Local-global compatibility for regular algebraic cuspidal automorphic representations when $\ell \neq p$

Ila Varma

October 31, 2022

Abstract

We prove the compatibility of local and global Langlands correspondences for GL_n up to semisimplification for the Galois representations constructed by Harris-Lan-Taylor-Thorne [10] and Scholze [18]. More precisely, let $r_p(\pi)$ denote an n-dimensional p-adic representation of the Galois group of a CM field F attached to a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$. We show that the restriction of $r_p(\pi)$ to the decomposition group of a place $v \nmid p$ of F corresponds up to semisimplification to $\mathrm{rec}(\pi_v)$, the image of π_v under the local Langlands correspondence. Furthermore, we can show that the monodromy of the associated Weil-Deligne representation of $r_p(\pi)|_{\mathrm{Gal}_{F_v}}$ is 'more nilpotent' than the monodromy of $\mathrm{rec}(\pi_v)$.

1 Introduction

Let F be an imaginary CM (or totally real) field and let π be a regular algebraic (i.e., π_∞ has the same infinitesimal character as an irreducible algebraic representation ρ_π of $\mathrm{RS}^F_{\mathbb{Q}}\,\mathrm{GL}_n$) cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. In Harris-Lan-Taylor-Thorne [10] and in Scholze [18], the authors construct a continuous semisimple representation (depending on a choice of a rational prime p and an isomorphism $\imath : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$)

$$r_{p,\imath}(\pi) : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p),$$

which satisfies the following: For every finite place $v \nmid p$ of F such that π and F are both unramified at v, $r_{p,\imath}(\pi)$ is unramified at v and

$$\mathrm{WD}(\, r_{p,\imath}(\pi)|_{G_{F_v}})^{ss} = \imath^{-1}\,\mathrm{rec}_{F_v}(\pi_v \otimes |\det|_v^{(1-n)/2})^{ss}. \tag{1.1}$$

Here rec_{F_v} as normalized in [11] denotes the local Langlands correspondence for F_v, and $\mathrm{WD}(r_v)$ denotes the Weil-Deligne representation associated to the the p-adic Galois representation r_v of the decomposition group $G_{F_v} := \mathrm{Gal}(\overline{F_v}/F_v)$. In this paper, we extend local-global compatibility up to semisimplification (1.1) to all primes $v \nmid p$ of F. In particular, we prove the following theorem:

Theorem 1. *Keeping the notation of the previous paragraph, let $v \nmid p$ be a prime of F. Then*

$$\mathrm{WD}(\, r_{p,\imath}(\pi)|_{G_{F_v}})^{ss} = \imath^{-1}\,\mathrm{rec}_{F_v}(\pi_v \otimes |\det|_v^{(1-n)/2})^{ss}.$$

In fact, our methods allow us to 'bound' the monodromy of $\mathrm{WD}(\, r_{p,\imath}(\pi)|_{G_{F_v}})^{\mathrm{Frob}-ss}$ by the monodromy of $\mathrm{rec}_{F_v}(\pi_v|\det|_v^{(1-n)/2})$. In the past, such versions of local-global compatibility have been used for proving the non-vanishing of certain Selmer groups (see for example, Bellaiche-Chenevier [3]). Using the notation introduced in Definition 8.2, we can generalize the above theorem to:

Theorem 2. *Keeping the notation of the first paragraph, let $v \nmid p$ be a prime of F. Then*

$$\mathrm{WD}(r_{p,\imath}(\pi)|_{G_{F_v}})^{\mathrm{Frob}-ss} \prec \imath^{-1}\,\mathrm{rec}_{F_v}(\pi_v \otimes |\det|_v^{(1-n)/2}),$$

where 'Frob-ss' denotes Frobenius semisimplification.

The above theorems are already known when such π are *conjugate self-dual*, by work of Caraiani [6], Shin [19], and Chenevier-Harris [9]. In particular, in [6, 19], the authors prove the stronger statement that the monodromy of $\mathrm{WD}(r_{p,\imath}(\pi)|_{G_{F_v}})^{\mathrm{Frob}-ss}$ is equal to that of $\mathrm{rec}_{F_v}(\pi_v \otimes |\det|_v^{(1-n)/2})$ under the added hypothesis that π is conjugate self-dual. When removing the "conjugate self-dual" hypothesis for a given π, one can no longer expect to find the corresponding Galois representations in the etale cohomology of Shimura varieties, and so the authors of [10] construct $r_{p,\imath}(\pi)$ instead using an p-adic interpolation argument. To prove Theorem 1, we must reconstruct the Galois representations $r_{p,\imath}(\pi)$ as in [10] while studying the Hecke action at all primes $v \nmid p$. We summarize the argument below.

Let π be a regular algebraic cuspidal automorphic representation on $\mathrm{GL}_n(\mathbb{A}_F)$. Let G denote the quasisplit unitary similitude group of signature (n,n) associated to F^{2n} and alternating form $\left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$, where the similitude factor GL_1 is defined over $\mathbb{Q}$ (not F). It has a maximal parabolic $P = \{\mathrm{GL}_1 \times \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \subset G$ with Levi $L = \{\mathrm{GL}_1 \times \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)\} \subset P$. However, note that $L \cong \mathrm{GL}_1 \times \mathrm{RS}^F_{\mathbb{Q}}\,\mathrm{GL}_n$. For all sufficiently large positive integers M, let

$$\Pi(M) = \mathrm{Ind}_{P(\mathbb{A}^{p,\infty})}^{G(\mathbb{A}^{p,\infty})}(1 \times \imath^{-1}(\pi \otimes ||\det||^M)^{p,\infty}),$$

(where Ind denotes unnormalized induction). The authors of [10] prove that $\Pi(M)$ is a subrepresentation of the space of overconvergent p-adic automorphic forms on G of some possibly non-classical weight and finite slope. Classical cusp forms on this space base change via the trace formula to GL_{2n} to isobaric sums of conjugate self-dual cuspidal automorphic representations and they have Galois representations satisfying full local-global compatibility. Now, at all primes $v \nmid p$ of F which split over F^+ (equivalently, at all primes away from p where G splits), take the Bernstein centers associated to a finite union of Bernstein components containing $\Pi(M)_v$ as the Hecke algebras acting on spaces of p-adic and classical cusp forms on G of arbitrary integral (not necessarily classical) weight. For each $\sigma \in W_{F_v}$, the image of the Bernstein centers contains Hecke operators whose eigenvalue on a p-adic cusp form Π' of G is equal to

$$\mathrm{tr}\,\mathrm{rec}_{F_v}(\Pi'_v \otimes |\det|_v^{(1-2n)/2})(\sigma).$$

If Π' is classical, then local-global compatibility is already known, and so the eigenvalue is also equal to

$$\mathrm{tr}\,\mathrm{WD}(r_p(\Pi')|_{G_{F_v}})^{ss}(\sigma),$$

where $r_p(\Pi') : G_F \to \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p)$ denotes the Galois representation associated to Π'. By showing that there are linear combinations of classical cusp forms of G whose Hecke eigenvalues are congruent mod p^k to those of $\Pi(M)$ for each positive k, we are able to construct a continuous pseudorepresentation $T : G_F \to \overline{\mathbb{Q}}_p$ satisfying the following: for every place $v \nmid p$ of F which is split over F^+ and each $\sigma_v \in W_{F_v}$,

$$T(\sigma_v) = \mathrm{tr}\,\mathrm{rec}_{F_v}(\Pi(M)_v \otimes |\det|_v^{(1-2n)/2})(\sigma_v).$$

This implies that there is a continuous semisimple Galois representation $r_{p,\imath}(\Pi(M)) : G_F \to \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p)$ whose trace is equal to T, and so for all primes v of F which are split over F^+ and lie above any rational prime other than p,

$$\mathrm{WD}(r_{p,\imath}(\Pi(M))|_{G_{F_v}})^{ss} \cong \imath^{-1}\,\mathrm{rec}_{F_v}(\Pi(M)_v \otimes |\det|_v^{(1-2n)/2})^{ss}.$$

thus if ϵ_p denotes the p-adic cyclotomic character, then $\mathrm{WD}(r_{p,\imath}(\Pi(M)) \otimes \epsilon_p^{-M}\big|_{G_{F_v}})^{ss}$ is isomorphic to

$$\imath^{-1}\,\mathrm{rec}_{F_v}(\pi_v|\det|_v^{(1-n)/2})^{ss} \oplus (\imath^{-1}\,\mathrm{rec}_{F_{c_v}}(\pi_{c_v}|\det|_{c_v}^{(1-n)/2})^{ss})^{\vee,c} \otimes \epsilon_p^{1-2n-2M}.$$

Because we construct $r_{p,\imath}(\Pi(M))$ for each sufficiently large positive integer M, it is now group theory to isolate an n-dimensional subquotient $r_{p,\imath}(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ satisfying

$$\mathrm{WD}(\, r_{p,\imath}(\pi)|_{G_{F_v}})^{ss} = \imath^{-1}\, \mathrm{rec}_{F_v}(\pi_v \otimes |\det|_v^{(1-n)/2})^{ss},$$

when $v \nmid p$ is a prime of F which is split over F^+. Using the patching lemma of Sorensen [21], we can remove the assumption that v must split over F^+ and therefore conclude Theorem 1^{ss}. We then use idempotents constructed by Schneider-Zink [17] and properties of $\wedge^k r_{p,\imath}(\Pi(M))$ and $\wedge^k \mathrm{rec}_{F_v}(\mathrm{BC}(\Pi(M))_v)$ to "bound" the monodromy of $\mathrm{WD}(r_{p,\imath}(\pi))^{\mathrm{Frob}-ss}$ by the monodromy of $\mathrm{rec}_{F_v}(\pi_v \otimes |\det|_v^{(1-n)/2})$.

Contents

Notation and Conventions.

Let F^+ be a totally real field and let F_0 denote an imaginary quadratic field. We set $F = F_0F^+$, and c will denote the nontrivial element of $\mathrm{Gal}(F/F^+)$. Let p denote a rational prime that splits in F_0. Let n denote a positive integer and if $F^+ = \mathbb{Q}$, assume $n > 2$. In the sequel, ℓ will always denote a rational prime such that $\ell \neq p$. Fix $\imath : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$.

For any field K, we will once and for all choose an algebraic closure $\overline{K}$ of K, and G_K will denote the absolute Galois group of $\overline{K}$ over K. If $K_0 \subset K$ is a subfield and S is a finite set of primes of K_0, then we will denote by G_K^S the maximal continuous quotient of G_K in which all primes of K not lying above an element of S are unramified.

If K is an arbitrary number field and v is a finite place of K, let ϖ_v denote the uniformizer of K_v and $k(v)$ is the residue field of v. Denote the absolute value on K associated to v by $|\cdot|_v$, which is normalized so that $|\varpi_v|_v = (\#k(v))^{-1}$. If v is a real place of K, then $|x|_v := \pm x$, and if v is complex, then $|x|_v = {}^c x x$. Let

$$||\cdot||_K = \prod_v |\cdot|_v : \mathbb{A}_K^\times \longrightarrow \mathbb{R}_{>0}^\times.$$

If $r : G_{K_v} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ denotes a continuous representation of G_{K_v} where $v \nmid p$ is finite, then we will write $\mathrm{WD}(r)$ for the corresponding Weil-Deligne representation of the Weil group W_{K_v} of K_v (see section 1 of Taylor-Yoshida [24]). A Weil-Deligne representation is denoted as $(r, V, N) = (r, N) = (V, N)$, where V is a finite-dimensional vector space over $\overline{\mathbb{Q}}_p$, $r : W_{F_v} \to \mathrm{GL}(V)$ is a representation with open kernel and $N(r) = N : V \to V$ is a nilpotent endomorphism satisfying

$$r(\sigma) N r(\sigma)^{-1} = |\operatorname{Art}_{F_v}^{-1}(\sigma)|_{F_v} N$$

(here, $\mathrm{Art}_{F_v} : F_v^\times \xrightarrow{\sim} W_{F_v}^{ab}$ denotes the local Artin map, normalized as in [24]). We say (r, V, N) is Frobenius semisimple if r is semisimple. We denote the Frobenius semisimplification of (r, V, N) by $(r, V, N)^{\mathrm{Frob}-ss}$ and the semisimplification of (r, V, N) is $(r, V, N)^{ss} = (r^{ss}, V, 0)$ (see Section 1 of [24]).

If π is an irreducible smooth representation of $\mathrm{GL}_n(K_v)$ over $\mathbb{C}$, we will write $\mathrm{rec}_{K_v}(\pi)$ for the Weil-Deligne representation of W_{K_v} corresponding to π by the local Langlands correspondence (see Harris-Taylor [11] or Henniart [12]). If π_1 and π_2 are irreducible smooth representations of $\mathrm{GL}_{n_1}(K_v)$ (resp. $\mathrm{GL}_{n_2}(K_v)$), then there is an irreducible smooth representation $\pi_1 \boxplus \pi_2$ of $\mathrm{GL}_{n_1+n_2}(K_v)$ over $\mathbb{C}$ satisfying

$$\mathrm{rec}_{K_v}(\pi_1 \boxplus \pi_2) = \mathrm{rec}_{K_v}(\pi_1) \oplus \mathrm{rec}_{F_v}(\pi_2).$$

Let G be a reductive group over K_v, and let P be a parabolic subgroup of G with unipotent radical N and Levi L. For a smooth representation π of $L(K_v)$ on a vector space V_π over a field Ω of characteristic 0, we define $\mathrm{Ind}_{P(K_v)}^{G(K_v)} \pi$ to be the representation of $G(K_v)$ by right translation on the set of locally constant functions $\varphi : G(K_v) \to V_\pi$ such that $\varphi(hg) = \pi(h)\varphi(g)$ for all $h \in P(F_v)$ and $g \in G(K_v)$. When $\Omega = \mathbb{C}$, define normalized induction as

$$\text{n-Ind}_{P(K_v)}^{G(K_v)} \pi = \mathrm{Ind}_{P(K_v)}^{G(K_v)} \pi \otimes |\det(\mathrm{ad}\ (h)|_{\mathrm{Lie}\, N})|_v^{1/2}.$$

2 Recollections

We recall the setup of Harris-Lan-Taylor-Thorne [10], including the unitary similitude group, and the Shimura variety (and various compactifications) associated to the unitary group, as well as their integral models. This will allow us to define automorphic vector bundles defined on these integral models, whose global sections will be the space of classical and p-adic automorphic forms.

2.1 Unitary Group

We define an integral unitary similitude group, which is associated to the following data. If Ψ_n denotes the $n \times n$ matrix with 1's on the anti-diagonal and 0's elsewhere then let J_n denote the following element of $\mathrm{GL}_{2n}(\mathbb{Z})$:

$$J_n = \begin{pmatrix} 0 & \Psi_n \\ -\Psi_n & 0 \end{pmatrix}.$$

Let $\mathcal{D}_F^{-1}$ denote the inverse different of $\mathcal{O}_F$, and define the $2n$-dimensional lattice $\Lambda = (\mathcal{D}_F^{-1})^n \oplus \mathcal{O}_F^n$. Let G be the group scheme over $\mathbb{Z}$ defined by

$$G(R) = \{(g, \mu) \in \mathrm{Aut}_{\mathcal{O}_F \otimes_{\mathbb{Z}} R}(\Lambda \otimes_{\mathbb{Z}} R) \times R^\times : {}^t g J_n {}^c g = \mu J_n\}$$

for any ring R. Over $\mathbb{Z}[1/\operatorname{Disc}(F/\mathbb{Q})]$, it is a quasi-split connected reductive group which splits over $\mathcal{O}_{\widetilde{F}}[1/\operatorname{Disc}(F/\mathbb{Q})]$ where $\widetilde{F}$ denotes the normal closure of $F/\mathbb{Q}$. Let $\nu : G \to \mathrm{GL}_1$ be the multiplier character sending $(g, \mu) \mapsto \mu$.

If $R = \Omega$ is an algebraically closed field of characteristic 0, then

$$G \times \operatorname{Spec} \Omega \cong \{(\mu, g_\tau) \in \mathbb{G}_m \times \mathrm{GL}_{2n}^{\mathrm{Hom}(F,\Omega)} : g_{\tau c} = \mu J_n {}^t g_\tau^{-1} J_n \quad \forall \tau \in \mathrm{Hom}(F, \Omega)\}.$$

Fix the lattice $\Lambda_{(n)} \cong (\mathcal{D}_F^{-1})^n$ consisting of elements of Λ whose last n coordinates are equal to 0, and define $\Lambda'_{(n)} \cong \mathcal{O}_F^n$ consisting of elements of Λ whose first n coordinates are equal to 0. Let $P^+_{(n)}$ denote the subgroup of G preserving $\Lambda_{(n)}$. Write $L_{(n),\mathrm{lin}}$ for the subgroup of $P^+_{(n)}$ consisting of elements with $\nu = 1$ which preserve $\Lambda'_{(n)}$, and write $L_{(n),\mathrm{herm}}$ for the subgroup of $P^+_{(n)}$ which act trivially on $\Lambda/\Lambda_{(n)}$ and preserve $\Lambda'_{(n)}$. Then $L_{(n),\mathrm{lin}} \cong \mathrm{RS}^{\mathcal{O}_F}_{\mathbb{Z}} \mathrm{GL}_n$ and $L_{(n),\mathrm{herm}} \cong \mathbb{G}_m$, and we can define $L_{(n)} := L_{(n),\mathrm{lin}} \times L_{(n),\mathrm{herm}}$.

Finally, let $G(\mathbb{A}^\infty)^{\mathrm{ord},\times} := G(\mathbb{A}^{p,\infty}) \times P^+_{(n)}(\mathbb{Z}_p)$, and $G(\mathbb{A}^\infty)^{\mathrm{ord}} = G(\mathbb{A}^{p,\infty}) \times \varsigma_p^{\mathbb{Z}_{\geq 0}} P^+_{(n)}(\mathbb{Z}_p)$, where $\varsigma_p \in L_{(n),\mathrm{herm}}(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times$ denotes the unique element with multiplier p^{-1}.

2.2 Level Structure

If $N_2 \geq N_1 \geq 0$ are integers, then let $U_p(N_1, N_2)$ be the subgroup of elements of $G(\mathbb{Z}_p)$ which $\bmod p^{N_2}$ lie in $P^+_{(n)}(\mathbb{Z}/p^{N_2}\mathbb{Z})$ and map to 1 in $L_{(n),\mathrm{lin}}(\mathbb{Z}/p^{N_1}\mathbb{Z})$. If U^p is an open compact subgroup of $G(\mathbb{A}^{p,\infty})$ we write $U^p(N_1, N_2)$ for $U^p \times U_p(N_1, N_2)$.

If $N \geq 0$ is an integer we write $U_p(N)$ for the kernel of the map $P^+_{(n)}(\mathbb{Z}_p) \to L_{(n),\mathrm{lin}}(\mathbb{Z}/p^N\mathbb{Z})$. In addition, $U_p(N)$ will also denote the image of this kernel inside $L_{(n),\mathrm{lin}}(\mathbb{Z}_p)$.

2.3 Shimura Variety

Fix a neat open compact subgroup U (as defined in section 0.6 of Pink [16]), and let S be a locally noetherian scheme over $\mathbb{Q}$. Recall from §3.1 in [10] that a polarized G-abelian scheme with U-level structure is an abelian scheme A over S of relative dimension $n \cdot [F : \mathbb{Q}]$ along with the following data:

- An embedding $\imath : F \hookrightarrow \mathrm{End}^0(A)$ such that $\mathrm{Lie}\, A$ is locally free of rank n over $F \otimes_{\mathbb{Q}} \mathcal{O}_S$.
- A polarization $\lambda : A \to A^\vee$
- U-level structure $[\eta]$.

For more precise definitions, see §3.1.1 of [10]. Denote by X_U the smooth quasi-projective scheme over $\mathbb{Q}$ which represents the functor that sends a locally noetherian scheme $S/\mathbb{Q}$ to the set of quasi-isogeny classes of polarized G-abelian schemes with U-level structure. Let $[(A^{\mathrm{univ}}, \imath^{\mathrm{univ}}, \lambda^{\mathrm{univ}}, [\eta^{\mathrm{univ}}])]$ denote the

universal equivalence class of polarized G-abelian varieties with U-level structure. Allowing U to vary, the inverse system $\{X_U\}$ has a right $G(\mathbb{A}^\infty)$ action, with finite etale transition maps $g : X_U \to X_{U'}$ whenever $U' \supset g^{-1}Ug$.

