CALCULUS IIIS/IVA: Section 1 – Ilya Kofman

Solutions to First Midterm Exam: Oct. 4, 2001

1. The ice cream cone is given in spherical coordinates by $0 \leq \rho \leq 3$, $0 \leq \phi \leq \phi_0, \quad 0 \leq \theta \leq 2\pi$ where ϕ_0 is the angle in the first quadrant whose cotangent is 1/2. The density written in spherical coordinates is $K\rho$. Thus the spherical integral computing the total mass is given by

$$\int_0^{2\pi}\int_0^{\phi_0}\int_0^3 K\rho(\rho^2\sin(\phi))d\rho d\phi d\theta = \int_0^{2\pi}\int_0^{\phi_0}\int_0^3 K\rho^3\sin(\phi)d\rho d\phi d\theta.$$

The ρ-integration yields

$$\int_0^{2\pi}\int_0^{\phi_0}\frac{81}{4}K\sin(\phi)d\phi d\theta.$$

The ϕ-integration yields

$$\int_0^{2\pi}\frac{81}{4}K(-\cos(\phi)|_0^{\phi_0})d\theta.$$

Since $\cot(\phi_0) = 1/2$, it follows that $\cos(\phi_0) = 1/\sqrt{5}$ and hence we have

$$\int_0^{2\pi}\frac{81}{4}K(1-\frac{1}{\sqrt{5}})d\theta = \frac{81K\pi}{2}(1-\frac{1}{\sqrt{5}}).$$

2. Interchanging the order of integration yields

$$\int_0^1\int_{\sqrt[3]{y}}^1\sqrt{x^4+1}dxdy = \int_0^1\int_0^{x^3}\sqrt{x^4+1}dydx.$$

The y-integration then produces

$$\int_0^1 x^3\sqrt{x^4+1}dx.$$

We subsitute $u = x^4+1$ so that $du = 4x^3$. The integral becomes:

$$\int_1^2 u^{1/2}\frac{1}{4}du = \frac{1}{6}u^{3/2}|_1^2 = \frac{1}{6}(2^{3/2}-1).$$

3. We view the plane P as the graph of the function $z = (2 - ax - by)/c$. We have $\frac{\partial z}{\partial x} = -a/c$ and $\frac{\partial z}{\partial y} = -b/c$. The region of the plane in the first quadrant is a triangle with vertices $(\frac{2}{a}, 0, 0)$, $(0, \frac{2}{b}, 0)$, $(0, 0\frac{2}{c})$. Its projection D to the xy-plane is the triangle with vertices $(0,0)$, $(\frac{2}{a}, 0)$ and $(0, \frac{2}{b})$. The equation of the hypotenuse of this triangle is $y = (2 - ax)/b$ Thus, the surface area of the region in question is given by

$$\int\int_D \sqrt{1 + (-a/c)^2 + (-b/c)^2} dA = \int_0^{2/a} \int_0^{(2-ax)/b} \sqrt{1 + (a/c)^2 + (b/c)^2} dy dx.$$

The first integration yields

$$\int_0^{2/a} \sqrt{1 + (a/c)^2 + (b/c)^2}((2 - ax)/b) dx,$$

which in turn integrates to

$$\sqrt{1 + (a/c)^2 + (b/c)^2}(\frac{4}{ab} - \frac{4a}{2a^2 b}) = \sqrt{1 + (a/c)^2 + (b/c)^2}\frac{2}{ab}$$

$$= \sqrt{a^2 + b^2 + c^2}\frac{2}{abc}.$$

4. The volume is given by

$$\int_D (x + 1) dA$$

where D is the projection of the cylinder onto the xy-plane. This projection is given by $D = \{(x, y) | x^2 + y^2 \leq 2y\}$. It is easiest to do this integral in polar coordinates. The equation of the boundary of D in polar coordinates in $r^2 = 2r\text{sin}(\theta)$ or $r = 2\text{sin}(\theta)$. The region D is given $0 \leq r \leq 2\text{sin}(\theta)$, $0 \leq \theta \leq \pi$. The integral above written in polar coordinates is

$$\int_0^{\pi} \int_0^{2\text{sin}(\theta)} (r\text{cos}(\theta) + 1) \cdot r dr d\theta = \int_0^{\pi} \int_0^{2\text{cos}(\theta)} (r^2\text{cos}(\theta) + r) dr d\theta.$$

The first integration yields

$$\int_0^{\pi} (\frac{8}{3}\text{sin}^3(\theta)\text{cos}(\theta) + 2\text{sin}^2(\theta)) d\theta.$$

The first term is integrated using $u = \sin(\theta)$ and hence $du = \cos(\theta)$. The second term is integrated either using the double angle formulas or symmetry. The θ-integration then yields:

$$\frac{2}{3}\sin^4(\theta)|_0^\pi + 2 \cdot \frac{1}{2}\pi = 0 + \pi = \pi.$$

Note that this problem can be greatly simplified by shifting the cylinder along the y-axis to be centered at the origin. The plane $z = x + 1$ is not affected by this shift $y \mapsto (y-1)$, so the volume of the resulting solid region remains the same. This approach is fine, but we required justification.

5. The change of variables is to set $u = x+y$ and $v = x-y$. Solving for x, y as functions of u, v gives $x = (u+v)/2$ and $y = (u-v)/2$. The region in the uv-plane that maps to the given region in the xy-plane is $-2 \le u \le 2,\ 0 \le v \le 1$. The function $x^2 - y^2 = uv$. The Jacobian determinant of the transformation from u, v-space to xy-space is

$$|\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}| = |\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix}| = \frac{1}{2}.$$

Thus, by the change of variables formula we have

$$\int\int_Q (x^2 - y^2) dx dy = \int_{-2}^{2} \int_0^1 \frac{1}{2} uv dv du.$$

The v-integration yields

$$\int_{-2}^{2} \frac{u}{4} du = \frac{u^2}{8}|_{-2}^{2} = 0.$$

5. Alternative solution: Let T be the linear transformation $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $T = \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{pmatrix}$, so the Jacobian is $|\begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{pmatrix}| = 1$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{u}{2} + v \\ -\frac{u}{2} + v \end{pmatrix}$$

Now, $x^2 - y^2 = (x+y)(x-y) = (2v)(u) = 2uv$.

$$\int\int_Q (x^2 - y^2) dx dy = \int_{-1}^{1}\int_{0}^{1} 2uv|1|\, du\, dv = \int_{-1}^{1} 2v\, dv = 0$$

Bonus Question:

The solid region given by the integral is a tetrahedron with only one vertex $(0,1,0)$ in the xy-plane. Its shadow in the xy-plane is the triangle $x + y \geq 1,\ x \leq 1,\ y \leq 1$. The slanted "floor" of the tetrahedron is the triangle in the plane $z = x$; its "roof" is the triangle in the plane $z = 1$; and its "back wall" is the triangle in the plane $x + y = 1$. Therefore, the equivalent integral is

$$\int_0^1 \int_{1-z}^{1} \int_{1-y}^{z} f(x,y,z)\, dx\, dy\, dz$$