Sample problems for Linear Algebra, Spring 2016, Exam 1

Problem 1. Give an example of the following, or write "impossible."

(a) A 3×3 matrix with no zeros but which is not invertible.

(b) A 2×2 matrix such that $A^2 = -I$.

(c) A 2×2 matrix such that $A^2 = 0$ but $A \neq 0$.

(d) Two 2×2 matrices such that $AB = -BA \neq 0$.

(e) Two 2×2 matrices such that $AB = 0$, but A and B have no zero entries.

(f) A system with two equations and three unknowns that is inconsistent.

(g) A system with two equations and three unknowns that has a unique solution.

(h) A system with two equations and three unknowns that has infinitely many solutions.

Problem 2. Justify these statements with a short general argument.

(a) If A and B are diagonal matrices then $AB = BA$.

(b) If A and B are symmetric matrices then $AB + BA$ is also symmetric.

(c) If A is any $n \times n$ matrix then $(A + A^T)$ is symmetric.

(d) If $A\mathbf{x} = \mathbf{b}$ has solutions $\mathbf{u}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$ then $\mathbf{u}_3 = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$ is also a solution.

(e) A system with two equations and three unknowns has infinitely many solutions.

(f) For square matrices A and B, if AB is invertible, then A is invertible.

(g) If A and B are invertible, then ABA^{-1} is invertible.

(h) If P is a permutation matrix, then $P^k = I$ for some positive integer k.

Problem 3. Solve the following linear system:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$$

Problem 4. Consider the following linear system:

$$\begin{cases} x_1 - x_2 + x_4 = 2 \\ x_1 - x_3 + 2x_4 = 0 \\ -x_2 + x_3 + x_4 = -6 \end{cases}.$$

(a) Write its associated augmented matrix.

(b) Reduce the matrix to its row-echelon form.

(c) Solve the system using part (b).

Problem 5. Consider the following linear system:

$$\begin{cases} -x_1 + 2x_2 - x_3 = 1 \\ -x_2 + 2x_3 = 0 \\ 2x_1 - x_2 + x_4 = 0 \end{cases}.$$

(a) Write its associated augmented matrix.

(b) Reduce the matrix to its row-echelon form.

(c) Solve the system using part (b).

Problem 6. Consider the following linear system:

$$\begin{cases} x_1 + x_2 - 2x_3 + 3x_4 = 6 \\ x_1 + 3x_2 + 2x_3 + 3x_4 = 8 \\ 2x_1 - 8x_3 + 5x_4 = 12 \end{cases}.$$

(a) Write its associated augmented matrix.

(b) Reduce the matrix to its row-echelon form.

(c) Solve the system using part (b).

Problem 7.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & k & 2 \end{bmatrix}$$

(a) For which values of k is A invertible?

(b) Use elementary operations to find the inverse of A when $k = -1$.

Problem 8. Find the LU factorization of the following matrix A.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Problem 9. Write down the matrix that exchanges the first and second rows of A, and adds twice the first row to the third row. Show that your answer is correct by doing the matrix multiplications.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Problem 10. Use row operations to find A^{-1} for the following matrix. Check your answer.

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -2 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

Problem 11. Use row operations to find A^{-1} for the following matrix. Check your answer.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 4 \\ 1 & -1 & 4 \end{bmatrix}$$