Surfaces and Knots, Sec. 1 Solutions to Midterm Exam (Odd)

Write all answers on this exam. Notation: χ means Euler characteristic. A *surface* may have holes. A graph G is *embedded* in a surface if its edges intersect only at vertices.

Write A, S, or N: **A**lways, **S**ometimes but not always, **N**ever

1. (5pts) A connected planar graph G has 12 vertices and 10 edges.

   N. $v - e + f = 2$ so then $f = 0$, which is impossible.

2. (5pts) For a surface F, $\chi(F) = 4$.

   N. $\chi(F) \leq 2$ for all surfaces.

3. (5pts) If a planar graph diagram G has $v = e$, then its dual graph has $v = 2$.

   A or S. For connected graphs, A, since $v - e + f = 2$, so $f(G) = 2$ which means that $v(G^*) = 2$. (S also acceptable since not true for disconnected graphs.)

4. (5pts) A graph G has an even number of odd-order vertices.

   A. This is the Handshake Principle.

5. (5pts) A graph G has an odd number of even-order vertices.

   S. Consider the triangle and the square graphs, for which all vertices have order 2.

Short answers, no partial credit

1. (8pts) Describe 3 surfaces that are not equivalent, but that all have $\chi = 0$.

   Any 3 of Torus, Klein bottle, Mobius band, Sphere with 2 holes.

2. (8pts) A soccer ball has pentagon and hexagon faces. Find the best inequality $Af \leq Be$.

$5f \le 2e$. This is given by the sum of the orders of the faces, which is between $5f$ and $6f$, and equals $2e$. Just for fun, every soccer ball has 12 pentagons and 20 hexagons, so $f = 32$, $v = (5 \cdot 12 + 6 \cdot 20)/3 = 60$, $e = (5 \cdot 12 + 6 \cdot 20)/2 = 90$. Therefore, $(5f = 160) < (2e = 180) < (6f = 192)$.

3. (8pts) If P_n is a polygon with n sides, then how many spanning trees does P_{17} have?

   17. Every spanning tree is given by omitting one edge, and there are 17 edges.

4. (a) (4pts) Find χ(Torus with a hole).

   -1. From $\chi(T) = 0$, subtract 1 for every hole.

   (b) (4pts) Find χ(Klein bottle with a hole).

   -1. From $\chi(K) = 0$, subtract 1 for every hole.

   (c) (4pts) Are these two surfaces equivalent?

   No. The torus is orientable, but the Klein bottle is not. With or without a hole, one surface contains a Mobius band, but the other does not.

<u>Show your work – **partial credit only for justified steps**</u>

1. (12pts) For the surface diagram shown, identify the surface it represents when all its holes are filled by discs. Use this fact: χ(surface diagram)$= -$(number of bands).

K or $P\#P$. There are 3 bands shown, so χ(surface diagram)$= -3$. The number of holes (i.e. boundary curves) is 3. So when all the holes are filled by discs, χ(surface)$= 3 - 3 = 0$. You can see a Mobius band (the band with a single twist), so the surface is not orientable. So we get the formula: $0 = 2 - g$, so $g = 2$ which means that the surface is a Klein bottle, i.e. $P\#P$.

2. (12pts) Identify the surface for the cell complex as a connect sum of some T, P or K.

   $P\#P\#P$ or $K\#P$ or $T\#P$. It is non-orientable since you can see two identified edge arrows pointing in the same direction, which gives a Mobius band. By marking identified vertices, $v = 1$. Also, $e = 3$ and $f = 1$, so $\chi = 1 - 3 + 1 = -1$. Therefore, $-1 = 2 - g$, so $g = 3$, which means that the surface is $P\#P\#P$ or $K\#P$ or $T\#P$.

3. (12pts) T is a torus. Show that K_{10} cannot be embedded in $T\#T\#T$.

   If K_{10} could be embedded, then $v = 10$, $e = \frac{10 \cdot 9}{2} = 45$, and we find f by using $\chi(T\#T\#T) = 2 - 2g = 2 - 6 = -4$. We get $v - e + f = -4$, so $10 - 45 + f = -4$, so $f = 31$. We get our contradiction by the inequality $3f \leq 2e$. Here, $3f = 3 \cdot 31 = 93$ but $2e = 2 \cdot 45 = 90$.

4. (12pts) Does K_{17} have an Euler circuit? Why or why not?

   Yes, K_{17} has an Euler circuit because every vertex has order 16. A graph has an Euler circuit if every vertex has even order.