Calculus III (Math 233) Exam 2

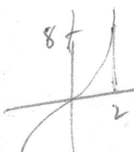
November 10, 2010 — Professor Ilya Kofman

Justify answers and show all work for full credit.

NAME: Key

Problem 1. Evaluate $\iint_D x^2\, dA$ where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

$$\int_{-1}^{1}\int_{2x^2}^{1+x^2} x^2\, dy\, dx = \int_{-1}^{1} x^2 - x^4\, dx = \frac{4}{15}$$



Problem 2. Find the volume enclosed by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$.

$D: x^2+y^2=3$

$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_{1+2r^2}^{7} r\, dz\, dr\, d\theta = 2\pi\int_0^{\sqrt{3}} 6r - 2r^3\, dr = 2\pi\left[3r^2 - \tfrac{1}{2}r^4\right]_0^{\sqrt{3}} = 9\pi$$

Problem 3. Change the order of integration to integrate $\int_0^8\int_{\sqrt[3]{y}}^{2} \sin(x^4)\, dx\, dy$.

$$\int_0^2\int_0^{x^3} \sin(x^4)\, dy\, dx = \int_0^2 x^3 \sin(x^4)\, dx = \tfrac{1}{4}\left(-\cos(x^4)\right)\Big|_0^2 = \frac{1-\cos(16)}{4}$$

Problem 4. Let R be the region bounded by the lines $y + x = 0$ and $y + x = 5$, $y - x = 0$ and $y - x = 3$. Use the change of variables $x = \frac{u-v}{2}$ and $y = \frac{u+v}{2}$ (i.e., $u = x + y$ and $v = y - x$) to evaluate $\iint_R 2(x+y)\, dA$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & +1/2 \end{vmatrix} = \tfrac{1}{4}+\tfrac{1}{4} = \tfrac{1}{2}$$

$$= \int_0^3\int_0^5 2u\left(\tfrac{1}{2}\right) du\, dv = \frac{25}{2}\cdot 3 = \frac{75}{2}$$

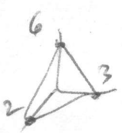
$= 6$

$$\int_0^2 9 - 9x + \tfrac{9}{4}x^2\, dx = 18 - 18 + \tfrac{9}{12}\cdot 2^3$$

$$\int_0^2 18 - 18x + \tfrac{9}{2}x^2 - (9 - 9x + \tfrac{9}{4}x^2)\, dx$$

$\frac{1}{3}\cdot 3\cdot 6$

Problem 5. Completely set up, **but do not evaluate**, the following integrals:



(a) The volume of the tetrahedron bounded by the plane $3x + 2y + z = 6$ in the first octant.

$$\int_0^2\int_0^{3-\frac{3}{2}x}\int_0^{6-3x-2y} dz\, dy\, dx = \int_0^2\int_0^{3-\frac{3}{2}x}(6-3x-2y)\, dy\, dx = \int_0^2 (6-3x)(3-\tfrac{3}{2}x) - (3-$$

(b) The volume of ice-cream bounded by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2 + y^2 + z^2 = 16$.

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^4 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi\left(\tfrac{1}{3}\cdot 4^3\right)\underbrace{\int_0^{\pi/4}\sin\varphi\, d\varphi}_{-\cos\varphi|_0^{\pi/4} = -\left(\frac{1}{\sqrt{2}}-1\right)} = \frac{128\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)$$

Problem 6. Evaluate $\iiint_E x^2 + y^2\, dV$ where E is the solid bounded by the paraboloids $z = 2 - x^2 - y^2$ and $z = x^2 + y^2 - 2$. $\Rightarrow x^2+y^2=2$

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2-2}^{2-r^2} (r^2)\, r\, dz\, dr\, d\theta = 2\pi\int_0^{\sqrt{2}} r^3(4-2r^2)\, dr = 2\pi\left(r^4 - \tfrac{1}{3}r^6\right)\Big|_0^{\sqrt{2}}$$

$$= 2\pi\left(4 - \tfrac{8}{3}\right)$$

Problem 7. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 + y^2 - 6x + 5$ on the ellipse $4x^2 + y^2 = 16$.

Problem 8. Find all the critical points of $f(x, y) = x^2 - y^2 + 4x + 6y - 16$, and classify them using the Second Derivative Test.

CHANGE OF VARIABLES FORMULAS:

$$\iint f(r,\theta)\, r\, dr\, d\theta, \text{ where } x = r\cos\theta,\ y = r\sin\theta$$

$$\iiint f(\rho,\phi,\theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta, \text{ where } x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$$

$$\iint f(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv, \text{ where } \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

7) $f_x = 2x - 6$ $\quad g_x = 8x$ $\quad \nabla f = \lambda \nabla g \Rightarrow$ $2x - 6 = \lambda \cdot 8x$ $\quad 2y = \lambda \cdot 2y$

$f_y = 2y$ $\quad g_y = 2y$ $\quad$ If $\lambda = 1$, $x = -1$, $y = \pm\sqrt{12}$ $\quad \lambda = 1$ or $y = 0$

If $y = 0$, $x = \pm 2$

$(4x^2 + y^2 = 16)$

CP: $(-1, \sqrt{12})$, $(-1, -\sqrt{12})$, $(-2, 0)$, $(2, 0)$

$f = 24$ $\quad f = 24$ $\quad f = 21$ $\quad f = -3$

8) $f_x = 2x + 4 = 0$ $\quad (-2, 3)$

$f_y = -2y + 6 = 0$

$f_{xx} = 2$ $\quad f_{xy} = 0$ $\quad D = -4$

$f_{yx} = 0$ $\quad f_{yy} = -2$ $\quad$ Saddle pt.