Review Problems for Exam 2 for Math 329 (Geometry) Spring 2009

Problem 1. Every isometry of $\mathbf{R}^2$ is either a rotation, translation, or a glide reflection (with possibly zero "glide"). Which type of isometry is each of the ones described below? Explain how you know.

(a) The composition of two translations $t_{\mathbf{v}} \circ t_{\mathbf{w}}$.

(b) The same glide reflection performed twice in a row.

(c) Rotation followed by translation: $t_{\mathbf{v}} \circ r_{\mathbf{0},\theta}$.

Problem 2.

(a) Let $r_{\pi/2}$ be the rotation of $\mathbf{R}^2$ about the origin by $\pi/2$. Let R be the reflection in the x–axis. Show that these two isometries do not commute.

[In general, unless the rotation r fixes the line ℓ, r does not commute with the reflection R_ℓ.]

(b) Give an example of two rotations whose product is a translation, and prove this claim for your example [consider three points].

(c) Show that reflection in two parallel lines is a translation $t_{\mathbf{v}}$ for any points in $\mathbf{R}^2$ [consider various cases]. How does $\mathbf{v}$ depend on the lines?

Problem 3. Prove each of the following claims, which together show that every isometry f of $\mathbf{R}^2$ is determined by the images of any three non-collinear points A, B, C.

(a) Any point P is uniquely determined by the three distances $|PA|$, $|PB|$, $|PC|$.

(b) $|f(P)f(A)| = |PA|$, $|f(P)f(B)| = |PB|$, $|f(P)f(C)| = |PC|$.

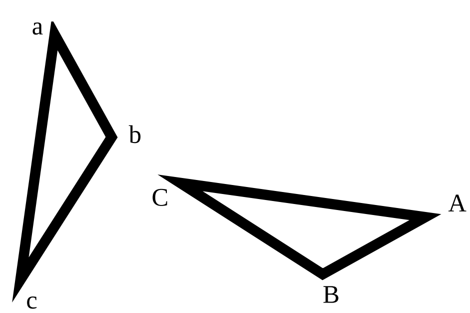
(c) $f(A)$, $f(B)$, $f(C)$ are non-collinear.

(d) $f(P)$ is uniquely determined by $f(A)$, $f(B)$, $f(C)$.

Problem 4. In the figure below, $\triangle abc \cong \triangle ABC$. Suppose f is an isometry that takes $\triangle abc$ to $\triangle ABC$.

(a) By general principles, how can you tell that f is a rotation?

(b) Find center P and angle θ such that $f = r_{P,\theta}$.



Problem 5. Problems for Chapter 4:
4.1.3–4.1.4, 4.2.1–4.2.2, 4.3.2–4.3.5, 4.4.1–4.4.2, 4.5.1–4.5.3

Problem 6. Linear transformations in $\mathbf{R}^2$:

(a) Which isometries are linear transformations?

(b) Show that the midpoint of any line segment is preserved by a linear transformation.

(c) Use vectors to prove that linear transformations preserve lines, and that they preserve parallel lines.

(d) Use matrices to prove that a product of rotations about $\mathbf{0}$ is also a rotation about $\mathbf{0}$.

Problem 7. Spherical geometry:

(a) Given that a reflection of $\mathbf{R}^3$ in a plane is an isometry of R^3, explain why a reflection of S^2 in a great circle is an isometry of S^2.

(b) Use vectors to show that the antipodal map takes great circles to great circles. [Hint: every great circle lies in a plane determined by its normal vector $\mathbf{n}$.]

(c) Explain why the "three reflections theorem" for S^2 implies that all isometries of S^2 are restrictions of isometries of $\mathbf{R}^3$ that fix $\mathbf{0}$.