

Midterm Exam 1

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Calculus II

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MATH V1102, Section 03

Name: _____

To receive full credit, you must explain your answers.

No calculators of any type are allowed.

Each question is worth 20 points.

**You may omit one part of one question for full credit (10 pts).
Just draw an \times through that part.**

1	
2	
3	
4	
5	
Total	

1. (a) Evaluate $\int_0^\pi \cos^4(3x) dx$.

$$\cos^4(3x) = (\cos^2(3x))^2 = \left(\frac{1}{2}(1 + \cos(6x))\right)^2$$

$$= \frac{1}{4}(1 + 2\cos(6x) + \cos^2(6x)) = \frac{1}{4}\left(1 + 2\cos(6x) + \frac{1}{2}(1 + \cos(12x))\right)$$

$$\int_0^\pi \cos^4(3x) dx = \frac{1}{4} \int (1 + 2\cos(6x) + \frac{1}{2} + \cos(12x)) dx.$$

$$= \left[\frac{3}{8}x + \frac{1}{12}\sin(6x) + \frac{1}{96}\sin(12x)\right]_0^\pi = \frac{3\pi}{8}$$

(b) Evaluate $\int \tan^3 x \sec^3 x dx$.

$$\int (\tan^2 x \sec^2 x) \sec x \tan x dx$$

$$\int (\sec^2 x - 1) (\sec^2 x) \sec x \tan x dx$$

Let $u = \sec x$. Then $du = \sec x \tan x dx$.

$$\int (u^2 - 1) u^2 du = \int u^4 - u^2 du$$

$$\frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$$

2. (a) Evaluate $\int_0^1 \frac{1}{x^2 + 6x + 16} dx$.

First, complete the square: $x^2 + 6x + 16 = (x + 3)^2 + 7$

Trig substitution: let $x + 3 = \sqrt{7} \tan \theta$. Then $dx = \sqrt{7} \sec^2 \theta d\theta$.

$$\begin{aligned} \int_0^1 \frac{dx}{x^2 + 6x + 16} &= \\ &= \left[\int \frac{\sqrt{7} \sec^2 \theta d\theta}{7(\tan^2 \theta + 1)} = \int \frac{1}{\sqrt{7}} d\theta = \frac{1}{\sqrt{7}} \theta = \frac{1}{\sqrt{7}} \tan^{-1}\left(\frac{x+3}{\sqrt{7}}\right) \right]_0^1 \\ &= \frac{1}{\sqrt{7}} \left(\tan^{-1}\left(\frac{4}{\sqrt{7}}\right) - \tan^{-1}\left(\frac{3}{\sqrt{7}}\right) \right) \end{aligned}$$

(b) Evaluate $\int \frac{1}{x^2 - 10x + 16} dx$.

Two approaches: 1.) Factor $x^2 - 10x + 16 = (x - 8)(x - 2)$, or

2.) Complete the square $x^2 - 10x + 21 = (x - 5)^2 - 9$.

Using (1.), write $\frac{1}{x^2 - 10x + 16} = \frac{A}{x-8} + \frac{B}{x-2}$. Then $A(x - 2) + B(x - 8) = 1$, so $6A = 1$ and $-6B = 1$.

$$\int \frac{1}{x^2 - 10x + 16} dx = \frac{1}{6} \int \frac{dx}{x-8} - \frac{1}{6} \int \frac{dx}{x-2} = \frac{1}{6} \ln |x - 8| - \frac{1}{6} \ln |x - 2| = \frac{1}{6} \ln \left| \frac{x-8}{x-2} \right| + C.$$

Using (2.), let $u = x - 5$, so $\int \frac{1}{x^2 - 10x + 16} dx = \int \frac{1}{u^2 - 9} du$

Write $\frac{1}{u^2 - 9} = \frac{A}{u-3} + \frac{B}{u+3}$, so $6A = 1$ and $-6B = 1$, continue as above.

3. (a) Evaluate $\int_0^3 x^3 e^{(x^2)} dx$. [Hint: start with a substitution.]

