\mathrm{Cl}_3^-(c)) = 1 + 4\cdot\zeta(2)\cdot \lim_{X\to\infty} \frac{\sum_{S\subseteq S_c} N_3^{(i)}(\mathcal{K}_S, X\cdot \prod_{p\in S} p^2)}{X},$$

where S_c is equal to the set of primes dividing c. By Theorem 3.2(a), we conclude that

$$\begin{aligned} \mathrm{Avg}^{(i)}(\mathrm{Cl}_3^-(c)) &= 1 + 4 \cdot \zeta(2) \cdot \sum_{S \subseteq S_c} \left(\prod_{p \in S} p^2 \cdot \frac{1}{n_i} \cdot \frac{1}{2 \cdot \zeta(2)} \cdot \prod_{p \in S} \frac{1}{p(p+1)} \right) \\ &= 1 + \frac{2}{n_i} \cdot \prod_{p \in S_c} \left(1 + \frac{p}{p+1} \right). \end{aligned}$$

This proves (a). We skip the proof of (b) as it is very similar to the proof of (c).

(c) Assume $9 \mid c$. By Proposition 2.10 and (8) (in conjunction with Proposition 8.4.1(1) of [5]), we have that

$$\begin{aligned} \mathrm{Avg}^{(i)}(\mathrm{Cl}_3^-(c)) = 1 + 4 \cdot \zeta(2) \cdot \Bigg(&\lim_{X \to \infty} \frac{\sum_{S \subseteq S_c \smallsetminus \{3\}} N_3^{(i)}(\mathcal{K}_S, X \cdot \prod_{p \in S} p^2) + N_3^{(i)}(\mathcal{K}^{(3)}_{S \cup \{3\}}, X \cdot \prod_{p \in S \cup \{3\}} p^2)}{X} \\ &+ \lim_{X \to \infty} \frac{\sum_{S \subseteq S_c \smallsetminus \{3\}} N_3^{(i)}(\mathcal{K}^{(9)}_{S \cup \{3\}}, 9X \cdot \prod_{p \in S \cup \{3\}} p^2)}{X} \Bigg), \end{aligned}$$

where S_c again denotes the set of primes dividing c. Theorem 3.2(a) and (c) then imply that $\mathrm{Avg}^{(i)}(\mathrm{Cl}_3^-(c))$ is equal to

$$\begin{aligned} &1 + 4 \cdot \zeta(2) \cdot \sum_{S \subseteq S_c \smallsetminus \{3\}} \left(\frac{1}{n_i} \cdot \frac{1}{2 \cdot \zeta(2)} \cdot \prod_{p \in S} \frac{p}{p+1} + \frac{1}{n_i} \cdot \frac{1}{3 \cdot \zeta(2)} \cdot \prod_{p \in S \cup \{3\}} \frac{p}{p+1} + \frac{3}{n_i} \cdot \frac{1}{2 \cdot \zeta(2)} \cdot \prod_{p \in S \cup \{3\}} \frac{p}{p+1} \right) \\ = \; &1 + \frac{30}{7 n_i} \cdot \prod_{p \in S_c} \left(1 + \frac{p}{p+1} \right). \end{aligned}$$

□

4.2 The asymptotic number of quadratic fields in certain acceptable families

In order to vary the family of quadratic fields we average over, we must first determine the asymptotics of these families. We first describe the asymptotic number of discriminants of quadratic fields that are relatively prime to a fixed integer.

Lemma 4.2. *Let c be a positive integer.*

$$\lim_{X \to \infty} \frac{\sum_{\substack{(\mathrm{Disc}(K_2), c) = 1 \\ 0 < (-1)^i \mathrm{Disc}(K_2) < X}} 1}{X} = \frac{1}{2 \cdot \zeta(2)} \cdot \prod_{p \mid c} \frac{p}{p+1}.$$

Proof. By Proposition 2.2 and (4.2) in [2], we have that the number of real (resp. imaginary) quadratic fields that are unramified away from c is asymptotically equal to

$$\frac{1}{2} \cdot \prod_{p \nmid c} \left(\frac{p-1}{p} \cdot \left(1 + \frac{1}{p} \right) \right) \cdot \prod_{p \mid c} \left(\frac{p-1}{p} \cdot 1 \right) \cdot X = \frac{1}{2 \cdot \zeta(2)} \cdot \left(\prod_{p \mid c} \frac{p}{p+1} \right) \cdot X.$$

□

Next, we determine the asymptotic number of quadratic fields with prescribed splitting at a finite number of primes.

Lemma 4.3. *Let $\underline{\mathcal{S}} = (S_+, S_-, S_0)$ be disjoint sets of primes, and let $\mathcal{K}_2(\underline{\mathcal{S}})$ denote the set of isomorphism classes of quadratic fields K_2 such that any prime $p \in S_+$ splits in K_2, any prime $p \in S_-$ remains inert in K_2, and any prime $p \in S_0$ ramifies in K_2. We then have:*

$$\lim_{X\to\infty} \frac{\displaystyle\sum_{\substack{K_2 \in \mathcal{K}_2(\underline{\mathcal{S}}) \\ 0<(-1)^i \operatorname{Disc}(K_2)<X}} 1}{X} = \frac{1}{2\cdot\zeta(2)} \cdot \prod_{p\in S_0} \frac{1}{p+1} \cdot \prod_{p\in S_\pm} \frac{p}{2(p+1)},$$

where $S_\pm = S_+ \cup S_-$.

