Calculus IIIS/IVA – Final Exam **Name:**____________________________

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WRITE YOUR NAME AND MY NAME ON EACH EXAM BOOKLET. Show all work. You can earn partial credit only if you justify your steps. No calculators are permitted. Start each problem on a new page.

1. **(10 points)** Solve the differential equation

$$y' + \frac{3y}{x} = 4.$$

2. **(10 points)** Let $x > 0$. Solve the differential equation

$$y' = \sqrt{xy}; \quad y(1) = 1.$$

3. **(10 points)** At what interest rate (expressed as a decimal) should money be invested if it is to double after 10 years, compounded continuously:

   (a) $\log_2(10)$ (b) $\ln(2)/10$ (c) $\sqrt[10]{2}-1$ (d) $\ln(10)/2$ (e) $\ln(\sqrt[10]{2}+1)$.

4. **(10 points)** Find the three complex numbers which are cube roots of -1.

5. **(10 points)** Use the Cauchy-Riemann equations to show that $f(z) = |z|^2$ is <u>not</u> holomorphic.

6. **(10 points)** Let C be the unit circle. Which of the following is equal to

$$\int_C \sin^2(z)\, dz$$

   (a) $2\pi i$ (b) π^2 (c) $\pi^2/3$ (d) $\sin^3(2\pi i)/3$ (e) 0.

7. **(10 points)** Let C be the circle of radius 2π, centered at the origin. Evaluate

$$\int_C \frac{\sin(z)}{z - (\pi/3)}\, dz$$

8. **(10 points)** In the region in the plane where it is defined (i.e., $\cos y \neq 0$) prove or disprove that $F(x, y) = (\tan y,\ x \sec^2 y)$ is conservative.

9. **(10 points)** Rewrite this integral by changing the order of integration to $dz\ dx\ dy$:

$$\int_0^1 \int_0^x \int_{x^2+y^2}^1 f(x,y,z)\ dz\ dy\ dx.$$

10. **(10 points)** Suppose that F is a vector field in 3-space everywhere perpendicular to a surface S with boundary C. Show that

$$\int\int_S (\nabla \times F) \cdot dS = 0$$

11. **(10 points)** Suppose that $\mathrm{div}(F) > 0$ inside the unit ball, $x^2 + y^2 + z^2 \leq 1$. Show that F cannot be everywhere tangent to the surface of the sphere.

LONGER QUESTIONS

12. **(15 points)** Let R be the square with vertices $(0,2), (1,1), (2,2), (1,3)$. Use the change of variables $u = x - y$ and $v = x + y$ to evaluate

$$\int\int_R (x-y)/(x+y)\ dA$$

13. **(20 points)** An open bottle B lies on the xy-plane (it fell since the last midterm). Its volume is 750 ml. Its lip (or boundary) is the circle $\{x^2 + (z-1)^2; \quad y = 10\}$. Let $F(x,y,z) = (x + y^2,\ y,\ x^2 + 1)$. Compute

$$\int\int_B F \cdot n\ dS$$

14. **(20 points)** Let $\rho(x,y,z) = z^2$ be the density of the cylinder $x^2 + y^2 \leq 1$ inside the sphere $x^2 + y^2 + z^2 \leq 4$ (including the curved caps of the cylinder). Compute the total mass.

15. **(15 points)** Let S be the surface parameterized by $r(u,v) = (u\cos v,\ u\sin v,\ u)$; $0 \leq u \leq 1; \quad 0 \leq v \leq 2\pi$. Compute

$$\int\int_S z(x^2 + y^2)\ dS$$

16. **(20 points)** **(a.)** Compute the following surface integral directly and **(b.)** verify the answer by applying one of our theorems. E is the solid cylinder $x^2 + y^2 \leq 1$; $0 \leq z \leq 1$. Let $F(x,y,z) = (x, y, -z)$. (Note: ∂E consists of the cylindrical side as well as the flat top and bottom.)

$$\int\int_{\partial E} F \cdot dS$$

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