Aspects Of Utility Maximization With Habit Formation: Dynamic Programming And Stochastic PDE's

Nikolaos Egglezos

Supervised by
Professor Ioannis Karatzas

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ABSTRACT

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This paper studies in detail the habit-forming preference problem of maximizing total expected utility from consumption net of the *standard of living*, a weighted-average of past consumption. We describe the effective state space of the corresponding optimal wealth and standard of living processes, while the associated value function is identified as a generalized utility function. In the case of deterministic coefficients, we exploit the interplay between dynamic programming and Feynman-Kac results, to obtain equivalent optimality conditions for the value function and its convex dual in terms of appropriate partial differential equations (PDE's) of parabolic type. The optimal portfolio/consumption pair is provided in feedback form as well.

In a more general context with random coefficients, this interrelation is established via the theory of random fields and stochastic PDE's. In fact, the resulting value random field of the optimization problem satisfies a non-linear, backward stochastic PDE of parabolic type, widely referred to as the stochastic Hamilton-Jacobi-Bellman equation. In addition, the dual value random field is characterized in terms of a linear, backward parabolic stochastic PDE. Employing the generalized Itô-Kunita-Wentzell formula, we present adapted versions of stochastic Feynman-Kac formulae, which lead to the formulation of stochastic feedback forms for the optimal portfolio and consumption choices.

Contents

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1 Introduction And Synopsis

An important question in financial mathematics is to explain the effect of past consumption patterns on current and future economic decisions. A useful tool in this effort has been the concept of habit formation. Namely, an individual who consumes portions of his wealth over time is expected to develop habits which will have a decisive impact on his subsequent consumption behavior. The nature and magnitude of this impact would presumably depend on his consumption preferences. Employed in a wide variety of economic applications (e.g. Hicks (1965), Pollak (1970), Ryder and Heal (1973)), habit formation was in turn considered by several authors in the classical utility optimization problem (e.g. Sundaresan (1989), Constantinides (1991), Detemple and Zapatero (1991, 1992), Heaton (1993), Chapman (1998), Schroder and Skiadas (2002)). In particular, fairly general utility preferences were introduced to accommodate a possible dependence not only on the investment and consumption decisions of a single agent, but also on his *standard of living*; i.e., an index-process that aggregates past consumption, reflecting the presence of habit persistence in the market.

The present thesis returns to the stochastic control problem described in Detemple and Zapatero (1992) and explores in detail particular aspects of portfolio/consumption optimization under habit formation in complete markets. We adopt non-separable von Neumann-Morgestern preferences over a given time-horizon $[0,T]$, and maximize total expected utility $E\int_0^T u(t,c(t)-z(t;c))dt$ from consumption $c(\cdot)$ in excess of standard of living $z(\cdot;c)$. Here the habit-index is defined as an average of past consumption, given by $z(t;c) \triangleq z\ e^{-\int_0^t \alpha(v)dv} + \int_0^t \delta(s)e^{-\int_s^t \alpha(v)dv}c(s)ds$, with $z \geq 0$ and nonnegative stochastic coefficients $\alpha(\cdot)$, $\delta(\cdot)$. It is now clear that an increase in the consumption strategy leads to a higher instantaneous utility, but also induces higher levels of standard of living, and eventually results in depressed future utilities. Moreover, by assuming infinite marginal utility at zero, i.e. $u'(t,0^+)=\infty$, we force consumption never to fall below the contemporaneous level of standard of living, thus triggering the development of "addictive" consumption patterns. As a lower bound of consumption, standard of living plays obviously the role of the addiction-regulator factor. Hence, our model is comparable to a real-life situation in which a consumer, who aims to maximize his total expected utility, displays addiction by being perennially sustained in excess of historical averages; in other words, he is constantly "forced" to consume more than he used to in the past. At $t=0$ the assumption $u'(t, 0^+) = \infty$ postulates the condition $x > wz$, specifying the half-plane $\mathcal{D}$ of Assumption 2.2 as the *domain of acceptability* for the initial wealth x and initial standard of living z. The quantity w stands for the cost per unit of standard of living of the *subsistence consumption*, the consumption policy that matches the standard of living at all times.

Existence of an optimal portfolio/consumption pair is proved in Detemple and Zapatero (1992) by establishing a recursive linear stochastic equation for the properly normalized marginal utility. In order to set up the mathematical background needed for further analysis, we present a brief formulation of their solution, based on the methodology of Detemple and Karatzas (2003) for the case of non-addictive habits. Our contribution starts by characterizing the *effective state space* of the corresponding optimal wealth $X_0(\cdot)$ and standard of living $z_0(\cdot)$ processes as the *random wedge* $\mathcal{D}_t$ (cf. (2.61)) determined by the stochastic evolution $\mathcal{W}(t)$ of w as a process. This result reveals the stochastic evolution of the imposed condition $x > wz$ over time, in the sense $X_0(t) > \mathcal{W}(t)z_0(t)$ for all $t \in [0, T)$. As a consequence, we are motivated to investigate in the sequel the *dynamic* aspects of our stochastic control problem. Straightforward computations yield a second representation (cf. (2.55)) for the "marginal" subsistence consumption cost w per unit of standard of living, in terms of the "normalized marginal utility" $\Gamma(\cdot)$ of (2.43), also referred to as the "adjusted" state-price density process. This second representation (2.55) provides a new mechanism for pricing economic features in our market, according to which $\Gamma(\cdot)$ is used as an alternative state-price density process; the role of the corresponding discount rate is played by $\alpha(\cdot)$. Such a mechanism has been identified in Schroder and Skiadas (2002) as an isomorphism between financial markets incorporating linear habit formation and markets without habit formation. The term "adjusted" in the terminology of $\Gamma(\cdot)$ comes from the fact that this process is the marginal utility of consumption (state-price density) augmented by the expected incremental impact on future utilities (cf. (2.37)). Furthermore, we define the value function V of the optimization problem as a mapping that depends on both x and z. Considering the latter as a pair of variables running on $\mathcal{D}$, we classify V in a broad family of utility functions; in fact, $V(\cdot, z)$ and the utility function $u(t, \cdot)$ exhibit similar analytic properties. This is carried out through the *convex dual* of the value function, defined in (2.67), in conjunction with differential techniques developed in Rockafellar (1970).

Duality methods in stochastic control were introduced in Bismut (1973) and elaborated further in Xu (1990), Karatzas and Shreve (1998). Detemple and Zapatero (1992) employ martingale methods (Cox and Huang (1989), Karatzas (1989), Karatzas, Lehockzy and Shreve (1987), and Pliska (1986)) to derive a closed-form solution for the optimal consumption policy, denoted by $c_0(\cdot)$. They also provide insights about the structure of the optimal portfolio investment $\pi_0(\cdot)$, that finances the policy $c_0(\cdot)$, via an application of the Clark (1970) formula due to Ocone and Karatzas (1991).

In order to describe quantitatively the dependence of the agent's optimal investment $\pi_0(\cdot)$ on his wealth $X_0(\cdot)$ and standard of living $z_0(\cdot)$, Detemple and Zapatero (1992) restrict the utility function to have either the logarithmic $u(t,x) = \log x$ or the power $u(t,x) = x^p/p$ form for a model with nonrandom coefficients. To amend this limitation, we pursue formulae for the optimal policies where now u can be any arbitrary utility function, by specializing our analysis to the case of deterministic coefficients. The ensuing Markovian setting and Feynman-Kac results permit us to express these policies in "feedback form" on the current levels of wealth and standard of living. Hence, the amount of available capital, at each time instant, together with the respective standard of living index, constitute a *sufficient statistic* for an economic agent who desires to invest and consume optimally in the financial market. Driven by ideas of dynamic programming, we characterize the value function V in terms of a non-linear, second-order parabolic partial differential equation, widely known as *Hamilton-Jacobi-Bellman equation*. This equation is derived from two *linear* Cauchy problems, which admit *unique* solutions subject to a certain growth condition. We establish a *linear*, second-order parabolic partial differential equation whose *unique* solution, subject to the same growth condition, is the convex dual $\widetilde{V}$ of V; compare with Xu (1990) as well, for "habit-free" markets. Thus, we obtain an alternative computational method for the value function of the maximization problem, since by solving the last equation for $\widetilde{V}$ and by inverting the dual transformation, V follows readily.

The use of dynamic programming techniques on stochastic control problems was originated by Merton (1969, 1971), in order to produce closed-form solutions in the special case of constant coefficients for models without habit formation. The infinite-horizon case was generalized by Karatzas, Lehoczky, Sethi and Shreve (1986). Karatzas, Lehoczky and Shreve (1987) coupled martingale with convexity methods to allow random, adapted model coefficients for general preferences; nonetheless, they reinstated the Markovian framework with constant coefficients to obtain the optimal portfolio in closed-form. A study on the case of deterministic coefficients in markets without habits can be found in Karatzas and Shreve (1998).

The work cited thus far indicates that the Hamilton-Jacobi-Bellman equation is inadequate for the analysis of a non-Markovian model. On the other hand, the dynamic evolution of domain $\mathcal{D}$, represented by the stochastically developing half-planes $\mathcal{D}_t$, hints that the principles of dynamic programming should be applicable in more general frameworks as well. Indeed, Peng (1992) considered a stochastic control problem with stochastic coefficients, and made use of Bellman's optimality principle to formulate an associated *stochastic* Hamilton-Jacobi-Bellman equation. The discussion in that paper was formal, due to insufficient regularity of the value function. This thesis culminates with an explicit application of Peng's idea to the utility maximization problem.

"Pathwise" stochastic control problems were recently studied by Lions and Souganidis (1998a, 1998b) who proposed a new notion of *stochastic* viscosity solutions for the associated fully non-linear *stochastic* Hamilton-Jacobi-Bellman equations. In two subsequent papers, Buckdahn and Ma (2001 Parts I, II) employ a Doss-Sussmann-type transformation to extend this notion in a "point-wise" manner, and obtain accordingly existence and uniqueness results for similar stochastic partial differential equations. A problem of "pathwise" stochastic optimization, that emerges from mathematical finance and concerns the optimality dependance on the paths of an exogenous noise, is considered in Buckdahn and Ma (2006).

Since stock prices and the money-market price are not Markovian processes anymore, we are now required to work with conditional expectations, which take into account the market history up to the present, and thereby lead to the consideration of *random fields*. In this context, an important role is played by certain linear, *backward* parabolic *stochastic* partial differential equations which characterize the resulting random fields as their *unique adapted* solutions; in other words, *adapted* versions of *stochastic* Feynman-Kac formulas are established. Results concerning the existence, uniqueness and regularity of the adapted solutions to stochastic partial differential equations of this sort were obtained in Ma and Yong (1997, 1999). Kunita (1990) contains a systematic study of semimartingales with spatial parameters, including the derivation of the generalized Itô-Kunita-Wentzell formula that is put to great use throughout our analysis.

