26

Calculus I (Math 231) Final Exam

Date: December 17, 2007 Professor Ilya Kofman

NAME: Key

Problem 1. Evaluate the following limits:

5 (a) $\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4}$ $= \lim_{x\to 4} \frac{x(x-4)}{(x+1)(x-4)} = \frac{4}{4+1} = \frac{4}{5}$

5 (b) $\lim_{x\to 0} \frac{8x}{\sin 2x}$ $= \lim_{x\to 0} 4\cdot\frac{2x}{\sin 2x} = 4$

5 (c) $\lim_{x\to -\infty} \frac{-5x^3+1}{17x^3+7x-11}$ $= \frac{-5}{17}$

Problem 2. Compute the first derivative for each of these functions:

5 (a) $f(x) = \frac{e^{3x}}{x^2+5}$ $f' = \frac{(x^2+5)(3e^{3x}) - (e^{3x})(2x)}{(x^2+5)^2}$

5 (b) $g(x) = \ln(6x)\sqrt{x^3+7x}$ $g' = (\ln 6x)\left(\frac{1}{2}(x^3+7x)^{-1/2}(3x^2+7)\right) + (x^3+7x)^{1/2}\left(\frac{6}{6x}\right)$

Problem 3. Evaluate

5 (a) $\int\left(\frac{7}{x^4}+3\sqrt{x}+e^{2x}\right)dx$ $= -\frac{7}{3}x^{-3} + 2x^{3/2} + \frac{1}{2}e^{2x} + C$

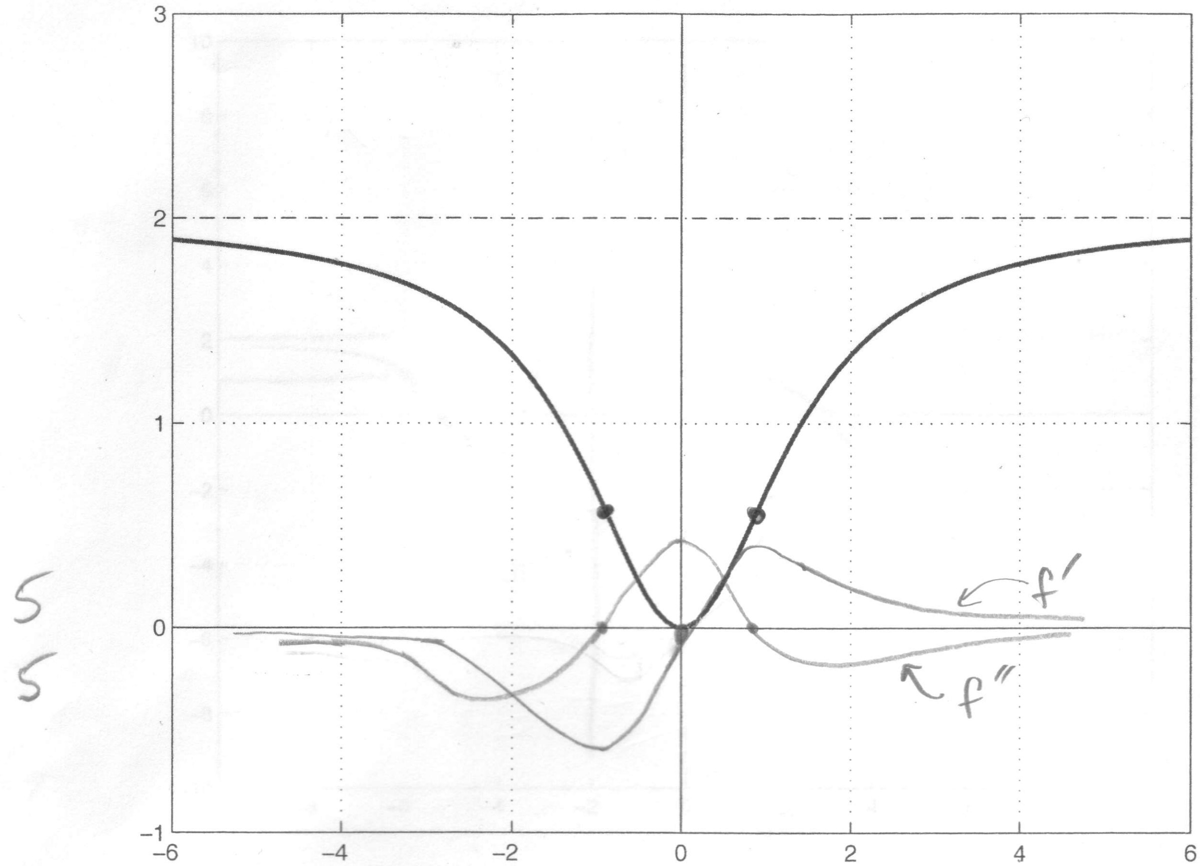
5 (b) $\int_1^3\left(6x^2+\frac{4}{x}+5\right)dx$ $= \left[2x^3 + 4\ln x + 5x\right|_1^3$

