Mth 232 Exam 2 4/27/2015 Solutions

(1) Converges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n^3}$ p-series, $p=3$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{7n^4+3} \cdot \frac{n^3}{1} = \lim_{n\to\infty} \frac{n^4}{7n^4+3} = \lim_{n\to\infty} \frac{1}{7+3/n^4} = \frac{1}{7}$$

(2) Converges by ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{10^{n+1}+5}{(n+1)!} \cdot \frac{n!}{10^n+5}\right| = \lim_{n\to\infty} \left|\frac{10^{n+1}+5}{10^n+5} \cdot \frac{1}{n+1}\right|$$

$$= \lim_{n\to\infty} \left(\frac{10+5/10^n}{1+5/10^n}\right)\left(\lim_{n\to\infty} \frac{1}{n+1}\right) = 10 \cdot 0 = 0$$

(3) Converges by integral test

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{R\to\infty} \int_2^R \frac{1}{x(\ln x)^3}\,dx$$

$$= \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{1}{u^3}\,du = \lim_{R\to\infty} -\frac{1}{2}u^{-2}\Big|_{\ln 2}^{R}$$

$$= 0 + \frac{1}{2}(\ln 2)^{-2} < \infty$$

(4) Diverges by n-th term test

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n e^n}{e^n+n} = \lim_{n\to\infty} \frac{(-1)^n}{1+n/e^n} = \lim_{n\to\infty} (-1)^n \text{ \underline{DNE}}$$

(5) Converges by alt. series test

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n^{3/2}+n^{2/3}} = 0$$

(Since denominator increases, terms are decreasing in abs. value)

(6) Diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series ($p=1$ series)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n^{3/2}+n^{2/3}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}+n^{2/3}} = \lim_{n\to\infty} \frac{1}{1+n^{-5/6}} = 1$$

(7) Diverges by comparison test with p-series, $p = \frac{1}{2}$

For $n \geq 3$, $\ln(3) > 1$ so $\frac{\ln n}{\sqrt{n}} > \frac{1}{\sqrt{n}}$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges ($p = \frac{1}{2}$, p-series) $\Rightarrow \sum_{n=3}^{\infty} \frac{\ln n}{\sqrt{n}}$ diverges

(8) Converges since telescoping series

$S_n = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \cdots + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \frac{1}{\sqrt{n+1}}$

Thus, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{\sqrt{n+1}} = 1$

(9) Geometric Series

$a_1 = \frac{\sqrt{5}^4}{e^3}$ $\quad r = \frac{\sqrt{5}}{e} < 1$, $\quad$ Sum $= \frac{a_1}{1-r} = \frac{25/e^3}{1 - \frac{\sqrt{5}}{e}} = \frac{25}{e^3 - \sqrt{5}e^2}$

(10) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(5-3x)^{n+1}}{2(n+1)+7} \cdot \frac{2n+7}{(5-3x)^n}\right| = \lim_{n\to\infty} |5-3x| \frac{2n+7}{2n+9} = |5-3x| < 1$

$-1 < 5-3x < 1 \Rightarrow -6 < -3x < -4 \Rightarrow 2 > x > \frac{4}{3}$ ie. $\underline{\frac{4}{3} < x < 2}$

Check endpts: $5-3x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{2n+7}$ Diverges by limit comparison with harmonic series

$5-3x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+7}$ Converges by alt. series test

Interval of convergence is $\underline{\underline{\left(\frac{4}{3}, 2\right]}}$

(11) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-4)^{n+1}}{(n+1)^2 5^{n+1}} \cdot \frac{n^2 5^n}{(x-4)^n}\right| = \lim_{n\to\infty} |x-4| \cdot \frac{1}{5} \cdot \frac{n^2}{(n+1)^2} = \frac{|x-4|}{5} < 1$

$|x-4| < 5 \Rightarrow -5 < x-4 < 5 \Rightarrow -1 < x < 9$

Check endpts: $x-4 = 5 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges (p-series, $p=2$)

$x-4 = -5 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ Converges by alt. series test

Interval of convergence is $[-1, 9]$

(12) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \Rightarrow \frac{1}{1+2x^3} = 1 - 2x^3 + (2x^3)^2 - (2x^3)^3 + \cdots$

$$\Rightarrow \frac{x^4}{1+2x^3} = x^4 - 2x^7 + 4x^{10} - 8x^{13} + \cdots = \sum_{n=0}^{\infty} (-1)^n 2^n x^{3n+4}$$

(valid for $|2x^3| < 1 \Rightarrow |x^3| < \frac{1}{2} \Rightarrow |x| < \frac{1}{\sqrt[3]{2}}$)

(13) $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + -$

$$\sin\left(\frac{1}{x^2}\right) = x^{-2} - \frac{x^{-6}}{3!} + \frac{x^{-10}}{5!} - \frac{x^{-14}}{7!} + \cdots$$

$$\int \sin\left(\frac{1}{x^2}\right) dx = \int \Big(\quad \downarrow \quad \Big) dx = -x^{-1} - \left(-\frac{1}{5}\right)\frac{x^{-5}}{3!} + \cdots$$

$$= (-1)x^{-1} - \left(-\frac{1}{5}\right)\frac{x^{-5}}{3!} + \left(-\frac{1}{9}\right)\frac{x^{-9}}{5!} - \left(-\frac{1}{13}\right)\frac{x^{-13}}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{-4n-2}}{(2n+1)!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{(-4n-1)} x^{-4n-1}$$

(14)

n	$f^{(n)}(x)$	$f^{(n)}(3)$	
0	$x^{1/2}$	$\sqrt{3}$	$= \sqrt{3}$
1	$\frac{1}{2}x^{-1/2}$	$1/2\sqrt{3}$	$= \frac{1}{2\sqrt{3}}$
2	$-\frac{1}{2^2}x^{-3/2}$	$-\frac{1}{2^2}\cdot 3^{-3/2}$	$= \frac{-1}{12\sqrt{3}}$
3	$+\frac{3}{2^3}\cdot x^{-5/2}$	$\frac{3}{2^3}\cdot 3^{-5/2}$	$= \frac{1}{24\sqrt{3}}$
4		$-\frac{3\cdot 5}{2^4}\cdot 3^{-7/2}$	

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sqrt{3} + \frac{1}{2\sqrt{3}}(x-3) - \frac{1}{24\sqrt{3}}(x-3)^2 + \frac{1}{144\sqrt{3}}(x-3)^3 - \cdots$$

$$= \sqrt{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \cdot \frac{(2n-3)!!}{2^n} \cdot 3^{-(2n-1)/2} (x-3)^n$$