By analogy with the Markovian case, we establish *stochastic* "feedback formulae" for the optimal portfolio/consumption decisions, in terms of the current levels of wealth and standard of living. This pair does not constitute a sufficient statistic for the maximization problem, any longer, for the "feedback formulae" convert from deterministic functions to random fields. Under reasonable assumptions on the utility preferences, the adapted *value field* of the stochastic control problem solves, in the classical sense, a non-linear, backward *stochastic* partial differential equation of parabolic type. To wit, the value field possesses sufficient smoothness, such that all the spatial derivatives involved in the equation exist almost surely. This equation is the stochastic Hamilton-Jacobi-Bellman equation one would expect, according to the theory of dynamic programming (Peng (1992)). Actually, apart from the classical linear/quadratic case discussed in Peng (1992), this work is, to the best of our knowledge, the first to illustrate explicitly and directly the role of backward stochastic partial differential equations in the study of stochastic control problems in any generality. We conclude by deducing a *necessary and sufficient* condition for the *dual* value random field as the unique adapted solution of a linear, backward parabolic stochastic partial differential equation.

Synopsis: An outline of the thesis is laid out as follows. In Chapter 2 we introduce the market model, and go briefly over the optimal portfolio-consumption solution of the stochastic control problem. Several aspects of this problem are discussed, concerning the state space of the corresponding optimal wealth and standard of living vector process, the classification of the value function in a set of generalized utility functions, and the dual of the value function. Chapter 3 specializes the preceding work to the case of deterministic model coefficients. Following the dynamic programming approach, we establish the optimal portfolio and consumption policies in "feedback form" on the current level of the associated optimal wealth and standard of living, and show that the value function satisfies a Hamilton-Jacobi-Bellman equation. An equivalent characterization for the dual value function and several examples are also provided. In Chapter 4 we investigate the dynamic programming reasoning in the model with random coefficients. Now the value function and the "feedback formulae" for the optimal portfolio/consumption pair are represented by random fields, and our analysis leads to the development of stochastic partial differential equations. Consequently, the value random field solves a stochastic Hamilton-Jacobi-Bellman equation, and its dual field is the unique solution of a linear, backward stochastic partial differential equation. The example of logarithmic utility is discussed in this case as well. Conclusions and several open problems related with more general utility preferences or incomplete markets are formulated in Chapter 5.

2 Optimal Portfolio-Consumption Policies With Habit Formation In Complete Markets

2.1 The Model

We adopt a model for the financial market $\mathcal{M}_0$ which consists of one riskless asset (money market) with price $S_0(t)$ given by:

$$dS_0(t) = r(t)S_0(t)dt, \qquad S_0(0) = 1, \tag{2.1}$$

and m risky securities (stocks) with prices per share $\{S_i(t)\}_{1\leq i\leq m}$, satisfying the equations

$$dS_i(t) = S_i(t)\left[b_i(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t)\right], \quad i = 1, ..., m. \tag{2.2}$$

Here $W(\cdot) = (W_1(\cdot), ..., W_d(\cdot))^*$ is a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathbb{F} = \{\mathcal{F}(t); 0 \leq t \leq T\}$ will denote the P-augmentation of the Brownian filtration $\mathcal{F}^W(t) \triangleq \sigma(W(s);\ s \in [0, t])$. We impose throughout that $d \geq m$, i.e., the number of sources of uncertainty in the model is at least as large as the number of stocks available for investment.

The interest rate $r(\cdot)$ as well as the instantaneous rate of return vector $b(\cdot) = (b_1(\cdot), ..., b_m(\cdot))^*$ and the volatility matrix $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1\leq i\leq m,\ 1\leq j\leq d}$ are taken to be $\mathbb{F}$-progressively measurable random processes and satisfy

$$\int_0^T \|b(t)\|dt < \infty, \qquad \int_0^T |r(t)|dt \leq \varrho \tag{2.3}$$

almost surely, for some given real constant $\varrho > 0$. It will be assumed in what follows that $\sigma(\cdot)$ is bounded and the matrix $\sigma(t)$ has full rank for every t . Under the latter assumption the matrix $\sigma(\cdot)\sigma^*(\cdot)$ is invertible, thus its inverse and the progressively measurable relative risk process

$$\vartheta(t) \triangleq \sigma^*(t)(\sigma(t)\sigma^*(t))^{-1}[b(t) - r(t)\mathbf{1}_m], \quad t \in [0, T] \tag{2.4}$$

are well defined; here we denote by $\mathbf{1}_k$ the k-dimensional vector whose every component is one. We make the additional assumption that $\vartheta(\cdot)$ satisfies the finite-energy condition

$$E \int_0^T \|\vartheta(t)\|^2 dt < \infty. \tag{2.5}$$

All processes encountered in this paper are defined on a finite time-horizon $[0, T]$ where T is the *terminal time* for our market.

Furthermore, the exponential local martingale process

$$Z(t) \triangleq \exp\left\{ -\int_0^t \vartheta^*(s) dW(s) - \frac{1}{2} \int_0^t \|\vartheta(s)\|^2 ds \right\}, \tag{2.6}$$

the discount process

$$\beta(t) \triangleq \exp\left\{ -\int_0^t r(s) ds \right\}, \tag{2.7}$$

their product, that is the so called state-price density process

$$H(t) \triangleq \beta(t) Z(t), \tag{2.8}$$

as well as the process

$$W_0(t) \triangleq W(t) + \int_0^t \vartheta(s) ds, \tag{2.9}$$

will be used quite often.

2.2 Portfolio and Consumption Processes

We envision an economic agent who starts with a given initial endowment $x > 0$, and whose actions cannot affect the market prices. At any time $t \in [0, T]$ the agent can decide both the proportion $\pi_i(t)$ of his wealth $X(t)$ to be invested in the ith stock ($1 \leq i \leq m$), and his consumption rate $c(t) \geq 0$. Of course, these decisions do not anticipate the future but must depend only on the current information $\mathcal{F}(t)$. The remaining amount $[1 - \sum_{i=1}^m \pi_i(t)] X(t)$ is invested in the money market. Here the investor is allowed both to sell stocks short, and to borrow money at the bond interest rate $r(\cdot)$; that is, the $\pi_i(\cdot)$ above are not restricted to take values only in $[0,1]$, and their sum may exceed 1.

The resulting *portfolio strategy* $\pi = (\pi_1, ..., \pi_m)^* : [0,T] \times \Omega \to \mathbb{R}^m$ and *consumption strategy* $c : [0,T] \times \Omega \to [0,\infty)$, are assumed to be $\mathbb{F}$-progressively measurable processes and to satisfy the integrability condition $\int_0^T \big(c(t) + \|\pi(t)\|^2\big) dt < \infty$, almost surely.

According to the model dynamics set forth in (2.1) and (2.2), the *wealth process* $X(\cdot) \equiv X^{x,\pi,c}(\cdot)$, corresponding to the portofolio/consumption pair (π, c) and initial capital $x \in (0,\infty)$, is the solution of the linear stochastic differential equation

$$\begin{aligned} dX(t) &= \sum_{i=1}^m \pi_i(t) X(t) \left\{ b_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t) dW_j(t) \right\} \\ &\quad + \left\{ 1 - \sum_{i=1}^m \pi_i(t) \right\} X(t) r(t) dt - c(t)dt \qquad (2.10) \\ &= [r(t)X(t) - c(t)]dt + X(t)\pi^*(t)\sigma(t)dW_0(t) \ , \end{aligned}$$

subject to the initial condition $X(0) = x > 0$. Equivalently, we have

$$\beta(t)X(t) + \int_0^t \beta(s)c(s)ds = x + \int_0^t \beta(s)X(s)\pi^*(s)\sigma(s)dW_0(s), \qquad (2.11)$$

and from Itô's lemma, applied to the product of Z and βX, we obtain

$$\begin{aligned} &H(t)X(t) + \int_0^t H(s)c(s)ds \\ &\qquad = x + \int_0^t H(s)X(s)[\sigma^*(s)\pi(s) - \vartheta(s)]^* dW(s) \ . \qquad (2.12) \end{aligned}$$

A portofolio/consumption pair process (π, c) is called *admissible for the initial capital* $x \in (0,\infty)$, if the agent's wealth remains nonnegative at all times, i.e., if

$$X(t) \geq 0, \qquad \text{for all } t \in [0,T], \qquad (2.13)$$

almost surely. We shall denote the family of admissible pairs (π, c) by $\mathcal{A}_0(x)$.

For any $(\pi, c) \in \mathcal{A}_0(x)$, the process

$$Y(t) = H(t)X(t) + \int_0^t H(s)c(s)ds$$

on the left-hand side of (2.12) is a continuous and nonnegative local martingale, thus a supermartingale. Consequently,

$$E\left(\int_0^T H(s)c(s)ds\right) \leq x \tag{2.14}$$

holds for every $(\pi, c) \in \mathcal{A}_0(x)$, since

$$x \geq E\left(H(T)X(T) + \int_0^T H(s)c(s)ds\right) \geq E\left(\int_0^T H(s)c(s)ds\right)$$

from the supermartingale property $Y(0) \geq E[Y(T)]$ and the condition (2.13) for $t = T$.

Let $\mathcal{B}_0(x)$ denote the set of consumption policies $c : [0, T] \times \Omega \to [0, \infty)$ which are progressively measurable and satisfy (2.14). Then we have just verified that $c(\cdot) \in \mathcal{B}_0(x)$, for all pairs $(\pi, c) \in \mathcal{A}_0(x)$. In a complete market, where the number of stocks available for trading matches exactly the dimension of the "driving" Brownian motion, the converse holds true as well, in the following sense.

Lemma 2.1. *Let the market model of (2.1), (2.2) be complete, namely $m = d$. Then, for every consumption process $c(\cdot) \in \mathcal{B}_0(x)$ there exists a portfolio process $\pi(\cdot)$ such that $(\pi, c) \in \mathcal{A}_0(x)$. The latter generate a wealth process $X(\cdot) \equiv X^{x,\pi,c}(\cdot)$ which is given by*

$$H(t)X(t) = x + E_t(D(t)) - E(D(0)), \quad t \in [0, T]$$

where $D(t) \triangleq \int_t^T H(s)c(s)ds$.