Proof. Similarly, by Proposition 2.2 and (4.2) in [2], we have that the asymptotic number of real (resp. imaginary) quadratic fields in $\mathcal{K}_2(\underline{\mathcal{S}})$ with (absolute) discriminant bounded by X is

$$\frac{1}{2} \cdot \prod_{p\notin\underline{\mathcal{S}}} \left(\frac{p-1}{p}\cdot\left(1+\frac{1}{p}\right)\right) \cdot \prod_{p\in S_\pm}\left(\frac{p-1}{p}\cdot\frac{1}{2}\right) \cdot \prod_{p\in S_0}\left(\frac{p-1}{p}\cdot\frac{1}{p}\right)\cdot X = \frac{1}{2\cdot\zeta(2)}\cdot\prod_{p\in S_0}\frac{1}{p+1}\cdot\prod_{p\in S_\pm}\frac{p}{2(p+1)}\cdot X$$

□

4.3 Averaging $\#\mathrm{Cl}_3^-(K_2, c)$ over quadratic fields unramified at c

We vary over only those quadratic fields whose discriminants are coprime to the choice of fixed conductor.

For shorthand, let

$$\mathrm{Avg}_c^{(i)}(\mathrm{Cl}_3^-(c)) := \lim_{X\to\infty} \frac{\displaystyle\sum_{\substack{(\operatorname{Disc}(K_2),c)=1 \\ 0<(-1)^i \operatorname{Disc}(K_2)<X}} \#\mathrm{Cl}_3^-(K_2,c)}{\displaystyle\sum_{\substack{(\operatorname{Disc}(K_2),c)=1 \\ 0<(-1)^i \operatorname{Disc}(K_2)<X}} 1}.$$

Proposition 4.4. *Fix a positive integer c, and let n denote the number of distinct primes dividing c. Recall that $n_0 = 6$ and $n_1 = 2$. Then*

(a) *If $3 \nmid c$, then $\mathrm{Avg}_c^{(i)}(\mathrm{Cl}_3^-(c)) = 1 + 2\cdot\dfrac{2^n}{n_i}$;*

(b) *If $3 \mid\mid c$, then $\mathrm{Avg}_c^{(i)}(\mathrm{Cl}_3^-(c)) = 1 + \dfrac{2^n}{n_i}$;*

(c) *If $9 \mid c$, then $\mathrm{Avg}_c^{(i)}(\mathrm{Cl}_3^-(c)) = 1 + 3\cdot\dfrac{2^n}{n_i}$.*

Proof. Let S_c denote the set of primes dividing c.
(a) Assume $3 \nmid c$. Proposition 2.10 combined with Lemma 4.2 implies that

$$\mathrm{Avg}_c^{(i)}(\mathrm{Cl}_3^-(c)) = 1 + 4\cdot\zeta(2)\cdot\left(\prod_{p\mid c}\frac{p+1}{p}\right)\cdot\lim_{X\to\infty}\frac{\sum_{S\subseteq S_c} N_3^{(i)}(\mathcal{K}_{S,S_c}, X\cdot\prod_{p\in S}p^2)}{X}.$$

By Theorem 3.2(d), we conclude that

$$\begin{aligned}\mathrm{Avg}_c^{(i)}(\mathrm{Cl}_3^-(c)) &= 1+4\cdot\zeta(2)\cdot\prod_{p\mid c}\frac{p+1}{p}\cdot\sum_{S\subseteq S_c}\left(\frac{1}{n_i}\cdot\frac{1}{2\cdot\zeta(2)}\cdot\prod_{p\in S}\frac{p}{p+1}\cdot\prod_{p\in S_c\smallsetminus S}\frac{p}{p+1}\right)\\ &= 1+\frac{2^{n+1}}{n_i}.\end{aligned}$$

(b) Note that for non-Galois cubic fields K_3 that are totally ramified at 3, $\mathrm{Disc}(K_3)$ is exactly divisible by 3^3, 3^4, or 3^5, and in order for the quadratic resolvent K_2 of K_3 to have discriminant relatively prime to 3, then $\mathrm{Disc}(K_3)=\mathrm{Disc}(K_2)f^2$ where either $3\nmid f$ or $9\mid\mid f$. Thus, if $3\mid\mid c$, any quadratic field K_2 that is unramified at 3 satisfies

$$\mathrm{Cl}^-(K_2,c)=\mathrm{Cl}^-(K_2,\tfrac{c}{3}).$$

Thus, Proposition 2.10, Lemma 4.2, and Theorem 3.2(d) together imply that

$$\begin{aligned}\mathrm{Avg}_c^{(i)}(\mathrm{Cl}_3^-(c)) &= 1+4\cdot\zeta(2)\cdot\left(\prod_{p\mid c}\frac{p+1}{p}\right)\cdot\lim_{X\to\infty}\frac{\sum_{S\subseteq S_c\smallsetminus\{3\}} N_3^{(i)}(\mathcal{K}_{S,S_c},X\cdot\prod_{p\in S}p^2)}{X}\\ &= 1+\frac{2^n}{n_i}.\end{aligned}$$