For each U, denote by $\Omega^1_{A^{\mathrm{univ}}/X_U}$ the sheaf of relative differentials on A^{univ}. Let Ω_U denote the Hodge bundle, i.e. the pullback by the identity section of $\Omega^1_{A^{\mathrm{univ}}/X_U}$. It is locally free of rank $n \cdot [F : \mathbb{Q}]$ and does not depend on A^{univ}.

For each neat open compact subgroup $U \subset G(\mathbb{A}^\infty)$, there is a normal projective scheme X^{min}_U over $\operatorname{Spec} \mathbb{Q}$ together with a $G(\mathbb{A}^\infty)$-equivariant dense open embedding

$$j_U : X_U \hookrightarrow X^{\mathrm{min}}_U,$$

which is known as the minimal compactification of X_U. Let the boundary be denoted by $\partial X^{\mathrm{min}}_U = X^{\mathrm{min}}_U \smallsetminus j_U X_U$. The inverse system $\{X^{\mathrm{min}}_U\}$ also has a right $G(\mathbb{A}^\infty)$-action. Furthermore, there is a normal projective flat $\mathbb{Z}_{(p)}$ scheme $\mathcal{X}^{\mathrm{min}}_U$ whose generic fiber is X^{min}_U. We will denote the ample line bundle on $\mathcal{X}^{\mathrm{min}}_U$ constructed in Propositions 2.2.1.2 and 2.2.3.1 in Lan [14] by ω_U. Its pullback to X_U is identified with $\wedge^{n[F:\mathbb{Q}]}\Omega_U$, and the system $\{\omega_U\}$ over $\{\mathcal{X}^{\mathrm{min}}_U\}$ has an action of $G(\mathbb{A}^{p,\infty} \times \mathbb{Z}_p)$. If we let $\overline{X}^{\mathrm{min}}_U = \mathcal{X}^{\mathrm{min}}_U \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p$, there is a canonical $G(\mathbb{A}^{p,\infty})$-invariant section $\mathrm{Hasse}_U \in H^0(\overline{X}^{\mathrm{min}}_U, \omega_U^{\otimes(p-1)})$ constructed in Corollaries 6.3.1.7-8 in [14] satisfying

$$g^* \mathrm{Hasse}_{g^{-1}Ug} = \mathrm{Hasse}_U \quad \forall g \in G(\mathbb{A}^{p,\infty} \times \mathbb{Z}_p).$$

Denote by $\overline{X}^{\mathrm{min}\,n-\mathrm{ord}}_U$ the zero locus in $\overline{X}^{\mathrm{min}}_U$ of Hasse_U.

Lemma 2.1. *The non-zero locus $\overline{X}^{\mathrm{min}}_U \backslash \overline{X}^{\mathrm{min}\,n-\mathrm{ord}}_U$ is relatively affine over $\overline{X}^{\mathrm{min}}_U$. Furthermore, it is affine over $\mathbb{F}_p$.*

Proof. The nonzero locus over $\overline{X}^{\mathrm{min}}_U$ is associated to the sheaf of algebras

$$\left(\oplus_{i=0}^\infty \omega_U^{\otimes(p-1)ai}\right)/(\mathrm{Hasse}^a_U - 1) \quad \forall a \in \mathbb{Z}_{>0},$$

Over $\mathbb{F}_p$, it is associated to the algebra

$$\left(\oplus_{i=0}^\infty H^0(\overline{X}^{\mathrm{min}}_U, \omega^{\otimes(p-1)ai})\right)/(\mathrm{Hasse}^a_U - 1) \quad \forall a \in \mathbb{Z}_{>0}.$$

We conclude the lemma. □

2.4 Ordinary Locus

Now let $\mathcal{S}$ denote a locally Noetherian scheme over $\mathbb{Z}_{(p)}$ and fix a neat open compact subgroup U^p along with two positive integers $N_2 \geq N_1$. Then the ordinary locus is a smooth quasi-projective scheme $\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2)}$ over $\mathbb{Z}_{(p)}$ representing the functor which sends $\mathcal{S}$ to the the set of prime-to-p quasi-isogeny classes of ordinary, prime-to-p quasi-polarized G-abelian schemes with $U^p(N_1, N_2)$-level structure as defined in §3.1 of [10]. It is a partial integral model of $X_{U^p(N_1,N_2)}$. Let $[\mathcal{A}^{\mathrm{univ}}, \imath^{\mathrm{univ}}, \lambda^{\mathrm{univ}}, [\eta^{\mathrm{univ}}]]/\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2)}$ denote the universal equivalence class of ordinary prime-to-p quasi-polarized G-abelian schemes with $U^p(N_1, N_2)$-level structure up to quasi-isogeny. Finally, let $\overline{X}^{\mathrm{ord}}_{U^p(N_1,N_2)} = \mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p$, which forms an inverse system each with a right $G(\mathbb{A}^\infty)^{\mathrm{ord}}$-action. Furthermore, the map

$$\varsigma_p : \overline{X}^{\mathrm{ord}}_{U^p(N_1,N_2+1)} \to \overline{X}^{\mathrm{ord}}_{U^p(N_1,N_2)}$$

is the absolute Frobenius map composed with the forgetful map $\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2+1)} \to \mathcal{X}^{\mathrm{ord}}_{U^p(N_2,N_2)}$ for any $N_2 \geq N_1 \geq 0$. If $N_2 > 0$, then ς_p defines a finite flat map

$$\varsigma_p : \mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2+1)} \to \mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2)}$$

with fibers of degree $p^{n^2[F^+:\mathbb{Q}]}$ (see §3.1 of [10]).

For each $U^p(N_1, N_2)$ such that U^p is neat, there is a partial minimal compactification of the ordinary locus $\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2)}$ denoted by $\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}$. By Theorem 6.2.1.1 in [14], this compactification of the ordinary locus is a normal quasi-projective scheme over $\mathbb{Z}_{(p)}$ together with a dense open $G(\mathbb{A}^\infty)^{\mathrm{ord}}$-equivariant embedding

$$j_{U^p(N_1,N_2)} : \mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2)} \hookrightarrow \mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}.$$

Its generic fiber is $X^{\mathrm{min}}_{U^p(N_1,N_2)}$ but unlike $\mathcal{X}^{\mathrm{min}}_{U^p(N_1,N_2)}$, it is not proper. Furthermore by Proposition 6.2.2.1 in [14], the induced action of $g \in G(\mathbb{A}^\infty)^{\mathrm{ord}}$ on $\{\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}\}$ is quasi-finite. Write $\partial\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)} = \mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)} - j_{U^p(N_1,N_2)}\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2)}$, for the boundary, and let $\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}$ be the formal completion along the special fiber of $\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}$. Note that by Corollary 6.2.2.8 and Example 3.4.5.5 in [14], the natural map

$$\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2')} \xrightarrow{\sim} \mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}$$

is an isomorphism, and so we will drop N_2 from notation. Define

$$\overline{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)} = \mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p.$$

For each $U^p(N_1, N_2)$ note that there are $G(\mathbb{A}^\infty)^{\mathrm{ord},\times}$-equivariant open embeddings

$$\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)} \hookrightarrow \mathcal{X}^{\mathrm{min}}_{U^p(N_1,N_2)}.$$

This induces a map on the special fibers

$$\overline{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)} \hookrightarrow \overline{X}^{\mathrm{min}}_{U^p(N_1,N_2)} \backslash \overline{X}^{\mathrm{min},n-\mathrm{ord}}_{U^p(N_1,N_2)}, \tag{2.1}$$

which is both an open and closed embedding by Proposition 6.3.2.2 of [14]. Note that only when the level is prime-to-p is the nonzero locus of $\mathrm{Hasse}_{U^p(N_1,N_2)}$ isomorphic to the special fiber of the minimally compactified ordinary locus. When $N_2 > 0$, the map in (2.1) is not an isomorphism.

2.5 Toroidal compactifications

We now introduce toroidal compactifications of X_U and $\mathcal{X}_{U^p(N_1,N_2)}$ which are parametrized by neat open compact subgroups of $G(\mathbb{A}^\infty)$ and certain cone decompositions defined in [14] and [10]. Let $\mathcal{J}^{\mathrm{tor}}$ be the indexing set of pairs (U, Δ) defined in Proposition 7.1.1.21 in [14] or on Pgs. 169-170 in [10], where U is a neat open compact subgroup and Δ is a U-admissible cone decomposition as defined in §5.2 of [10]. We will not recall the definition here as it is not necessary for any argument.

If $(U, \Delta) \in \mathcal{J}^{\mathrm{tor}}$ then by Theorem 1.3.3.15 of [14], there is a smooth projective scheme $X_{U,\Delta}$ and a divisor with simple normal crossings $\partial X_{U,\Delta} \subset X_{U,\Delta}$ equipped with an isomorphism

$$j_{U,\Delta} : X_U \xrightarrow{\sim} X_{U,\Delta} \smallsetminus \partial X_{U,\Delta}$$

and a projection $\pi_{\mathrm{tor}\,/\,\mathrm{min}} : X_{U,\Delta} \to X^{\mathrm{min}}_U$ such that the following diagram commutes:

$$\begin{array}{ccc} X_U & \hookrightarrow & X_{U,\Delta} \\ \downarrow & & \downarrow \\ X_U & \hookrightarrow & X^{\mathrm{min}}_U. \end{array}$$

The collection $\{X_{U,\Delta}\}_{\mathcal{J}^{\mathrm{tor}}}$ becomes a system of schemes with a right $G(\mathbb{A}^\infty)$-action via the maps $\pi_{(U,\Delta)/(U',\Delta')} : X_{U,\Delta} \to X_{U',\Delta'}$ whenever $(U, \Delta) \geq (U', \Delta')$ (see Pg. 166 of [10] for the definition of $\geq$ in this context).

If $(U^p(N_1,N_2),\Delta)\in\mathcal{J}^{\mathrm{tor}}$, then by Theorem 7.1.4.1 of [14], there is a smooth quasi-projective scheme $\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}$ and a divisor with simple normal crossings $\partial\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}\subset\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}$ equipped with an isomorphism

$$j^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}:\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2)}\xrightarrow{\sim}\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}\smallsetminus\partial\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta};$$

$$\pi^{\mathrm{ord}}_{\mathrm{tor}\,/\,\min}:\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}\to\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}$$

such that the following diagram commutes:

$$\begin{array}{ccc}\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2)} & \hookrightarrow & \mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}\\ \downarrow & & \downarrow\\ \mathcal{X}^{\mathrm{ord}}_{U^p)N_1,N_2)} & \hookrightarrow & \mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}.\end{array}$$

The collection $\{\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}\}_{\mathcal{J}^{\mathrm{tor}}}$ becomes a system of schemes with a right $G(\mathbb{A}^\infty)^{\mathrm{ord}}$-action via the maps $\pi_{(U^p(N_1,N_2),\Delta)/(U^{p\prime}(N_1',N_2'),\Delta')}:\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}\to\mathcal{X}^{\mathrm{ord}}_{U^{p'}(N_1',N_2),\Delta'}$ whenever $(U^p(N_1,N_2),\Delta)\geq(U^{p'}(N_1',N_2),\Delta')$ (see Pg. 167 of [10] for the definition of $\geq$ in this context).

3 Automorphic Bundles

We first define the coherent sheaves on $\mathcal{X}^{\min}$ whose global sections are what we consider to be the finite part of classical cuspidal automorphic forms on G. They are sheaves originally defined over the toroidal compactifications $X_{U,\Delta}$ (where they are locally free) and are then pushed forward to $X^{\min}$ via $\pi_{\mathrm{tor,min}}$. We start by recalling some differential sheaves that have already been defined.

3.1 Automorphic Bundles on compactifications of the Shimura Variety

Recall from the previous section that we have a locally free sheaf Ω_U on X_U, which is the pullback by the identity section of the sheaf of relative differentials from A^{univ}, the universal abelian variety over X_U. On $\mathcal{X}^{\min}_U$, the normal integral model of the minimal compactification of X_U, there is an ample line bundle ω_U whose pullback to X_U is identified with $\wedge^{n[F:\mathbb{Q}]}\Omega_U$.

Any universal abelian variety A^{univ}/X_U extends to a semi-abelian variety $A_\Delta/X_{U,\Delta}$ (see remarks 1.1.2.1 and 1.3.1.4 of [14]). Define $\Omega_{U,\Delta}$ as the pullback by the identity section of the sheaf of relative differentials on A_Δ. Note that when restricting to the Shimura variety X_U, the sheaf $\Omega_{U,\Delta}\big|_{X_U}$ is canonically isomorphic to Ω_U. Let $\mathcal{O}_{X_{U,\Delta}}(||\nu||)$ denote the structure sheaf with $G(\mathbb{A}^\infty)$-action twisted by $||\nu||$.

Let $\mathcal{E}^{\mathrm{can}}_{U,\Delta}$ denote the principal $L_{(n)}$-bundle on $X_{U,\Delta}$ defined as follows: For a Zariski open W, $\mathcal{E}^{\mathrm{can}}_{U,\Delta}(W)$ is the set of pairs of isomorphisms

$$\xi_0:\mathcal{O}_{X_{U,\Delta}}(||\nu||)\big|_W\xrightarrow{\sim}\mathcal{O}_W\qquad\text{and}\qquad\xi_1:\Omega_{U,\Delta}\xrightarrow{\sim}\mathrm{Hom}_{\mathbb{Q}}(V/V_{(n)},\mathcal{O}_W),$$

where $V=\Lambda\otimes\mathbb{Q}=F^{2n}$ and $V_{(n)}=\Lambda_{(n)}\otimes\mathbb{Q}\cong F^n$. There is an action of $h\in L_{(n)}$ on $\mathcal{E}^{\mathrm{can}}_{U,\Delta}$ by

$$h(\xi_0,\xi_1)=(\nu(h)^{-1}\xi_0,\xi_1\circ h^{-1}).$$

The inverse system $\{\mathcal{E}^{\mathrm{can}}_{U,\Delta}\}$ has an action of $G(\mathbb{A}^\infty)$.

Let R be any $\mathbb{Q}$-algebra. Fix a representation ρ of $L_{(n)}$ on a finite, locally free R-module W_ρ. Define the locally free sheaf $\mathcal{E}^{\mathrm{can}}_{U,\Delta,\rho}$ over $X_{U,\Delta}\times\operatorname{Spec}R$ as follows: For a Zariski open W, let $\mathcal{E}^{\mathrm{can}}_{U,\Delta,\rho}(W)$ be the set of $L_{(n)}(\mathcal{O}_W)$-equivariant maps of Zariski sheaves of sets,

$$\mathcal{E}^{\mathrm{can}}_{U,\Delta}\big|_W\to W_\rho\otimes_R\mathcal{O}_W.$$

With fixed ρ, the system of sheaves $\{\mathcal{E}^{\mathrm{can}}_{U,\Delta,\rho}\}$ has a $G(\mathbb{A}^\infty)$-action. If Std denotes the representation over $\mathbb{Z}$ of $L_{(n)}$ on $\Lambda/\Lambda_{(n)}$, then let $\omega_{U,\Delta} := \mathcal{E}^{\mathrm{can}}_{U,\Delta,\wedge^{n[F:\mathbb{Q}]}\mathrm{Std}^\vee}$. We will write $\mathcal{I}_{\partial X_{U,\Delta}}$ for the ideal sheaf in $\mathcal{O}_{X_{U,\Delta}}$ defining the boundary $\partial X_{U,\Delta}$. Define the subcanonical extension

$$\mathcal{E}^{\mathrm{sub}}_{U,\Delta,\rho} = \mathcal{E}^{\mathrm{can}}_{U,\Delta,\rho} \otimes \mathcal{I}_{\partial X_{U,\Delta}}.$$

Recall the projection $\pi_{\mathrm{tor}\,/\,\mathrm{min}} : X_{U,\Delta} \to X^{\mathrm{min}}_U$, and define $\mathcal{E}^{\mathrm{sub}}_{U,\rho} = \pi_{\mathrm{tor}\,/\,\mathrm{min}\,*}\mathcal{E}^{\mathrm{sub}}_{U,\Delta,\rho}$. The coherent sheaves defined on X^{min}_U are independent of the choice of Δ. If we fix ρ, there is an action of $G(\mathbb{A}^\infty)$ on the system $\{\mathcal{E}^{\mathrm{sub}}_{U,\rho}\}$ indexed by neat open compact subgroups.

Now let ρ_0 be a representation of $L_{(n)}$ on a finite locally free $\mathbb{Z}_{(p)}$-module. By Definition 8.3.5.1 of [14], there is a system of coherent sheaves associated to ρ_0 over $\{\mathcal{X}^{\mathrm{min}}_U\}$ with $G(\mathbb{A}^\infty)^{\mathrm{ord},\times}$-action whose pull-back to $\{X^{\mathrm{min}}_U\}$ is $G(\mathbb{A}^\infty)$-equivariantly identified with $\{\mathcal{E}^{\mathrm{sub}}_{U,\rho_0\otimes\mathbb{Q}}\}$. We will also refer to these sheaves by $\mathcal{E}^{\mathrm{sub}}_{U,\rho_0}$. Note that over $\mathcal{X}^{\mathrm{min}}_U$,

$$\mathcal{E}^{\mathrm{sub}}_{U,\rho_0} \otimes \omega_U \cong \mathcal{E}^{\mathrm{sub}}_{U,\rho_0\otimes(\wedge^{n[F:\mathbb{Q}]}\mathrm{Std}^\vee)},$$

where ω_U denotes the ample line bundle defined on $\mathcal{X}^{\mathrm{min}}_U$.

3.2 Automorphic Bundles on the Ordinary Locus

We now define automorphic vector bundles on the system of integral models of the minimally compactified ordinary locus $\{\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}\}$ as well as its formal completion along the special fiber $\{\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1)}\}$. The global sections of these coherent sheaves will consist of what we consider cuspidal p-adic automorphic forms. We first recall some definitions of sheaves defined on $\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}$.

Any universal abelian variety $\mathcal{A}^{\mathrm{univ}}/\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2)}$ extends uniquely to a semi-abelian variety $\mathcal{A}_\Delta/\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}$ by Remarks 1.1.2.1 and 1.3.1.4 of [14]. Define $\Omega^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}$ as the pullback by the identity section of the sheaf of relative differentials on $\mathcal{A}_\Delta$. The inverse system $\{\Omega^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}\}$ has an action of $G(\mathbb{A}^\infty)^{\mathrm{ord},\times}$. There is also a natural map

$$\varsigma_p : \varsigma_p^*\Omega^{\mathrm{ord}}_{U^p(N_1,N_2-1),\Delta} \to \Omega^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}.$$

Denote by $\mathcal{O}_{\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}}(||\nu||)$ the structure sheaf $\mathcal{O}_{\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}}$ with $G(\mathbb{A}^\infty)^{\mathrm{ord}}$-action twisted by $||\nu||$ (recall that $\{\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}\}$ has a right $G(\mathbb{A}^\infty)^{\mathrm{ord}}$-action).