Here and in the sequel, $E_t[\,\cdot\,]$ denotes conditional expectation $E[\,\cdot\,|\mathcal{F}(t)]$ with respect to the probability measure P, given the σ-algebra $\mathcal{F}(t)$.

Proof: For a given $c(\cdot) \in \mathcal{B}_0(x)$, we define the continuous, positive process $X(\cdot)$ via

$$H(t)X(t) \triangleq x + E_t\left[\int_t^T H(s)c(s)ds\right] - E\left[\int_0^T H(s)c(s)ds\right], \quad 0 \le t \le T.$$

This process satisfies $X(0) = x,\ X(T) \ge 0$.

By the representation property of Brownian martingales as stochastic integrals (Karatzas and Shreve (1991), Theorem 3.4.15 and Problem 3.4.16), the process

$$\begin{aligned} M(t) &\triangleq H(t)X(t) + \int_0^t H(s)c(s)ds \\ &= x + E_t\left[\int_0^T H(s)c(s)ds\right] - E\left[\int_0^T H(s)c(s)ds\right] \end{aligned}$$

can be expressed as

$$M(t) = x + \int_0^t \psi^*(s)dW(s)$$

for a suitable $\mathbb{F}$-progressively measurable, $\mathbb{R}^d$-valued process $\psi(\cdot)$ that satisfies $\int_0^T \|\psi(s)\|^2 ds < \infty$, almost surely. Thus,

$$H(t)X(t) + \int_0^t H(s)c(s)ds = x + \int_0^t \psi^*(s)dW(s)$$

and comparing with (2.12) we see that the portfolio process $\pi(\cdot)$ which, together with $x > 0$ and $c(\cdot)$, generates $X(\cdot)$ as its wealth process $X(\cdot) \equiv X^{x,\pi,c}(\cdot)$, is given by

$$\pi(t) = \left(\sigma(t)\sigma^*(t)\right)^{-1}\sigma(t)\left[\frac{\psi(t)}{X(t)H(t)} + \vartheta(t)\right].$$

Clearly, the square integrability of $\pi(\cdot)$ follows from the square integrability of $\psi(\cdot)$, the continuity of $X(\cdot)$ and condition (2.5). $\Diamond$

This argument ensures that any consumption strategy satisfying the budget restriction (2.14) can be financed by a portfolio policy. For this reason, (2.14) can be interpreted as a "budget constraint".

2.3 Utility Functions

A *utility function* is a jointly continuous mapping $u : [0,T] \times (0,\infty) \to \mathbb{R}$ such that, for every $t \in [0,T]$, the function $u(t,\cdot)$ is strictly increasing, strictly concave, of class $C^1((0,\infty))$, and satisfies

$$u'(t,0^+) = \infty, \qquad u'(t,\infty) = 0, \tag{2.15}$$

where $u'(t,x) \triangleq \frac{\partial}{\partial x} u(t,x)$.

Due to these assumptions, the inverse $I(t,\cdot) : (0,\infty) \to (0,\infty)$ of the function $u'(t,\cdot)$ exists for every $t \in [0,T]$, and is continuous and strictly decreasing with

$$I(t,0^+) = \infty, \qquad I(t,\infty) = 0. \tag{2.16}$$

Furthermore, one can see that the stronger assertion

$$\lim_{x\to\infty} \max_{t\in[0,T]} u'(t,x) = 0 \tag{2.17}$$

holds. Indeed, fix an $\varepsilon > 0$ and for each $n \in \mathbb{N}$ consider the set $K_n(\varepsilon) \triangleq \{t \in [0,T];\ u'(t,n) \geq \varepsilon\}$. This leads to a nested sequence $\{K_n(\varepsilon)\}_{n\in\mathbb{N}}$ of compact sets whose intersection is empty. Therefore, there exists a positive integer n such that $K_n(\varepsilon) = \emptyset$.

Let us now introduce, for each $t \in [0,T]$, the Legendre-Fenchel transform $\tilde{u}(t,\cdot) : (0,\infty) \to \mathbb{R}$ of the convex function $-u(t,-x)$, namely

$$\tilde{u}(t,y) \triangleq \max_{x>0}[u(t,x) - xy] = u(t,I(t,y)) - yI(t,y), \qquad 0 < y < \infty. \tag{2.18}$$

The function $\tilde{u}(t,\cdot)$ is strictly decreasing, strictly convex, and satisfies

$$\frac{\partial}{\partial y}\tilde{u}(t,y) = -I(t,y), \qquad 0 < y < \infty, \tag{2.19}$$

$$u(t,x) = \min_{y>0}[\tilde{u}(t,y) + xy] = \tilde{u}(t,u'(t,x)) + xu'(t,x), \qquad 0 < x < \infty. \tag{2.20}$$

As consequences of (2.18) and (2.20), we have the following useful inequalities

$$u(t,I(t,y)) \geq u(t,x) + y[I(t,y) - x], \tag{2.21}$$

$$\tilde{u}(t, u'(t,x)) + x[u'(t,x) - y] \leq \tilde{u}(t,y) \tag{2.22}$$

for every $x > 0,\ y > 0$. It is also not hard to verify that

$$\tilde{u}(t,\infty) = u(t,0^+), \qquad \tilde{u}(t,0^+) = u(t,\infty). \tag{2.23}$$

We note here that $\tilde{u} : [0,T] \times (0,\infty) \to \mathbb{R}$ is jointly continuous as well.

2.4 The Maximization Problem

For a given utility function u and a given initial capital $x > 0$, we shall consider von Neumann-Morgenstern preferences with expected utility

$$J(z;\pi,c) \equiv J(z;c) \triangleq E\left[\int_0^T u(t, c(t) - z(t;c))dt\right], \tag{2.24}$$

corresponding to any given pair $(\pi, c) \in \mathcal{A}_0(x)$ and its associated index-process $z(\cdot) \equiv z(\cdot;c)$ defined in (2.26), (2.28) below. This process represents the "standard of living" of the decision-maker, an index that captures past consumption behavior and condition the current consumption felicity by developing "habits". Of course, in order to ensure that the above expectation exists and is finite, we shall take into account only consumption strategies $c(\cdot)$ that satisfy

$$c(t) - z(t;c) > 0, \quad \forall\ 0 \leq t \leq T, \tag{2.25}$$

almost surely. This additional budget specification insists that consumption must always exceed the standard of living, establishing incentives for a systematic built-up of habits over time and leading to "addiction patterns".

We shall stipulate that the standard of living follows the dynamics

$$\begin{aligned} dz(t) &= \big(\delta(t)c(t) - \alpha(t)z(t)\big)dt, \qquad t \in [0,T], \\ z(0) &= z, \end{aligned} \tag{2.26}$$

where $\alpha(\cdot)$ and $\delta(\cdot)$ are nonnegative, bounded and $\mathbb{F}$-adapted processes and $z \geq 0$ is a given real number. Thus, there exist constants $A > 0$ and $\Delta > 0$ such that

$$0 \leq \alpha(t) \leq A, \quad 0 \leq \delta(t) \leq \Delta, \qquad \forall\ t \in [0,T], \tag{2.27}$$

hold almost surely. Equivalently, (2.26) stipulates

$$z(t) \equiv z(t;c) = z\ e^{-\int_0^t \alpha(v)dv} + \int_0^t \delta(s)e^{-\int_s^t \alpha(v)dv}c(s)ds \tag{2.28}$$

and expresses $z(\cdot)$ as an exponentially-weighted average of past consumption.

In light of the constraint (2.25), we see that consumption $c(\cdot)$ must always exceed the "subsistence consumption" $\hat{c}(\cdot)$ for which $\hat{c}(\cdot) = z(\cdot;\hat{c})$, namely, that consumption pattern which barely meets the standard of living. From (2.26), this subsistence consumption satisfies

$$\begin{aligned} d\hat{c}(t) &= \big(\delta(t) - \alpha(t)\big)\hat{c}(t)dt, \qquad t \in [0,T], \\ \hat{c}(0) &= z, \end{aligned}$$

and therefore with $\hat{z}(\cdot) \equiv z(\cdot;\hat{c})$ we have

$$c(t) > \hat{c}(t) = \hat{z}(t) = z\ e^{\int_0^t (\delta(v)-\alpha(v))dv}, \quad \forall\ t \in [0,T].$$

Back into the budget constraint (2.14), this inequality gives

$$z\ E\left(\int_0^T e^{\int_0^s (\delta(v)-\alpha(v))dv} H(s)ds\right) \le x.$$

Therefore, keeping in mind the strict inequality of (2.25), we need to impose the following restriction on the initial capital x and the initial standard of living level z.

Assumption 2.2. $(x,z) \in \mathcal{D} \triangleq \Big\{(x',z') \in (0,\infty) \times [0,\infty);\ x' > wz'\Big\}$, *where*

$$w \triangleq E\left[\int_0^T e^{\int_0^t (\delta(v)-\alpha(v))dv} H(t)dt\right] \tag{2.29}$$

represents the "marginal" cost of subsistence consumption per unit of standard of living.

Definition 2.3. (Dynamic Optimization). The dynamic optimization problem is to maximize the expression of (2.24) over the class $\mathcal{A}_0'(x,z)$ of *admis-*

sible portfolio/consumption pairs $(\pi, c) \in \mathcal{A}_0(x)$ that satisfy (2.25) and

$$E\left[\int_0^T u^-(t, c(t) - z(t;c))dt\right] < \infty. \tag{2.30}$$

(Here and in the sequel, b^- denotes the negative part of the real number $b : b^- = \max\{-b, 0\}$.) The value function of this problem will be denoted by

$$V(x,z) \triangleq \sup_{(\pi,c)\in\mathcal{A}_0'(x,z)} J(z;\pi,c), \qquad (x,z) \in \mathcal{D}. \tag{2.31}$$

Definition 2.4. (Static Optimization). The static optimization problem is to maximize the expression (2.24) over the class $\mathcal{B}_0'(x,z)$ of *consumption processes* $c(\cdot) \in \mathcal{B}_0(x)$ that satisfy (2.25) and (2.30). The value function of this problem will be denoted by

$$U(x,z) \triangleq \sup_{c(\cdot)\in\mathcal{B}_0'(x,z)} J(z;c), \qquad (x,z) \in \mathcal{D}. \tag{2.32}$$

We obtain immediately from (2.14) that

$$V(x,z) \leq U(x,z), \quad \forall\ (x,z) \in \mathcal{D}.$$

In fact, equality prevails above: it suffices to solve only the static maximization problem, since for a *static* consumption optimizer process $c_0(\cdot) \in \mathcal{B}_0'(x,z)$ in (2.32) we can always construct, according to Lemma 2.1, a portfolio process $\pi_0(\cdot)$ such that $(\pi_0, c_0) \in \mathcal{A}_0'(x,z)$ satisfies

$$U(x,z) = J(z;c_0) = J(z;c_0,\pi_0) = V(x,z), \quad \forall\ (x,z) \in \mathcal{D}$$

and thus constitutes a *dynamic* portfolio/consumption maximizer pair process for (2.31).