(c) If $9\mid c$, by Proposition 2.10 and Lemma 4.2, we have

$$\begin{aligned}\mathrm{Avg}_c^{(i)}(\mathrm{Cl}_3^-(c)) &= 1+4\cdot\zeta(2)\cdot\left(\prod_{p\mid c}\frac{p+1}{p}\right)\cdot\left(\lim_{X\to\infty}\frac{\sum_{S\subseteq S_c\smallsetminus\{3\}} N_3^{(i)}(\mathcal{K}_{S,S_c},X\cdot\prod_{p\in S}p^2)}{X}\right.\\ &\qquad\left.+\lim_{X\to\infty}\frac{\sum_{S\subseteq S_c\smallsetminus\{3\}} N_3^{(i)}(\mathcal{K}^{(9)}_{S\cup\{3\},S_c},9X\cdot\prod_{p\in S\cup\{3\}}p^2)}{X}\right).\end{aligned}$$

Finally, by Theorem 3.2(d) and (e), we conclude that

$$\mathrm{Avg}_c^{(i)}(\mathrm{Cl}_3^-(c))=1+3\cdot\frac{2^n}{n_i}.$$

□

4.4 Averaging $\#\mathrm{Cl}_3^-(K_2,c)$ over quadratic fields with prescribed splitting at a finite number of primes

Next, we vary over quadratic fields in $\mathcal{K}_2(\underline{S})$. Let $\underline{S}=(S_+,S_-,S_0)$ be disjoint sets of primes, and recall that $\mathcal{K}_2(\underline{S})$ denotes the set of isomorphism classes of quadratic fields K_2 such that any prime $p\in S_+$ splits in K_2, any prime $p\in S_-$ remains inert in K_2, and any prime $p\in S_0$ ramifies in K_2. Recall that we set $S_\pm=S_+\cup S_-$.

For shorthand, let

$$\operatorname{Avg}_{\underline{\mathcal{S}}}^{(i)}(\operatorname{Cl}_3^-(c)) := \lim_{X\to\infty} \frac{\displaystyle\sum_{\substack{K_2\in\mathcal{K}_2(\underline{\mathcal{S}})\\ 0<(-1)^i \operatorname{Disc}(K_2)<X}} \#\operatorname{Cl}_3^-(K_2,c)}{\displaystyle\sum_{\substack{K_2\in\mathcal{K}_2(\underline{\mathcal{S}})\\ 0<(-1)^i \operatorname{Disc}(K_2)<X}} 1}.$$

Proposition 4.5. *Fix a positive integer c coprime to 3, and let $\underline{\mathcal{S}} = (S_+, S_-, S_0)$ be disjoint sets of primes such that any prime $p \mid c$ is not contained in S_0. If S_+^{good} (respectively, S_-^{good}) denote the subset consisting of all primes p in S_+ (resp., in S_-) that are congruent to $1 \bmod 3$ (resp., $2 \bmod 3$) and $p \mid c$. We then have*

$$\operatorname{Avg}_{\underline{\mathcal{S}}}^{(i)}(\operatorname{Cl}_3^-(c)) = 1 + \frac{2}{n_i}\cdot 3^{\#(S_+^{\mathrm{good}}\cup S_-^{\mathrm{good}})}\cdot \prod_{\substack{p\mid c\\ p\notin S_\pm}}\left(1+\frac{p}{p+1}\right)$$

Proof. In order to determine $\operatorname{Avg}_{\underline{\mathcal{S}}}^{(i)}(\operatorname{Cl}_3^-(c))$, by Proposition 2.10, we must compute the asymptotic number of non-cyclic c-valid cubic fields whose quadratic resolvent field is in $\mathcal{K}_2(\underline{\mathcal{S}})$. If $p \in S_+$ and so $K_2 \in \mathcal{K}_2(\underline{\mathcal{S}})$ splits at p, then in any non-cyclic c-valid cubic field K_3 with quadratic resolvent K_2, p remains inert in K_3, p splits completely, or p is totally ramified. Additionally, if $p \in S_-$, then $p\mathcal{O}_{K_3}$ either decomposes into a product of exactly two distinct prime ideals or totally ramifies. Finally, we recall that a prime p totally ramifies in K_3 with discriminant $\operatorname{Disc}(K_2)f^2$ if and only if $p \mid f$.