Let $w = x^2$. Then $dw = 2x dx$.

$$\int_0^3 x^3 e^{(x^2)} dx = \frac{1}{2} \int_0^9 w e^w dw$$

Integration by parts: let $u = w$ and $dv = e^w dw$.

$$= \frac{1}{2} [w e^w - \int e^w dw]_0^9 = \frac{1}{2} (8e^9 + 1).$$

(b) Evaluate $\int \sin(\ln x) dx$.

Integration by parts twice: let $u_1 = \sin(\ln x)$ and $dv_1 = dx$.

$$\int \sin(\ln x) dx = x \sin(\ln x) - \int \cos(\ln x) dx$$

Let $u_2 = \cos(\ln x)$ and $dv_2 = dx$.

$$\int \sin(\ln x) dx = x \sin(\ln x) - x \cos(\ln x) + \int \sin(\ln x) dx.$$

$$\int \sin(\ln x) dx = \frac{1}{2} (x \sin(\ln x) - x \cos(\ln x)) + C$$

4. (a) Evaluate $\int \frac{x^2 + 1}{x^3 - x^2} dx$.

Partial fractions: $\frac{x^2 + 1}{x^3 - x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1}$.

$A(x^2 - x) + B(x - 1) + Cx^2 = x^2 + 1$. Plug in $x = 0, 1$ to get $B = -1$ and $C = 2$. Then compare coefficients on x^2 to get $A + C = 1$.

$$\begin{aligned} \int \frac{x^2 + 1}{x^3 - x^2} dx &= - \int \frac{dx}{x} - \int \frac{dx}{x^2} + \int \frac{2 dx}{x - 1} \\ &= -\ln|x| + \frac{1}{x} + 2 \ln|x - 1| + C = \ln \left| \frac{(x-1)^2}{x} \right| + \frac{1}{x} + C \end{aligned}$$

(b) Evaluate $\int \frac{x^2}{\sqrt{9 - 16x^2}} dx$.

Trig substitution: Want $16x^2 = 9 \sin^2 \theta$, so let $x = \frac{3}{4} \sin \theta$, then $dx = \frac{3}{4} \cos \theta d\theta$.

$$\begin{aligned} \int \frac{x^2}{\sqrt{9 - 16x^2}} dx &= \int \frac{\frac{9}{16} \sin^2 \theta}{\sqrt{9 \cos^2 \theta}} \frac{3}{4} \cos \theta d\theta \\ &= \frac{9}{16} \frac{3}{4} \frac{1}{3} \int \sin^2 \theta d\theta = \frac{9}{64} \int \frac{1}{2} (1 - \cos(2\theta)) d\theta \\ &= \frac{9}{128} (\theta - \frac{1}{2} \sin(2\theta)) = \frac{9}{128} (\theta - \sin \theta \cos \theta) \\ &= \frac{9}{128} \left(\sin^{-1}(4x/3) - (4x/3) \frac{\sqrt{9-16x^2}}{3} \right) + C \end{aligned}$$

5. (a) Evaluate $\lim_{x \rightarrow \infty} x \tan(1/x)$.

L'Hospital's Rule: $\lim_{x \rightarrow \infty} x \tan(1/x) = \lim_{x \rightarrow \infty} \frac{\tan(1/x)}{1/x}$ which has $\frac{0}{0}$ -form.

$$\begin{aligned} \text{Therefore, } \lim_{x \rightarrow \infty} \frac{\tan(1/x)}{1/x} &= \lim_{x \rightarrow \infty} \frac{\sec^2(1/x)(-x^{-2})}{-x^{-2}} = \lim_{x \rightarrow \infty} \sec^2(1/x) \\ &= \sec^2(0) = 1. \end{aligned}$$

(b) The widths of a pool were measured at 2-meter intervals as shown. Use Simpson's rule to estimate the area of the pool.

$$A = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) =$$

$$\frac{2}{3}(2 + 4 \cdot 3 + 2 \cdot 1 + 4 \cdot 2 + 3) = \frac{2}{3}(27) = 18.$$