We also note that the $\mathcal{B}_0'(x,z)$ is convex, thanks to the linearity of $c \mapsto z(t;c)$ and the concavity of $x \mapsto u(t,x)$.

2.5 Solution of the Optimization Problem in Complete Markets

The static optimization problem of Definition 2.4 is treated as a typical maximization problem with constraints (2.14) and (2.25) in the case $m = d$ of a complete market, and admits a solution derived in Detemple and Zapatero (1992). In this section, we shall follow their analysis, obtaining further results associated with the value function V and related features. More precisely, we shall identify the *effective state space* of the optimal wealth/standard of living vector process, generated by the optimal portfolio/consumption pair, as a *random wedge*, spanned by the temporal variable $t \in [0, T]$ and a family of suitable random half-planes (cf. Theorem 2.12). Theorem 2.17 below describes the relation of the value function V with a utility function as defined in Section 2.3, and begins the study of its dual value function $\widetilde{V}$. An alternative representation for the quantity w of (2.29) is provided as well.

We shall deal with maximization problems of this sort by the standard method of introducing the *Lagrange multiplier* $y \in (0, \infty)$ to enforce the static constraint (2.14). Hence for any such multiplier, we consider the auxiliary functional

$$\mathcal{R}(x, z, c; y) \triangleq E\left[\int_0^T u(t, c(t) - z(t; c))dt\right] + y\left[x - E\left(\int_0^T H(s)c(s)ds\right)\right]. \tag{2.33}$$

One would expect an additional Lagrange multiplier, related to the restriction (2.25). Nonetheless, this consideration turns out to be redundant, since condition (2.25) is satisfied by the optimal nonnegative consumption process $c_0(\cdot)$, thanks to the property of infinite marginal utility imposed in (2.15); cf. (2.46)-(2.50). In other words, the constraint (2.25) remains practically in the shadow, once Assumption 2.2 has been made.

For every consumption process $c(\cdot) \in \mathcal{B}_0'(x, z)$, namely, any progressively measurable process $c(\cdot) \geq 0$ that satisfies (2.14), (2.25) and (2.30), we have from (2.33) that $\mathcal{R}(x, z, c; y) \geq J(z; c)$ holds, and with equality if and only if the condition

$$E\left(\int_0^T H(s)c(s)ds\right) = x \tag{2.34}$$

is satisfied. As a direct consequence of these observations comes the following result.

Lemma 2.5. *Assume that for any given $y > 0$ as above, there exists a nonnegative progressively measurable process $c^y(\cdot)$ that satisfies the conditions (2.34), (2.25), (2.30) and maximizes the functional (2.33), namely*

$$\mathcal{R}(x, z, c^y; y) \geq \mathcal{R}(x, z, c; y), \qquad \forall\ c(\cdot) \in \mathcal{B}_0'(x, z). \tag{2.35}$$

Then

$$J(z; c^y) = \mathcal{R}(x, z, c^y; y) \geq \mathcal{R}(x, z, c; y) \geq J(z; c), \qquad \forall\ c(\cdot) \in \mathcal{B}_0'(x, z), \tag{2.36}$$

so this $c^y(\cdot)$ is optimal for the static problem.

Obviously, under our assumptions, the process $c^y(\cdot) \geq 0$ solves the static optimization problem, thus the dynamic one as well.

Optimality in the static problem: Analysis

In finding the solution of the auxiliary optimization problem (2.35), a prominent role will be played by the "adjusted" state-price density process

$$\Gamma(t) = H(t) + \delta(t) \cdot E_t \left(\int_t^T e^{\int_t^s (\delta(v) - \alpha(v))dv} H(s)ds \right), \qquad t \in [0, T] \tag{2.37}$$

introduced in Detemple and Zapatero (1992). The process $\Gamma(\cdot)$ is the state-price density process $H(\cdot)$ compensated by an additional term that reflects the effect of habits. We shall also make use of the following helpful notation

$$F^y(t) \triangleq e^{\int_0^t (\alpha(v) - \delta(v))dv} I(t, y\Gamma(t)), \qquad t \in [0, T] \tag{2.38}$$

for a suitable Lagrange multiplier $y = y_0 > 0$.

Ansatz 2.6. *Suppose, for some given $y > 0$, that there exists a process $c^y(\cdot) \in \mathcal{B}_0'(x, z)$ which solves the auxiliary problem (2.35), namely,*

$$\infty > \mathcal{R}(x, z, c^y; y) \geq \mathcal{R}(x, z, c; y), \quad \text{for all } \ c(\cdot) \in \mathcal{B}_0'(x, z).$$

Then we have

$$c^y(t) = e^{\int_0^t (\delta(v)-\alpha(v))dv} \left[F^y(t) + z + \int_0^t \delta(s) F^y(s) ds \right] \tag{2.39}$$

and

$$\begin{aligned} z^y(t) \equiv z(t; c^y) &= c^y(t) - e^{\int_0^t (\delta(v)-\alpha(v))dv} F^y(t) \\ &= e^{\int_0^t (\delta(v)-\alpha(v))dv} \left[z + \int_0^t \delta(s) F^y(s) ds \right], \end{aligned} \tag{2.40}$$

in the notation of (2.37), (2.38).

Discussion: The optimality of $c^y(\cdot)$, in conjunction with the convexity of $\mathcal{B}_0'(x,z)$, implies

$$\overline{\lim}_{\epsilon \downarrow 0} \frac{1}{\epsilon} \cdot \Big[\mathcal{R}\big(x, z, c^y + \epsilon(c - c^y); y\big) - \mathcal{R}(x, z, c^y; y) \Big] \leq 0$$

for every $c(\cdot) \in \mathcal{B}_0'(x,z)$, or equivalently

$$\overline{\lim}_{\epsilon \downarrow 0} \frac{1}{\epsilon} \cdot E \int_0^T \Big[u\big(t, c^y(t) + \epsilon(c(t) - c^y(t)) - z(t; c^y + \epsilon(c - c^y))\big) - u\big(t, c^y(t) - z(t; c^y)\big) \Big] dt$$

$$\leq y \cdot E \left[\int_0^T H(t)(c(t) - c^y(t)) dt \right].$$

Consider now only those processes $c(\cdot) \in \mathcal{B}_0'(x,z)$ that satisfy the condition $\sup_{0 \leq t \leq T} |c(t) - c^y(t)| \leq 1$, and denote the resulting class by $\mathcal{B}_0^y(x,z)$. Then the concavity of the utility function $u(t,\cdot)$, the assumption $\mathcal{R}(x,z,c^y;y) < \infty$, the linearity of $c \mapsto z(t;c)$ and the dominated convergence theorem, lead from the above inequality to

$$E \left[\int_0^T u'(t, c^y(t) - z(t; c^y)) \cdot \Big[(c(t) - c^y(t)) - (z(t;c) - z(t;c^y)) \Big] dt \right]$$

$$\leq y \cdot E \left[\int_0^T H(t)(c(t) - c^y(t)) dt \right].$$

In other words, we have

$$
\begin{aligned}
& E\left[\int_0^T \left\{u'(t, c^y(t) - z(t; c^y)) - yH(t)\right\} \cdot [c(t) - c^y(t)]dt\right] \\
\leq \; & E\left[\int_0^T u'(t, c^y(t) - z(t; c^y)) \cdot (z(t; c) - z(t; c^y))dt\right]
\end{aligned} \tag{2.41}
$$

for every $c(\cdot) \in \mathcal{B}_0^y(x, z)$. According to (2.28), we can rewrite the right-hand side of the last inequality as

$$
\begin{aligned}
& E\left[\int_0^T u'(t, c^y(t) - z(t; c^y)) \cdot \left(\int_0^t \delta(s) e^{-\int_s^t \alpha(v)dv} (c(s) - c^y(s))ds\right) dt\right] \\
= \; & E\left[\int_0^T \delta(s) \left(\int_s^T e^{-\int_s^t \alpha(v)dv} u'(t, c^y(t) - z(t; c^y))dt\right) (c(s) - c^y(s))ds\right] \\
= \; & E\left[\int_0^T \delta(t) \cdot E_t \left(\int_t^T e^{-\int_t^s \alpha(v)dv} u'(s, c^y(s) - z(s; c^y))ds\right) (c(t) - c^y(t))dt\right].
\end{aligned}
$$

Through substitution back into (2.41), we come to the conclusion that the “utility-gradient”

$$
\mathcal{G}(t; c^y) \triangleq u'(t, c^y(t) - z(t; c^y)) - \delta(t) \cdot E_t \left(\int_t^T e^{-\int_t^s \alpha(v)dv} u'(s, c^y(s) - z(s; c^y))ds\right)
$$

satisfies

$$
E\left[\int_0^T \{\mathcal{G}(t; c^y) - yH(t)\} \cdot [c(t) - c^y(t)]dt\right] \leq 0, \qquad \forall \; c(\cdot) \in \mathcal{B}_0^y(x, z).
$$

This strongly suggests the relationship $\mathcal{G}(t, c^y) = yH(t)$, or equivalently the fact that $c^y(\cdot)$ satisfies the equation

$$
u'(t, c^y(t) - z(t; c^y)) - \delta(t) \cdot E_t \left(\int_t^T e^{-\int_t^s \alpha(v)dv} u'(s, c^y(s) - z(s; c^y))ds\right) = yH(t) \tag{2.42}
$$

for all $t \in [0, T]$. It is then straightforward that the “normalized marginal utility” process

$$
\Gamma(t) \triangleq \frac{1}{y} u'(t, c^y(t) - z(t; c^y)), \qquad t \in [0, T] \tag{2.43}
$$

appearing in (2.42), solves the *recursive linear stochastic equation*

$$\Gamma(t) = H(t) + \delta(t) \cdot E_t \left(\int_t^T e^{-\int_t^s \alpha(v)dv} \Gamma(s)ds \right), \qquad t \in [0,T]. \tag{2.44}$$

As was shown by Detemple and Karatzas (2003) (Appendix), the solution of (2.44) is provided by the process $\Gamma(\cdot)$ of (2.37). Inverting (2.43), we deduce that the optimal consumption in excess of the standard of living is given by

$$\begin{aligned} c^y(t) - z(t;c^y) &= I(t, y\Gamma(t)) \\ &= e^{\int_0^t (\delta(v)-\alpha(v))dv} F^y(t), \quad \text{in the notation of (2.38),} \end{aligned}$$

and substituting into (2.26) we obtain the dynamics

$$dz^y(t) = \big[\delta(t) I(t, y\Gamma(t)) + (\delta(t) - \alpha(t)) z^y(t)\big] dt, \qquad z^y(0) = z, \tag{2.45}$$

for the standard of living process $z^y(\cdot) \equiv z(\cdot\,; c^y)$.