Let S_c denote the set of primes dividing c, and recall that by assumption, $S_c \cap S_0 = \emptyset$. Additionally, let S_c^{good} denote the set of all primes $p \mid c$ such that if $p \in S_+$ (respectively, if $p \in S_-$), then p is congruent to $1 \bmod 3$ (resp., to $2 \bmod 3$), and set $c_{\mathrm{good}} = \prod_{p\in S_c^{\mathrm{good}}} p$. Proposition 2.10 combined with Lemma 4.3 implies that

$$\operatorname{Avg}_{\underline{\mathcal{S}}}^{(i)}(\operatorname{Cl}_3^-(c_{\mathrm{good}})) = 1 + 4\cdot\zeta(2)\cdot\left(\prod_{p\in S_\pm}\frac{2(p+1)}{p}\right)\cdot\left(\prod_{p\in S_0} p+1\right)\cdot\lim_{X\to\infty}\frac{\displaystyle\sum_{S\subseteq S_c^{\mathrm{good}}} N_3^{(i)}(\mathcal{K}_3^S(\underline{\mathcal{S}}), X\cdot\prod_{p\in S}p^2)}{X}.$$

By Theorem 3.2(a), we conclude that $\operatorname{Avg}_{\underline{\mathcal{S}}}^{(i)}(\operatorname{Cl}_3^-(c_{\mathrm{good}}))$ is equal to

$$\begin{aligned}
& 1 + 4\cdot\zeta(2)\cdot\left(\prod_{p\in S_\pm}\frac{2(p+1)}{p}\right)\cdot\sum_{S\subseteq S_c^{\mathrm{good}}}\left(\frac{1}{n_i}\cdot\frac{1}{2\cdot\zeta(2)}\cdot\prod_{p\in S}\frac{p}{p+1}\cdot\prod_{p\in S_\pm\smallsetminus(S\cap S_\pm)}\frac{p}{2(p+1)}\right)\\
= \;& 1+\frac{2}{n_i}\cdot\sum_{S\subseteq S_c^{\mathrm{good}}}\left(\prod_{p\in S\smallsetminus(S\cap S_\pm)}\frac{p}{p+1}\cdot\prod_{p\in S\cap S_\pm}2\right)\\
= \;& 1+\frac{2}{n_i}\cdot 3^{\#(S_c^{\mathrm{good}}\cap S_\pm)}\cdot\prod_{p\in S_c^{\mathrm{good}}\smallsetminus(S_c^{\mathrm{good}}\cap S_\pm)}\left(1+\frac{p}{p+1}\right).
\end{aligned}$$

We finish the proof by verifying that $\operatorname{Avg}_{\underline{\mathcal{S}}}^{(i)}(\operatorname{Cl}_3^-(c_{\mathrm{good}})) = \operatorname{Avg}_{\underline{\mathcal{S}}}^{(i)}(\operatorname{Cl}_3^-(c))$. Indeed, the 3-torsion subgroups of ray class groups of conductor $\frac{c}{c_{\mathrm{good}}}$ are trivial for quadratic fields in $\mathcal{K}_2(\underline{\mathcal{S}})$. If a prime $p \mid c$ and $p \nmid c_{\mathrm{good}}$, then either $p \in S_+$ and $p \equiv 2 \bmod 3$ or $p \in S_-$ and $p \equiv 1 \bmod 3$. In both of these cases, there are no cubic extensions of a quadratic field in $\mathcal{K}_2(\underline{\mathcal{S}})$ that are totally ramified at p. ∎

Remark 4.6. *Let c be a positive integer exactly divisible by 3. If $\underline{S} = (S_+, S_-, S_0)$ is as in Proposition 4.5, but we further assume $3 \notin S_+ \cup S_-$, then*

$$\mathrm{Avg}^{(i)}_{\underline{S}}(\mathrm{Cl}_3^-(c)) \quad = \quad 1 + \frac{12}{7n_i} \cdot 3^{\#(S_+^{\mathrm{good}} \cup S_-^{\mathrm{good}})} \cdot \prod_{\substack{p \mid c \\ p \notin S_\pm}} \left(1 + \frac{p}{p+1}\right),$$

where $S_\pm^{\mathrm{good}}$ are as in Proposition 4.5.

5 Proofs of Theorems 1-2 and Corollary 4

We put together the results of the previous sections in order to conclude Theorems 1 and 2, as well as Corollary 4. For Theorem 1, we will first allow K_2 to vary over all quadratic fields of bounded discriminant and use the combination of Propositions 2.10, 3.1, and 4.1. Afterwards, we only vary over the discriminants of quadratic fields that are coprime to the fixed conductor, and we combine Propositions 2.10, 3.1, and 4.4 in order to conclude Theorem 2. Corollary 4 is then derived from a generalization of Theorem 1.