Let $\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2),\Delta}$ denote the principal $L_{(n)}$-bundle on $\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}$ in the Zariski topology defined as follows: For a Zariski open W, $\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2),\Delta}(W))$ is the set of pairs of isomorphisms

$$\xi_0 : \mathcal{O}_{\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}}(||\nu||)\Big|_W \xrightarrow{\sim} \mathcal{O}_W \qquad \text{and} \qquad \xi_1 : \Omega^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\Lambda/\Lambda_{(n)}, \mathcal{O}_W).$$

(Recall that $\Lambda_{(n)}$ is the sublattice of $\Lambda = (\mathcal{D}_F^{-1})^n \oplus \mathcal{O}_F^n$ consisting of elements whose last n coordinates are equal to 0.) There is an action of $h \in L_{(n)}$ on $\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2),\Delta}$ by

$$h(\xi_0, \xi_1) = (\nu(h)^{-1}\xi_0, \xi_1 \circ h^{-1}).$$

The inverse system $\{\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2),\Delta}\}$ has an action of $G(\mathbb{A}^\infty)^{\mathrm{ord}}$. Let R be a $\mathbb{Z}_{(p)}$-algebra. Fix a representation ρ of $L_{n,(n)}$ on a finite, locally free R-module W_ρ. Denote the canonical extension to $\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta,\rho} \times \mathrm{Spec}\, R$ of the automorphic vector bundle on $\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2)}$ associated to ρ by $\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2),\Delta,\rho}$, which is defined as follows: For any Zariski open W, $\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2),\Delta,\rho}(W)$ is the set of $L_{(n)}(\mathcal{O}_W)$-equivariant maps of Zariski sheaves of sets

$$\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2),\Delta}\Big|_W \to W_\rho \otimes_R \mathcal{O}_W.$$

When ρ is fixed, the system of sheaves $\{\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2),\Delta,\rho}\}$ has an action of $G(\mathbb{A}^\infty)^{\mathrm{ord}}$. Furthermore, the inverse of ς_p^* gives a map

$$(\varsigma_p^*)^{-1}:\varsigma_{p*}\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2),\Delta,\rho}\xrightarrow{\sim}\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2-1),\Delta,\rho}\otimes_{\mathcal{O}_{\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2-1),\Delta}}}\varsigma_{p*}\mathcal{O}_{\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}}.$$

Composing $(\varsigma_p^*)^{-1}$ with $1\otimes\mathrm{tr}_{\varsigma_p}:\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2-1),\Delta,\rho}\otimes\varsigma_{p*}\mathcal{O}_{\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}}\to\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2-1),\Delta,\rho}$ gives a $G(\mathbb{A}^\infty)^{\mathrm{ord},\times}$-equivariant map

$$\mathrm{tr}_F:\varsigma_{p*}\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2),\Delta,\rho}\to\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2-1),\Delta,\rho}$$

satisfying $\mathrm{tr}_F\circ\varsigma_p^*=p^{n^2[F^+:\mathbb{Q}]}$. If Std denotes the representation over $\mathbb{Z}$ of $L_{(n)}$ on $\Lambda/\Lambda_{(n)}$, then let $\omega_{U^p(N_1,N_2),\Delta}:=\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2),\Delta,\wedge^{n[F:\mathbb{Q}]}\mathrm{Std}^\vee}$ denote the pullback of ω_U to $\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}$. We will write $\mathcal{I}_{\partial\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}}$ for the ideal sheaf in $\mathcal{O}_{\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}}$ defining the boundary $\partial\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}$. Define the subcanonical extension as

$$\mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1,N_2),\Delta,\rho}=\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2),\Delta,\rho}\otimes\mathcal{I}_{\partial\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}}.$$

Again, the inverse of ς_p^* gives a map

$$(\varsigma_p^*)^{-1}:\varsigma_{p*}\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2),\Delta,\rho}\xrightarrow{\sim}\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2-1),\Delta,\rho}\otimes_{\mathcal{O}_{\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2-1),\Delta}}}\varsigma_{p*}\mathcal{I}_{\partial\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}}.$$

Composing $(\varsigma_p^*)^{-1}$ with $1\otimes\mathrm{tr}_{\varsigma_p}:\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2-1),\Delta,\rho}\otimes\varsigma_{p*}\mathcal{I}_{\partial\mathcal{X}^{\mathrm{ord}}_{U^p(N_1,N_2),\Delta}}\to\mathcal{E}^{\mathrm{ord,can}}_{U^p(N_1,N_2-1),\Delta,\rho}$ gives another $G(\mathbb{A}^\infty)^{\mathrm{ord},\times}$-equivariant map

$$\mathrm{tr}_F:\varsigma_{p*}\mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1,N_2),\Delta,\rho}\to\mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1,N_2-1),\Delta,\rho}$$

satisfying $\mathrm{tr}_F\circ\varsigma_p^*=p^{n^2[F^+:\mathbb{Q}]}$ and compatible with the analogous map defined on $\{\mathcal{E}^{\mathrm{ord,can}}_{U,\Delta,\rho}\}_U$.

Denote the pushforward by $\mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1,N_2),\rho}=\pi^{\mathrm{ord}}_{\mathrm{tor}\,/\,\min\,*}\mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1,N_2),\Delta,\rho}$. These coherent sheaves defined on $\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}$ are independent of the choice of Δ by Proposition 1.4.3.1 and Lemma 8.3.5.2 in [14]. Note that

$$\mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1,N_2),\rho}\otimes\omega_{U^p(N_1,N_2)}\cong\mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1,N_2),\rho\otimes(\wedge^{n[F:\mathbb{Q}]}\mathrm{Std}^\vee)},$$

and by Lemma 5.5 in [10], the pullback of $\mathcal{E}^{\mathrm{sub}}_{U^p(N_1,N_2),\rho}$ to $\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2),\rho}$ is $\mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1,N_2),\rho}$.

Abusing notation, denote the pullback of $\mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1,N_2),\rho}$ to $\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1)}$ by $\mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1),\rho}$. It is independent of N_2, and thus, tr_F induces a $G(\mathbb{A}^\infty)^{\mathrm{ord},\times}$-equivariant map

$$\mathrm{tr}_F:\varsigma_{p*}\mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1),\rho}\to\mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1),\rho}$$

over $\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1)}$, and also induces an endomorphism on global sections.

4 Classical and p-adic automorphic forms

Before we define cuspidal automorphic representations on $G(\mathbb{A}^\infty)$, $L_{(n)}(\mathbb{A})$ and $\mathrm{GL}_m(\mathbb{A}_F)$, we first recall some facts about highest weights of algebraic representations of $L_{(n)}$ and G.

4.1 Weights

For each integer $0\le i\le n$, let $\Lambda_{(i)}$ denote the elements of Λ for which the last $2n-i$ coordinates are zero, and let B_n denote the Borel of G preserving the chain $\Lambda_{(n)}\supset\Lambda_{(n-1)}\supset\ldots\supset\Lambda_{(0)}$. Let T_n denote the subgroup of diagonal matrices of G.

Let $X^*(T_{n/\Omega}) := \mathrm{Hom}(T_n \times \mathrm{Spec}\,\Omega, \mathbb{G}_m \times \mathrm{Spec}\,\Omega)$ and denote by $\Phi_n \subset X^*(T_{n/\Omega})$ the set of roots of T_n on $\mathrm{Lie}\,G$. The subset of positive roots with respect to B_n will be denoted Φ_n^+ and Δ_n will denote the set of simple positive roots. For any ring $R \subset \mathbb{R}$, let $X^*(T_{n/\Omega})^+_R$ will denote the subset of elements $X^*(T_n/\Omega) \otimes_{\mathbb{Z}} R$ which pair non-negatively with the simple coroots $\check{\alpha} \in X_*(T_n/\Omega) = \mathrm{Hom}(\mathbb{G}_m \times \mathrm{Spec}\,\Omega, T_n \times \mathrm{Spec}\,\Omega)$ corresponding to the elements of $\alpha \in \Delta_n$.

Let $\Phi_{(n)} \subset \Phi_n$ denote the set of roots of T_n on $\mathrm{Lie}\,L_{(n)}$ and set $\Phi^+_{(n)} = \Phi_{(n)} \cap \Phi_n^+$ as well as $\Delta_{(n)} = \Delta_n \cap \Phi_{(n)}$. If $R \subset \mathbb{R}$ is a subring, then $X^*(T_{n/\Omega})^+_{(n),R}$ will denote the subset of $X^*(T_{n/\Omega})_{(n)} \otimes_{\mathbb{Z}} R$ consisting of elements which pair non-negatively with the simple coroot $\check{\alpha} \in X_*(T_{n/\Omega})_{(n)}$ corresponding to each $\alpha \in \Delta_{(n)}$.

Recall that $L_{(n)} \times \mathrm{Spec}\,\Omega \cong \mathrm{GL}_1 \times \mathrm{GL}_n^{\mathrm{Hom}(F,\Omega)}$, which induces an identification

$$T_n \times \mathrm{Spec}\,\Omega \cong \mathrm{GL}_1 \times (\mathrm{GL}_1^n)^{\mathrm{Hom}(F,\Omega)},$$

and hence $X^*(T_{n/\Omega}) \cong \mathbb{Z} \bigoplus (\mathbb{Z}^n)^{\mathrm{Hom}(F,\Omega)}$. Under this isomorphism, the image of $X^*(T_{n/\Omega})^+_{(n)}$ is the set

$$\{(b_0, (b_{\tau,i})) \in \mathbb{Z} \oplus (\mathbb{Z}^n)^{\mathrm{Hom}(F,\Omega)} : b_{\tau,1} \geq b_{\tau,2} \geq \ldots \geq b_{\tau,n} \quad \forall \tau\}.$$

Furthermore, $X^*(T_{n/\Omega})^+$ is identified with

$$\{(b_0, (b_{\tau,i})) \in \mathbb{Z} \oplus (\mathbb{Z}^n)^{\mathrm{Hom}(F,\Omega)} : b_{\tau,1} \geq b_{\tau,2} \geq \ldots \geq b_{\tau,n} \text{ and } b_{\tau,1} + b_{\tau c,1} \leq 0 \quad \forall \tau\}.$$

Denote by Std the representation of $L_{(n)}$ on $\Lambda/\Lambda_{(n)}$ over $\mathbb{Z}$. Note that the representation $\wedge^{n[F:\mathbb{Q}]}\,\mathrm{Std}^\vee$ is irreducible with highest weight $(0, (-1, \ldots, -1)_\tau)$. If ρ is an irreducible algebraic representation of $L_{(n)}$ over $\overline{\mathbb{Q}}_p$, then its highest weight lies in $X^*(T_{n/\overline{\mathbb{Q}}_p})^+_{(n)}$ and uniquely up to isomorphism identifies ρ. Thus, for any $\underline{b} \in X^*(T_{n/\overline{\mathbb{Q}}_p})^+_{(n)}$, let $\rho_{\underline{b}}$ denote the $L_{(n)}$-representation over $\overline{\mathbb{Q}}_p$ with highest weight $\underline{b}$.

Define the set of *classical* highest weights $X^*(T_{n/\overline{\mathbb{Q}}_p})^+_{\mathrm{cl}}$ as any $\underline{b} = (b_0, (b_{\tau,i})_{\tau \in \mathrm{Hom}(F,\overline{\mathbb{Q}}_p)}) \in X^*(T_{n/\overline{\mathbb{Q}}_p})^+_{(n)}$ such that $b_{\tau,1} + b_{\tau c,1} \leq -2n$.

We next turn to local components of automorphic representations, i.e. smooth representations of $G(\mathbb{Q}_\ell)$ when $\ell \neq p$. We relate them to smooth representations of $\mathrm{GL}_{2n}(\mathbb{Q}_\ell)$ via local base change defined below.

4.2 Local Base Change

For a rational prime $\ell \neq p$, denote the primes of F^+ above $\mathbb{Q}$ as $u_1, \cdots, u_r, v_1 \cdots v_s$, where each $u_i = w_i{}^c w_i$ splits in F and none of the v_j split in F. Note that

$$G(\mathbb{Q}_\ell) \cong \prod_{i=1}^r \mathrm{GL}_{2n}(F_{w_i}) \times H,$$

where

$$H = \left\{ (\mu, g_i) \in \mathbb{Q}_\ell^\times \times \prod_{i=1}^s \mathrm{GL}_{2n}(F_{v_i}) : {}^t g_i J_n {}^c g_i = \mu J_n \quad \forall i \right\}.$$

Here, H contains a product $\prod_{i=1}^s G^1(F^+_{v_i})$, where G^1 denotes the group scheme over $\mathcal{O}_{F^+}$ defined by

$$G^1(R) = \{g \in \mathrm{Aut}_{\mathcal{O}_F \otimes_{\mathcal{O}_{F^+}} R}(\Lambda \otimes_{\mathcal{O}_{F^+}} R) : {}^t g J_n {}^c g = J_n\}.$$

Note that $\ker \nu \cong \mathrm{RS}^{\mathcal{O}_{F^+}}_{\mathbb{Z}} G^1$. If Π is an irreducible smooth representation of $G(\mathbb{Q}_\ell)$, then

$$\Pi = (\otimes_{i=1}^r \Pi_{w_i}) \otimes \Pi_H.$$

Define $\mathrm{BC}(\Pi)_{w_i} := \Pi_{w_i}$ and $\mathrm{BC}(\Pi)_{cw_i} := \Pi^{c,\vee}_{w_i}$. This does not depend on the choice of w_i. We call Π *unramified* at v_i if v_i is unramified over F^+ and

$$\Pi^{G^1(\mathcal{O}_{F^+,v_i})} \neq (0).$$

Let B^1 denote the Borel subgroup of G^1 consisting of upper triangular matrices and T^1 the torus subgroup consisting of diagonal matrices.

If Π is unramified at v_i, then there is a character χ of $T^1(F^+_{v_i})/T^1(\mathcal{O}_{F^+,v_i})$ such that $\Pi|_{G^1(F^+_{v_i})}$ and $\text{n-Ind}_{B^1(F^+_{v_i})}^{G^1(F^+_{v_i})}\chi$ share an irreducible subquotient with a $G^1(\mathcal{O}_{F^+,v_i})$-fixed vector. Define a map between the torus of diagonal matrices of $\mathrm{GL}_{2n}(F_{v_i})$ and $G^1(\mathcal{O}_{F^+,v_i})$:

$$\begin{aligned} \mathbb{N} : T_{\mathrm{GL}_{2n}}(F_{v_i}) &\to T^1(F^+_{v_i}), && (4.1)\\ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & t_{2n} \end{pmatrix} &\mapsto \begin{pmatrix} t_1/{}^c t_{2n} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & t_{2n}/{}^c t_1 \end{pmatrix}. && (4.2) \end{aligned}$$

We define $\mathrm{BC}(\Pi)_{v_i}$ to be the unique subquotient of

$$\text{n-Ind}_{B_{\mathrm{GL}_{2n}}(F_{v_i})}^{\mathrm{GL}_{2n}(F_{v_i})}\chi\circ\mathbb{N}$$

with a $\mathrm{GL}_{2n}(\mathcal{O}_{F,v_i})$-fixed vector, where $B_{\mathrm{GL}_{2n}}(F_{v_i})$ denote the Borel subgroup of upper triangular matrices.

Lemma 4.1 (Lemma 1.1 in [10])**.** *Suppose that $\psi\otimes\pi$ is an irreducible smooth representation of*

$$L_{(n)}(\mathbb{Q}_\ell)\cong L_{(n),\mathrm{herm}}(\mathbb{Q}_\ell)\times L_{(n),\mathrm{lin}}(\mathbb{Q}_\ell)=\mathbb{Q}_\ell^\times\times\mathrm{GL}_n(F_\ell).$$

1. *If v is unramified over F^+ and π_v is unramified then* $\text{n-Ind}_{P_{(n)}(\mathbb{Q}_q)}^{G(\mathbb{Q}_q)}(\psi\otimes\pi)$ *has a subquotient Π which is unramified at v. Moreover* $\mathrm{BC}(\Pi)_v$ *is the unramified irreducible subquotient of* $\text{n-Ind}_{B_{\mathrm{GL}_{2n}}(F_v)}^{\mathrm{GL}_{2n}(F_v)}(\pi_v^{c,\vee}\otimes\pi_v)$.
2. *If v is split over F^+ and Π is an irreducible subquotient of the normalized induction* $\text{n-Ind}_{P_{(n)}(\mathbb{Q}_q)}^{G(\mathbb{Q}_q)}(\psi\otimes\pi)$ *then* $\mathrm{BC}(\Pi)_v$ *is an irreducible subquotient of* $\text{n-Ind}_{B_{\mathrm{GL}_{2n}}(F_v)}^{\mathrm{GL}_{2n}(F_v)}(\pi_{c_v}^{c,\vee}\otimes\pi_v)$.

Note that in both cases $\mathrm{BC}(\Pi_v)$ *does not depend on v.*

4.3 Cuspidal automorphic representations

Here, we define automorphic representations on $G(\mathbb{A})$ whose finite parts will be realized in the space of global sections of $\mathcal{E}^{\mathrm{sub}}_{U,\rho}$ on $X^{\mathrm{min}}_{U,\rho}$. We first recall a few definitions. Let $U(n)\subset\mathrm{GL}_n(\mathbb{C})$ denote the subgroup of matrices g satisfying ${}^t g^c g=1_n$. Define

$$\mathcal{K}_{n,\infty}=(U(n)\times U(n))^{\mathrm{Hom}(F^+,\mathbb{R})}\rtimes S_2,$$

where S_2 acts by permuting $U(n)\times U(n)$. We can embed $\mathcal{K}_{n,\infty}$ in

$$G(\mathbb{R})\ \subset\ \mathbb{R}^\times\times\prod_{\tau\in\mathrm{Hom}(F^+,\mathbb{R})}GL_{2n}(F\otimes_{F^+,\tau}\mathbb{R})$$

via the map sending

$$(g_\tau,h_\tau)_{\tau\in\mathrm{Hom}(F^+,\mathbb{R})}\mapsto\left(1,\begin{pmatrix}(g_\tau+h_\tau)/2 & (g_\tau-h_\tau)\Psi_n/2i\\ \Psi_n(g_\tau-h_\tau)/2i & \Psi_n(g_\tau+h_\tau)\Psi_n/2\end{pmatrix}_{\tau\in\mathrm{Hom}(F^+,\mathbb{R})}\right),$$

and sending the nontrivial element of S_2 to $\left(-1,\left(\begin{smallmatrix}-1_n & 0\\ 0 & 1_n\end{smallmatrix}\right)_{\tau\in\mathrm{Hom}(F^+,\mathbb{R})}\right)$. This forces $\mathcal{K}_{n,\infty}$ to be a maximal compact subgroup of $G(\mathbb{R})$ such that $\mathcal{K}_{n,\infty}\cap P_{(n)}(\mathbb{R})$ is a maximal compact of $L_{(n)}(\mathbb{R})$. Let $\mathfrak{g}=(\mathrm{Lie}\,G(\mathbb{R}))_{\mathbb{C}}$,

and denote by A_n the image of $\mathbb{G}_m$ in G via the embedding $t \mapsto t \cdot 1_{2n}$. We define a *cuspidal automorphic representation of* $G(\mathbb{A})$ to be an irreducible admissible $G(\mathbb{A}^\infty) \times (\mathfrak{g}, \mathcal{K}_{n,\infty})$-submodule of the space of cuspidal automorphic forms on the double coset space $G(\mathbb{Q})\backslash G(\mathbb{A})/A_n(\mathbb{R})^0$. Furthermore, a *square-integrable automorphic representation of* $G(\mathbb{A})$ is the twist by a character on $\mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}^\times_{>0}$ of an irreducible admissible $G(\mathbb{A}^\infty) \times (\mathfrak{g}, \mathcal{K}_{n,\infty})$-module that occurs discretely in the space of square integrable automorphic forms on $G(\mathbb{A})\backslash G(\mathbb{A})/A_n(\mathbb{R})^0$.

Now let $\mathfrak{l} = (\mathrm{Lie}\, L_{(n)}(\mathbb{R}))_{\mathbb{C}}$ and let $A_{(n)}$ denote the maximal split torus in the center of $L_{(n)}$. A *cuspidal automorphic representation of* $L_{(n)}(\mathbb{A})$ is an irreducible admissible $L_{(n)}(\mathbb{A}^\infty) \times (\mathfrak{l}, \mathcal{K}_{n,\infty} \cap L_{(n)}(\mathbb{R}))$-submodule of the space of cuspidal automorphic forms of $L_{(n)}(\mathbb{A})$ on the double coset space $L_{(n)}(\mathbb{Q})\backslash L_{(n)}(\mathbb{A})/A_{(n)}(\mathbb{R})^0$.

For a number field K and any positive integer m, let $\mathcal{K}_{K,\infty}$ denote a maximal compact subgroup of $\mathrm{GL}_m(K_\infty)$ and let $\mathfrak{gl} = (\mathrm{Lie}\, \mathrm{GL}_m(K_\infty))_{\mathbb{C}}$. Define a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_K)$ as an irreducible admissible $\mathrm{GL}_m(\mathbb{A}_K^\infty) \times (\mathfrak{gl}, \mathcal{K}_{K,\infty})$-submodule of the space of cuspidal automorphic forms on the double coset space $\mathrm{GL}_m(K)\backslash \mathrm{GL}_m(\mathbb{A}_K)/\mathbb{R}^\times_{>0}$. Finally, by a square-integrable automorphic representation of $\mathrm{GL}_m(\mathbb{A}_K)$, we shall mean the twist by a continuous character on $K^\times/\mathbb{A}_K^\times/\mathbb{R}^\times_{>0}$ of an irreducible admissible $\mathrm{GL}_m(\mathbb{A}_K^\infty) \times (\mathfrak{gl}, \mathcal{K}_{K,\infty})$-module that occurs discretely in the space of square integrable automorphic forms on $\mathrm{GL}_m(\mathbb{A}_K)$.