Eventually, by solving the first-order linear ordinary differential equation (2.45) we arrive at the expression given by (2.40), and the formula (2.39) for the optimal consumption process $c^y(\cdot)$ follows immediately. $\Diamond$

Optimality in the static problem: Synthesis

Let us follow now the steps of our previous analysis backwards, and provide constructive arguments to establish the existence of an optimal consumption policy in the static problem. For this procedure to work, we shall need to impose the following conditions:

Assumption 2.7. *It will be assumed throughout that*

$$E\left(\int_0^T H(t) I(t, y\Gamma(t)) dt \right) < \infty, \qquad \forall\ y \in (0,\infty),$$

$$E\left(\int_0^T \big| u\big(t, I(t, y\Gamma(t))\big) \big| dt \right) < \infty, \qquad \forall\ y \in (0,\infty).$$

In the sequel, we shall provide conditions on both the utility preferences and the model coefficients, that ensure the validity of the above assumption; cf. Remarks 2.15, 2.16 and 3.4.

We recall the process $\Gamma(\cdot)$ of (2.37), as well as the process $F^y(\cdot)$ of (2.38), parametrized by $y \in (0,\infty)$. Similarly, we define for each $y \in (0,\infty)$ the processes $c^y(\cdot)$, $z^y(\cdot)$ as in (2.39), (2.40), and observe

$$c^y(t) - z^y(t) = e^{\int_0^t (\delta(v)-\alpha(v))dv} F^y(t) = I(t, y\Gamma(t)) > 0 \quad \text{for } t \in [0,T]. \tag{2.46}$$

We shall select the scalar Lagrange multiplier $y > 0$ so that the budget constraint (2.14) is satisfied without slackness, i.e.,

$$E\int_0^T H(t)c^y(t)dt = x$$

as mandated by Lemma 2.5. Recall the notation of (2.29), and write this requirement as $\mathcal{X}(y) = x - wz$, where

$$\begin{aligned}
\mathcal{X}(y) &\triangleq E\left[\int_0^T H(t)c^y(t)dt\right] - wz \qquad (2.47)\\
&= E\left[\int_0^T H(t)\left(c^y(t) - ze^{\int_0^t (\delta(v)-\alpha(v))dv}\right)dt\right]\\
&= E\left[\int_0^T e^{\int_0^t (\delta(v)-\alpha(v))dv} H(t)\left(F^y(t) + \int_0^t \delta(s)F^y(s)ds\right)dt\right]\\
&= E\Bigg[\int_0^T e^{\int_0^t (\delta(v)-\alpha(v))dv} H(t)F^y(t)dt\\
&\qquad\qquad + \int_0^T \delta(s)F^y(s)\left(\int_s^T e^{\int_0^t (\delta(v)-\alpha(v))dv} H(t)dt\right)ds\Bigg]\\
&= E\Bigg[\int_0^T e^{\int_0^t (\delta(v)-\alpha(v))dv} F^y(t)H(t)dt\\
&\qquad\qquad + \int_0^T \delta(t)F^y(t)\cdot E_t\left(\int_t^T e^{\int_0^s (\delta(v)-\alpha(v))dv} H(s)ds\right)dt\Bigg]\\
&= E\left[\int_0^T e^{\int_0^t (\delta(v)-\alpha(v))dv}\Gamma(t)F^y(t)dt\right]\\
&= E\left[\int_0^T \Gamma(t)I(t, y\Gamma(t))dt\right], \qquad 0 < y < \infty \qquad (2.48)
\end{aligned}$$

[the third equation comes from (2.39), and the next-to-last equation from the expression for $\Gamma(\cdot)$ in (2.37)].

Under Assumption 2.7, the function $\mathcal{X}(\cdot)$ inherits from $I(t,\cdot)$ its continuity and strict decrease, as well as $\mathcal{X}(0^+) = \infty$ and $\mathcal{X}(\infty) = 0$. We shall denote the (continuous, strictly decreasing, onto) inverse of this function by $\mathcal{Y}(\cdot)$. Obviously then, Assumption 2.2 ensures the existence of a scalar $y_0 \triangleq \mathcal{Y}(x - wz)$ that satisfies the condition

$$\mathcal{X}(y_0) = x - wz. \tag{2.49}$$

It is straightforward that according to this $y_0 > 0$, we can now consider the consumption policy

$$c_0(\cdot) \equiv c^{y_0}(\cdot) \tag{2.50}$$

as in (2.39), and note that we have satisfied the budget constraint without slackness, namely

$$E\left[\int_0^T H(t)c_0(t)dt\right] = x, \tag{2.51}$$

due to the special choice of y_0.

The following result concludes our construction of an optimal consumption rate process for the static problem.

Theorem 2.8. *The consumption process $c_0(\cdot)$ of (2.50) solves the static optimization problem; that is, $c_0(\cdot) \in \mathcal{B}_0'(x,z)$, and $J(z;c) \le J(z;c_0) < \infty$ holds for any $c(\cdot) \in \mathcal{B}_0'(x,z)$.*

Proof: Substituting in the formula (2.40) we obtain the process

$$z_0(\cdot) \equiv z^{y_0}(\cdot). \tag{2.52}$$

From Assumption 2.7, we have clearly $J(z;c_0) < \infty$. On the other hand, (2.21) implies

$$\begin{aligned} u(t, c_0(t) - z_0(t)) &\ge u(t,1) + y_0\Gamma(t)\left[I(t, y_0\Gamma(t)) - 1\right] \\ &\ge -|u(t,1)| - y_0\Gamma(t)\ ; \end{aligned}$$

through the observation

$$\begin{aligned} E[\Gamma(t)] &\le E[H(t)] + \Delta e^{\Delta T} \cdot \left(\int_t^T E[H(s)]ds\right) \\ &\le e^{\varrho}\left(1 + \Delta T e^{\Delta T}\right) < \infty, \end{aligned} \tag{2.53}$$

where we have used (2.3), (2.27) and the supermartingale property of $Z(\cdot)$, it is apparent that $c_0(\cdot)$ satisfies condition (2.30) and therefore $c_0(\cdot) \in \mathcal{B}_0'(x, z)$. For any $c(\cdot) \in \mathcal{B}_0(x)$, the concavity of $u(t, \cdot)$ implies

$$u(t, c_0(t) - z_0(t)) - u(t, c(t) - z(t)) \geq u'(t, c_0(t) - z_0(t)) \cdot [c_0(t) - c(t) - (z_0(t) - z(t))],$$

so, using also (2.28), (2.51), (2.14), (2.43) and (2.44), we have

$$\begin{aligned}
J(z; c_0) - J(z; c) \geq E \int_0^T u'(t, c_0(t) - z_0(t)) \Big\{ c_0(t) - c(t) \\
- \int_0^t \delta(s)(c_0(s) - c(s)) e^{-\int_s^t \alpha(v)dv} ds \Big\} dt
\end{aligned}$$

$$\begin{aligned}
&= E \int_0^T u'(t, c_0(t) - z_0(t)) \cdot (c_0(t) - c(t))\, dt \qquad (2.54) \\
&\quad - E \int_0^T \delta(s) \left(\int_s^T u'(t, c_0(t) - z_0(t)) e^{-\int_s^t \alpha(v)dv} dt \right) (c_0(s) - c(s)) ds \\
&= E \int_0^T u'(t, c_0(t) - z_0(t)) \cdot (c_0(t) - c(t))\, dt \\
&\quad - E \int_0^T \delta(t) \cdot E_t \left(\int_t^T u'(s, c_0(s) - z_0(s)) e^{-\int_t^s \alpha(v)dv} ds \right) (c_0(t) - c(t)) dt \\
&= E \int_0^T \left[y_0 \Gamma(t) - \delta(t) \cdot E_t \left(\int_t^T y_0 \Gamma(s) e^{-\int_t^s \alpha(v)dv} ds \right) \right] (c_0(t) - c(t)) dt \\
&= y_0 E \int_0^T H(t)(c_0(t) - c(t)) dt = y_0 \left[x - E \int_0^T H(t) c(t) dt \right] \geq 0. \qquad \diamondsuit
\end{aligned}$$

Remark 2.9. *From (2.47), (2.48), (2.46), (2.44) and (2.28), we have for every $y > 0$ and $z > 0$ the computations*

$$\begin{aligned}
zw &= E \int_0^T H(t) c^y(t) dt - \mathcal{X}(y) \\
&= E \int_0^T \big[H(t) c^y(t) - \Gamma(t) \big(c^y(t) - z^y(t) \big) \big] dt \\
&= E \int_0^T \left[-\delta(t) E_t \left(\int_t^T e^{-\int_t^s \alpha(v)dv} \Gamma(s) ds \right) c^y(t) + z^y(t) \Gamma(t) \right] dt
\end{aligned}$$

$$= E\left[-\int_0^T \Gamma(s)\left(\int_0^s \delta(t)e^{-\int_t^s \alpha(v)dv}c^y(t)dt\right)ds + \int_0^T z^y(t)\Gamma(t)dt\right]$$
$$= E\int_0^T \left(z^y(t) - \int_0^t \delta(s)e^{-\int_s^t \alpha(v)dv}c^y(s)ds\right)\Gamma(t)dt$$
$$= z \cdot E\int_0^T e^{-\int_0^t \alpha(v)dv}\Gamma(t)dt.$$

We obtain the expression

$$w = E\left[\int_0^T e^{-\int_0^t \alpha(v)dv}\Gamma(t)dt\right], \tag{2.55}$$

which re-casts the subsistence consumption cost per unit of standard of living w *of (2.29), as a weighted average of the "adjusted" state-price density process* $\Gamma(\cdot)$ *of (2.37), discounted at the rate* $\alpha(\cdot)$*. Due to this representation of* w*, the terminology "adjusted" state-price density for* $\Gamma(\cdot)$ *becomes now quite intuitive: namely, a comparison of (2.55) with (2.29), which involves only the density process* $H(\cdot)$*, clearly connotes the significance of* $\Gamma(\cdot)$ *as an alternative state-price density process, according to a different discount rate, for the evaluation of economic features in our market.*