5.1 Proof of Theorem 1

We combine Propositions 4.1 and 3.1 of the previous sections to prove Theorem 1. Let c be an integer, and assume that there are m primes dividing c that are congruent to 1 mod 3. Define k to satisfy $3^k \mid\mid c$, and recall that $\mathrm{Cl}_3^+(K_2, c)$ only depends on m and k. It is independent of the choice of quadratic field, so in particular, we have by Proposition 3.1,

$$\lim_{X \to \infty} \frac{\displaystyle\sum_{0 < (-1)^i \,\mathrm{Disc}(K_2) < X} \#\mathrm{Cl}_3(K_2, c)}{\displaystyle\sum_{0 < (-1)^i \,\mathrm{Disc}(K_2) < X} 1} \quad = \quad \#\mathrm{Cl}_3^+(K_2, c) \cdot \lim_{X \to \infty} \frac{\displaystyle\sum_{0 < (-1)^i \,\mathrm{Disc}(K_2) < X} \#\mathrm{Cl}_3^-(K_2, c)}{\displaystyle\sum_{0 < (-1)^i \,\mathrm{Disc}(K_2) < X} 1}.$$

We conclude by Propositions 3.1 and 4.1 in conjunction with (6) that

$$\lim_{X \to \infty} \frac{\displaystyle\sum_{0 < (-1)^i \,\mathrm{Disc}(K_2) < X} \#\mathrm{Cl}_3(K_2, c)}{\displaystyle\sum_{0 < (-1)^i \,\mathrm{Disc}(K_2) < X} 1} = \begin{cases} 3^m \cdot \left(1 + \frac{2}{n_i} \cdot \prod_{p \mid c} \left(1 + \frac{p}{p+1}\right)\right) & \text{if } k = 0; \\ 3^m \cdot \left(1 + \frac{12}{7n_i} \cdot \prod_{p \mid c} \left(1 + \frac{p}{p+1}\right)\right) & \text{if } k = 1; \\ 3^{m+1} \cdot \left(1 + \frac{30}{7n_i} \cdot \prod_{p \mid c} \left(1 + \frac{p}{p+1}\right)\right) & \text{if } k \geq 2. \end{cases} \tag{9}$$

5.2 Proof of Theorem 2

In order to compute the average number of 3-torsion elements in ray class groups of fixed conductor of quadratic fields with discriminant that is both bounded and coprime to the choice of conductor, we combine Propositions 3.1 and 4.4. Let c be an integer, and assume that there are n distinct primes dividing c, m of which are congruent to 1 mod 3. Define k to satisfy $3^k \mid\mid c$. By Proposition 3.1 and (6),

$$\lim_{X\to\infty} \frac{\displaystyle\sum_{\substack{(\mathrm{Disc}(K_2),c)=1\\ 0<(-1)^i \mathrm{Disc}(K_2)<X}} \#\mathrm{Cl}_3(K_2,c)}{\displaystyle\sum_{\substack{(\mathrm{Disc}(K_2),c)=1\\ 0<(-1)^i \mathrm{Disc}(K_2)<X}} 1} = \#\mathrm{Cl}_3^+(K_2,c)\cdot \lim_{X\to\infty} \frac{\displaystyle\sum_{\substack{(\mathrm{Disc}(K_2),c)=1\\ 0<(-1)^i \mathrm{Disc}(K_2)<X}} \#\mathrm{Cl}_3^-(K_2,c)}{\displaystyle\sum_{\substack{(\mathrm{Disc}(K_2),c)=1\\ 0<(-1)^i \mathrm{Disc}(K_2)<X}} 1}.$$

Combining with Proposition 4.4, we obtain

$$\lim_{X\to\infty} \frac{\displaystyle\sum_{\substack{(\mathrm{Disc}(K_2),c)=1\\ 0<(-1)^i \mathrm{Disc}(K_2)<X}} \#\mathrm{Cl}_3(K_2,c)}{\displaystyle\sum_{\substack{(\mathrm{Disc}(K_2),c)=1\\ 0<(-1)^i \mathrm{Disc}(K_2)<X}} 1} = \begin{cases} 3^m\cdot\left(1+\dfrac{2^{n+1}}{n_i}\right) & \text{if } k=0,\\ 3^m\cdot\left(1+\dfrac{2^{n}}{n_i}\right) & \text{if } k=1, \text{ and}\\ 3^{m+1}\cdot\left(1+3\cdot\dfrac{2^{n}}{n_i}\right) & \text{if } k\geq 2.\end{cases} \tag{10}$$

5.3 Generalizing Theorem 1(a) and the proof of Corollary 4

Before turning to the proof of Corollary 4, we first generalize Theorem 1(a) when $(6,c)=1$.

Theorem 5.1. *Let c be an integer coprime to 3, and let $\underline{S}=(S_+,S_-,S_0)$ be a disjoint set of primes such that no prime $p\mid c$ is contained in S_0. Let m denote the number of primes $p\mid c$ that are congruent to* 1 mod 3*, and let S_+^{good} (respectively, S_-^{good}) denote the subset of primes $p\mid c$ that are contained in S_+ (resp., S_-) and congruent to* 1 mod 3 *(resp.,* 2 mod 3*).*

(a) *The average size of the 3-torsion subgroups in ray class groups of conductor c of real quadratic fields that are split at primes in S_+, inert at primes in S_-, and ramified at primes in S_0 is*

$$3^m\cdot\left(1+3^{\#(S_+^{\mathrm{good}}\cup S_-^{\mathrm{good}})-1}\cdot\prod_{\substack{p\mid c\\ p\notin S_+\cup S_-}}\left(1+\frac{p}{p+1}\right)\right)$$

when these quadratic fields are ordered by their discriminant.