$= (2\cdot 3^3 + 4\ln 3 + 15) - (2 + 4\cdot 0 + 5)$

$= 62 + 4\ln 3 \approx 66.394\ldots$

35 + 4

Problem 4. Let $f(x)$ be the function defined by the following graph,



10 (a) $f'(x) < 0$ for which x? X<0 2

$f'(x) > 0$ for which x? X>0. 2

(b) $\lim_{x \to \infty} f(x) =$ 2 and $\lim_{x \to -\infty} f(x) =$ 2. 4

(c) Sketch a graph of $f'(x)$ on the figure.

(d) Label the approximate locations of all points of inflection of $f(x)$. 2

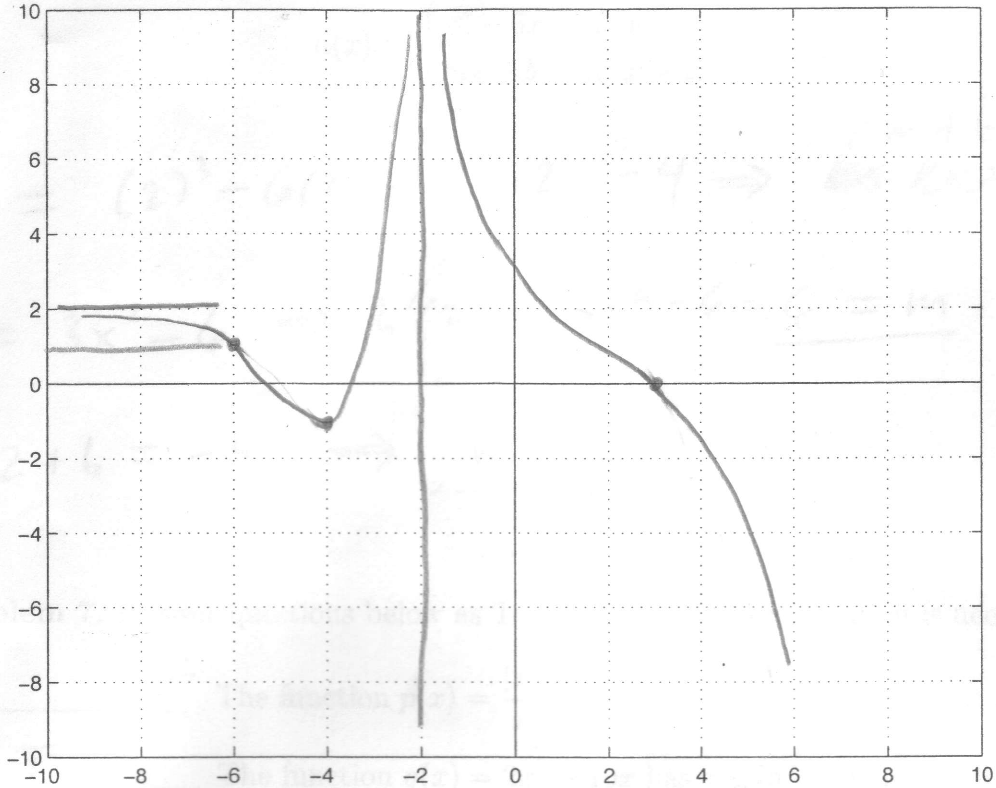
(e) Sketch a graph of $f''(x)$ on the figure.

- Make sure your sketches are clearly labeled above!

4 **BONUS:** $\lim_{x \to \infty} f'(x) =$ 0 and $\lim_{x \to \infty} f''(x) =$ 0.

20 + 4

Problem 5. Sketch the graph of a differentiable function $f(x)$ with all of the properties below.



- The domain of f is $(-\infty, -2) \cup (-2, \infty)$.
- $f(-6) = 1$, $f(-4) = -1$, and $f(3) = 0$.
- $\lim_{x \to -2} f(x) = \infty$.
- $\lim_{x \to -\infty} f(x) = 1$ and $\lim_{x \to \infty} f(x) = -\infty$.
- $f'(x) > 0$ for $-4 < x < -2$.
- $f'(x) < 0$ for $x < -4$ and for $x > -2$.
- $f''(x) > 0$ for $-6 < x < -2$ and for $-2 < x < 3$.
- $f''(x) < 0$ for $x < -6$ and for $x > 3$.

