CALCULUS IIIS/IVA: Section 2 – John Morgan

Second Midterm Exam: Nov. 13, 2001

Solutions

1. The following are the ones that are defined and zero:

1. curl grad(f)
2. div curl(F)
3. $\int_C$ grad(f) $\cdot\, dr$
4. $\int\int_S$ curl(F) $\cdot\, ndA$

2. Let D be the region of the plane inclusde by C and let C be oriented as the boundary of this region. By Green's theorem

$$\int_C -ydx + xdy = \int\int_D \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} dA = \int\int_D 2dA = 2\text{area}(D).$$

3.(a) The boundary C is the unit circle with the orientation being clockwise. Thus, a parameterization of C is given by $(\cos(t), -\sin(t), 0)$ for $0 \leq t \leq 2\pi$. Hence, we have

$$\int_C F \cdot dr = \int_0^{2\pi} (-3\sin(t), 0, 0) \cdot (-\sin(t), \cos(t), 0)dt = \int_0^{2\pi} 3\sin^2(t)dt = 3\pi.$$

(b) Let Σ be the lower unit hemisphere oriented by the downward unit normal. By Stokes' theorem

$$\int_C F \cdot dr = \int\int_\Sigma \text{curl}(F) \cdot ndS.$$

We have $\mathrm{curl}(F) = (0, 0, -3)$. We parameterize Σ by

$$r(\phi, \theta) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$$

for $0 \leq \theta \leq 2\pi; 0 \leq \phi \leq \pi$. Then

$$\frac{\partial r}{\partial \phi} = (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), -\sin(\phi))$$

$$\frac{\partial r}{\partial \theta} = (-\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0).$$

Hence, the corresponding normal vector is

$$\det\begin{pmatrix} i & j & k \\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi)) \\ -\sin(\theta)\sin(\phi) & \cos(\theta)\sin(\phi) & 0 \end{pmatrix}$$

$$= (\cos(\theta)\sin^2(\phi), -\sin(\theta)\sin^2(\phi), \cos(\theta)\sin(\phi)).$$

Notice that since $\pi/2 \leq \phi \leq \pi$ we have that $\sin(\phi) \geq 0$ and $\cos(phi) \leq 0$, so that the last coordinate is ≤ 0 and hence is a downward pointing normal. Now we have

$$\begin{aligned}
&\int\int_\Sigma F \cdot n dA = \\
&= \int_0^{2\pi}\int_{\pi/2}^{\pi} (0, 0, -3) \cdot (\cos(\theta)\sin^2(\phi), -\sin(\theta)\sin^2(\phi), \cos(\theta)\sin(\phi)) d\phi d\theta \\
&= \int_0^{2\pi}\int_{\pi/2}^{\pi} -3\cos(\theta)\sin(\phi) d\phi d\theta \\
&= (2\pi)\int_{\pi/2}^{\pi} -3\cos(\theta)\sin(\phi) d\phi = (2\pi)(\frac{3}{2}\cos^2(\phi)\Big|_{\pi/2}^{\pi} = 3\pi.
\end{aligned}$$

4. The endpoints of the curve C are $(1, 0, 2\pi)$ and $(1, 0, 0)$. The vector field $F = (ze^{xz}, 0, xe^{xz})$ is conservative, and in fact $F = \nabla f$ where $f(x, y, z) = e^{xz}$. Thus, by the fundamental theorem for line integrals, we have

$$\int_C F \cdot dr = \int_C \nabla(f) \cdot dr = f(1, 0, 2\pi) - f(1, 0, 0) = e^{2\pi} - 1.$$

5. Let D be the unit disk in the plane parallel to the xy-plane whose boundary is the boundary of the open bottle B. Let R be the region of

3-space bounded by the bottle and the planar cap D. The divergence of $F = (x + y^2, x(z^2 + 1), 1)$ is 1. Thus, by the divergence theorem we have

$$1 \cdot \mathrm{vol}(R) = 750 = \int\int\int_R \mathrm{div}(F) dvol = \int\int_B F \cdot ndA + \int\int_D F \cdot ndA.$$

Now let us compute $\int\int_D F \cdot ndA$. The outward unit normal to D is the vector $(0, 0, 1)$ and the vector field has third component 1. Hence $F \cdot n = 1$, and thus this integral is equal to the area of D which is π. Hence

$$\int\int_B F \cdot ndA = 750 - \pi.$$

Bonus Question:

The differential ω in question is the sum of two differentials and thus the line integral around the ellipse is the sum of two line integrals – the first being the line integral of

$$\omega_1 = \frac{xdy - ydx}{x^2 + y^2}$$

and the second being the line integral of

$$\omega_2 = \frac{(x-1)dy - ydx}{(x-1)^2 + y^2}.$$

The first differential has a singularity at $(0, 0)$ which is inside the ellipse. We set $P = \frac{-y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$ and compute

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$

Thus, by Green's theorem applied to the region inside the ellipse and outside a small circle centered at the origin, the line integral of ω_1 around the ellipse is equal to its integral around a small circle centered at $(0, 0)$, say the circle parameterized by

$$r(t) = (\epsilon \cos(t), \epsilon \sin(t)).$$

Clearly, then $dr = (-\epsilon \sin(t), \epsilon \cos(t))dt$ and $F(r(t)) = (-\epsilon \cos(t), \epsilon \sin(t))/\epsilon^2$. Thus, the line integral becomes:

$$\int_0^{2\pi} \cos^2(t) + \sin^2(t) dt = 2\pi.$$

The other differential ω_2 has a singularity at $(1, 0)$ which is also inside the ellipse. The change of variables replacing $x - 1$ by x changes this differential into the first and thus its cross partials also cancel out, and hence by Green's theorem the line integral of ω_2 around the ellipse is equal to the line integral of ω_2 around a small circle centered at $(1, 0)$. Computations similar to the above evaluate the line integral around a small circle centered at $(1, 0)$ to be 2π. It follows that the integral of ω_2 around C is also equal to 2π. Hence, the line integral of $\omega = \omega_1 + \omega_2$ around C is $2\pi + 2\pi = 4\pi$.