(b) *The average size of the 3-torsion subgroups in ray class groups of conductor c of imaginary quadratic fields that are split at primes in S_+, inert at primes in S_-, and ramified at primes in S_0 is*

$$3^m\cdot\left(1+3^{\#(S_+^{\mathrm{good}}\cup S_-^{\mathrm{good}})}\cdot\prod_{\substack{p\mid c\\ p\notin S_+\cup S_-}}\left(1+\frac{p}{p+1}\right)\right)$$

when these quadratic fields are ordered by their discriminant.

Proof. In order to compute the average number of 3-torsion elements in ray class groups of fixed conductor of quadratic fields with prescribed splitting, we combine Propositions 3.1 and 4.5 using (6). We obtain:

$$\begin{aligned}
\lim_{X\to\infty} \frac{\displaystyle\sum_{\substack{K_2\in\mathcal{K}_2(\underline{S})\\ 0<(-1)^i \operatorname{Disc}(K_2)<X}} \#\mathrm{Cl}_3(K_2,c)}{\displaystyle\sum_{\substack{K_2\in\mathcal{K}_2(\underline{S})\\ 0<(-1)^i \operatorname{Disc}(K_2)<X}} 1} &= \#\mathrm{Cl}_3^+(K_2,c)\cdot \lim_{X\to\infty} \frac{\displaystyle\sum_{\substack{K_2\in\mathcal{K}_2(\underline{S})\\ 0<(-1)^i \operatorname{Disc}(K_2)<X}} \#\mathrm{Cl}_3^-(K_2,c)}{\displaystyle\sum_{\substack{K_2\in\mathcal{K}_2(\underline{S})\\ 0<(-1)^i \operatorname{Disc}(K_2)<X}} 1} \\
&= 3^m\cdot\left(1+\frac{2}{n_i}\cdot 3^{\#(S_+^{\mathrm{good}}\cup S_-^{\mathrm{good}})}\prod_{\substack{p\mid c\\ p\notin S_+\cup S_-}}\left(1+\frac{p}{p+1}\right)\right).
\end{aligned}$$

□

The above theorem (along with Remark 4.6) directly implies that as long as S is disjoint from $S_+\cup S_-$, the mean size of the 3-torsion subgroups in ray class groups of conductor c are independent of the family $\mathcal{K}_2(\underline{S})$ of quadratic fields one averages over. More precisely:

Corollary 5.2. *Let c be an integer such that $9\nmid c$, and let $\underline{S}=(S_+,S_-,S_0)$ be a disjoint set of primes such that no prime $p\mid c$ is contained in $S_+\cup S_-\cup S_0$. If m denotes the number of primes $p\mid c$ that are congruent to $1 \bmod 3$, then the average size of the 3-torsion subgroups in ray class groups of conductor c of quadratic fields with i pairs of complex embeddings that are split at primes in S_+, inert at primes in S_-, and ramifies in S_0 is equal to*

$$3^m\cdot\left(1+\frac{2}{n_i}\cdot\prod_{p\mid c}\left(1+\frac{p}{p+1}\right)\right) \qquad \text{if } 3\nmid c,$$

$$3^m\cdot\left(1+\frac{12}{7n_i}\cdot\prod_{p\mid c}\left(1+\frac{p}{p+1}\right)\right) \qquad \text{if } 3\mid c.$$

when these quadratic fields are ordered by discriminant.

This allows for the generalization of Theorem 3 given in Corollary 4, whose proof we turn to next.

Proof of Corollary 4. We use Corollary 5.2 to compute lower bounds for the proportion $P_i(\underline{S},c)$ of quadratic fields in $\mathcal{K}_2(\underline{S})$ with i pairs of complex embeddings whose ray class groups of conductor c have trivial 3-torsion subgroup.

We assume $(3,c)=1$. If m denotes the distinct number of primes dividing c that are congruent to $1 \bmod 3$, we have by Theorem 5.1,

$$3^m\cdot\left(1+\frac{2}{n_i}\cdot\prod_{p\mid c}\left(1+\frac{p}{p+1}\right)\right)\geq 1\cdot P_i(\underline{S},c)+3\cdot(1-P_i(\underline{S},c)).$$

Hence, $P_i(\underline{S},c)>0$ if and only if $m=0$ and

$$n_i>\prod_{p\mid c}\left(1+\frac{p}{p+1}\right).$$

Thus, we conclude automatically that for any conductor c of the form $c=p$ where $p\equiv 2 \bmod 3$, a positive proportion of real (resp. imaginary) quadratic fields have trivial 3-torsion subgroup in their ray class groups of conductor c. Additionally, for any conductor c of the form $c = p_1p_2$ where $p_i \equiv 2 \bmod 3$, we see that a positive proportion of real quadratic fields have trivial 3-torsion subgroup in their ray class groups of conductor c.