Corollary 2.10. *There exists a portfolio process* $\pi_0(\cdot)$ *such that the pair of policies* $(\pi_0, c_0) \in \mathcal{A}'_0(x,z)$ *attains the supremum of* $J(z;\pi,c)$ *over* $\mathcal{A}'_0(x,z)$ *in (2.31) and the corresponding wealth process* $X_0(\cdot) \equiv X^{x,\pi_0,c_0}(\cdot)$ *is given by*

$$X_0(t) = \frac{1}{H(t)}E_t\left[\int_t^T H(s)c_0(s)ds\right], \quad t \in [0,T]. \tag{2.56}$$

This optimal investment $\pi_0(\cdot)$ *has the characterization*

$$\pi_0(t) = (\sigma(t)\sigma^*(t))^{-1}\sigma(t)\left[\frac{\psi_0(t)}{X_0(t)H(t)} + \vartheta(t)\right], \tag{2.57}$$

in terms of the $\mathbb{R}^d$*-valued,* $\mathbb{F}$*-progressively measurable and almost surely square-integrable process* $\psi_0(\cdot)$ *that represents the martingale*

$$M_0(t) \triangleq E_t\left[\int_0^T H(s)c_0(s)ds\right], \quad t \in [0,T] \tag{2.58}$$

as the stochastic integral $M_0(t) = x + \int_0^t \psi_0^*(s)dW(s)$*.*
Furthermore, the value function V *of the dynamic maximization problem (2.31) is captured as*

$$V(x,z) = G(\mathcal{Y}(x - wz)), \quad (x,z) \in \mathcal{D}; \tag{2.59}$$

here $\mathcal{Y}(\cdot)$ *is the inverse of the function* $\mathcal{X}(\cdot)$*, defined in (2.47), and*

$$G(y) \triangleq E\left[\int_0^T u\big(t, I(t, y\Gamma(t))\big)dt\right], \quad y \in (0,\infty). \tag{2.60}$$

Proof: The existence of the optimal portfolio $\pi_0(\cdot)$ is an immediate consequence of Lemma 2.1, along with the validation of (2.56), (2.57) and (2.58). From the optimality of (π_0, c_0) we get

$$V(x,z) = E\left[\int_0^T u(t, c_0(t) - z_0(t))dt\right], \quad (x,z) \in \mathcal{D},$$

and (2.59) follows readily from (2.47), (2.51). $\diamondsuit$

Note that, under the optimal policies (π_0, c_0), the investor goes bankrupt at time $t = T$: $X_0(T) = 0$, almost surely. This is natural, since utility is desired here only from consumption, not from terminal wealth.

Remark 2.11. *Both inequalities in (2.54) are strict, unless* $c(\cdot) = c_0(\cdot)$*. Hence,* $c_0(\cdot)$ *is the* unique *optimal consumption process, and thereby* $\pi_0(\cdot)$ *is the* unique *optimal portfolio process, up to almost-everywhere equivalence under the product of Lebesgue measure and* P*.*

Assumption 2.2 determines the "domain of acceptability" $\mathcal{D}$ for the initial values of wealth and standard of living. The next reasonable issue to be explored is the temporal evolution of these quantities as random processes, under the optimal pair policy (π_0, c_0) and for all times $t \in [0,T]$.

Theorem 2.12. *The effective state space of optimal wealth/standard of living process* $\big(X_0(\cdot), z_0(\cdot)\big)$ *is given by the family of random half-planes*

$$\begin{aligned} \mathcal{D}_t &\triangleq \Big\{(x', z') \in (0,\infty)\times[0,\infty);\ x' > \mathcal{W}(t)z'\Big\}, \quad 0 \le t < T, \\ \mathcal{D}_T &\triangleq \Big\{(0, z');\ z' \in [0,\infty)\Big\}, \end{aligned} \tag{2.61}$$

where

$$\mathcal{W}(t) \triangleq \frac{1}{H(t)} E_t\left[\int_t^T e^{\int_t^s (\delta(v)-\alpha(v))dv} H(s)ds\right], \quad 0 \le t \le T \tag{2.62}$$

stands for the cost of subsistence consumption per unit of standard of living, at time t. In other words,

$$\big(X_0(t), z_0(t)\big) \in \mathcal{D}_t, \quad \textit{for all} \ \ t \in [0,T], \tag{2.63}$$

almost surely.

Note that $\mathcal{W}(0) = w$ and $\mathcal{D}_0 = \mathcal{D}$, the quantities of Assumption 2.2; thus, the *random wedge* $\mathcal{D}_t$ determines dynamically, over time, the half-planes where the vector process of wealth/standard of living $\big(X_0(\cdot), z_0(\cdot)\big)$ takes values under the optimal regime.

Proof of Theorem 2.12: Consider the Lagrange multiplier $y = y_0 > 0$, which gives rise to the optimal pair (π_0, c_0) and the resulting standard of living $z_0(\cdot)$ processes, specified by (2.57), (2.50) and (2.52), successively. Recalling the definitions of (2.37) and (2.62), the corresponding wealth process $X_0(\cdot)$ of (2.56) may be reformulated as

$$\begin{aligned} X_0(t) = \frac{1}{H(t)}\ E_t\bigg[\int_t^T H(s)\Big\{ & I(s, y_0\Gamma(s)) + ze^{\int_0^s (\delta(v)-\alpha(v))dv} \\ & + \int_0^s \delta(\theta) e^{\int_\theta^s (\delta(v)-\alpha(v))dv} I(\theta, y_0\Gamma(\theta))d\theta\Big\}ds\bigg] \\ = \frac{1}{H(t)}\ E_t\bigg[\int_t^T H(s)\Big\{ & ze^{\int_0^s (\delta(v)-\alpha(v))dv} \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \int_0^t \delta(\theta) e^{\int_\theta^s (\delta(v)-\alpha(v))dv} I(\theta, y_0\Gamma(\theta)) d\theta \Big\} ds \\
&\qquad + \int_t^T H(s) I(s, y_0\Gamma(s)) ds \\
&\qquad + \int_t^T \delta(\theta) I(\theta, y_0\Gamma(\theta)) \left(\int_\theta^T e^{\int_\theta^s (\delta(v)-\alpha(v))dv} H(s) ds \right) d\theta \Bigg] \\
&= \frac{1}{H(t)}\, E_t \Bigg[z_0(t) \int_t^T e^{\int_t^s (\delta(v)-\alpha(v))dv} H(s) ds \\
&\qquad + \int_t^T \left\{ H(s) + \delta(s) E_s \left(\int_s^T H(\theta) e^{\int_s^\theta (\delta(v)-\alpha(v))dv} d\theta \right) \right\} I(s, y_0\Gamma(s)) ds \Bigg] \\
&= \mathcal{W}(t) z_0(t) + \frac{1}{H(t)} E_t \left[\int_t^T \Gamma(s) I(s, y_0\Gamma(s)) ds \right], \quad 0 \le t \le T.
\end{aligned}
$$

Therefore,

$$
X_0(t) - \mathcal{W}(t) z_0(t) = \frac{1}{H(t)} E_t \left[\int_t^T \Gamma(s) I(s, y_0\Gamma(s)) ds \right] > 0, \quad \forall\, t \in [0, T),
$$

almost surely, and (2.63) holds on $[0, T)$. The remaining assertions of the theorem follow directly from (2.56). $\diamondsuit$

Example 2.13. *(Logarithmic utility).* Consider $u(t, x) = \log x$, $\forall\ (t, x) \in [0, T] \times (0, \infty)$. Then $I(t, y) = 1/y$ for $(t, y) \in [0, T] \times (0, \infty)$, $\mathcal{X}(y) = T/y$ for $y \in (0, \infty)$, and $\mathcal{Y}(x) = T/x$ for $x \in (0, \infty)$. The optimal consumption, standard of living, and wealth processes are as follows:

$$
c_0(t) = z e^{\int_0^t (\delta(v)-\alpha(v))dv} + \frac{x - wz}{T} \left[\frac{1}{\Gamma(t)} + \int_0^t \frac{\delta(s)}{\Gamma(s)} e^{-\int_s^t (\delta(v)-\alpha(v))dv} ds \right],
$$

$$
z_0(t) = z e^{\int_0^t (\delta(v)-\alpha(v))dv} + \frac{x - wz}{T} \int_0^t \frac{\delta(s)}{\Gamma(s)} e^{-\int_s^t (\delta(v)-\alpha(v))dv} ds
$$

and

$$
X_0(t) = \frac{1}{H(t)} \left[z_0(t) E_t \left(\int_t^T e^{\int_t^s (\delta(v)-\alpha(v))dv} H(s) ds \right) + \frac{T - t}{T}(x - wz) \right]
$$

for $0 \leq t \leq T$. Moreover,

$$G(y) = -T \log y - E\left[\int_0^T \log \Gamma(t) dt\right], \quad y \in (0, \infty),$$

and the value function is

$$V(x, z) = T \log\left(\frac{x - wz}{T}\right) - E\left[\int_0^T \log \Gamma(t) dt\right], \quad (x, z) \in \mathcal{D}.$$

Note here that the conditions of Assumption 2.7 are satisfied; the first holds trivially, and the second is implied by the observation

$$E\big(\log \Gamma(t)\big) \leq \log\big(E(\Gamma(t))\big) \leq \varrho + \log\big(1 + \Delta T e^{\Delta T}\big) < \infty, \quad 0 \leq t \leq T,$$

where we used Jensen's inequality, (2.3) and the supermartingale property of $Z(\cdot)$. Finally, one may ascertain an explicit stochastic integral representation for $M_0(\cdot)$, defined in (2.58), under the additional assumption of *deterministic* model coefficients; cf. Example 3.13. The optimal portfolio process $\pi_0(\cdot)$ follows then by (2.57).

