

Midterm Exam 2

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Calculus II

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MATH V1102, Section 04

Name: _____

To receive full credit, you must explain your answers.

No calculators of any type are allowed.

Each question is worth 20 points.

1	
2	
3	
4	
5	
Total	

1. A group of people received treatment for a deadly disease. Let t be the survival time, the number of years a person lives after receiving treatment. The density function giving the distribution of t is $p(t) = Ce^{-Ct}$ for some constant $C > 0$.

(a) Suppose that 20% of the population died within two years of the treatment. Find the value of C .

$$\int_0^2 Ce^{-Ct} dt = 0.20$$

$$-e^{-Ct} \Big|_0^2 = 0.20$$

$$1 - e^{-2C} = 0.20$$

$$0.8 = e^{-2C}$$

$$\ln 0.8 = -2C$$

$$C = -(\ln 0.8)/2 \quad \text{or} \quad \ln(\sqrt{5/4})$$

(b) Find the mean survival time.

Let μ be the mean survival time.

$$\mu = \int_0^{\infty} tCe^{-Ct} dt$$

Integrate by parts, $u = t$ and $dv = Ce^{-Ct}$

$$\mu = -te^{-Ct} \Big|_0^{\infty} + \int_0^{\infty} e^{-Ct} dt$$

$$\mu = 0 + -\frac{1}{C}e^{-Ct} \Big|_0^{\infty} = \frac{1}{C}$$

$$\mu = \frac{1}{C} = -2/(\ln 0.8)$$

2. (a) The curve $\{y = \sqrt[3]{x}, 1 \leq y \leq 2\}$ is rotated about the y -axis.
Find the area of the surface.

$$SA = \int 2\pi r \ell = \int_1^2 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \quad \text{or} \quad \int_1^8 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$SA = \int_1^2 2\pi y^3 \sqrt{1 + (3y^2)^2} dy \quad \text{or} \quad \int_1^8 2\pi x \sqrt{1 + (x^{-2/3}/3)^2} dx$$

The left integral is much easier: let $u = 1 + 9y^4$, and integrate $\int \sqrt{u} du$.

$$SA = \frac{\pi}{27} (1 + 9y^4)^{3/2} \Big|_1^2$$

- (b) Evaluate $\int_0^\infty \frac{1}{x^2} \sin \frac{1}{x} dx$.

Both limits of integration are improper, so split up the integral:

$$\int_0^\infty \frac{1}{x^2} \sin \frac{1}{x} dx = \int_0^1 \frac{1}{x^2} \sin \frac{1}{x} dx + \int_1^\infty \frac{1}{x^2} \sin \frac{1}{x} dx$$

Let $u = \frac{1}{x}$.

$$\int_0^\infty \frac{1}{x^2} \sin \frac{1}{x} dx = \int_\infty^1 -\sin u du + \int_1^0 -\sin u du$$

The left integral diverges:

$$\lim_{b \rightarrow \infty} \int_1^b \sin u du = \lim_{b \rightarrow \infty} (-\cos u) \Big|_1^b \quad \text{Does not exist.}$$

So the original integral diverges.

3. (a) Evaluate $\int_0^1 \frac{1}{3y-1} dy$.

Vertical asymptote at $y = 1/3$.

$$\int_0^1 \frac{1}{3y-1} dy = \int_0^{1/3} \frac{1}{3y-1} dy + \int_{1/3}^1 \frac{1}{3y-1} dy$$

$$\lim_{b \rightarrow \frac{1}{3}^-} \frac{1}{3} \ln |3y-1| \Big|_0^b + \lim_{b \rightarrow \frac{1}{3}^+} \frac{1}{3} \ln |3y-1| \Big|_b^1$$

Both integrals diverge, so original integral diverges.

(b) Determine whether the following series converges conditionally, absolutely, or diverges: $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt[3]{n}}{\sqrt{n}}$

Note that $\frac{\sqrt[3]{n}}{\sqrt{n}} = \frac{1}{\sqrt[6]{n}} = \frac{1}{n^{1/6}}$.

Alternating series converges by the alternating series test:

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/6}} = 0 \text{ and decreasing : } \frac{1}{(n+1)^{1/6}} < \frac{1}{n^{1/6}}$$

Absolute value series diverges by p -series test, $p = 1/6 < 1$.

So the series is conditionally convergent.

4. Determine whether the following series converge or diverge:

(a)
$$\sum_{n=1}^{\infty} \frac{3^n}{n! + n^3}$$

Compare this series: $\frac{3^n}{n! + n^3} < \frac{3^n}{n!}$

The series $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ converges by the ratio test, since $\lim_{n \rightarrow \infty} \frac{3}{n+1} = 0 < 1$.

So by the comparison test, the original series converges.

(b)
$$\sum_{n=1}^{\infty} \frac{1}{n + \cos n}$$

Limit comparison test with the harmonic series:

$$\lim_{n \rightarrow \infty} \frac{1/n}{1/(n + \cos n)} = \lim_{n \rightarrow \infty} \frac{n + \cos n}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{\cos n}{n}\right) = 1 + 0 = 1$$

So this series diverges just like the harmonic series.

(c)
$$\sum_{n=1}^{\infty} \frac{n^2}{(\ln \pi)^n}$$

Ratio test or root test both give after taking the limit, $\frac{1}{\ln \pi}$.

Since $\pi > e$, then $\ln \pi > \ln e = 1$. Thus, $\frac{1}{\ln \pi} < 1$, so series converges.

5. The prime numbers $\{2, 3, 5, 7, 11, 13, 17, 19, 23, \dots\}$ are still a very mysterious sequence. If p_n is the n^{th} prime, it is a profound theorem that

$$\lim_{n \rightarrow \infty} \frac{p_n}{n \ln n} = 1$$

- (a) What can you conclude about the series $\sum_{n=1}^{\infty} \frac{1}{p_n}$? Justify your answer.

Use the given limit to do a limit comparison test with the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

This series diverges by the integral test:

$$\int_2^{\infty} \frac{1}{x \ln x} dx \stackrel{u=\ln x}{=} \int_{\ln 2}^{\infty} \frac{1}{u} du = \ln u \Big|_{\ln 2}^{\infty} \text{ diverges}$$

So the original series diverges.

- (b) Suppose another sequence $\{q_n\}$ satisfies

$$\lim_{n \rightarrow \infty} \frac{q_n}{n(\ln n)^{1.01}} = 1.01$$

What can you conclude about the series $\sum_{n=1}^{\infty} \frac{1}{q_n}$? Justify your answer.

Use the given limit to do a limit comparison test with the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.01}}$.

This series converges by the integral test:

$$\int_2^{\infty} \frac{1}{x(\ln x)^{1.01}} dx \stackrel{u=\ln x}{=} \int_{\ln 2}^{\infty} \frac{1}{u^{1.01}} du = \frac{1}{u^{0.01}} \Big|_{\ln 2}^{\infty} = \frac{1}{(\ln 2)^{0.01}}$$

So the original series converges.