We will now relate the finite parts of these automorphic representations to the global sections of the automorphic bundles defined previously.

4.4 Global sections of automorphic bundles over the Shimura variety

Let ρ be a representation of $L_{(n)}$ on a finite $\mathbb{Q}$-vector space. Define the admissible $G(\mathbb{A}^\infty)$-module

$$H^0(X^{\min}, \mathcal{E}^{\mathrm{sub}}_\rho) = \varinjlim_U H^0(X^{\min}_U, \mathcal{E}^{\mathrm{sub}}_{U,\rho}).$$

Note that for any neat open compact U, $H^0(X^{\min}, \mathcal{E}^{\mathrm{sub}}_\rho)^U = H^0(X^{\min}_U, \mathcal{E}^{\mathrm{sub}}_{U,\rho})$ (see Lemma 5.5 of [10] or Proposition 8.3.6.9 of [14].

Proposition 4.2 (Corollary 5.12 in [10])**.** *Suppose that* $\underline{b} \in X^*(T_{n/\overline{\mathbb{Q}}_p})^+_{\mathrm{cl}}$*, and* $\rho_{\underline{b}}$ *is the irreducible representation of* $L_{(n)}$ *with highest weight* $\underline{b}$*. Then* $H^0(X^{\min}, \mathcal{E}^{\mathrm{sub}}_{\rho_{\underline{b}}})$ *is a semisimple* $G(\mathbb{A}^\infty)$ *module, and if* Π *is an irreducible subquotient of* $H^0(X^{\min}, \mathcal{E}^{\mathrm{sub}}_{\rho_{\underline{b}}})$*, then there is a continuous representation*

$$R_p(\Pi) : G_F \to \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p),$$

which is de Rham above p *and has the following property: Suppose that* $v \nmid p$ *is a prime of* F *above a rational prime* ℓ *such that*

- *either* ℓ *splits in* F_0*,*
- *or* F *and* Π *are unramified above* ℓ*;*

then

$$\mathrm{WD}(R_p(\Pi)|_{G_{F_v}})^{\mathrm{Frob}-ss} \cong \mathrm{rec}_{F_v}(\mathrm{BC}(\Pi_\ell)_v |\det|_v^{(1-2n)/2}),$$

where ℓ *is the rational prime below* v*.*

Proof. Each irreducible subquotient Π of $H^0(X^{\min}, \mathcal{E}^{\mathrm{sub}}_{\rho_{\underline{b}}})$ is the finite part of a cohomological cuspidal $G(\mathbb{A})$-automorphic representation π by Lemma 5.11 in [10], and furthermore, $H^0(X^{\min}, \mathcal{E}^{\mathrm{sub}}_{\rho_{\underline{b}}})$ is a semisimple $G(\mathbb{A}^\infty)$-module. For such π, by Shin [20] and Moeglin-Waldspurger [15], there is a decomposition into positive integers

$$2n = m_1 n_1 + \ldots + m_r n_r,$$

and cuspidal conjugate self-dual automorphic representations $\tilde{\pi}_i$ of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$ such that for each $i \in [1,r]$, $\tilde{\pi}_i || \det ||^{(m_i+n_i-1)/2}$ is cohomological and satisfies the following at all primes v of F which are split over F^+

$$\pi_v = \boxplus_{i=1}^{r} \boxplus_{j=0}^{n_i-1} \tilde{\pi}_{i,v} | \det |_v^{(n_i-1)/2-j}.$$

These $\tilde{\pi}_i$ are automorphic representations which have Galois representations associated to them satisfying full local-global compatibility, results due to many people including [9, 19, 6, 2] (for a summary, see [1]). □

We will refer to irreducible subquotients of $H^0(X^{\min}, \mathcal{E}^{\mathrm{sub}}_{\rho_{\underline{b}}})$ as classical cuspidal G-automorphic forms of weight $\rho_{\underline{b}}$.

4.5 p-adic (cuspidal) G-automorphic forms

Now let ρ be a representation of $L_{(n)}$ on a finite locally free $\mathbb{Z}_{(p)}$-module. Let $H^0(\mathfrak{X}^{\mathrm{ord,min}}, \mathcal{E}^{\mathrm{ord,sub}}_{\rho})$ denote the smooth $G(\mathbb{A}^\infty)^{\mathrm{ord}}$-module defined as

$$H^0(\mathfrak{X}^{\mathrm{ord,min}}, \mathcal{E}^{\mathrm{ord,sub}}_{\rho}) := \varinjlim_{U^p, N_1} H^0(\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1)}, \mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1),\rho}).$$

For each positive integer r, define

$$H^0(\mathcal{X}^{\mathrm{ord,min}}, \mathcal{E}^{\mathrm{ord,sub}}_{\rho} \otimes \mathbb{Z}/p^r\mathbb{Z}) := \varinjlim_{U^p(N_1,N_2)} H^0(\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1,N_2),\rho} \otimes \mathbb{Z}/p^r\mathbb{Z}).$$

It is a smooth $G(\mathbb{A}^\infty)^{\mathrm{ord}}$-module of p-adic cuspidal G-automorphic forms of weight ρ, with the property that

$$H^0(\mathcal{X}^{\mathrm{ord,min}}, \mathcal{E}^{\mathrm{ord,sub}}_{\rho} \otimes \mathbb{Z}/p^r\mathbb{Z})^{U^p(N_1,N_2)} = H^0(\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1,N_2),\rho} \otimes \mathbb{Z}/p^r\mathbb{Z}).$$

Note that mod p^M, there is a $G(\mathbb{A}^\infty)^{\mathrm{ord}}$-equivariant embedding

$$H^0(\mathfrak{X}^{\mathrm{ord,min}}, \mathcal{E}^{\mathrm{ord,sub}}_{\rho}) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^M\mathbb{Z} \hookrightarrow H^0(\mathcal{X}^{\mathrm{ord,min}}, \mathcal{E}^{\mathrm{ord,sub}}_{\rho} \otimes \mathbb{Z}/p^M\mathbb{Z}).$$

Fix a neat open compact subgroup $U^p \subset G(\mathbb{A}^{p,\infty})$ and integers $N_2 \geq N_1 \geq 0$, and recall that there is a canonical section $\mathrm{Hasse}_U \in H^0(\overline{X}^{\min}_U, \omega_U^{\otimes(p-1)})$ which is $G(\mathbb{A}^{p,\infty} \times \mathbb{Z}_p)$-invariant. Let $\widetilde{\mathrm{Hasse}}_U$ denote the noncanonical lift of Hasse_U over an open subset of $\mathcal{X}^{\min}_U$. For each positive integer M, the powers $\widetilde{\mathrm{Hasse}}_U^{p^{M-1}} \bmod p^M$ are canonical despite the noncanonical choice of $\widetilde{\mathrm{Hasse}}_U$, and hence they glue with each other and give a canonical $G(\mathbb{A}^{\infty,p} \times \mathbb{Z}_p)$-invariant section $\mathrm{Hasse}_{U,M}$ of $\omega^{\otimes(p-1)p^{M-1}}$ over $\mathcal{X}^{\min}_U \times \mathrm{Spec}\, \mathbb{Z}/p^M\mathbb{Z}$.

Fix ρ a representation of $L_{(n)}$ on a finite free $\mathbb{Z}_{(p)}$-module then for each integer i, define the $G(\mathbb{A}^\infty)^{\mathrm{ord},\times}$-equivariant map,

$$\begin{aligned} H^0(\mathcal{X}^{\min}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{sub}}_{\rho} \otimes \omega_U^{ip^{M-1}(p-1)}) &\to H^0(\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{ord,sub}}_{\rho} \otimes \mathbb{Z}/p^M\mathbb{Z}), \\ f &\mapsto (f|_{\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}}) / \mathrm{Hasse}^i_{U^p(N_1,N_2),M}\,. \end{aligned}$$

Using the map defined above, Harris-Lan-Taylor-Thorne [10] prove the following density theorem relating p-adic and classical cuspidal automorphic forms.

Lemma 4.3 (Lemma 6.1 in [10]). *Let ρ be an irreducible representation of $L_{(n)}$ on a finite free $\mathbb{Z}_p$-module. The induced map*

$$\bigoplus_{j=r}^{\infty} H^0(\mathcal{X}^{\min}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{sub}}_{U^p(N_1,N_2),\rho\otimes(\wedge^{n[F:\mathbb{Q}]}\mathrm{Std}^\vee)^{jp^{M-1}(p-1)}}) \to H^0(\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2),\rho}, \mathcal{E}^{\mathrm{sub,ord}}_{U^p(N_1,N_2),\rho} \otimes \mathbb{Z}/p^M\mathbb{Z})$$

is surjective for any integer r.

5 The U_p-operator and the Main Theorem of [10]

The map $\mathrm{tr}_F : \varsigma_{p*}\mathcal{E}^{\mathrm{sub}}_{U^p(N_1),\rho} \to \mathcal{E}^{\mathrm{sub}}_{U^p(N_1),\rho}$ over $\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1)}$ induces an endomorphism $U_p = \mathrm{tr}_F$ in the endomorphism algebra of $H^0(\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1)}, \mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1),\rho})_{\overline{\mathbb{Q}}_p} := H^0(\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1)}, \mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1),\rho}) \otimes \overline{\mathbb{Q}}_p$ satisfying $U_p \circ \varsigma_p = p^{n^2[F^+:\mathbb{Q}]}$. The subspace of overconvergent automorphic forms $H^\dagger$ in $H^0(\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1)}, \mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1),\rho})_{\overline{\mathbb{Q}}_p}$ defined in §6.4 of [10] admits a slope decomposition for U_p in the sense of §6.2 of [10]. This means that for each $a \in \mathbb{Q}$, there is a U_p-preserving decomposition

$$H^\dagger_{\leq a} \oplus H^\dagger_{>a} = H^\dagger \subseteq H^0(\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1)}, \mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1),\rho})_{\overline{\mathbb{Q}}_p}$$

such that $H^\dagger_{\leq a}$ is finite-dimensional and satisfies:

1. There is a nonzero polynomial $f(X) \in \overline{\mathbb{Q}}_p[X]$ with slopes $\leq a$ (i.e., $f(x) \neq 0$ and every root of $f(x)$ has p-adic valuation at most equal to a) such that the endomorphism $f(U_p)$ restricts to 0 on $H^\dagger_{\leq a}$;
2. If the roots of $f(X) \in \overline{\mathbb{Q}}_p[X]$ have slopes $\leq a$, then the endomorphism $f(U_p)$ restricts to an automorphism of $H^\dagger_{>a}$.

Additionally, $H^\dagger_{\leq a}$ is an admissible $G(\mathbb{A}^\infty)^{\mathrm{ord},\times}$-module. Fix an isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$.

Theorem 5.1. *Assume that $n > 1$ and that ρ is an irreducible algebraic representation of $L_{(n),\mathrm{lin}}$ on a finite-dimensional $\overline{\mathbb{Q}}_p$-vector space. Suppose that π is a cuspidal automorphic representation of $L_{(n),\mathrm{lin}}(\mathbb{A})$ such that π_∞ has the same infinitesimal character as $\rho^\vee$, and suppose also that ψ is a continuous $\overline{\mathbb{Q}}_p$-character of $\mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}^\times_{>0}$ such that $\psi|_{\mathbb{Z}_p^\times} = 1$. Then for all $M \in \mathbb{Z}_{>0}$ sufficiently large and for each irreducible subquotient π_j of $\mathrm{Ind}^{G(\mathbb{A}^{p,\infty})}_{P^+_{(n)}(\mathbb{A}^{p,\infty})}(\pi^\infty||\det||^M \times \psi^\infty)$, there exist a representation $\rho(M)$ of $L_{(n)}$ over $\mathbb{Z}_{(p)}$, a corresponding scalar $a(M) \in \mathbb{Q}$, and an admissible representation Π' of $H^\dagger_{\leq a} \subseteq H^0(\mathfrak{X}^{\mathrm{ord,min}}, \mathcal{E}^{\mathrm{sub}}_{\rho(M)})_{\overline{\mathbb{Q}}_p}$ such that π_j is a subquotient of Π'.*

Proof. Combine Corollary 1.9, Lemma 6.12, Corollary 6.17, Lemma 6.20, and Corollary 6.25 in [10]. □

Our next step is to consider properties of the Galois representations associated to the irreducible $G(\mathbb{A}^\infty)^{\mathrm{ord},\times}$-subquotients of $H^\dagger_{\leq a}$, as constructed in Corollary 6.13 in [10]. In order to prove local-global compatibility at all primes above ℓ such that $\ell \neq p$, we strengthen the construction of Galois representations associated to irreducible admissible $G(\mathbb{A}^\infty)^{\mathrm{ord},\times}$-subquotients of $H^0(\mathfrak{X}^{\mathrm{ord,min}}, \mathcal{E}^{\mathrm{sub}}_\rho)_{\overline{\mathbb{Q}}_p}$, i.e., Galois representations associated to p-adic cuspidal G-automorphic forms of weight ρ (see Proposition 6.5 of [10]). These Galois representations are constructed using the following two facts we have already recalled:

1. Proposition 4.2: Classical cuspidal G-automorphic forms of classical weight ρ have Galois representations associated to them; furthermore, they satisfy full local-global compatibility at all primes ℓ such that $\ell \neq p$.
2. Lemma 4.3: For any integer M, every p-adic cuspidal G-automorphic form of any weight ρ "is congruent mod p^M to" some classical cuspidal G automorphic form of classical weight ρ' which is of the form $\rho' = \rho \otimes (\wedge^{n[F:\mathbb{Q}]}\mathrm{Std}^\vee)^{(p-1)p^{M-1}j}$ for some integer j.

To prove local-global compatibility when $\ell \neq p$, we will use these two results to reconstruct the Galois representations associated to p-adic cuspidal automorphic forms on G of weight ρ, but we will consider the action of a larger Hecke algebra than in [10] on the p-adic automorphic spaces $H^0(\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{ord,sub}}_\rho \otimes \mathbb{Z}/p^M\mathbb{Z})$ and $H^0(\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1)}, \mathcal{E}^{\mathrm{ord,sub}}_\rho)$ as well as the classical automorphic spaces $H^0(\mathcal{X}^{\mathrm{min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{sub}}_\rho)$.

6 Hecke Algebras away from p

Let S denote the set of "bad" rational primes consisting of p and the primes ℓ which ramify in F but do *not* split in F_0. Let S_{ram} denote the set of rational primes $\ell\,(\neq p)$ that ramify in F and split in F_0. Let S_{ur} denote the set of rational primes $\ell\,(\neq p)$ that are unramified in F and split in F_0; note that $S_{\mathrm{spl}} := S_{\mathrm{ur}} \sqcup S_{\mathrm{ram}}$ contains all rational primes away from p that split in F_0. Let $Q = S_{\mathrm{ur}} \sqcup S_{\mathrm{ram}} \sqcup S$, and let $Q^p = Q\backslash\{p\}$. Finally, let $S^p = S\backslash\{p\}$.

For each conjugate pair of primes $\{v, {}^cv\}$ of F above a rational prime $\ell \in S_{\mathrm{spl}}$, choose exactly one of $\{v, {}^cv\}$ to put into a set $\underline{\mathcal{S}}_{\mathrm{spl}}$ and the other in $\underline{\mathcal{S}}^c_{\mathrm{spl}}$. For $\ell \in S_{\mathrm{spl}}$, identify

$$G(\mathbb{Q}_\ell) \cong \prod_{\substack{v \in \underline{\mathcal{S}}_{\mathrm{spl}} \\ v|\ell}} \mathrm{GL}_{2n}(F_v).$$

6.1 At unramified primes

We recall the definition of the unramified Hecke algebra. Fix a neat open compact subgroup $U^p = G(\widehat{\mathbb{Z}}^Q) \times U_{Q^p} \subset G(\mathbb{A}^{p,\infty})$. Suppose that v is a place of F above a rational prime $\ell \notin S$ and let $i \in \mathbb{Z}$.

By work of Bernstein-Deligne [4] building on Satake, there is an element $T_v^{(i)} \in \mathbb{Q}[G(\mathbb{Z}_\ell)\backslash G(\mathbb{Q}_\ell)/G(\mathbb{Z}_\ell)]$ such that if Π_ℓ is an unramified representation of $G(\mathbb{Q}_\ell)$ and then its eigenvalue on $\Pi_\ell^{G(\mathbb{Z}_\ell)}$ is equal to

$$\mathrm{tr}\,\mathrm{rec}_{F_v}(\mathrm{BC}(\Pi_\ell)_v)|\det|_v^{(1-2n)/2}(\mathrm{Frob}_v^i).$$

(For more details on this construction, see Pg. 196-7 in [10].) If v is an unramified prime of F which splits over F^+, then we can write the Hecke operator $T_v^{(1)}$ as the double coset

$$G(\mathbb{Z}_\ell)\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \varpi_v \end{pmatrix} G(\mathbb{Z}_\ell),$$

where ϖ_v denotes a uniformizer of F_v.

For each unramified prime v of F and each integer $i \in \mathbb{Z}$, there exists an integer $d_v^{(i)} \in \mathbb{Z}$ such that

$$d_v^{(i)} T_v^{(i)} \in \mathbb{Z}[G(\mathbb{Z}_\ell)\backslash G(\mathbb{Q}_\ell))/G(\mathbb{Z}_\ell)].$$

Let $\mathcal{H}^{\mathrm{ur}}_{\mathbb{Z}_p} := \mathbb{Z}_p[G(\widehat{\mathbb{Z}}^Q)\backslash G(\mathbb{A}^Q)/G(\widehat{\mathbb{Z}}^Q)]$ denote the abstract unramified Hecke algebra. Let N_1 and N_2 be two integers $N_2 \geq N_1 \geq 0$, and let ρ be a representation of $L_{(n)}$ over $\mathbb{Z}_{(p)}$. The Hecke algebra $\mathcal{H}^{\mathrm{ur}}_{\mathbb{Z}_p}$ has an action on the classical and p-adic spaces $H^0(\mathcal{X}^{\mathrm{min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{sub}}_{U^p(N_1,N_2),\rho})$, $H^0(\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1)}, \mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1),\rho})$, and $H^0(\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1,N_2),\rho} \otimes \mathbb{Z}/p^M\mathbb{Z})$ induced from the action of $G(\mathbb{A}^S)$. Denote by $\mathbb{T}^{\mathrm{ur}}_{U^p(N_1,N_2),\rho}$ the image of $\mathcal{H}^{\mathrm{ur}}_{\mathbb{Z}_p}$ in the endomorphism algebra

$$\mathrm{End}_{\mathbb{Z}_p}(H^0(\mathcal{X}^{\mathrm{min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{sub}}_{U^p(N_1,N_2),\rho})).$$

Furthermore, if $W \subset H^0(\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1)}, \mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1),\rho})$ (respectively, if $W \subset H^0(\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{ord,sub}}_{U^p(N_1,N_2),\rho} \otimes \mathbb{Z}/p^M\mathbb{Z})$) is a finitely-generated $\mathbb{Z}_p$-submodule invariant under the action of the algebra $\mathcal{H}^{\mathrm{ur}}_{\mathbb{Z}_p}$, then let $\mathbb{T}^{\mathrm{ord,ur}}_{U^p(N_1,N_2),\rho}(W)$ (respectively, let $\mathbb{T}^{\mathrm{ord,ur}}_{U^p(N_1,N_2),\rho,M}(W)$) denote the image of $\mathcal{H}^{\mathrm{ur}}_{\mathbb{Z}_p}$ in $\mathrm{End}_{\mathbb{Z}_p}(W)$.

For each v, let $\tilde{T}_v^{(i)}$ denote the image of $d_v^{(i)} T_v^{(i)}$ in any $\mathcal{H}^{\mathrm{ur}}_{\mathbb{Z}_p}$-algebra $\mathbb{T}$ via the canonical map $\mathcal{H}^{\mathrm{ur}}_{\mathbb{Z}_p} \to \mathbb{T}$.

