Mathematics V1205x, Calculus IIIS/IVA, Midterm #2 Solutions

1. Suppose $F = 2xyi + (x^2 + byz)j + y^2k$.

A. For what number b does $\text{div}(F) = \text{curl}(F) \cdot \mathbf{i}$?

Note that $\text{div}(F) = 2y + bz$ and $\text{curl}(F) = (2y - by)\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$. So $\text{div}(F) = \text{curl}(F) \cdot \mathbf{i}$ when $b = 0$.

B. For what (different) number b is F conservative? Since F is conservative when $\text{curl}(F) = 0$ we need $b = 2$.

C. For this second number b, find f so that $\nabla(f) = F$. One choice is $f = x^2y + y^2z$.

2. A. Rewrite the following integral as an iterated integral in the order $dydxdz$:

$$\int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} f(x, y, z)dzdydx.$$

The answer is

$$\int_{0}^{1} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} f(x, y, z)dydxdz.$$

B. Rewrite the integral in spherical coordinates.

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} zdzdydx.$$

The answer is

$$\int_{0}^{2\pi} \int_{0}^{\pi/4} \int_{0}^{2} \rho^3 \cos(\phi) \sin(\phi) d\rho d\phi d\theta.$$

C. Evaluate $\int\int_R e^{9x^2+4y^2} dA$ where R is the region bounded by $9x^2+4y^2 = 1$.

Let $x = u/3$ and $y = v/2$. Let S be the region in the (u, v)-plane bounded by $u^2 + v^2 = 1$. Then the Jacobian of this transformation is $J = 1/6$. So the answer is

$$\int\int_S \frac{1}{6} e^{u^2+v^2} dA = \int_{0}^{2\pi} \int_{0}^{1} \frac{1}{6} e^{r^2} r dr d\theta = e\pi/6.$$

3. A. Let $F = \langle 1 + \tan(x), x^2 + e^y \rangle$ be a force field. Let C be the boundary of the region enclosed by the parabola $x = y^2$ and the lines $x = 1$ and $y = 0$. Find the work done by F as a particle travels once around C in the counterclockwise direction.

The work is $W = \int_C F \cdot dr = \int_C (1 + \tan(x))dx + (x^2 + e^y)dy$. By Green's Theorem, $W = \int_0^1 \int_{y^2}^1 2x dx dy = \int_0^1 (1 - y^4) dy = 4/5$.

B. (5 points) Find a vector field F such that $\int_C F \cdot dr = 0$ whenever the endpoints of C both lie on the curve $y = x^3 + x + 1$.

If $f(x, y) = y - x^3 - x - 1$, then $F = \nabla(f) = (-3x^2 - 1)\mathbf{i} + \mathbf{j}$. Suppose C starts at $a = (x_1, y_1)$ and ends at $b = (x_2, y_2)$ where a and b lie on this curve and thus $f(a) = f(b) = 0$. By the Fundamental Theorem of line integrals, $\int_C F \cdot dr = f(b) - f(a) = 0 - 0$.

4. Consider the surface S in $\mathbb{R}^3$ given parametrically by $x = u\cos(v)$, $y = u\sin(v)$, and $z = u$. Let (u, v) range through the domain $D = \{(u, v) | 0 \leq u \leq 1, 0 \leq v \leq 2\pi\}$.

A. Graph S. Mark the grid curves $u = 1$ and $v = 0$.

See picture V on page 1091 for a picture of this cone. The curve $u = 1$ is a circle and $v = 1$ is a line.

B. Find the surface area (for $(u, v) \in D$).

Let $r(u, v) = \langle u\cos(v), u\sin(v), u \rangle$. Then $r_u = \langle \cos(v), \sin(v), 1 \rangle$ and $r_v = \langle -u\sin(v), u\cos(v), 0 \rangle$. Thus $r_u \times r_v = \langle -u\cos(v), -u\sin(v), u \rangle$. Thus $|r_u \times r_v| = \sqrt{2}u$. The surface area is $SA = \int_0^1 \int_0^{2\pi} \sqrt{2} u dv du$. Thus $SA = (2\pi)\sqrt{2}/2 = \sqrt{2}\pi$.

C. Let C be the grid curve $v = 0$, $0 \leq u \leq 1$. Find $\int_C 1 ds$. What physical quantity does this integral represent?

Here $r(u, 0) = \langle u, 0, u \rangle$ and $r'(u) = \langle 1, 0, 1 \rangle$ for $0 \leq u \leq 1$. Note $r'(u)$ has length $\sqrt{2}$. By page 1053, $\int_C 1 ds = \int_0^1 \sqrt{2} du = \sqrt{2}$ is the arclength.