Label all horizontal and vertical asymptotes, local extrema, and inflection points.

10 **Problem 6.** Find the values of the constants m and b such that the following function is differentiable everywhere:

$$h(x) = \begin{cases} x^3 - 6x & \text{if } x \le 2 \\ mx + b & \text{if } x > 2 \end{cases}$$

$2m + b = -4$

$h(2) = (2)^3 - 6(2) = 8 - 12 = -4 \Rightarrow$

$h'(x) = 3x^2 - 6 \Rightarrow h'(2) = 3 \cdot 4 - 6 = 6 = m$

$6 \cdot 2 + b = -4 \Rightarrow b = -16$

20 **Problem 7.** Answer questions below as True or False. (No explanation is needed.)

(a) F The function $p(x) = \frac{|x|}{x}$ has a removable discontinuity at $x = 0$.

(b) T The function $q(x) = 2x^5 - 10x$ has a zero in the interval $(1, 2)$.

(c) T The function $r(x) = x^{1/3}$ has a vertical tangent line at the origin.

(d) F If $s'(2) = 0$ then $x = 2$ is a local max or min of $s(x)$.

(e) F A rational function can have at most two vertical asymptotes.

(f) F $\int_0^5 f(x)\,dx = -\int_{-5}^0 f(x)\,dx$ for all integrable $f(x)$.

(g) T $\frac{d}{dx}\left(\int_0^x t^{\sqrt{2}}\,dt\right) = x^{\sqrt{2}}$.

(h) T $\int_0^{2\pi} |\sin x|\,dx = 2\int_0^{\pi} \sin x\,dx$.

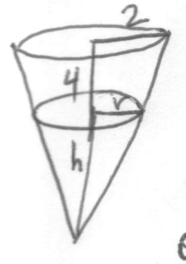
(i) T $\int_0^{\pi} \sin^2 x\,dx = \int_{\pi}^{2\pi} \sin^2 x\,dx$.

(j) T $\int_{-4}^{4} (x^5 + 7x)^{13}\,dx = 0$.

30

CHOOSE ANY TWO PROBLEMS ON THIS PAGE

Problem 8. A paper cup in the shape of a circular cone has radius $r = 2\,\text{cm}$ and height $h = 4\,\text{cm}$. Water is poured into the cup at a rate of $2\,\text{cm}^3/\text{sec}$. Find the rate at which the water level is rising when the water is 3 cm deep. (Hint: $V = \frac{1}{3}\pi r^2 h$)



$V = \frac{1}{3}\pi r^2 h$, $\frac{h}{4} = \frac{r}{2} \Rightarrow h = 2r$ $\qquad h = 3 \Rightarrow r = \frac{3}{2}$

OR $V = \frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3$ $\qquad \frac{dh}{dt} = 2\frac{dr}{dt}$

$\frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt} \Rightarrow 2 = 2\pi\left(\frac{3}{2}\right)^2 \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{\pi\left(\frac{3}{2}\right)^2} = \frac{4}{9\pi}$ $\qquad \frac{dh}{dt} = \frac{8}{9\pi}$

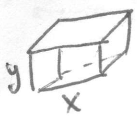
$V = \frac{\pi h^3}{12}$

$V' = \frac{\pi}{4} h^2 \frac{dh}{dt}$

$2 = \frac{\pi}{4} \cdot 9 \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{8}{9\pi}$

Problem 9. An open box with a total surface area of 300 in^2 and with a square base is to be made from sheet metal. Find the dimensions of the box that will maximize its volume.



$x^2 + 4xy = 300$

$V = x^2 y = x^2\left(\frac{300 - x^2}{4x}\right) = 75x - \frac{x^3}{4}$

$V' = 75 - \frac{3}{4}x^2 \overset{\text{set}}{=} 0$

$x^2 = \frac{4}{3}(75) = 100$

$x = 10, \quad y = 5 \qquad (V = 500)$

Problem 10. Consider the curve described by the relation $x^4 + y^4 = 32$. Find the equation of the tangent line to the curve at the point $(-2, 2)$.

$x^4 + y^4 = 32$

$4x^3 + 4y^3 \cdot \frac{dy}{dx} = 0$

$-8 + 8y' = 0 \Rightarrow y' = 1$

$y - 2 = 1(x + 2)$

$y = x + 4$

30