6.2 At primes which are split in F_0

Suppose that $v \in \underline{\mathcal{S}}_{\mathrm{spl}} \sqcup \underline{\mathcal{S}}_{\mathrm{spl}}^c$ is a place of F above a rational prime ℓ, and let σ_v denote an element of W_{F_v}, the Weil group of F_v. Let $\mathcal{B}$ denote a fixed Bernstein component; it is a subcategory of the smooth representations of $\mathrm{GL}_{2n}(F_v)$. Every component $\mathcal{B}$ is uniquely associated to an inertial equivalence class (M, ω), where M denotes a Levi subgroup of $\mathrm{GL}_{2n}(F_v)$ and ω is a supercuspidal representation of M. (Recall that two inertial classes (M, ω) and (M', ω') are equivalent if there exists $g \in G$ and an unramified character χ of M' such that $M = g^{-1}Mg$ and $\omega' = \chi \otimes \omega(g \cdot g^{-1})$.) Then, $\mathcal{B}$ is defined to be the full subcategory of smooth representations of $\mathrm{GL}_{2n}(F_v)$ consisting of those representations all of whose irreducible subquotients have *inertial support* equivalent to (M, ω). This implies that there exists some $(M', \omega') \sim (M, \omega)$ such that π occurs as a composition factor of the parabolic induction $\mathrm{Ind}_{P_M}^{\mathrm{GL}_{2n}(F_v)}(\omega')$ where ω' is an irreducible supercuspidal representation and P_M is a parabolic subgroup of $\mathrm{GL}_{2n}(F_v)$ with Levi M.

Let $\mathfrak{z}_{\mathcal{B}} = \mathfrak{z}_{[M,\omega]}$ denote the Bernstein center of $\mathcal{B}$, which is the image under the idempotent $e_{\mathcal{B}}$ associated to $\mathcal{B}$ of

$$\varprojlim_{K} \mathcal{Z}(\mathbb{C}[K \backslash \mathrm{GL}_{2n}(F_v)/K]),$$

the inverse limit over open compact subgroup K of the centers of the complex Hecke algebra for $\mathrm{GL}_{2n}(F_v)$.

Proposition 6.1 (Proposition 3.11 in Chenevier [8])**.** *For an inertial equivalence class $[M, \omega]$, there is a representative (M, ω) which can be defined over $\overline{\mathbb{Q}}$. Let $E \subset \overline{\mathbb{Q}}$ denote a sufficiently large finite-degree normal field over which ω, $\mathrm{rec}(\omega)$, $\mathcal{B}_{[M,\omega]}$, $\mathfrak{z}_{[M,\omega]}$ are all defined over E. Let $E[\mathcal{B}_{[M,\omega]}]$ denote the affine coordinate ring of the variety associated to $\mathcal{B}_{[M,\omega]}$. Then there exists a unique pseudocharacter of dimension $2n$*

$$T^{\mathcal{B}} = T^{[M,\omega]} : W_{F_v} \to E[\mathcal{B}] = \mathfrak{z}_{\mathcal{B}}$$

such that for all irreducible smooth representations π of $\mathcal{B}$ and $\sigma_v \in W_{F_v}$,

$$T^{\mathcal{B}}(\sigma_v)(\pi) = \mathrm{tr}\, \mathrm{rec}_{F_v}(\pi)(\sigma_v).$$

For a Bernstein component $\mathcal{B}$ and $\sigma \in W_{F_v}$, let $T_{v,\mathcal{B},\sigma}$ denote the twist of $T^{\mathcal{B}}(\sigma)$ such that $T_{v,\mathcal{B},\sigma}(\pi) = \mathrm{tr}\, \mathrm{rec}_{F_v}(\pi |\det|_v^{(1-2n)/2})(\sigma)$ if π is a smooth irreducible representation in $\mathcal{B}$. Multiplying $T_{v,\mathcal{B},\sigma}$ by $e_{\mathcal{B}}$ if necessary, we may suppose that $T_{v,\mathcal{B},\sigma}$ acts as 0 on all irreducible $\pi \notin \mathcal{B}$.

Theorem 6.2 (Bernstein [4])**.** *For each prime $v \in \underline{\mathcal{S}}_{\mathrm{spl}}$. Let $\mathcal{B}_v = \mathcal{B}$ be a Bernstein component, and let $e_{\mathcal{B}}$ denote the projector element such that for any smooth irreducible representation π of $\mathrm{GL}_{2n}(F_v)$, $e_{\mathcal{B}}(\pi) = \pi$ if and only if $\pi \in \mathcal{B}$.*

There is a compact open subgroup K of $\mathrm{GL}_{2n}(F_v)$ for which we may find a finite union of Bernstein components $\mathfrak{B} = \mathfrak{B}_v$ containing $\mathcal{B}_v$ with the following property: If π_v is an irreducible smooth representation of $\mathrm{GL}_{2n}(F_v)$, then π_v^K is nonzero if and only if π_v belongs to one of the Bernstein components in $\mathfrak{B}$.

Proof. For the first statement, see Proposition 2.10 in [4]. For the second statement see Proposition 3.8 and Corollary 3.9(i) of [4]. Also, see §2.3 and 2.5 of [5]. □

We will denote this compact open subgroup by $K_{\mathfrak{B}} = K_{\mathfrak{B}_v}$; note that all irreducible smooth representations inside $\mathcal{B}$ have a fixed vector under $K_{\mathfrak{B}}$. More generally, for every $\mathcal{B}' \subset \mathfrak{B}$, $\mathfrak{z}_{\mathcal{B}'}$ embeds in the center of $\mathcal{H}(\mathrm{GL}_{2n}, K_{\mathfrak{B}})_{\mathbb{C}} = \mathbb{C}[K_{\mathfrak{B}} \backslash \mathrm{GL}_{2n}(F_v)/K_{\mathfrak{B}}]$ via multiplication by the characteristic function of $K_{\mathfrak{B}}$. Let $\mathfrak{z}_{\mathfrak{B}_v} = \mathfrak{z}_{\mathfrak{B}} := \mathrm{im}(\prod_{\mathcal{B}' \subset \mathfrak{B}} \mathfrak{z}_{\mathcal{B}'} \hookrightarrow \mathcal{H}(\mathrm{GL}_{2n}, K_{\mathfrak{B}})_{\mathbb{C}})$. Note that $\mathfrak{z}_{\mathfrak{B}}$ is the center of $\mathcal{H}(\mathrm{GL}_{2n}, K_{\mathfrak{B}})_{\mathbb{C}}$.

For $\ell \in S_{\mathrm{spl}}$, assume K_ℓ is an open compact subgroup of $G(\mathbb{Q}_\ell)$ such that under the identification $G(\mathbb{Q}_\ell) \cong \prod_{\underline{\mathcal{S}}_{\mathrm{spl}} \ni v|\ell} \mathrm{GL}_{2n}(F_v)$, we can decompose

$$K_\ell = \prod_{\underline{\mathcal{S}}_{\mathrm{spl}} \ni v|\ell} K_{\mathfrak{B}_v}.$$

If $v \in \underline{\mathcal{S}}_{\mathrm{spl}}$ divides the rational prime ℓ and $\mathfrak{B}$ is a Bernstein component, then for any $\sigma \in W_{F_v}$ we can find an element of $\mathfrak{z}_{\mathfrak{B}}$, which we will denote by $T_{v,\mathfrak{B},\sigma}$, such that its eigenvalue on the K_ℓ-fixed vectors of an irreducible representation π of $G(\mathbb{Q}_\ell)$ in $\mathcal{B}$ is

$$\operatorname{tr} \operatorname{rec}_{F_v}(\pi_v |\det|_v^{(1-2n)/2})(\sigma).$$

(On the other hand, if $\pi_v \notin \mathfrak{B}$, then π^{K_ℓ} is trivial and $T_{v,\mathfrak{B},\sigma}$ acts as 0.) This element $T_{v,\mathfrak{B},\sigma}$ is the image in $\mathfrak{z}_{\mathfrak{B}}$ of $\prod_{\mathcal{B}' \subset \mathfrak{B}} T_{v,\mathcal{B}',\sigma} \in \prod_{\mathcal{B}' \subset \mathfrak{B}} \mathfrak{z}_{\mathcal{B}'}$. It is independent of π. Furthermore, for each $\varphi \in \mathrm{Aut}(\mathbb{C})$, we have that ${}^{\varphi}\mathfrak{B} = \mathfrak{B}$ and additionally,

$${}^{\varphi} \operatorname{rec}_{F_v}(\pi_v |\det|_v^{(1-2n)/2}) \cong \operatorname{rec}_{F_v}({}^{\varphi}(\pi_v |\det|_v^{(1-2n)/2})).$$

Thus, we have that ${}^{\varphi}T_{v,\mathfrak{B},\sigma} = T_{v,\mathfrak{B},\sigma}$, and so $T_{v,\mathfrak{B},\sigma} \in \mathbb{Q}[K_\ell \backslash G(\mathbb{Q}_\ell)/K_\ell]$.

Define

$$\mathfrak{z}_\ell^0 := \prod_{\underline{\mathcal{S}}_{\mathrm{spl}} \ni v | \ell} (\mathfrak{z}_{\mathfrak{B}_v} \cap \mathbb{Z}[K_{\mathfrak{B}} \backslash \mathrm{GL}_{2n}(F_v)/K_{\mathfrak{B}}]).$$

Then $\mathfrak{z}_\ell^0$ lies in the center of $\mathbb{Z}[K_\ell \backslash G(\mathbb{Q}_\ell)/K_\ell]$. Note that for any element $T \in \mathfrak{z}_{\mathfrak{B}} \cap \mathbb{Q}[K_\ell G(\mathbb{Q}_\ell)/K_\ell]$, there exists a nonzero integer $d(T) \in \mathbb{Z}$ such that $d(T)T \in \mathfrak{z}_\ell^0$, where $v \mid \ell$. Thus, we can choose $d(T_{v,\mathfrak{B},\sigma}) \in \mathbb{Z} \smallsetminus \{0\}$ such that

$$d(T_{v,\mathfrak{B},\sigma}) T_{v,\mathfrak{B},\sigma} \in \mathbb{Z}[K_\ell \backslash G(\mathbb{Q}_\ell)/K_\ell],$$

so $d(T_{v,\mathfrak{B},\sigma}) T_{v,\mathfrak{B},\sigma} \in \mathfrak{z}_\ell^0$.

For each $v \in \underline{\mathcal{S}}_{\mathrm{spl}}$, fix a Bernstein component $\mathcal{B}_v$ and let $\mathfrak{B}_v$ be the disjoint union as defined in Theorem 6.2. We will make the further assumption that $U^p = \prod_{\ell \neq p} U_\ell$ is a neat open compact subgroup of $G(\mathbb{A}^{p,\infty})$ such that

$$U_\ell = \prod_{\substack{v \in \underline{\mathcal{S}}_{\mathrm{spl}} \\ v | \ell}} K_{\mathfrak{B}_v}. \tag{6.1}$$

Let $\mathcal{H}_{\mathrm{spl},\mathbb{Z}_p} := \left(\bigotimes_{\ell \in S_{\mathrm{spl}}} \mathfrak{z}_\ell^0 \right)$, be the abstract ramified Hecke algebra. For any two integers $N_2 \geq N_1 \geq 0$ and any algebraic representation ρ of $L_{(n)}$ over $\mathbb{Z}_{(p)}$, recall that the classical space $H^0(\mathcal{X}^{\min}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{sub}}_{U^p(N_1,N_2),\rho})$ has an action of $G(\mathbb{A}^{p,\infty})$ which induces an action of $\mathcal{H}_{\mathrm{spl},\mathbb{Z}_p}$, and similarly, the p-adic spaces $H^0(\mathfrak{X}^{\mathrm{ord},\min}_\rho, \mathcal{E}^{\mathrm{ord},\mathrm{sub}}_\rho)$ and $H^0(\mathcal{X}^{\mathrm{ord},\min}_\rho, \mathcal{E}^{\mathrm{ord},\mathrm{sub}}_\rho \otimes \mathbb{Z}/p^M\mathbb{Z})$ have an action of $G(\mathbb{A}^{p,\infty})$, which similarly induces an action of $\mathcal{H}_{\mathrm{spl},\mathbb{Z}_p}$. Let $\mathbb{T}^p_{U^p(N_1,N_2),\rho}$ denote the image of $\mathcal{H}^p_{\mathbb{Z}_p} := \mathcal{H}^{\mathrm{ur}}_{\mathbb{Z}_p} \otimes \mathcal{H}_{\mathrm{spl},\mathbb{Z}_p}$ in

$$\mathrm{End}_{\mathbb{Z}_p}(H^0(\mathcal{X}^{\min}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{sub}}_{U^p(N_1,N_2),\rho})).$$

Furthermore, if $W \subset H^0(\mathfrak{X}^{\mathrm{ord},\min}_{U^p(N_1)}, \mathcal{E}^{\mathrm{ord},\mathrm{sub}}_{U^p(N_1),\rho})$ (resp., $W \subset H^0(\mathcal{X}^{\mathrm{ord},\min}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{ord},\mathrm{sub}}_{U^p(N_1,N_2),\rho} \otimes \mathbb{Z}/p^M\mathbb{Z})$) is a finitely generated $\mathbb{Z}_p$-submodule invariant under the action of the algebra $\mathcal{H}^p_{\mathbb{Z}_p}$, then let $\mathbb{T}^{\mathrm{ord},p}_{U^p(N_1,N_2),\rho}(W)$ (resp. $\mathbb{T}^{\mathrm{ord},p}_{U^p(N_1,N_2),\rho,M}(W)$) denote the image of $\mathcal{H}^p_{\mathbb{Z}_p}$ in $\mathrm{End}_{\mathbb{Z}_p}(W)$.

For each $v \in \underline{\mathcal{S}}_{\mathrm{spl}} \sqcup \underline{\mathcal{S}}^c_{\mathrm{spl}}$, let $\tilde{T}_{v,\mathfrak{B},\sigma}$ denote the image of $d(T_{v,\mathfrak{B},\sigma}) T_{v,\mathfrak{B},\sigma}$ in any $\mathcal{H}^p_{\mathbb{Z}_p}$-algebra $\mathbb{T}$.

7 Interpolating the Hecke action

The main goal of this section is to prove the following proposition.

Proposition 7.1. *Let ρ be an algebraic representation of $L_{(n)}$ over $\mathbb{Z}_{(p)}$. Suppose that Π is an irreducible quotient of an admissible $G(\mathbb{A}^\infty)^{\mathrm{ord},\times}$-submodule Π' of $H^0(\mathfrak{X}^{\mathrm{ord},\min}, \mathcal{E}^{\mathrm{ord},\mathrm{sub}}_\rho) \otimes \overline{\mathbb{Q}}_p$. Then there is a continuous semisimple representation*

$$R_p(\Pi) : G_F \to \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p)$$

with the following property: If $\ell \neq p$ is a rational prime such that either ℓ splits in F_0, or both F and Π are unramified above ℓ, and $v \mid \ell$ is a prime of F, then

$$\mathrm{WD}(R_p(\Pi)_{G_{F_v}})^{ss} \cong \mathrm{rec}_{F_v}((\Pi_\ell)_v |\det|_v^{(1-2n)/2})^{ss}.$$

Proposition 6.5 in [10] proves the existence of $R_p(\Pi)$ and its local-global compatibility at primes above $\ell \notin S_{\mathrm{ram}} \sqcup S$ such that Π is unramified at ℓ. We extend the local-global compatibility results to primes above $\ell \in S_{\mathrm{ram}} \sqcup S_{\mathrm{ur}}$.

Fix ρ, Π, and Π' as in the proposition. For each $v \in \underline{\mathcal{S}}_{\mathrm{spl}}$, let $\mathcal{B}_v$ denote the Bernstein component containing $\mathrm{BC}(\Pi_\ell)_v$. Let $\mathfrak{B}_v$ be a disjoint union of Bernstein components containing $\mathcal{B}_v$ such that there is an open compact subgroup $K_{\mathfrak{B}_v}$ of $\mathrm{GL}_{2n}(F_v)$ and an irreducible representation of $\mathrm{GL}_{2n}(F_v)$ with a nontrivial $K_{\mathfrak{B}_v}$-fixed vector is contained in $\mathfrak{B}_v$. Choose a neat open compact subgroup $U^p = \prod_{\ell \neq p} U_\ell$ of $G(\mathbb{A}^{p,\infty})$ such that $U_\ell = K_\ell$ for each $\ell \in S_{\mathrm{spl}}$ as well as an integer N such that $\Pi^{U^p(N)} \neq (0)$. Recall that $\mathcal{H}_{\mathrm{spl},\mathbb{Z}_p} := (\bigotimes_{\ell \in S_{\mathrm{spl}}} \mathfrak{z}_\ell^0)$, where $\mathfrak{z}_\ell^0$ is associated to the Bernstein components $\mathcal{B}_v$ and disjoint unions $\mathfrak{B}_v$ and open compact subgroups $K_{\mathfrak{B}_v}$ fixed above, for each $v \in \underline{\mathcal{S}}_{\mathrm{spl}}$, and let $\mathcal{H}^p_{\mathbb{Z}_p} = \mathbb{Z}_p[G(\widehat{\mathbb{Z}}^Q)\backslash G(\mathbb{A}^Q)/G(\widehat{\mathbb{Z}}^Q)] \otimes_{\mathbb{Z}_p} \mathcal{H}_{\mathrm{spl},\mathbb{Z}_p}$ as before.

We first show the existence and local-global compatibility of a Galois representations associated to irreducible subquotients of the classical space $H^0(\mathcal{X}^{\min}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{sub}}_{\rho \otimes (\wedge^{n[F:\mathbb{Q}]}\mathrm{Std}^\vee)^{\otimes(p-1)t}})$ for t sufficiently large. It will be most relevant to write this result in terms of pseudorepresentations.

Lemma 7.2. *For t sufficiently large, there is a continuous pseudorepresentation*

$$T_t : G_F^S \to \mathbb{T}^p_{U^p(N_1,N_2),\rho \otimes (\wedge^{n[F:\mathbb{Q}]}\mathrm{Std}^\vee)^{\otimes(p-1)t}},$$

where if $v \mid \ell \notin S$:

$$\begin{cases} d(T_{v,\mathfrak{B}_v,\sigma})T_t(\sigma) = \tilde{T}_{v,\mathfrak{B}_v,\sigma} \text{ for all } \sigma \in W_{F_v} & \text{if } v \mid \ell \in S_{\mathrm{spl}} \\ d_v^{(i)} T_t(\mathrm{Frob}_v^i) = \tilde{T}_v^{(i)} \text{ for all } i \geq 0 & \text{if } v \mid \ell \notin Q. \end{cases}$$

for all positive integers i and for all $\sigma \in W_{F_v}$.

Proof. First, assume that $\rho \otimes \overline{\mathbb{Q}}_p$ is irreducible. Let $(b_0, (b_{\tau,i})) \in X^*(T_{n/\overline{\mathbb{Q}}_p})^+_{(n)}$ denote the highest weight of $\rho \otimes \overline{\mathbb{Q}}_p$. If $t \in \mathbb{Z}$ satisfies the inequality

$$-2n \geq (b_{\tau,1} - t(p-1)) + (b_{\tau c,1} - t(p-1)),$$

and $\rho_t := \rho \otimes (\wedge^{n[F:\mathbb{Q}]}\mathrm{Std})^{\otimes(p-1)t}$, then by Lemma 5.11 of [10],

$$\mathbb{T}^p_{U^p(N_1,N_2),\rho_t} \otimes \overline{\mathbb{Q}}_p \cong \bigoplus_{\Pi} \overline{\mathbb{Q}}_p,$$

where the sum runs over irreducible admissible representations of $G(\mathbb{A}^\infty)$ with $\Pi^{U^p(N_1,N_2)} \neq (0)$ that occur in $H^0(X^{\min} \times \mathrm{Spec}\,\overline{\mathbb{Q}}_p, \mathcal{E}^{\mathrm{sub}}_{\rho_t})$. Further, from Proposition 4.2, we deduce that there is a continuous representation

$$r_{\rho_t} : G_F^S \to \mathrm{GL}_{2n}(\mathbb{T}^p_{U^p(N_1,N_2),\rho_t} \otimes \overline{\mathbb{Q}}_p) \tag{7.1}$$

satisfying for $v \mid \ell \notin S$,

$$\begin{cases} \mathrm{tr}\, r_{\rho_t}(\mathrm{Frob}_v^i) = T_v^{(i)} \text{ for all } i \geq 0 & \text{if } v \mid \ell \in S_{\mathrm{spl}} \\ \mathrm{tr}\, r_{\rho_t}(\sigma) = T_{v,\mathfrak{B},\sigma} \text{ for all } \sigma \in W_{F_v} & \text{if } v \mid \ell \notin Q. \end{cases} \tag{7.2}$$

Let $T_t := \mathrm{tr}\, r_{\rho_t}$. Note that if $v \mid \ell \notin S$, then $T_t(\mathrm{Frob}_v) = T_v^{(1)} \in \mathbb{T}^p_{U^p(N_1,N_2),\rho_t}$, thus by Cebotarev density theorem, $T_t : G_F^S \to \mathbb{T}^p_{U^p(N_1,N_2),\rho_t}$.

