Remarks for Exam 2 in Linear Algebra

Span, linear independence and basis

The span of a set of vectors is the set of all linear combinations of the vectors. A set of vectors is linearly independent if the only solution to $c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = 0$ is $c_i = 0$ for all i.

Given a set of vectors, you can determine if they are linearly independent by writing the vectors as the columns of the matrix A, and solving $A\mathbf{x} = 0$. If there are any non-zero solutions, then the vectors are linearly dependent. If the only solution is $\mathbf{x} = 0$, then they are linearly independent.

A basis for a subspace S of $\mathbf{R}^n$ is a set of vectors that spans S and is linearly independent. There are many bases, but every basis must have exactly $k = \dim(S)$ vectors. A spanning set in S must contain at least k vectors, and a linearly independent set in S can contain at most k vectors. A spanning set in S with exactly k vectors is a basis. A linearly independent set in S with exactly k vectors is a basis.

Rank and nullity

The span of the rows of matrix A is the row space of A. The span of the columns of A is the column space $C(A)$. The row and column spaces always have the same dimension, called the rank of A.

Let $r = \text{rank}(A)$. Then r is the maximal number of linearly independent row vectors, and the maximal number of linearly independent column vectors. So if $r < n$ then the columns are linearly dependent; if $r < m$ then the rows are linearly dependent.

Let $R = \text{rref}(A)$. Then $r = \#$pivots of R, as both A and R have the same rank. The dimensions of the four fundamental spaces of A and R are the same. The null space $N(A) = N(R)$ and the row space $\text{Row}(A) = \text{Row}(R)$, but the column space $C(A) \neq C(R)$. The pivot columns of A form a basis for $C(A)$.

Let A be an $m \times n$ matrix with rank r. The null space $N(A)$ is in $\mathbf{R}^n$, and its dimension (called the nullity of A) is $n - r$. In other words, $\text{rank}(A) + \text{nullity}(A) = n$. Any basis for the row space together with any basis for the null space gives a basis for $\mathbf{R}^n$. If $\mathbf{u}$ is in $\text{Row}(A)$ and $\mathbf{v}$ is in $N(A)$, then $\mathbf{u} \perp \mathbf{v}$. If $r = n$ (A has full column rank) then the columns of A are linearly independent. If $r = m$ (A has full row rank) then the columns of A span $\mathbf{R}^m$.

If $\text{rank}(A) = \text{rank}([A|\mathbf{b}])$ then the system $A\mathbf{x} = \mathbf{b}$ has a solution.
If $\text{rank}(A) = \text{rank}([A|\mathbf{b}]) = n$ then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.
If $\text{rank}(A) = \text{rank}([A|\mathbf{b}]) < n$ then the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.
If $\text{rank}(A) < \text{rank}([A|\mathbf{b}])$ then the system $A\mathbf{x} = \mathbf{b}$ is inconsistent; i.e., $\mathbf{b}$ is not in $C(A)$.

Let A be an $n \times n$ matrix. The following statements are equivalent:

1. A is invertible
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^n$.
3. $A\mathbf{x} = 0$ has only the solution $\mathbf{x} = 0$.
4. $\text{rref}(A) = I_{n\times n}$.
5. $\text{rank}(A) = n$.
6. $\text{nullity}(A) = 0$.
7. The column vectors of A span $\mathbf{R}^n$.
8. The column vectors of A form a basis for $\mathbf{R}^n$.
9. The column vectors of A are linearly independent.
10. The row vectors of A span $\mathbf{R}^n$.
11. The row vectors of A form a basis for $\mathbf{R}^n$.
12. The row vectors of A are linearly independent.