If $3 \mid\mid c$, and m denotes the distinct number of primes dividing c that are congruent to $1 \bmod 3$, we similarly have that $P(i, c) > 0$ if and only if $m = 0$ and

$$\frac{7n_i}{6} > \prod_{p|c}\left(1 + \frac{p}{p+1}\right).$$

Thus, for real quadratic fields, if c is a product of 3 and a prime p which is congruent to 2 mod 3, a positive proportion of real quadratic fields have trivial 3-torsion subgroup in their ray class groups of conductor c. Additionally, a positive proportion of imaginary quadratic fields have trivial 3-torsion subgroup in their ray class groups of conductor 3. □

Remark 5.3. *Similarly, one can show that if* $3 \nmid c$, *at least* 50% *of real quadratic fields have trivial* 3*-torsion subgroup in their ray class groups of prime conductor* $c \equiv 2 \bmod 3$. *If* $3 \mid\mid c$, *at least* 50% *of real quadratic fields have trivial* 3*-torsion subgroup in their ray class groups of conductor* 3 *or* $3p$ *where* $p \equiv 2 \bmod 3$.

6 Second main term and the proof of Theorem 5

To compute the second main term for the mean number of 3-torsion elements in ray class groups of quadratic fields of bounded discriminant, we use a refinement of Theorem 3.2. For any set of primes S not containing 3, recall that $\mathcal{K}_S$ denotes the set of isomorphism classes of cubic fields that are totally ramified *exactly* at the primes $p \in S$. We first introduce some notation from [3] and [17]. For a free $\mathbb{Z}_p$-module M, define $M^{\mathrm{Prim}} \subset M$ by $M^{\mathrm{Prim}} := M \setminus \{pM\}$. Also, for any element x in a cubic order, let $i(x) := [R : \mathbb{Z}_p[x]]$. As in the proof of Theorem 3.2, let Σ^S denote the set of all isomorphism classes of rings of integers of cubic fields in $\mathcal{K}_S$. Then, Σ^S is *strongly acceptable* as defined in [3]. Thus, if $N_3^{(i)}(\Sigma^S, X)$ denotes the number of cubic orders $R \in \Sigma^S$ satisfying $0 < (-1)^i \operatorname{Disc}(R) < X$, Theorem 1.3 of [17] determines the asymptotic count with two main terms:

$$\begin{aligned} N_3^{(i)}(\Sigma^S; X) \;=\; &\frac{1}{2n_i} \cdot \prod_p \Big(\frac{p-1}{p} \cdot \sum_{R \in \Sigma_p} \frac{1}{\operatorname{Disc}_p(R)} \cdot \frac{1}{\#\operatorname{Aut}(R)}\Big) \cdot X \\ &+ \frac{c_2^{(i)}}{\zeta(2)} \cdot \prod_p \Big((1-p^{-1/3}) \cdot \sum_{R \in \Sigma_p} \frac{1}{\operatorname{Disc}_p(R)} \cdot \frac{1}{\#\operatorname{Aut}(R)} \int_{(R/\mathbb{Z}_p)^{\mathrm{Prim}}} i(x)^{2/3} dx\Big) \cdot X^{5/6} \\ &+ O_\epsilon(X^{5/6 - 7/138 + \epsilon}), \end{aligned} \tag{11}$$

where dx assigns measure 1 to $(R/\mathbb{Z}_p)^{\mathrm{Prim}}$, and additionally,

$$c_2^{(i)} = \begin{cases} \dfrac{\sqrt{3}\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{30\Gamma(2/3)} & \text{if } i = 0, \\ \dfrac{\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{10\Gamma(2/3)} & \text{if } i = 1; \end{cases} \qquad \text{and} \qquad \Sigma_p = \begin{cases} \Sigma_p^{\mathrm{ntr}} & \text{if } p \notin S, \\ \Sigma_p^{\mathrm{tr}} & \text{if } p \in S. \end{cases}$$

(Recall that for any prime p, Σ_p^{tr} denotes the set of all isomorphism classes of maximal cubic orders over $\mathbb{Z}_p$ that are totally ramified, and Σ_p^{ntr} denotes the set of all isomorphism classes of maximal cubic orders over $\mathbb{Z}_p$ that are not totally ramified.)

In order to compute the second main term's constant, we combine Table 1, Lemma 28, and Lemma 37 in [3] to determine

$$\sum_{R\in\Sigma_p} \frac{1}{\mathrm{Disc}_p(R)} \cdot \frac{1}{\#\,\mathrm{Aut}(R)} \int_{(R/\mathbb{Z})^{\mathrm{Prim}}} i(x)^{2/3} dx = \begin{cases} \dfrac{1}{p(p+1)} + \dfrac{1}{p^{4/3}(p+1)} & \text{if } p \in S, \text{ and} \\ \dfrac{p^{1/3}}{p^{1/3}-1} - \dfrac{p^{2/3}+p^{1/3}}{p(p+1)(p^{1/3}-1)} & \text{if } p \notin S. \end{cases}$$