For general ρ, recall that algebraic representations of $L_{(n)}(\mathbb{Z}_p)$ over $\overline{\mathbb{Q}}_p$ are semisimple, and so we can construct from the Galois representations associated to the irreducible constituents of $\rho \otimes \overline{\mathbb{Q}}_p$ a continuous representation $r : G_F^S \to \mathrm{GL}_{2n}(\mathbb{T}^p_{U^p(N_1,N_2),\rho \otimes (\wedge^{n[F:\mathbb{Q}]}\mathrm{Std})^{\otimes(p-1)t}} \otimes \overline{\mathbb{Q}}_p)$ for sufficiently large t whose trace satisfies the desired properties. □

Combining the above lemma with the congruences properties established in Lemma 4.3, we have the following corollaries.

Corollary 7.3. *If W is a finitely generated $\mathcal{H}^p_{\mathbb{Z}_p}$-invariant submodule of either*

$$H^0(\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{ord,sub}}_\rho \otimes \mathbb{Z}/p^M\mathbb{Z}) \text{ or } H^0(\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N)}, \mathcal{E}^{\mathrm{ord,sub}}_\rho),$$

then there is a continuous pseudorepresentation

$$T : G_F^S \to \mathbb{T}^{\mathrm{ord},p}_{U^p(N_1,N_2),\rho,M}(W) \qquad s.t. \qquad \begin{cases} d(T_{v,\mathfrak{B}_v,\sigma})T(\sigma) = \tilde{T}_{v,\mathfrak{B}_v,\sigma} \text{ for all } \sigma \in W_{F_v} & \text{if } v \mid \ell \in S_{\mathrm{spl}}, \text{ and} \\ d_v^{(i)} T(\mathrm{Frob}_v^i) = \tilde{T}_v^{(i)} \text{ for all } i \geq 0 & \text{if } v \mid \ell \notin Q. \end{cases}$$

Proof. It suffices to show that for finitely generated

$$W \subset H^0(\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{ord,sub}}_\rho \otimes \mathbb{Z}/p^M\mathbb{Z}),$$

such a pseudorepresentation exists since there is an $G(\mathbb{A}^\infty)^{\mathrm{ord}}$-equivariant embedding

$$H^0(\mathfrak{X}^{\mathrm{ord,min}}_{U^p(N_1)}, \mathcal{E}^{\mathrm{ord,sub}}_\rho) \otimes \mathbb{Z}/p^M\mathbb{Z} \hookrightarrow H^0(\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{ord,sub}}_\rho \otimes \mathbb{Z}/p^M\mathbb{Z}).$$

Since W is finitely generated, there exists $k \in \mathbb{Z}$ such that

$$W \subset \mathrm{Im}\Big(\bigoplus_{j=r}^{k} H^0(\mathcal{X}^{\mathrm{min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{sub}}_{U^p(N_1,N_2),\rho_{jp^{M-1}(p-1)}}) \to H^0(\mathcal{X}^{\mathrm{ord,min}}_{U^p(N_1,N_2)}, \mathcal{E}^{\mathrm{sub,ord}}_{U^p(N_1,N_2),\rho} \otimes \mathbb{Z}/p^M\mathbb{Z})\Big).$$

Since the above map is $G(\mathbb{A}^\infty)^{\mathrm{ord},\times}$-equivariant, we see that for r sufficiently large, by Lemma 4.3, there is a continuous pseudorepresentation $T_r : G_F^S \to \mathbb{T}^p_{U^p(N_1,N_2),\rho_{rp^{M-1}(p-1)}}$. If we take r to be sufficiently large, then we can compose to get

$$T : G_F^S \to \mathbb{T}^{\mathrm{ord},p}_{U^p(N),\rho}(W)$$

such that

$$\begin{cases} d(T_{v,\mathfrak{B}_v,\sigma})T(\sigma) = \tilde{T}_{v,\mathfrak{B}_v,\sigma} \text{ for all } \sigma \in W_{F_v} & \text{if } v \in \underline{\mathcal{S}}_{\mathrm{spl}}, \text{ and} \\ d_v^{(i)} T(\mathrm{Frob}_v^i) = \tilde{T}_v^{(i)} \text{ for all } i \geq 0 & \text{if } v \mid \ell \notin Q. \end{cases}$$

□

We use the pseudorepresentations constructed in Lemma 7.3 to finish the proof of Proposition 7.1.

Proof of Proposition 7.1. Since $(\Pi')^{U^p(N)}$ is finite dimensional, it is a closed subspace of

$$H^0(\mathfrak{X}^{\mathrm{ord,min}}, \mathcal{E}^{\mathrm{ord,sub}}_\rho) \otimes \overline{\mathbb{Q}}_p$$

that is preserved by the action of $\mathcal{H}^p_{\mathbb{Z}_p}$; thus, we have by Corollary 7.3 that there is a continuous pseudorepresentation

$$T : G_F^S \to \mathbb{T}^{\mathrm{ord},p}_{U^p(N_1,N_2),\rho}((\Pi')^{U^p(N_1,N_2)}) \quad \text{s.t.} \quad \begin{cases} d(T_{v,\mathfrak{B}_v,\sigma})T(\sigma) = \tilde{T}_{v,\mathfrak{B}_v,\sigma} \text{ for all } \sigma \in W_{F_v} & \text{if } v \in \underline{\mathcal{S}}_{\mathrm{spl}}, \text{ and} \\ d_v^{(i)} T(\mathrm{Frob}_v^i) = \tilde{T}_v^{(i)} \text{ for all } i \geq 0 & \text{if } v \mid \ell \notin Q. \end{cases}$$

Since there is a $\mathcal{H}^p_{\mathbb{Z}_p}$-equivariant map $(\Pi')^{U^p(N)} \twoheadrightarrow \Pi^{U^p(N)}$, there is a map

$$\varphi_\Pi : \mathbb{T}^{\mathrm{ord},p}_{U^p(N_1,N_2),\rho}((\Pi')^{U^p(N_1,N_2)}) \to \overline{\mathbb{Q}}_p$$

sending a Hecke operator to its eigenvalue on $(\Pi)^{U^p(N_1,N_2)}$. Composing $\varphi_\Pi \circ T =: T_\Pi$ gives a pseudorepresentation

$$T_\Pi : G_F^S \to \overline{\mathbb{Q}}_p, \tag{7.3}$$

which by work of Taylor [23] is the trace of a continuous semisimple Galois representation satisfying the semi-simplified local-global compatibility at the primes away from $\underline{\mathcal{S}}^{\mathrm{nspl}}$ (and away from the primes above p). The proposition then follows from the main theorem on pseudorepresentations (see again [23]). □

8 Bounding the monodromy

Let $\ell \neq p$ be distinct prime that splits in F_0 and v a prime of F above ℓ, i.e., $\ell \in S_{\mathrm{spl}}$ and $v \in \underline{\mathcal{S}}_{\mathrm{spl}} \sqcup \underline{\mathcal{S}}^c_{\mathrm{spl}}$. The main result of this section is as follows.

Proposition 8.1. *Suppose ρ is an algebraic representation of $L_{(n)}$ over $\mathbb{Z}_{(p)}$ and that Π is an irreducible quotient of an admissible $G(\mathbb{A}^\infty)^{\mathrm{ord},\times}$-submodule Π' of $H^0(\mathfrak{X}^{\mathrm{ord,min}}, \mathcal{E}_\rho^{\mathrm{ord,sub}}) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$. Then the continuous semi-simple representation $R_{p,\imath}(\Pi)$ satisfies for $v \mid \ell \in S_{\mathrm{spl}}$, i.e., for all primes of F above ℓ (away from p) which splits in F_0:*

$$\mathrm{WD}(\, R_{p,\imath}(\Pi)|_{W_{F_v}})^{\mathrm{Frob}\,-ss} \prec \mathrm{rec}_{F_v}(\mathrm{BC}(\Pi_\ell)_v |\det|_v^{(1-2n)/2}),$$

where $\prec$ is defined below.

Let (σ, N) be a Weil-Deligne representation of W_{F_v} over $\overline{\mathbb{Q}}_p$, where $\sigma : W_{F_v} \to \mathrm{GL}(V)$ and $N \in \mathrm{End}(V)$. Let $\mathcal{W}$ denote the set of equivalence classes of irreducible representations of W_{F_v} over $\overline{\mathbb{Q}}_p$ with open kernel, where two representations s, s' of W_{F_v} are in the same equivalence if $s \cong s' \otimes \chi$ for some unramified character χ. We can decompose any Weil-Deligne representation into isotypic components indexed by these equivalence classes of $\mathcal{W}$, i.e.

$$\sigma \cong \bigoplus_{\omega \in \mathcal{W}} \sigma[\omega], \quad \text{and} \quad V \cong \bigoplus_{\omega \in \mathcal{W}} V[\omega],$$

where $\sigma[\omega] : W_{F_v} \to \mathrm{GL}(V[\omega])$ is a Weil representation with all irreducible subquotients lying in $\omega \in \mathcal{W}$. The operator N preserves isotypic components of σ, thus it preserves $V[\omega]$. If $N[\omega]$ denotes N restricted to $V[\omega]$, then $(\sigma[\omega], N[\omega])$ is a Weil-Deligne representation. Recall from Tate [22] that there is an indecomposable Weil-Deligne representation $\mathrm{Sp}(m)$ of dimension m with nilpotent matrix of degree exactly m. Explicitly, we have for all $\tau \in W_{F_v}$:

$$\mathrm{Sp}(m)(\tau) = \begin{pmatrix} |\tau|^{\frac{m-1}{2}} & & & & \\ & |\tau|^{\frac{m-3}{2}} & & & \\ & & \ddots & & \\ & & & |\tau|^{\frac{3-m}{2}} & \\ & & & & |\tau|^{\frac{1-m}{2}} \end{pmatrix},$$

where

$$N(\mathrm{Sp}(m)) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}.$$

It is well known that every indecomposable Frobenius-semisimple Weil-Deligne representation is isomorphic to one of the form $s \otimes \mathrm{Sp}(m)$, where s is an irreducible representation of W_{F_v} and $N(s) = 0$ (see [22, 4.1.5]). If (σ, N) and (σ', N') are two Weil-Deligne representations of the same dimension, then for each $\omega \in \mathcal{W}$, we can compare the dimensions of $\mathrm{Sp}(\cdot)$ in the decomposition of $\sigma[\omega]^{\mathrm{Frob}\,-ss}$ and $\sigma'[\omega]^{\mathrm{Frob}\,-ss}$ into indecomposable representations using the following ordering:

Definition 8.2. *For each $\omega \in \mathcal{W}$, and for each Weil-Deligne representation (σ, N), there exists a unique decreasing sequence of non-negative integers $m_{1,\omega}(\sigma, N) \geq m_{2,\omega}(\sigma, N) \geq \ldots$ with an associated sequence of $s_1, s_2, \ldots \in \omega$ such that*

$$\sigma[\omega]^{\mathrm{Frob}\,-ss} \cong \bigoplus_{s_i \in \omega} s_i \otimes \mathrm{Sp}(m_{i,\omega}(\sigma, N)).$$

The sequence $(m_{i,\omega}(\sigma,N))_i$ *is a partition of the integer* $\dim(\sigma[\omega])/\dim(s_i)$ *for any* $s_i \in \omega$. *If* (σ',N') *is another Weil-Deligne representation, then we say*

$$(\sigma,N) \prec (\sigma',N')$$

if and only if $\forall \omega \in \mathcal{W}$ *and* $i \geq 1$,

$$m_{1,\omega}(\sigma,N)+\cdots+m_{i,\omega}(\sigma,N) \leq m_{1,\omega}(\sigma',N')+\cdots+m_{i,\omega}(\sigma',N').$$

In particular, $(\sigma,N) \prec (\sigma',N')$ if and only if $N[\omega]$ is "more nilpotent" than $N'[\omega]$ for each $\omega \in \mathcal{W}$. Denote by I_v the inertia subgroup of the Weil group W_{F_v} at v, and let $\mathcal{I}$ denote the set of isomorphism classes of irreducible representations of I_v with open kernel. For every $\theta \in \mathcal{I}$, define $\sigma[\theta]$ to be the isotypic component of $\sigma|_{I_v}$, whose irreducible subquotients are isomorphic to θ. Since N commutes with the image of I_v, these isotypic components are preserved by the monodromy operator, and thus we can define $N[\theta]$ as the restriction of N to $V[\theta]$.

Definition 8.3. *Let* (σ,N) *be a Weil-Deligne representations of* W_{F_v} *over* $\overline{\mathbb{Q}}_p$. *For each* $\theta \in \mathcal{I}$, *we can define a unique decreasing sequence of non-negative integers* $n_{1,\theta}(\sigma,N) \geq n_{2,\theta}(\sigma,N) \geq \ldots$ *which determines the conjugacy class of the monodromy operator* $N[\theta]$. *It is a partition of the integer* $\dim(r[\theta])/\dim(\theta)$. *If* (σ',N') *is another Weil-Deligne representation, then we say*

$$(\sigma,N) \prec_I (\sigma',N')$$

if and only if $\sigma|_{I_v} \cong \sigma'|_{I_v}$ *and* $\forall \theta \in \mathcal{I}$ *and* $i \geq 1$,

$$n_{1,\theta}(\sigma,N)+\ldots+n_{i,\theta}(\sigma,N) \leq n_{1,\theta}(\sigma',N')+\ldots+n_{i,\theta}(\sigma',N').$$

We have the following lemma relating the two dominance relations $\prec$ and $\prec_I$ defined above. For any sequence of integers $(m_i)_{i\in\mathbb{Z}_{>0}}$ and $d \in \mathbb{Z}_{>0}$, let $d\cdot(m_i)_i$ be the sequence of integers $(m_1,m_1,\ldots,m_1, m_2,m_2,\ldots,m_2,\ldots)$ where each m_i occurs d times.

Lemma 8.4 (Lemma 6.5.3 in [3])**.** *Let* (σ,N) *be a Weil-Deligne representation of* W_{F_v}.

1. *Let* $\omega \in \mathcal{W}$ *and* θ *an irreducible constituent of* $s|_{I_v}$ *for any* $s \in \omega$. *Then* $\sigma[s'] \cap \sigma[\theta] = 0$ *if* s' *is not an unramified twist of* s. *Furthermore, if* $d = \dim(s)/\dim(\theta)$, *then*

$$(n_{1,\theta}(\sigma,N), n_{2,\theta}(\sigma,N),\ldots) = d\cdot(m_{1,\omega}(\sigma,N), m_{2,\omega}(\sigma,N),\ldots).$$

2. *If* (σ',N') *is another Weil-Deligne representation of* F_v *such that* $\sigma^{ss} \cong \sigma'^{ss}$, *then* $(\sigma,N) \prec (\sigma',N') \Leftrightarrow (\sigma,N) \prec_I (\sigma',N')$.

From Lemma 8.4 and Proposition 7.1, it suffices to prove that

$$\mathrm{WD}(r_{p,\iota}(\Pi)|_{W_{F_v}})^{\mathrm{Frob}-ss} \prec_I \mathrm{rec}_{F_v}(\mathrm{BC}(\Pi_\ell)_v|\det|_v^{(1-2n)/2})$$

in order to conclude the proposition. We start by characterizing irreducible representations of I_v with open kernel.

Definition 8.5. *If* (θ,V) *is a representation of* I_v *and* τ *is an irreducible representation of a subgroup* H *of* I_v, *set* $(\theta[\tau],V[\tau])$ *to be the* τ*-isotypical component of the* H*-representation* $(\theta|_H, V|_H)$. *Furthermore, if* N *is a commuting nilpotent endomorphism of* V, *then set* $N[\tau] = N \cap V[\tau]$.

Let P denote a Sylow pro-p-subgroup of I_v. Recall that there is a map $t_p : I_v \to \mathbb{Z}_p$ since $v \nmid p$ and let $I_v^p := \ker t_p$. Recall that there is also an identification of P with I_v/I_v^p. Let $\mathcal{I}^p$ denote the set of isomorphism classes of representations of I_v^p with open kernel; there is a canonical action on $\mathcal{I}^p$ by I_v/I_v^p acting by conjugation. For $i \in I_v$, let c_i denote the conjugation map $I_v \to I_v$ where $x \mapsto ixi^{-1}$, and abusing notation, we let c_i also denote restrictions of c_i to certain subgroups of I_v.

Let $\mathcal{I}_0^p$ denote the subset of elements of $\mathcal{I}^p$ with open stabilizer in I_v/I_v^p. For $\eta \in \mathcal{I}_0^p$, set $I^\eta = \mathrm{Stab}_{I_v}(\eta) = \{i \in I_v : \eta \circ c_i \cong \eta\}$, which is open in I_v. Additionally, fix a choice of topological generator g_η of $P \cap I^\eta$ such that $I^\eta = \overline{\langle I_v^p, g_\eta \rangle}$. Note that g_η has pro-p-order and can be chosen so that $g_{\eta \circ c_g} = g_\eta$ for all $g \in P$.

Lemma 8.6. *If $\eta \in \mathcal{I}_0^p$, there exists an irreducible representation $\tilde{\eta}$ of I^η with open kernel such that $\tilde{\eta}|_{I_v^p} \cong \eta$.*

Proof. Since $\eta \in \mathcal{I}_0^p$, we have that $I_v^p/\ker(\eta)$ is finite order, and conjugation by g_η induces an automorphism of the quotient. This automorphism must have finite order as well, and since g_η has pro-p-order in I_v, conjugating by g must have p-power order as an automorphism of $I_v^p/\ker(\eta)$. This implies that there is some nonnegative integer n such that $g_\eta^{p^n}$ centralizes $I_v^p/\ker(\eta)$. Let A_{g_η} be an invertible matrix such that $\eta \circ c_{g_\eta} = A_{g_\eta} \circ \eta \circ A_{g_\eta}^{-1}$. Then $A_{g_\eta}^{p^n}$ centralizes η and therefore must be a scalar since η is irreducible; thus, we may suppose that $A_{g_\eta}^{p^n} = 1$. We can then define the representation $\tilde{\eta} : I^\eta \to \mathrm{GL}_{\dim \eta}(\overline{\mathbb{Q}}_p)$ sending

$$i_0 g_\eta^k \mapsto \eta(i_0) A_{g_\eta}^k, \qquad \text{where } i_0 \in I_v^p.$$

Furthermore, since η is irreducible, $\tilde{\eta}$ is also irreducible. □

For each $\eta \in \mathcal{I}_0^p$, choose once and for all a lift $\tilde{\eta}$ to I^η such that $\widetilde{\eta \circ c_g} = \tilde{\eta} \circ c_g$ for all $g \in P$. If $\eta \in \mathcal{I}_0^p$ and χ is a character of I^η with open kernel containing I_v^p, set $\theta_{\eta,\chi} := \mathrm{Ind}_{I^\eta}^{I_v}(\tilde{\eta} \otimes \chi)$.

Lemma 8.7. *If $\eta \in \mathcal{I}_0^p$ and χ is a character of I^η with open kernel containing I_v^p then:*

1. *$\theta_{\eta,\chi}$ is irreducible and $\theta_{\eta,\chi}|_{I^\eta} \cong \bigoplus_{[i] \in I_v/I^\eta} \tilde{\eta} \circ c_i \otimes \chi$.*
2. *$\theta_{\eta,\chi} \cong \theta_{\eta',\chi'}$ if and only if $\chi = \chi'$ and $\eta' \cong \eta \circ c_i$ for some $i \in I_v$.*
3. *Every irreducible representation of I_v with open kernel arises in this way.*

Proof. 1. For any character $\chi : I^\eta \to \overline{\mathbb{Q}}_p^\times$ with open kernel containing I_v^p, $\tilde{\eta} \otimes \chi$ is irreducible since η is. Thus, we can prove that $\theta_{\eta,\chi}$ is irreducible using Mackey's Criterion: Consider some element $i \in I_v \smallsetminus I^\eta$. We want to show that $\theta_{\eta,\chi}$ and $\theta_{\eta,\chi} \circ c_i$ are disjoint representations of I^η, i.e. have no irreducible component in common. It is enough to see that they are disjoint on I_v^p. Since $\theta_{\eta,\chi} \circ \mathrm{id}|_{I_v^p} = \eta$ and $\theta_{\eta,\chi} \circ c_i|_{I_v^p} = \eta \circ c_i$ for $i \notin I^\eta$, these are not isomorphic irreducible representations, thus they must be disjoint. The second part follows from Frobenius reciprocity and the definition of $\theta_{\eta,\chi}$ as an induced representation from the stabilizer of η in I_v to I_v.