Example 2.14. *(Power utility).* Consider $u(t, x) = x^p/p, \ \forall\ (t, x) \in [0, T] \times (0, \infty)$, where $p < 1,\ p \neq 0$. Then $I(t, y) = y^{1/(p-1)}$ for $(t, y) \in [0, T] \times (0, \infty)$, and

$$\mathcal{X}(y) = y^{\frac{1}{p-1}} E\left[\int_0^T (\Gamma(t))^{p/(p-1)} dt\right] = \mathcal{X}(1) y^{\frac{1}{p-1}}, \quad y \in (0, \infty),$$

$$\mathcal{Y}(x) = \left(\frac{x}{\mathcal{X}(1)}\right)^{p-1}, \quad x \in (0, \infty).$$

The optimal consumption, standard of living, and wealth process are given by

$$c_0(t) = z e^{\int_0^t (\delta(v) - \alpha(v)) dv} + \frac{x - wz}{\mathcal{X}(1)} \Bigg[(\Gamma(t))^{1/(p-1)} + \int_0^t \delta(s) e^{-\int_s^t (\delta(v) - \alpha(v)) dv} (\Gamma(s))^{1/(p-1)} ds\Bigg],$$

$$z_0(t) = ze^{\int_0^t (\delta(v)-\alpha(v))dv} + \frac{x-wz}{\mathcal{X}(1)}\left[\int_0^t \delta(s)e^{-\int_s^t(\delta(v)-\alpha(v))dv}(\Gamma(s))^{1/(p-1)}ds\right],$$

and

$$X_0(t) = \frac{1}{H(t)}\Bigg[z_0(t)E_t\left(\int_t^T e^{\int_t^s(\delta(v)-\alpha(v))dv}H(s)ds\right) + \frac{x-wz}{\mathcal{X}(1)}E_t\left(\int_t^T (\Gamma(s))^{p/(p-1)}ds\right)\Bigg]$$

for every $0 \le t \le T$. In addition,

$$G(y) = \frac{1}{p}\mathcal{X}(1)y^{\frac{p}{p-1}}, \quad y \in (0,\infty),$$
$$V(x,z) = \frac{1}{p}\mathcal{X}(1)^{1-p}(x-wz)^p, \quad (x,z) \in \mathcal{D}.$$

As in the previous example, a concrete formula for the optimal portfolio process $\pi_0(\cdot)$ can be obtained in the case of deterministic coefficients (cf. Example 3.14).

Remark 2.15. *Given any fixed* $p \in (-,\infty) \cup (0,1)$*, we set* $(H(t))^{p/(p-1)} = m(t)L(t)$*, in terms of*

$$m(t) \triangleq \exp\left\{\frac{p}{1-p}\int_0^t r(v)dv + \frac{p}{2(1-p)^2}\int_0^t \|\vartheta(v)\|^2 dv\right\},$$
$$L(t) \triangleq \exp\left\{\frac{p}{1-p}\int_0^t \vartheta^*(v)dW(v) - \frac{p^2}{2(1-p)^2}\int_0^t \|\vartheta(v)\|^2 dv\right\}$$

for $0 \le t \le T$*. In case* $\vartheta(\cdot)$ *is bounded uniformly on* $[0,T]\times\Omega$*,* but not necessarily deterministic*,* $m(\cdot)$ *is also bounded uniformly in* (t,ω) *and* $Z(\cdot)$*,* $L(\cdot)$ *are martingales, thanks to (2.3) and Novikov condition. Then, Assumption 2.7 is valid for the utility function of Example 2.14 as well, since*

$$\mathcal{X}(1) = E\left[\int_0^T m(t)L(t)\left\{1 + \delta(t)\cdot E_t\left(\int_t^T e^{\int_t^s(\delta(v)-\alpha(v))dv}\frac{H(s)}{H(t)}ds\right)\right\}^{p/(p-1)}dt\right]$$
$$\le \left(1 \vee \left[1 + \Delta T e^{\varrho+\Delta T}\right]^{p/(p-1)}\right) E\left[\int_0^T m(t)L(t)dt\right] < \infty.$$

Remark 2.16. *Consider utility functions such that*

$$\sup_{0\le t\le T} I(t,y) \le \kappa y^{-\rho}, \quad \forall\ y \in (0,\infty), \tag{2.64}$$

holds for some $\kappa > 0$, $\rho > 0$. *Then, the first condition of Assumption 2.7 holds under* at least one *of the subsequent conditions:*

$$0 < \rho \le 1, \tag{2.65}$$

or

$$\vartheta(\cdot) \text{ is bounded uniformly on } [0,T]\times\Omega. \tag{2.66}$$

In particular, (2.64) and (2.65) yield

$$\mathcal{X}(y) \le \kappa y^{-\rho} E\left[\int_0^T (1 \vee \Gamma(t))\right] < \infty, \quad y \in (0,\infty).$$

Otherwise, use (2.66) to set $(H(t))^{1-\rho} = m(t)L(t)$, *in terms of*

$$m(t) \triangleq \exp\left\{(\rho-1)\int_0^t r(v)dv + \frac{1}{2}\rho(\rho-1)\int_0^t \|\vartheta(v)\|^2 dv\right\},$$

and the martingale

$$L(t) \triangleq \exp\left\{(\rho-1)\int_0^t \vartheta^*(v)dW(v) - \frac{1}{2}(\rho-1)^2\int_0^t \|\vartheta(v)\|^2 dv\right\}.$$

As in Remark 2.15, the boundedness of $m(\cdot)$ *and (2.64) imply that*

$$\mathcal{X}(y) \le \kappa y^{-\rho}\left(1 + \Delta T e^{\varrho + \Delta T}\right)^{(1-\rho)} E\left[\int_0^T m(t)L(t)dt\right] < \infty, \quad y \in (0,\infty).$$

The function $V(\cdot, z)$ satisfies all the conditions of a utility function as defined in Section 2.3, for any given $z \ge 0$; we formalize this aspect of the value function in the result that follows, leading to the notion of a *generalized utility function* and to the explicit computation of its *convex dual*

$$\widetilde{V}(y) \triangleq \sup_{(x,z)\in\mathcal{D}} \left\{V(x,z) - (x - wz)y\right\}, \quad y \in \mathbb{R}. \tag{2.67}$$

Theorem 2.17. *The function $V : \mathcal{D} \to \mathbb{R}$ is a generalized utility function, in the sense of being strictly concave and of class $C^{1,1}(\mathcal{D})$; it is strictly increasing in its first argument, strictly decreasing in the second, and satisfies $V_x((wz)^+, z) = \infty$, $V_x(\infty, z) = 0$ for any $z \geq 0$. Additionally, for all pairs $(x, z) \in \mathcal{D}$, we have that*

$$\lim_{(x,z)\to(\chi,\zeta)} V(x,z) = \int_0^T u(t, 0^+)dt, \quad \forall\ (\chi, \zeta) \in \partial\mathcal{D}, \tag{2.68}$$

where $\partial\mathcal{D} = \Big\{(x', z') \in [0, \infty)^2;\ x' = wz'\Big\}$ is the boundary of $\mathcal{D}$. Furthermore,

$$V_x(x, z) = \mathcal{Y}(x - wz), \quad \forall\ (x, z) \in \mathcal{D}, \tag{2.69}$$

$$V_z(x, z) = -w\mathcal{Y}(x - wz), \quad \forall\ (x, z) \in \mathcal{D}, \tag{2.70}$$

$$\begin{aligned} \widetilde{V}(y) &= G(y) - y\mathcal{X}(y) \\ &= E\int_0^T \tilde{u}(t, y\Gamma(t))dt, \quad \forall\ y > 0, \end{aligned} \tag{2.71}$$

$$\widetilde{V}'(y) = -\mathcal{X}(y), \quad \forall\ y > 0, \tag{2.72}$$

with $\mathcal{X}(\cdot)$, $G(\cdot)$ given by (2.48), (2.60), respectively.

Proof: We show first the strict concavity of V. Let $(x_1, z_1), (x_2, z_2) \in \mathcal{D}$ and $\lambda_1, \lambda_2 \in (0, 1)$ such that $\lambda_1 + \lambda_2 = 1$. For each (x_i, z_i) consider the optimal portfolio/consumption policy $(\pi_i, c_i) \in \mathcal{A}'_0(x_i, z_i)$ which generates the corresponding wealth process $X^{x_i,\pi_i,c_i}(\cdot)$, and the standard of living process $z_i(\cdot)$, $i = 1, 2$. Define now the portfolio/consumption plan $(\pi, c) \triangleq (\lambda_1\pi_1 + \lambda_2\pi_2, \lambda_1 c_1 + \lambda_2 c_2)$, denoting by $X^{x,\pi,c}(\cdot)$, $z(\cdot)$ the corresponding wealth and standard of living with $x \triangleq \lambda_1 x_1 + \lambda_2 x_2$ and $z \triangleq \lambda_1 z_1 + \lambda_2 z_2$. It is then easy to see that $(\pi, c) \in \mathcal{A}'_0(x, z)$ and

$$X^{x,\pi,c}(\cdot) = \lambda_1 X^{x_1,\pi_1,c_1}(\cdot) + \lambda_2 X^{x_2,\pi_2,c_2}(\cdot),$$

$$z(\cdot) = \lambda_1 z_1(\cdot) + \lambda_2 z_2(\cdot)$$

hold almost surely. Therefore, the strict concavity of $u(t, \cdot)$ implies

$$\begin{aligned}
&\lambda_1 V(x_1,z_1) + \lambda_2 V(x_2,z_2) \\
&= \lambda_1 E\left[\int_0^T u(t,c_1(t)-z_1(t))dt\right] + \lambda_2 E\left[\int_0^T u(t,c_2(t)-z_2(t))dt\right] \\
&< E\left[\int_0^T u(t,c(t)-z(t))dt\right] \\
&\le V(x,z) = V(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 z_1 + \lambda_2 z_2).
\end{aligned}$$

As a real-valued concave function on $\mathcal{D}$, V is continuous on its domain.

To establish (2.68), we consider pairs $(x,z)\in\mathcal{D}$, and observe from (2.59) that $\lim_{(x,z)\to(\chi,\zeta)} V(x,z) = \lim_{y\to\infty} G(y)$ holds for any $(\chi,\zeta)\in\partial\mathcal{D}$. But (2.16) indicates that $\lim_{y\to\infty} I(t,y\Gamma(t)) = 0$ for $0\le t\le T$, and Assumption 2.7 ensures that $G(y)$ of (2.60) is finite for any $y\in(0,\infty)$; thus, (2.68) becomes a direct consequence of the monotone convergence theorem.