We then calculate that $\prod_p \left(1 - p^{-1/3}\right) \cdot \prod_{R\in\Sigma_p} \frac{1}{\mathrm{Disc}_p(R)} \cdot \frac{1}{\#\,\mathrm{Aut}(R)} \int_{(R/\mathbb{Z})^{\mathrm{Prim}}} i(x)^{2/3} dx$ is equal to

$$\begin{aligned} & \prod_p \left(1 - p^{-1/3}\right) \left(\frac{p^{1/3}}{p^{1/3}-1} - \frac{p^{2/3}+p^{1/3}}{p(p+1)(p^{1/3}-1)}\right) \cdot \prod_{p\in S} \frac{\dfrac{1}{p(p+1)} + \dfrac{1}{p^{4/3}(p+1)}}{\dfrac{p^{1/3}}{p^{1/3}-1} - \dfrac{p^{2/3}+p^{1/3}}{p(p+1)(p^{1/3}-1)}} \\ = & \prod_p \left(1 - p^{-1/3}\right) \cdot \left(\frac{p^{1/3}p(p+1) - p^{2/3} - p^{1/3}}{p(p+1)(p^{1/3}-1)}\right) \cdot \prod_{p\in S} \frac{p^{2/3}-1}{p^{8/3}+p^{5/3}-p-p^{2/3}} \\ = & \prod_p 1 - \frac{p^{1/3}+1}{p(p+1)} \cdot \prod_{p\in S} \frac{1}{p(p+1)} \cdot \frac{1-p^{-2/3}}{1 - \dfrac{p^{1/3}+1}{p(p+1)}}. \end{aligned}$$

We can thus conclude the following refinement of Theorem 3.2.

Theorem 6.1. *Let S denote a set of primes not containing 3, and let $n_i = |\,\mathrm{Aut}(\mathbb{R}^{3-2i} \oplus \mathbb{C}^i)|$ for $i = 0$ or 1. Let $\mathcal{K}_S$ denote the set of isomorphism classes of cubic fields that are totally ramified* exactly *at the primes $p \in S$.*

$$\begin{aligned} N_3^{(i)}(\mathcal{K}_S, X) = & \frac{1}{n_i} \cdot \frac{1}{2\cdot\zeta(2)} \cdot \prod_{p\in S} \frac{1}{p(p+1)} \cdot X \\ & + \frac{c_2^{(i)}}{\zeta(2)} \cdot \prod_p \left(1 - \frac{p^{1/3}+1}{p(p+1)}\right) \prod_{p\in S} \left(\frac{1}{p(p+1)} \cdot \frac{1-p^{-2/3}}{1-\frac{p^{1/3}+1}{p(p+1)}}\right) \cdot X^{5/6} \\ & + O_\epsilon(X^{5/6-7/138+\epsilon}), \end{aligned}$$

where

$$c_2^{(i)} = \begin{cases} \dfrac{\sqrt{3}\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{30\Gamma(2/3)} & \text{if } i = 0, \text{ and} \\ \dfrac{\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{10\Gamma(2/3)} & \text{if } i = 1. \end{cases}$$

We are now ready to prove Theorem 5. Let c be a positive integer coprime to 3, and let S_c denote the set of primes dividing c. Proposition 2.10 combined with Theorem 6.1 implies that

$$\begin{aligned}\sum_{0<(-1)^i\,\mathrm{Disc}(K_2)<X} \#\mathrm{Cl}_3^-(K_2,c) &= 1+2\cdot\sum_{S\subseteq S_c} N_3^{(i)}(\mathcal{K}_S, X\cdot\prod_{p\in S}p^2) \\ &= 1+2\cdot\left[\frac{1}{n_i}\cdot\frac{1}{2\cdot\zeta(2)}\cdot\sum_{S\subseteq S_c}\left(\prod_{p\in S}\frac{1}{p(p+1)}\cdot X\cdot\prod_{p\in S}p^2\right.\right. \\ &\quad + \frac{c_2^{(i)}}{\zeta(2)}\cdot\prod_p\left(1-\frac{p^{1/3}+1}{p(p+1)}\right)\cdot\prod_{p\in S}\left(\frac{1}{p(p+1)}\cdot\frac{1-p^{-2/3}}{1-\frac{p^{1/3}+1}{p(p+1)}}\right)\cdot X^{5/6}\cdot\prod_{p\in S}p^{5/3} \\ &\quad \left.\left. + O_\epsilon(X^{5/6-7/138+\epsilon})\right)\right].\end{aligned}$$

Simplifying, we conclude

$$\begin{aligned}\sum_{0<(-1)^i\,\mathrm{Disc}(K_2)<X} \#\mathrm{Cl}_3^-(K_2,c) &= 1+2\cdot\left[\frac{1}{n_i}\cdot\prod_{p\in S}\left(1+\frac{p}{p+1}\right)\cdot\sum_{0<(-1)^i\,\mathrm{Disc}(K_2)<X} 1\right. \\ &\quad \left. + \frac{c_2^{(i)}}{\zeta(2)}\cdot\prod_p\left(1-\frac{p^{1/3}+1}{p(p+1)}\right)\cdot\prod_{p\in S}\left(1+\frac{p(1-p^{1/3})}{1-\frac{p(p+1)}{p^{1/3}+1}}\right)\cdot X^{5/6}\right] \\ &\quad + O_{\epsilon,c}(X^{5/6-7/138+\epsilon}).\end{aligned}$$

Combining with Proposition 3.1 and (6), we deduce Theorem 5.

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