2. Next, we prove that $\theta_{\eta,\chi}$ and $\theta_{\eta',\chi'}$ are isomorphic if and only if for some $i \in I_v$, $\eta \cong \eta' \circ c_i$ and $\chi = \chi'$. One direction follows from the first part of the lemma. To prove the converse, assume $\theta_{\eta,\chi}$ and $\theta_{\eta',\chi'}$ are isomorphic. Restricting to I_v^p, we have:

$$\bigoplus_{[i] \in I_v/I^\eta} \eta \circ c_i \cong \theta_{\eta,\chi}|_{I_v^p} \cong \theta_{\eta',\chi'}|_{I_v^p} \cong \bigoplus_{[i] \in I_v/I^{\eta'}} \eta' \circ c_i,$$

thus $\eta \cong \eta' \circ c_i$ for some $[i] \in I_v/I_v^p$. This further implies $I^\eta \cong I^{\eta'}$ where the isomorphism is given by conjugation by i since for any element $g \in I^\eta$,

$$\eta' \circ c_{igi^{-1}} \cong \eta \circ c_{ig} \cong \eta \circ c_i \cong \eta'.$$

In fact, since I_v/I_v^p is abelian, we have proven that $I^\eta = I^{\eta'}$.

It remains to show that $\chi \cong \chi'$. By Frobenius reciprocity,

$$\mathrm{Hom}_{I_v}(\theta_{\eta,\chi'}, \theta_{\eta,\chi}) = \mathrm{Hom}_{I^\eta}(\tilde{\eta} \otimes \chi', \bigoplus_{[i]\in I_v/I^\eta} \tilde{\eta} \circ c_i \otimes \chi).$$

Since $\tilde{\eta} \circ c_i \otimes \chi$ is irreducible, it remains to check that $\tilde{\eta} \otimes \chi \ncong \tilde{\eta}$ as representations of I^η for nontrivial χ. Let $\chi(g_\eta) = \lambda_{g_\eta}$ and note that if $\tilde{\eta}(g_\eta) = (\tilde{\eta} \otimes \chi)(g_\eta) = \lambda_{g_\eta}\tilde{\eta}(g_\eta)$, thus either $\lambda_{g_\eta} = 1$ or $\mathrm{tr}(\tilde{\eta}(g_\eta))$ is zero; however, since $\tilde{\eta}$ is irreducible, for any $h \in I_v^p$, we have $\tilde{\eta}(g_\eta h) = \lambda_{g_\eta}\tilde{\eta}(g_\eta h)$, and for some h, $\mathrm{tr}(\tilde{\eta}(g_\eta h)) \neq 0$. Thus, $\mathrm{tr}(\tilde{\eta}(g_\eta)) \neq 0$, and so we must have that $\lambda_{g_\eta} = 1$. Thus, we conclude that $\theta_{\eta,\chi} \neq \theta_{\eta',\chi'}$ when $\chi \neq \chi'$ or η and η' are not in the same orbit of $\mathcal{I}_0^p$ under the action of I_v/I_v^p (or equivalently, I_v/I^η).

3. Finally, we show that any irreducible (finite-dimensional) representation of I_v arises as $\theta_{\eta,\chi}$ for some η and χ. Let $\theta : I \to \mathrm{GL}(V)$ be an irreducible representation, and restrict to I_v^p. Let $\oplus_{\eta\in\mathcal{I}_0^p} V[\eta]$ denote the decomposition of $\theta|_{I_v^p}$ into its isotypic components. For each η, $I^\eta = \mathrm{Stab}_I(\eta)$ acts on $V[\eta]$ and furthermore, each $i \in I$ induces an identification of $V[\eta]$ and $V[\eta \circ c_i]$. This implies that $\mathrm{Ind}_{I^\eta}^I V[\eta] \cong V$, and thus as a representation of I^η, $V[\eta]$ is irreducible. There is an isomorphism as $\overline{\mathbb{Q}}_p$-vector spaces,

$$\mathrm{Hom}_{I_v^p}(\tilde{\eta}|_{I_v^p}, V[\eta]|_{I_v^p}) \otimes \tilde{\eta} \xrightarrow{\sim} V[\eta]. \tag{8.1}$$

The space $\mathrm{Hom}_{I_v^p}(\tilde{\eta}|_{I_v^p}, V[\eta]|_{I_v^p})$ has an action of $i \in I^\eta$ by conjugation, and I_v^p acts trivially. With this action, (8.1) is indeed an isomorphism of I^η-representations. However, since $V[\eta]$ is irreducible, $\mathrm{Hom}_{I_v^p}(\tilde{\eta}|_{I_v^p}, V[\eta]|_{I_v^p})$ must be irreducible over I^η/I_v^p, which is abelian. Letting $\chi = \mathrm{Hom}_{I_v^p}(\tilde{\eta}|_{I_v^p}, V[\eta]|_{I_v^p})$, we conclude that $\theta = \theta_{\eta,\chi}$. □

We now consider a more useful version of Definition 8.3 to all representations of I_v with open kernel and commuting nilpotent endomorphism.

Proposition 8.8. *If (σ, V, N) and (σ', V', N') are two Weil-Deligne representations, then $(\sigma, V, N) \prec_I (\sigma', V', N)$ if and only if $\sigma|_{I_v} \cong \sigma'|_{I_v}$ and*

$$\dim(\ker(N^j) \cap V[\theta_{\eta,\chi}]) \geq \dim(\ker(N'^j) \cap V'[\theta_{\eta,\chi}])$$

for all $j \in \mathbb{Z}_{>0}$, $\eta \in \mathcal{I}_0^p$, and χ a character of I^η/I_v^p with open kernel.

Proof. Note that for any $\theta \in \mathcal{I}$, the conjugacy class of $N[\theta]$ (resp. $N'[\theta]$) is determined by the partition of $\dim(\sigma[\theta])/\dim(\theta)$ (resp. $\dim(\sigma'[\theta])/\dim(\theta)$) given by $(n_{i,\theta}(\sigma, N))_{i\geq 1}$ (resp. $(n_{i,\theta}(\sigma', N'))_{i\geq 1}$). The condition

$$n_{1,\theta}(\sigma, N) + \ldots + n_{i,\theta}(\sigma, N) \leq n_{1,\theta}(\sigma', N') + \ldots + n_{i,\theta}(\sigma', N') \qquad \forall i \geq 1$$

is equivalent to the condition

$$\mathrm{rk}\, N[\theta]^j \leq \mathrm{rk}(N'[\theta])^j \qquad \forall j \geq 0.$$

Since we require $\sigma|_{I_v} \cong \sigma'|_{I_v}$ in both definitions, we have that their dimensions are equal, thus $\mathrm{rk}\, N[\theta]^j \leq \mathrm{rk}(N'[\theta])^j$ is equivalent to

$$\dim \ker N[\theta]^j \geq \dim \ker N'[\theta]^j.$$

By Lemma 8.7, we know that all $\theta \in \mathcal{I}$ are of the form $\theta_{\eta,\chi}$ where $\eta \in \mathcal{I}_0^p$ and χ is a character of I^η with open kernel containing I_v^p and so we are done. □

Furthermore, given $j \in \mathbb{Z}_{>0}$, $\eta \in \mathcal{I}_0^p$, and χ a character of I^η/I_v^p with open kernel, then using the fact that $\dim \ker N = [I : I^\eta] \dim \ker N|_{\theta_{\eta,\chi}[\tilde{\eta}\otimes\chi]}$ (coming from the Lemma 8.7(1)), we can conclude

$$\dim(\ker N^j \cap V[\theta_{\eta,\chi}]) \geq \dim(\ker N'^j \cap V'[\theta_{\eta,\chi}]) \Leftrightarrow \dim(\ker N^j \cap V[\tilde{\eta} \otimes \chi]) \geq \dim(\ker N'^j \cap V'[\tilde{\eta} \otimes \chi]).$$

If η denotes a representation of I_v^p with open kernel and $f : I_v^p \to \overline{\mathbb{Q}}_p$ is a locally constant function, then let $\eta(f) := \int_{I_v^p} f(i)\eta(i)di$, where di denotes the Haar measure on I_v^p (normalized so that $\mathrm{vol}(I_v^p) = 1$). Since I_v^p is compact, this integral is in fact a finite sum. Recall that for each η, we fixed a choice of topological generator g_η of $P \cap I^\eta$ such that $I^\eta = \overline{\langle I_v^p, g_\eta \rangle}$. The following lemma describes the existence of projection operators for representations of I_v^p and the relationship between the image of $\tilde{\eta}$ and η (both irreducible).

Lemma 8.9. *If (η, V) and $(\eta', V') \in \mathcal{I}_0^p$, then:*

1. *There exists a locally constant function $\epsilon_\eta : I_v^p \to \overline{\mathbb{Q}}_p$ sending $i \mapsto \frac{\mathrm{tr}(\eta^\vee(i))}{\dim \eta}$ (where $\eta^\vee$ denotes the dual representation) such that $\eta(\epsilon_\eta) = 1$ but $\eta'(\epsilon_\eta) = 0$ for all $\eta' \ncong \eta$.*
2. *There exists a locally constant function $a_\eta : I_v^p \to \overline{\mathbb{Q}}_p$ such that $\tilde{\eta}(g_\eta) = \eta(a_\eta)$ but $\eta'(a_\eta) = 0$ if $\eta \ncong \eta'$.*

Proof. The first part is clear. As for the second part, since η has open kernel, there is a finite quotient $I_p^v/\ker(\eta)$ through which it factors. Furthermore, η is irreducible, and thus the matrix $\tilde{\eta}(g_\eta) \in \mathrm{Hom}(V)$ can be written as a sum $\sum_{g \in I_p^v/\ker(\eta)} a_{\eta,g}\eta(g)$. Define a_η by sending $g \mapsto a_{\eta,g}$. By orthogonality, we have that $\eta'(a_\eta) = 0$ for $\eta' \not\equiv \eta$. Recall that the Peter-Weyl theorem gives an isomorphism

$$\mathrm{Hom}(I_p^v/\ker(\eta), \overline{\mathbb{Q}}_p) \xrightarrow{\sim} \bigoplus_{(r,V) \in \mathrm{Irr}(I_p^v/\ker(\eta))} \mathrm{End}_{\overline{\mathbb{Q}}_p}(V),$$

and thus a_η pulls back to a locally constant function of I_v^p. □

For each $\eta \in \mathcal{I}_0^p$, fix a choice of ϵ_η and a_η as described in Lemma 8.9. If (σ, V, N) (resp. (σ', V', N')) uniquely determine (local) Galois representations ρ_v (resp. ρ_v') of G_{F_v} acting on the same underlying vector space V (resp. V'), then recall that the defining relation between ρ_v and (σ, V, N) is

$$\rho_v(i) = \sigma(i)\exp(t_p(i)N) \qquad \text{for } i \in I_v.$$

If $i \in I_v$ is an element such that $t_p(g)$ is nonzero, then we can write $\log(\sigma(i)^{-1}\rho_v(i)) = t_p(i)N$. Additionally, for all positive j, $\mathrm{rk}(t_p(i)N)^j = \mathrm{rk}\, N^j$ and for any unipotent matrix U, $\mathrm{rk}(\log U)^j = \mathrm{rk}(U-1)^j$, and thus

$$\mathrm{rk}\left(\sigma(g)^{-1}\rho_v(g) - \mathrm{id}\right)^j = \mathrm{rk}\, N^j.$$

This implies that

$$\mathrm{rk}(\, N|_{V[\tilde{\eta}\otimes\chi]})^j = \mathrm{rk}(\, (\rho_v(g_\eta) - \sigma(g_\eta))^j\big|_{V[\tilde{\eta}\otimes\chi]}),$$

and we have that $(\sigma, N) \prec_I (\sigma', N')$ if and only if $\sigma|_{I_v} \cong \sigma'|_{I_v}$ and for all $j \in \mathbb{Z}_{>0}$, $\eta \in \mathcal{I}_0^p$, and χ a character of I^η/I_v^p with open kernel,

$$\dim(\ker\, (\rho(g_\eta) - \sigma(g_\eta))^j\big|_{V[\tilde{\eta}\otimes\chi]}) \geq \dim(\ker\, (\rho'(g_\eta) - \sigma'(g_\eta))^j\big|_{V'[\tilde{\eta}\otimes\chi]}). \tag{8.2}$$

Additionally, since $\ker\, (\rho(g_\eta) - \sigma(g_\eta))^j\big|_{V[\tilde{\eta}\otimes\chi]} = \ker(\rho(g_\eta) - \rho(a_\eta)\chi(g_\eta))^j$, we can then conclude:

Lemma 8.10. *If $(\rho, V), (\rho, V')$ are two continuous m-dimensional representations of I_v (arising from continuous G_{F_v}-representations), then $(\sigma_\rho, N) \prec_I (\sigma_{\rho'}, N')$ if and only if $\rho|_{I_v^p} \cong \rho'|_{I_v^p}$ and*

$$\wedge^k(\rho'(g_\eta) - \rho'(a_\eta)\zeta)^j = 0 \Rightarrow \wedge^k(\rho(g_\eta) - \rho(a_\eta)\zeta)^j = 0 \tag{8.3}$$

for all $j, k \in \mathbb{Z}_{>0}$, $\eta \in \mathcal{I}_0^p$, and p-power root of unity ζ.

Proof. This follows from (8.2) and the fact that for any $A \in \mathrm{End}(V)$, $\dim\ker A = \dim V + 1 - \min\{k \in \mathbb{Z}_{>0} : \wedge^k A = 0\}$. □

Suppose ρ'_v is a local p-adic G_{F_v}-Galois representation of dimension m, and ρ is a semisimple continuous m-dimensional global Galois representations of $G_F \supset I_v$. Then $\wedge^k\rho$ is also semisimple, and $\mathrm{WD}(\rho|_{G_{F_v}})^{\mathrm{Frob}-ss} \prec_I \mathrm{WD}(\rho'_v)^{\mathrm{Frob}-ss}$ if and only if for all $j,k \in \mathbb{Z}_{>0}$, $\eta \in \mathcal{I}_0^p$, ζ a p-power root of unity,

$$\wedge^k(\rho'(g_\eta) - \rho'(a_\eta)\zeta)^j = 0 \quad \Rightarrow \quad \mathrm{tr}(\wedge^k(\rho(g_\eta) - \rho(a_\eta)\zeta)^j\rho(\tau)) = 0 \quad \forall \tau \in G_F$$

because trace is a non-degenerate bilinear form on the image of semisimple representation. For any ρ, we can extend it by linearity to $\rho : \overline{\mathbb{Q}}_p[G_F] \to \mathrm{GL}(V)$, and let $b_{\eta,\zeta} := g_\eta - \zeta \cdot a_\eta \in \overline{\mathbb{Q}}_p[G_F]$. Then $\mathrm{WD}(\rho)^{\mathrm{Frob}-ss} \prec_I \mathrm{WD}(\rho')^{\mathrm{Frob}-ss}$ if and only if for all $k,j \in \mathbb{Z}_{>0}$, $\eta \in \mathcal{I}_0^p$, and p-power roots of unity ζ,

$$\wedge^k(\rho'(g_\eta) - \rho'(a_\eta)\zeta)^j = 0 \quad \Rightarrow \quad \mathrm{tr}\wedge^k\rho(b_{\eta,\zeta}^j\tau) = 0 \quad \forall \tau \in G_F. \tag{8.4}$$

Now, if T denotes a $2n$-dimensional continuous pseudocharacter of G_F, then by extending linearly and using the recursive formula for a matrix A, $\mathrm{tr}\wedge^k A = \frac{1}{k}\sum_{m=1}^k (-1)^{m-1}\mathrm{tr}(A^m)\,\mathrm{tr}\wedge^{k-m}(A)$, we can define

$$\wedge^k T : \overline{\mathbb{Q}}_p[G_F] \to \overline{\mathbb{Q}}_p \quad \text{by} \quad g \mapsto \frac{1}{k}\sum_{m=1}^k (-1)^{m-1}T(g^m)\wedge^{k-m} T(g)$$

for $k \le 2n$. In the sequel, we will be interested in whether the following function

$$B_{\eta,\zeta}^{k,j}(T) : G_F \to \overline{\mathbb{Q}}_p \quad \tau \mapsto \wedge^k T(b_{\eta,\zeta}^j\tau)$$

is identically zero.

8.1 Proof of $\prec$

In this section we prove Proposition 8.1.

Proof. Fix $\ell \in S_{\mathrm{spl}}$ and let $v \mid \ell$ be a prime of F in $\underline{\mathcal{S}}_{\mathrm{spl}} \sqcup \underline{\mathcal{S}}_{\mathrm{spl}}^c$. We have already seen that for Π satisfying the hypothesis of the proposition,

$$\mathrm{WD}(R_{p,\imath}(\Pi)|_{W_{F_v}})^{ss} \cong \mathrm{rec}_{F_v}(\mathrm{BC}(\Pi_\ell)_v|\det|_v^{(1-2n)/2})^{ss}.$$

By Lemma 8.4, it therefore remains to show that

$$\mathrm{WD}(R_{p,\imath}(\Pi)|_{W_{F_v}})^{\mathrm{Frob}-ss} \prec_I \mathrm{rec}_{F_v}(\mathrm{BC}(\Pi_\ell)_v|\det|_v^{(1-2n)/2}).$$

For ease of notation, let the p-adic local Galois representation associated to the Frobenius semisimple Weil-Deligne representation $\mathrm{rec}_{F_v}(\mathrm{BC}(\Pi_\ell)_v|\det|_v^{(1-2n)/2})$ be denoted $\rho_{\Pi,v}^{\mathrm{rec}}$. We want to show that for all $\eta \in \mathcal{I}_0^p$, p-power roots of unity ζ, and $j,k \in \mathbb{Z}_{>0}$,

$$\wedge^k\rho_{\Pi,v}^{\mathrm{rec}}(b_{\eta,\zeta}^j) = 0 \quad \Rightarrow \quad \wedge^k R_{p,\imath}(\Pi)(b_{\eta,\zeta}^j) = 0.$$

Recall from (7.3) that for $T_\Pi := \varphi_\Pi \circ T$ constructed in the proof of Proposition 7.1, there is a function $B_{\eta,\zeta}^{k,j}$ for each $j,k \in \mathbb{Z}_{>0}$, $\eta \in \mathcal{I}_0^p$, and p-power root of unity ζ such that

$$B_{\eta,\zeta}^{k,j}(T_\Pi)(\tau) = \mathrm{tr}\wedge^k(r_{p,\imath}(\Pi)(\epsilon_\eta g_\eta) - R_{p,\imath}(\Pi)(a_\eta)\zeta)^j R_{p,\imath}(\Pi)(\tau).$$

By (8.4), we want to show that

$$\wedge^k\left(\rho_{\Pi,v}^{\mathrm{rec}}(g_\eta) - \zeta\cdot\rho_{\Pi,v}^{\mathrm{rec}}(a_\eta)\right)^j = 0 \quad \Rightarrow \quad B_{\eta,\zeta}^{k,j}(T_\Pi) = 0.$$

Let $\mathcal{B}_v$ denote the Bernstein component containing $\mathrm{BC}(\Pi_\ell)_v$. By Proposition 6.2 in [17], associated to Π, there exists an idempotent $e_{\Pi,\mathcal{B}_v}$ inside the Bernstein center $\mathfrak{z}_{\mathcal{B}_v}$ associated to $\mathcal{B}_v$ such that:

- $e_{\Pi,\mathcal{B}_v}(\mathrm{BC}(\Pi_\ell)_v) \neq 0$
- $e_{\Pi,\mathcal{B}_v}(\Pi_0) \neq 0 \Rightarrow \mathrm{rec}(\Pi_0) \prec_I \mathrm{rec}(\mathrm{BC}(\Pi_\ell)_v)$ for all irreducible Π_0 of $\mathrm{GL}_{2n}(F_v)$,

If e_Π denotes the image of $e_{\Pi,\mathcal{B}_v} \in \mathfrak{z}_{\mathfrak{B}_v}$, where $\mathfrak{B}_v$ is the disjoint union of Bernstein components containing $\mathcal{B}_v$ defined in Theorem 6.2, let $\tilde{e}_\Pi := d(e_\Pi)e_\Pi \in \mathfrak{z}^0_\ell$ and abusing notation, let $\tilde{e}_\Pi$ also denote its own image in $\mathcal{H}^{\mathrm{spl}}_{\mathbb{Z}_p}$ and $\mathrm{End}(H^0(\mathcal{X}^{\min}_U, \mathcal{E}^{\mathrm{sub}}_{U,\rho_{\underline{b}}}))$ for any $\underline{b} \in X^*(T_{n/\overline{\mathbb{Q}}_p})^+_{\mathrm{cl}}$ and any neat open compact $U = \prod U_\ell$ such that $U_v = K_{\mathfrak{B}_v}$.

