Math 233 Exam 3 12/8/2010 Solutions (Kofman)

① $\int_C P\,dx + Q\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$P = \sqrt{1+x^3} \quad Q = 2xy \Rightarrow \int_D (2y-0)\,dA = \int_0^1 \int_0^{3x} 2y\,dy\,dx = \int_0^1 y^2\Big|_0^{3x} dx$

$= \int_0^1 (3x)^2\,dx = \frac{1}{3} 9x^3\Big|_0^1 = 3$

② a) $\nabla f = (f_x, f_y, f_z) = (ze^{xz}, 0, xe^{xz})$

$f(x,y,z) = \int f_x\,dx = \int ze^{xz}\,dx = e^{xz} + g(y,z)$ $\Rightarrow f(x,y,z) = e^{xz}$

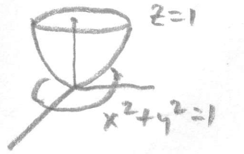
$f_z = xe^{xz} + \cancel{g_z} \to 0$

b) Since $F = \nabla f$, by FT for line integrals of ∇f,

t	C
0	(1,0,0)
2π	$(1,0,2\pi)$

$\int_C F \cdot d\vec{s} = \int_C \nabla f \cdot d\vec{s} = f(1,0,2\pi) - f(1,0,0)$

$= e^{1\cdot 2\pi} - e^{1\cdot 0} = e^{2\pi} - 1$

③ 

$\int_S \nabla \times F \cdot d\vec{S} = \int_C F \cdot d\vec{s}$

$F = (y^2, x, z^2) \Rightarrow \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x & z^2 \end{vmatrix} = (0, 0, 1-2y) = (1-2y)\vec{k}$

$z = f(x,y) = x^2+y^2 \Rightarrow \vec{n} = (-f_x, -f_y, 1) = (-2x, -2y, 1)$

[OR, $f(x,y,z) = x^2+y^2-z = 0 \Rightarrow \vec{n} = -\nabla f = (-2x,-2y,1)$]

$\int_S \nabla \times F \cdot d\vec{S} = \int_S (\nabla \times F)\cdot \vec{n}\,dS = \int_D (1-2y)\,dA = \int_0^{2\pi}\int_0^1 (1 - 2r\sin\theta)\,r\,dr\,d\theta$

$= \int_0^{2\pi} \left(\frac{1}{2}r^2\Big|_0^1 - (\sin\theta)\frac{2}{3}r^3\Big|_0^1\right) d\theta = \int_0^{2\pi} \left(\frac{1}{2} - \cancel{\frac{2}{3}\sin\theta}\right) d\theta$ $\to 0$ full period

$= \frac{1}{2}\cdot 2\pi = \pi$

$\int_C F \cdot d\vec{s} = \int_0^{2\pi} (\sin^2\theta, \cos\theta, 1)\cdot(-\sin\theta, \cos\theta, 0)\,d\theta \quad \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = 1 \end{cases}$

$= \int_0^{2\pi} -\sin^3\theta + \cos^2\theta\,d\theta = 0 + \frac{1}{2}\cdot 2\pi = \pi$

(4)  $F=(x,y,-z)$

a) $\int_{\partial E} F\cdot d\vec{S} = \int_{top} F\cdot\vec{k}\,dA + \int_{bottom} F\cdot(-\vec{k})\,dA + \int_{side} F\cdot(x,y,0)\,dS$

$= \int_D \underset{\uparrow z=3}{(-3)}\,dA + \int_D \underset{\uparrow z=0}{(0)}\,dA + \int_{cyl.} \underbrace{(x^2+y^2)}_{=1}\,dS$

$= -3\pi + 0 + (2\pi)(3)$ ← SA of cylinder

$= 3\pi$

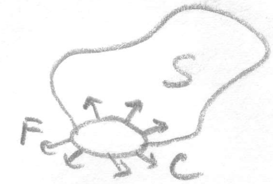
b) $\int_{\partial E} F\cdot dS = \int_E \nabla\cdot F\,dV = \int_E (1+1-1)\,dV = Vol(cyl.) = 3\pi$

(5) $\iint_B F\cdot d\vec{S} + \underbrace{\iint_{cap} F\cdot d\vec{S}}_{} = \iiint_{\substack{solid\\Bottle}} \nabla\cdot F\,dV = \iiint (2+0+0)\,dV$

$= 2\cdot 750$

$\iint_D F\cdot\vec{j}\,dA = \iint_D 3\,dA = 3\pi$

$\Rightarrow \iint_B F\cdot d\vec{S} = 2\cdot 750 - 3\pi$

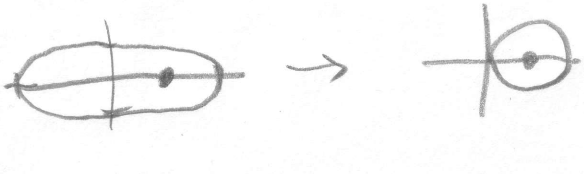


(6) $F\perp S \Rightarrow F\perp C=\partial S$

$\int_S \nabla\times F\cdot dS = \int_C F\cdot d\vec{s} = \int_C \overbrace{F\cdot T}^{=0}\,ds = 0$

(Also, $F\perp S \Rightarrow \nabla\times F$ is tangent to $S \Rightarrow (\nabla\times F)\cdot\vec{n}=0$)

(7) $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y} \Rightarrow$ Can take C' to be $(x-1)^2+y^2=1$ instead of ellipse



Now, change variables $u=x-1$

$\int_C \frac{-y\,dx+(x-1)\,dy}{(x-1)^2+y^2} = \int_{C'} \frac{-y\,dx+x\,dy}{x^2+y^2} = 2\pi$