

Mathematics V1205x, Calculus IIIS/IVA

Midterm Examination #1 Answers: February 14, 2001.

1. A ladybug is hovering at the point $(x, y, z) = (0, 1, 2)$. The temperature is given by the function $f(x, y, z) = ze^{xy}$.
 - A. The temperature increases most rapidly in the direction of $\nabla f = \langle yze^{xy}, xze^{xy}, e^{xy} \rangle$. So the direction is $\langle 2, 0, 1 \rangle$ with rate $|\nabla f| = \sqrt{5}$
 - B. What is the rate of increase in the direction $u = \langle 1, 0, 0 \rangle$?
The dot product of ∇f with the unit vector u is 2.
 - C. The ladybug can move in any linear combination of the directions $\langle 0, 1, 0 \rangle$ and $\langle \frac{1}{2}, 0, -1 \rangle$ without changing temperature.

2. Suppose $f(x, y) = 2x^2 + 3y^2 - 4x - 5$.

- A. Since $f_x = 4x - 4$ and $f_y = 6y$, the critical point for $f(x, y)$ is $(1, 0)$. Note that $f(1, 0) = -7$.
- B. Use the method of Lagrange multipliers to find the absolute maximum and minimum of $f(x, y)$ on the region $x^2 + y^2 \leq 16$.
If $g(x, y) = x^2 + y^2$, then $\nabla f = \lambda \nabla g$ implies that $4x - 4 = \lambda 2x$ and $6y = \lambda 2y$. So $\lambda = 3$, $x = -2$ and $y = \pm 2\sqrt{3}$ or $y = 0$ and $x = \pm 4$. The maximum is $f(-2, \pm 2\sqrt{3}) = 47$ and the minimum is $f(1, 0) = -7$.

3. Evaluate the integral by reversing the order of integration: $\int_0^1 \int_{3y}^3 e^{x^2} dx dy$.

This equals $\int_0^3 \int_0^{x/3} e^{x^2} dy dx = \int_0^3 \frac{x}{3} e^{x^2} dx = (e^9 - 1)/6$.

4. A. Suppose $f(x, y) = -16 + x^2 + 4y^2$. Find the region D for which $\int \int_D f(x, y) dA$ is a minimum. Why is it a minimum for this region? Hint: draw a graph.
 $D = \{(x, y) | x^2 + 4y^2 \leq 16\}$ because this is the region for which the function is negative.

- B. Suppose $f(x, y) = 5 - x$. Let $D = \{(x, y) | 0 \leq y \leq 3, 0 \leq x \leq 5\}$. Which of the following expressions is equal to $\int \int_D f(x, y) dA$?

iv)

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n \left(5 - \frac{5j}{n}\right) \frac{5}{n} \frac{3}{m}$$

This is because $\Delta x = \frac{5}{n}$, $\Delta y = \frac{3}{m}$ and the function evaluated at the upper (or lower) right hand of each little box is $5 - x = 5 - \frac{5j}{n}$.

5. Find the volume of icecream which lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

These intersect when $1 = 2(x^2 + y^2)$ or at $r = 1/\sqrt{2}$.

$$V = \int_0^{2\pi} \int_0^{1/\sqrt{2}} [\sqrt{1-r^2} - \sqrt{r^2}] r dr d\theta$$

With integration by substitution, this equals

$$V = 2\pi \left[\frac{-1}{3} (1-r^2)^{3/2} - \frac{1}{3} r^3 \right]_0^{1/\sqrt{2}}$$

$$V = \frac{\pi}{3} (2 - \sqrt{2}).$$

6. Find the surface area of the part of the plane $3x + 2y + z = 6$ which lies in the first octant.

$$A(S) = \int \int_D \sqrt{1 + (-3)^2 + (-2)^2} dA = \sqrt{14} \text{Area}(D) = 3\sqrt{14}$$

Bonus: For positive constants a, b , and c find the surface area of the part of the plane $ax + by + z = c$ which lies in the first octant.

$$A(S) = \sqrt{1 + a^2 + b^2} \text{Area}(D) = \sqrt{1 + a^2 + b^2} \frac{c^2}{2ab}$$

7. Let x be the number of bags of candy hearts that a student can eat in an hour. Let y be the number of miles that a student can run in an hour. The joint probability density function representing the continuous random variables x and y is

$$f(x, y) = \frac{1}{15} e^{-x/3} e^{-y/5} \text{ if } x \geq 0 \text{ and } y \geq 0 \text{ and } f(x, y) = 0 \text{ otherwise.}$$

- A. Find the probability that a student can run less than one mile in an hour.

$$P = \int_0^\infty \frac{1}{15} e^{-x/3} dx \int_0^1 e^{-y/5} dy = 1 - e^{-1/5}.$$

- B. Find the expected value (mean value) of x .

$$\mu_x = \int_0^\infty \frac{1}{15} e^{-y/5} dy \int_0^\infty x e^{-x/3} dx$$

With integration by parts,

$$\mu_x = \frac{1}{3} (-3x e^{-x/3} - 9e^{-x/3})_0^\infty = 3$$