Lemma 8.11. *Let $\underline{b} \in X^*(T_{n/\overline{\mathbb{Q}}_p})^+_{\mathrm{cl}}$, and let $\mathbb{T}^p_{U,\underline{b}}$ denote the image in $\mathcal{H}^p_{\mathbb{Z}_p}$ in*

$$\mathrm{End}_{\mathbb{Z}_p}(H^0(\mathcal{X}^{\min}_U, \mathcal{E}^{\mathrm{sub}}_{U,\underline{b}})),$$

where $U = \prod U_\ell$ and U_ℓ satisfies (6.1) for every $\ell \in S_{\mathrm{spl}}$. There is a continuous representation $r_{\underline{b}} : G^S_F \to \mathrm{GL}_{2n}(\mathbb{T}_{\underline{b}} \otimes \overline{\mathbb{Q}}_p)$ described in (7.1) for every $\underline{b}$, and let $T_{\underline{b}} = \mathrm{tr}\, r_{\underline{b}}$. Assume that $\eta \in \mathcal{I}^p_0$, ζ a p-power root of unity, and $k, j \in \mathbb{Z}_{>0}$ are such that

$$\wedge^k \left(\rho^{\mathrm{rec}}_{\Pi,v}(g_\eta) - \zeta \cdot \rho^{\mathrm{rec}}_{\Pi,v}(a_\eta)\right)^j = 0.$$

For each $\underline{b}$, the map $\tilde{e}_\Pi B^{k,j}_{\eta,\zeta}(T_{\underline{b}}) : G^S_F \to \mathbb{T}_{\underline{b}}$ is identically zero.

Proof. Recall that $\mathbb{T}_{\underline{b}} \cong \oplus_{\Pi_0} \overline{\mathbb{Q}}_p$ where the sum runs over irreducible admissible representations of $G(\mathbb{A}^{p,\infty} \times \mathbb{Z}_p)$ with $\Pi_0^U \neq (0)$ which occur in $H^0(\mathcal{X}^{\min}, \mathcal{E}^{\mathrm{sub}}_{\underline{b}})$. We will prove that for each Π_0, the composition

$$\varphi_{\Pi_0} \circ \tilde{e}_\Pi B^{k,j}_{\eta,\zeta}(T_{\underline{b}}) : G^S_F \to \mathbb{T}_{\underline{b}} \overset{\varphi_{\Pi_0}}{\to} \overline{\mathbb{Q}}_p$$

is zero. Assume $\tilde{e}_\Pi(\mathrm{BC}(\Pi_{0,\ell})_v) \neq 0$ for some $\Pi_0 \in H^0(\mathcal{X}^{\min}, \mathcal{E}^{\mathrm{sub}}_{\underline{b}})$. Then $\mathrm{rec}(\mathrm{BC}(\Pi_{0,\ell})_v) \prec_I \mathrm{rec}(\mathrm{BC}(\Pi_\ell)_v)$, and so by Lemma 8.10, for $\eta \in \mathcal{I}^p_0$

$$\wedge^k \left(\rho^{\mathrm{rec}}_{\Pi,v}(\epsilon_\eta g_\eta) - \zeta \cdot \rho^{\mathrm{rec}}_{\Pi,v}(a_\eta)\right)^j = 0 \Rightarrow \wedge^k \left(\rho^{\mathrm{rec}}_{\Pi_0,v}(g_\eta) - \zeta \cdot \rho^{\mathrm{rec}}_{\Pi_0,v}(a_\eta)\right)^j = 0.$$

By Corollary 7.1 and Lemma 8.10, we know this implies that

$$\wedge^k \left(R_{p,\imath}(\Pi_0)(g_\eta) - \zeta \cdot R_{p,\imath}(\Pi_0)(a_\eta)\right)^j = 0,$$

thus $\varphi_{\Pi_0} \circ \tilde{e}_\Pi B^{k,j}_{\eta,\zeta}(T_{\underline{b}}) = 0$. □

Continuing the proof of the proposition, since T_Π is constructed in terms of $T_{\underline{b}}$, if $\tilde{e}_\Pi(\mathrm{BC}(\Pi_\ell)_v)B^{k,j}_{\eta,\zeta}(T_{\underline{b}})$ is identically zero for all $\underline{b} \in X^*(T_{n/\overline{\mathbb{Q}}_p})^+_{\mathrm{cl}}$, then $\tilde{e}_\Pi B^{k,j}_{\eta,\zeta}(T_\Pi)$ is also identically zero. Since $\tilde{e}_\Pi(\mathrm{BC}(\Pi_\ell)_v) \neq 0$, we can conclude that $B^{k,j}_{\eta,\zeta}(T_\Pi) = 0$ if $\eta \in \mathcal{I}^p_0$, ζ a p-power root of unity, and $k, j \in \mathbb{Z}_{>0}$ are such that

$$\wedge^k \left(\rho^{\mathrm{rec}}_{\Pi,v}(g_\eta) - \zeta \cdot \rho^{\mathrm{rec}}_{\Pi,v}(a_\eta)\right)^j = 0.$$

This implies that $\mathrm{WD}(R_{p,\imath}(\Pi)|_{W_{F_v}})^{\mathrm{Frob}-ss} \prec_I \mathrm{rec}_{F_v}(\mathrm{BC}(\Pi_\ell)_v |\det|_v^{(1-2n)/2})$. Thus, we conclude

$$\mathrm{WD}(r_{p,\imath}(\Pi)|_{W_{F_v}})^{\mathrm{Frob}-ss} \prec \mathrm{rec}_{F_v}(\mathrm{BC}(\Pi_\ell)_v |\det|_v^{(1-2n)/2}).$$

□

Proposition 8.1 in conjunction with Theorem 5.1 then allows us to conclude the following;

Corollary 8.12. *Assume that $n > 1$, that ρ is an irreducible algebraic representation of $L_{(n),\mathrm{lin}}$ on a finite-dimensional $\overline{\mathbb{Q}}_p$-vector space. Suppose that π is a cuspidal automorphic representation of $L_{(n),\mathrm{lin}}(\mathbb{A})$ such that π_∞ has the same infinitesimal character as $\rho^\vee$. Then, for all sufficiently large integers M, there is a continuous semi-simple representation*

$$R_{p,\imath}(\pi, M) : G_F \to \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_p)$$

with the following property: if $\ell \neq p$ is a rational prime in S_{spl}, then for all primes $v \mid \ell$:

$$\mathrm{WD}(R_{p,\imath}(\pi, M)|_{W_{F_v}})^{\mathrm{Frob}-ss} \prec \mathrm{rec}_{F_v}(\pi_v |\det|_v^{(1-n)/2}) \oplus \mathrm{rec}_{F_{c_v}}(\pi_{c_v} |\det|_{c_v}^{(1-n)/2})^{\vee,c} \otimes \epsilon_p^{1-2n-2M}.$$

Proof. Let Π be an irreducible subquotient of the induced representation $\mathrm{Ind}_{P_{(n)}^+(\mathbb{A}^{p,\infty})}^{G(\mathbb{A}^{p,\infty})}(\pi^\infty ||\det||^M \times 1)$ with the property that at any $v \mid \ell \in S_{\mathrm{spl}}$,

$$\Pi_v = \pi_v |\det|^M \boxplus \pi_{v^c}^{c,\vee} |\det|^{-M}.$$

Then set

$$R_p(\pi, M) = R_p\left(\imath^{-1}\Pi\right) \otimes \epsilon_p^{-M}$$

□

9 Group Theory

Let Γ be a topological group and let $\mathfrak{F}$ be a dense set of elements of Γ. Let k be an algebraically closed, topological field of characteristic 0 and let $d \in \mathbb{Z}_{>0}$. Let $\mu : \Gamma \to k^\times$ be a continuous homomorphism such that $\mu(f)$ has infinite order for all $f \in \mathfrak{F}$. For $f \in \mathfrak{F}$ let $\mathcal{E}_f^1$ and $\mathcal{E}_f^2$ be two d-elements multi set of elements of $k^\times$. Let $\mathcal{M}$ be an infinite subset of $\mathbb{Z}$. For $m \in \mathcal{M}$ suppose that

$$\rho_m : \Gamma \to \mathrm{GL}_{2d}(k)$$

is a continuous semi-simple representation such that for every $f \in \mathfrak{F}$ the multi-set of roots of the characteristic polynomial of $\rho_m(f)$ equals

$$\mathcal{E}_f^1 \bigsqcup \mathcal{E}_f^2 \mu(f)^m.$$

Proposition 9.1 (Proposition 7.12 in [10])**.** *There are continuous semi simple representations*

$$\rho^i : \Gamma \to \mathrm{GL}_d(k)$$

for $i = 1, 2$ such that for all $f \in \mathfrak{F}$ the multi set of roots of the characteristic polynomial of $\rho^i(f)$ equals $\mathcal{E}_f^i$.

Theorem 9.2. *Suppose that π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ such that π_∞ has the same infinitesimal character as an algebraic representation of $\mathrm{RS}_{\mathbb{Q}}^F \mathrm{GL}_n$. Then there is a continuous semi simple representation*

$$r_{p,\imath}(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

such that if $v \nmid p$ is a prime of F above a rational prime ℓ satisfying either

1. *ℓ is split over F_0, or*
2. *π and F are unramified at all primes above ℓ,*

then

$$\mathrm{WD}(r_{p,\imath}(\pi)|_{W_{F_v}})^{\mathrm{Frob}-ss} \prec \mathrm{rec}_{F_v}(\pi_v |\det|_v^{(1-n)/2}).$$

In particular, if π and F are unramified at v, then $r_{p,\imath}(\pi)$ is unramified.

Proof. Assume that $n > 1$. Recall that S contains p and the rational primes that are not split in F_0 but ramified in F; and let G_F^S denote the Galois group over F of the maximal extension of F unramified outside S. Let $\Gamma = G_{F,S}$ and $k = \overline{\mathbb{Q}}_p$, and $\mu = \epsilon_p^{-2}$, and $\mathcal{M}$ consisting of all sufficiently large integers m, and choose an irreducible subquotient Π of the induced representation $\mathrm{Ind}_{P_{(n)}^+(\mathbb{A}^{p,\infty})}^{G(\mathbb{A}^{p,\infty})}(\pi^\infty||\det||^m \times 1)$ satisfying for $v \mid \ell \in S_{\mathrm{spl}}$,

$$\Pi_v = \pi_v|\det|^m \boxplus \pi_{v^c}^{c,\vee}|\det|^{-m}.$$

Then, we set

$$\rho_m = R_p(\pi, m) = R_p\left(\imath^{-1}\Pi\right) \otimes \epsilon_p^{-m} \qquad m \in \mathcal{M}$$

For each v and let $k(v)$ denote the residue field of F_v. Let $\mathfrak{F}$ contain all elements $\sigma_v \in W_{F_v}$ which projects to a power of Frobenius under the map $W_{F_v} \to \mathrm{Gal}(\overline{k(v)}/k(v))$, where $v \notin S'$. Denote by $\sigma_{^cv}$ the image of σ_v under the isomorphism $W_{F_v} \cong W_{F_{^cv}}$ induced by conjugation c. Define $\mathcal{E}^1_{\sigma_v}$ to be the multiset of roots of the characteristic polynomial $\imath^{-1}\,\mathrm{rec}_{F_v}(\pi_v|\det|_v^{(1-n)/2})(\sigma_v)$ and $\mathcal{E}^2_{\sigma_v}$ equal to the multiset of roots of the characteristic polynomial of $\imath^{-1}\,\mathrm{rec}_{F_{^cv}}(\pi_{^cv}|\det|_{^cv}^{(-1+3n)/2})(\sigma_{^cv}^{-1})$. We can then conclude

$$(\,r_{p,\imath}(\pi)|_{W_{F_v}})^{ss} \cong \imath^{-1}\,\mathrm{rec}_{F_v}(\pi_v|\det|_v^{(1-n)/2})^{ss}. \tag{9.1}$$

By Proposition 9.1, we have that for a sufficiently large integer M in the sense of Theorem 5.1,

$$R_p(\pi, M) \cong r_{p,\imath}(\pi) \oplus r_{p,\imath}({}^c\pi)^{c,\vee} \otimes \epsilon_p^{1-2n-2M}. \tag{9.2}$$

Now, choose a finite order Hecke character ψ on $\mathbb{A}_F^\times$ such that

- ψ is unramified at v,
- ψ highly ramified at v^c, and
- $\mathrm{rec}_{F_v}((\pi\otimes\psi)_v|\det|_v^{(1-n)/2})$ and $\mathrm{rec}_{F_v}((\pi\otimes\psi)_{^cv}|\det|_{^cv}^{(1-n)/2})^{c,\vee}$ have no common irreducible constituents, even after restricting to I_v.

From (9.2) applied to $\pi\otimes\psi$ in conjunction with Corollary 8.12, after untwisting we obtain

$$\mathrm{WD}(\,r_{p,\iota}(\pi\otimes\psi)|_{W_{F_v}})^{\mathrm{Frob}-ss} \oplus \mathrm{WD}(\,r_{p,\iota}({}^c\pi\otimes{}^c\psi)^{c,\vee}|_{W_{F_v}})^{\mathrm{Frob}-ss} \prec \mathrm{rec}_{F_v}((\pi\otimes\psi)_v|\det|_v^{(1-n)/2}) \oplus \mathrm{rec}_{F_{^cv}}((\pi\otimes\psi)_{^cv}|\det|_{^cv}^{(1-n)/2})$$

Additionally, by (9.1),

$$r_{p,\iota}(\pi\otimes\psi)|_{W_{F_v}})^{ss} \cong \mathrm{rec}_{F_v}((\pi\otimes\psi)_v|\det|_v^{(1-n)/2})^{ss}.$$

Since $\prec$ is defined component-by-component, we can conclude

$$\mathrm{WD}(r_{p,\iota}(\pi\otimes\psi)|_{W_{F_v}})^{\mathrm{Frob}-ss} \prec \mathrm{rec}_{F_v}((\pi\otimes\psi)_v).$$

Since the relation $\prec$ is compatible with twisting, we conclude the theorem.

□

Corollary 9.3. *Suppose that E is a totally real or CM field and that π is a cuspidal automorphic representation such that π_∞ has the same infinitesimal character as an algebraic representation of $\mathrm{RS}_{\mathbb{Q}}^E \mathrm{GL}_n$. Then there is a continuous semi simple representation*

$$r_{p,\imath} : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

such that, if $\ell \neq p$ is a prime and if $v \mid \ell$ is a prime of E, then

$$\mathrm{WD}(\,r_{p,\imath}(\pi)|_{W_{E_v}})^{\mathrm{Frob}-ss} \prec \mathrm{rec}_{E_v}(\pi_v|\det|_v^{(1-n)/2}).$$

Proof. This can be deduced from Theorem 9.2 in conjunction with Lemma 1 of [21] using the same argument as in Theorem VII.1.9 of [11]. □

Acknowledgements

The author would like to thank Richard Taylor for suggesting this problem and for his invaluable assistance throughout this project; she would also like to thank Ana Caraiani, Gaetan Chenevier, David Geraghty, and Jay Pottharst for answering questions. The author would like to thank the anonymous referees for their careful suggestions and patience, especially with the proof of Theorem 9.2.

References

[1] T. Barnet-Lamb, T. Gee, D. Geraghty, R. Taylor, *Local-global compatibility for $\ell = p$, II,* Ann. Sci. Éc. Norm. Supr. **47** 4 (2014), no. 1, 165–179.

[2] T. Barnet-Lamb, D. Geraghty, M. Harris, R. Taylor, *A family of Calabi-Yau varieties and potential automorphy II,* P.R.I.M.S. **47** (2011) 29–98.

[3] J. Bellaiche, G. Chenevier, *Families of Galois representations and Selmer groups,* Asterisque **324**, Soc. Math. France, 2009.

[4] J. Bernstein, P. Deligne *Le "centre" de Bernstein,* Representations of reductive groups over a local field, Travaux en Cours (1984) 1-32.

[5] J. Bernstein, P. Deligne, D. Kazhdan *Trace Paley-Weiner Theorem for reductive p-adic groups* J. Analyse Math. **47** (1986) 180-192.

[6] A. Caraiani, *Local-global compatibility and the action of monodromy on nearby cycles,* Duke Math. Journal **161** (2012), no. 12, 2311-2413.

[7] C. L. Chai, G. Faltings *Degeneration of abelian varieties,* Springer 1990.

[8] G. Chenevier, *Une application des variétés de Hecke des groupes unitaires,* Shimura Varieties, London Math. Soc. Lecture Note Series **457** (2020) 266-296.
(preprint available at http://gaetan.chenevier.perso.math.cnrs.fr/articles/famgal.pdf)

[9] G. Chenevier, M. Harris *Construction of automorphic Galois representations, II* Cambridge Math. Journal **1** (2013) 53-73.

[10] M. Harris, K. Lan, R. Taylor, J. Thorne, *On the rigid cohomology of certain Shimura varieties,* Research in the Mathematical Sciences **3** 37 (2016).

[11] M. Harris, R. Taylor, *The geometry and cohomology of some simple Shimura varieties,* Annals of Math. Studies, PUP **151** (2001).

[12] G. Henniart, *Une preuve simple des conjectures de Langlands pour* GL(n) *sur un corps p-adique,* Inventiones Mathematicae, **139** 2 (2000).

[13] K.-W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, Monographs of the London Math. Society **36** (2013).

[14] K.-W. Lan, *Compactifications of PEL-type Shimura varieties and Kuga families with ordinary loci,* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.
(preprint available at www.math.umn.edu/∼kwlan/academic.html)

[15] C. Moeglin, J. -L. Waldspurger, *Le spectre résiduel de GL(n)*, Ann. Sci. Éc. Norm. Sup **22** (1989), 605-674.

[16] R. Pink, *Arithmetical compactifications of mixed Shimura varieties,* Bonner Mathematische Schriften **209**, Bonn (1990).

[17] P. Schneider, E. W. Zink, *K-types for the tempered components of a p-adic general linear group*, Journal für die reine ind angel. Math. **517** (1999), 161-208.

[18] P. Scholze, *On torsion in the cohomology of locally symmetric varieties*, Ann. of Math. **182** (2015), no. 3, 945–1066.

[19] S. W. Shin, *Galois representations arising from some compact Shimura varieties,* Ann. of Math. **173** (2011) 1645-1741.

[20] S. W. Shin, *On the cohomological base change for unitary similitude groups,* Appendix to: W.Goldring, "Galois representations associated to holomorphic limits of discrete series I: Unitary Groups," Compositio Math. **150** (2014), no.2, 191–228.

[21] C. Sorensen, *A patching lemma,* to appear in "On the Stabilization of the Trace Formula, Volume 2." preprint available at http://www.math.ucsd.edu/∼csorense/patch.pdf

[22] J. Tate, *Number-theoretic Background,* Automorphic Forms, Representations, and *L*-functions, Proc. Sympos. Pure Math., Part 2, Corvallis **33** (1979), no. 2, 3-26.

[23] R. Taylor, *Galois representations associated to Siegel modular forms of low weight,* Duke Math. J. **63** (1991) 281-322.

[24] R. Taylor, T. Yoshida *Compatibility of local and global Langlands correspondences,* J. Amer. Math. Soc. **20** (2007), no. 2, 467-493.