We next undertake (2.71). Its second equality is checked algebraically via (2.28), (2.48) and (2.60). Turning now to the first, for every $(x,z)\in\mathcal{D}$, $y>0$ and $(\pi,c)\in\mathcal{A}_0'(x,z)$, the relation of (2.18) gives

$$u(t,c(t)-z(t)) \;\le\; \tilde{u}(t,y\Gamma(t)) + y\Gamma(t)(c(t)-z(t)). \tag{2.73}$$

Taking expectations, we employ (2.28), (2.44), (2.55) and the budget constraint (2.14) to obtain

$$\begin{aligned}
&E\int_0^T u(t,c(t)-z(t))dt \le E\int_0^T \Big[\tilde{u}(t,y\Gamma(t)) + y\Gamma(t)(c(t)-z(t))\Big]dt \\
&= E\int_0^T \tilde{u}(t,y\Gamma(t))dt \\
&\quad + y\cdot E\int_0^T \Gamma(t)\left(c(t) - ze^{-\int_0^t \alpha(v)dv} - \int_0^t \delta(s)e^{-\int_s^t \alpha(v)dv}c(s)ds\right)dt \\
&= E\int_0^T \tilde{u}(t,y\Gamma(t))dt - ywz \\
&\quad + y\cdot E\left[\int_0^T \Gamma(t)c(t)dt - \int_0^T \delta(s)\left(\int_s^T e^{-\int_s^t \alpha(v)dv}\Gamma(t)dt\right)c(s)ds\right]
\end{aligned}$$

$$
\begin{aligned}
&= E\int_0^T \tilde{u}(t, y\Gamma(t))dt - ywz \\
&\quad + y\cdot E\int_0^T \left\{\Gamma(t) - \delta(t)E_t\left(\int_t^T e^{-\int_t^s \alpha(v)dv}\Gamma(s)ds\right)\right\}c(t)dt \\
&= E\int_0^T \tilde{u}(t, y\Gamma(t))dt - ywz + y\cdot E\int_0^T H(t)c(t)dt \\
&\leq E\int_0^T \tilde{u}(t, y\Gamma(t))dt + y(x - wz) = G(y) - y\mathcal{X}(y) + y(x - wz). \qquad (2.74)
\end{aligned}
$$

The inequalities in (2.74) will hold as equalities, if and only if

$$
c(t) - z(t) = I(t, y\Gamma(t)) \qquad (2.75)
$$

and

$$
E\int_0^T H(t)c(t)dt = x.
$$

Setting $Q(y) \triangleq G(y) - y\mathcal{X}(y)$ and maximizing over $(\pi, c) \in \mathcal{A}_0'(x, z)$, it follows from (2.74) that $V(x, z) \leq Q(y) + (x - wz)y$ for every $(x, z) \in \mathcal{D}$, and thereby $\widetilde{V}(y) \leq Q(y)$ for every $y > 0$. Conversely, (2.74) becomes an equality, if (2.75) is satisfied and if $\mathcal{X}(y) = x - wz$, so $Q(y) = V(\mathcal{X}(y) + wz, z) - \mathcal{X}(y)y \leq \widetilde{V}(y)$. Hence (2.71) is established, and clearly the supremum in (2.67) is attained if $x - wz = \mathcal{X}(y)$.

We argue now (2.72) by bringing to our attention the identity

$$
\begin{aligned}
yI(t, y) - hI(t, h) - \int_h^y I(t, \lambda)d\lambda &= yI(t, y) - hI(t, h) + \tilde{u}(t, y) - \tilde{u}(t, h) \\
&= u(t, I(t, y)) - u(t, I(t, h)), \qquad (2.76)
\end{aligned}
$$

which holds for any utility function u and $0 \leq t \leq T$, $0 < h < y < \infty$; recall (2.18) and (2.19). This enables us to compute

$$
\begin{aligned}
&y\mathcal{X}(y) - h\mathcal{X}(h) - \int_h^y \mathcal{X}(\xi)d\xi \\
&= E\int_0^T \left[yH(t)I(t, yH(t)) - hH(t)I(t, hH(t)) - \int_{hH(t)}^{yH(t)} I(t, \lambda)d\lambda\right] dt
\end{aligned}
$$

$$= E\int_0^T \Big[u\big(t, I(t, yH(t))\big) - u\big(t, I(t, hH(t))\big)\Big]dt$$
$$= G(y) - G(h), \tag{2.77}$$

which in conjunction with (2.71) leads to

$$\widetilde{V}(y) - \widetilde{V}(h) = -\int_h^y \mathcal{X}(\xi)d\xi, \quad 0 < h < y < \infty, \tag{2.78}$$

and (2.72) follows.

Finally, let us rewrite (2.67) in the more suggestive form

$$\widetilde{V}(y) = \sup_{(x,z)\in\mathcal{D}} \big\{V(x,z) - (x,z)\cdot(y,-wy)\big\}, \quad y \in \mathbb{R},$$

where $v_1 \cdot v_2$ stands for the dot product between any two vectors v_1 and v_2. We recall that for $(x^*, z^*) \in \mathcal{D}$ and $y > 0$, we have $(y, -wy) \in \partial V(x^*, z^*)$ if and only if the maximum in the above expression is attained by (x^*, z^*) (e.g., Rockafellar (1970), Theorem 23.5). However, we have already shown that this maximum is attained by the pair (x^*, z^*) only if $x^* - wz^* = \mathcal{X}(y)$, implying

$$\partial V(x^*, z^*) = \big\{\big(\mathcal{Y}(x^* - wz^*), -w\mathcal{Y}(x^* - wz^*)\big)\big\}.$$

Therefore, (2.69), (2.70) are proved (e.g. Theorem 23.4 loc. cit.), and imply that $V_x(\cdot, z)$ is continuous, positive (thus $V(\cdot, z)$ strictly increasing), strictly decreasing on (wz, ∞), with $\lim_{x\downarrow wz} V_x(x,z) = \lim_{x\downarrow wz} \mathcal{Y}(x - wz) = \infty$ and $\lim_{x\uparrow\infty} V_x(x,z) = \lim_{x\uparrow\infty} \mathcal{Y}(x - wz) = 0$; while, $V_z(x, \cdot)$ is continuous, negative, and so $V(x, \cdot)$ decreases strictly. Consequently, V is a generalized utility function since it satisfies all its aforementioned properties. ♢

Remark 2.18. *We note that given any $z \in [0, \infty)$, (2.59) can be written as $G(y) = V(\mathcal{X}(y) + wz, z)$ for every $y \in (0, \infty)$. Thus, if $\mathcal{X}(\cdot)$ is differentiable, then $G(\cdot)$ is also differentiable with*

$$G'(y) = V_x(\mathcal{X}(y) + wz, z)\mathcal{X}'(y) = y\mathcal{X}'(y), \quad y \in (0, \infty) \tag{2.79}$$

by (2.69).

3 Representation Of Optimal Strategies In A Markovian Setting, With Habit-Forming Preferences And Complete Markets

3.1 Deterministic Coefficients

In Chapter 2 we established the existence and uniqueness, up to almost-everywhere equivalence, of a solution to our habit-modulated utility maximization problem in the case of a complete security market. The analysis resulted in a concrete representation for the optimal consumption process $c_0(\cdot)$, given by (2.50), but not for the optimal portfolio strategy $\pi_0(\cdot)$; we provided for it no useful expression aside from (2.57). In this chapter we shall confront this issue by revisiting the model in the special case of continuous, *deterministic* coefficients $r(\cdot) : [0,T] \to \mathbb{R}$, $\vartheta(\cdot) : [0,T] \to \mathbb{R}^d$, $\sigma(\cdot) : [0,T] \to L(\mathbb{R}^d;\mathbb{R}^d)$, the set of $d \times d$ matrices, $\alpha(\cdot) : [0,T] \to [0,\infty)$ and $\delta(\cdot) : [0,T] \to [0,\infty)$. Under this additional assumption, asset prices have the Markov property. This Markovian framework will allow us to characterize the value function of the dynamic problem in (2.31) as a solution of a nonlinear, second-order parabolic Hamilton-Jacobi-Bellman (HJB) partial differential equation.

We shall provide the optimal portfolio $\pi_0(t)$ and consumption policy $c_0(t)$ in closed, "feedback forms" on the current level of wealth $X_0(t)$ and the standard of living $z_0(t)$. In other words, we derive appropriate functions $C : [0,T) \times (0,\infty) \times [0,\infty) \to (0,\infty)$ and $\Pi : [0,T) \times (0,\infty) \times [0,\infty) \to \mathbb{R}^d$, such that

$$c_0(t) = C(t, X_0(t), z_0(t)), \quad \pi_0(t) = \Pi(t, X_0(t), z_0(t)), \quad 0 \le t < T. \tag{3.1}$$

It is then evident that the decision-maker, responsible for choosing his trading strategy and consumption rate at each instant $t \in [0,T]$, needs only to be aware of his current level of wealth $X_0(t)$ and standard of living $z_0(t)$, without keeping track of the entire history of the market up to time t; in other words, the current level of these processes constitutes a *sufficient statistic* for the optimization problem (2.31).

The following two assumptions will be adopted throughout this chapter.

Assumption 3.1. *Suppose that the market coefficients* $r(\cdot)$, $\vartheta(\cdot)$, $\sigma(\cdot)$ *and the standard of living weights* $\alpha(\cdot)$, $\delta(\cdot)$ *are non-random, continuous (and thence bounded) functions on* $[0,T]$. *In fact, let* $\|\vartheta(\cdot)\|$ *be Hölder continuous; that is, there exist some* $k>0$ *and* $\rho\in(0,1)$ *such that*

$$\big|\|\vartheta(t_1)\|-\|\vartheta(t_2)\|\big| \;\le k|t_1-t_2|^{\rho}$$

stands for every $t_1,t_2\in[0,T]$. *Additionally, assume that* $\delta(\cdot)$ *is differentiable and* $\|\vartheta(\cdot)\|$ *is bounded away from zero and infinity, i.e.*

$$\exists\;\; k_1,k_2>0 \;\;\textit{such that}\;\;\; 0<k_1\le\|\vartheta(t)\|\le k_2<\infty,\quad \forall\; t\in[0,T]. \tag{3.2}$$

Since the market price of risk $\vartheta(\cdot)$ is bounded, the local martingale $Z(\cdot)$ of (2.6) is a martingale. Thus, by Girsanov's theorem, the process $W_0(\cdot)$ of (2.9) is standard, d-dimensional Brownian motion under the new probability measure

$$P^0(A)\triangleq E[Z(T)\mathbf{1}_A],\quad A\in\mathcal{F}(T). \tag{3.3}$$

We shall refer to P^0 as the *equivalent martingale measure* of the financial market $\mathcal{M}_0$, and denote expectation under this measure by E^0.

Assumption 3.2. *Suppose that the utility function* u *satisfies*

(i) *polynomial growth of* I *:*

$$\exists\;\;\gamma>0\quad\textit{such that}\quad I(t,y)\le\gamma+y^{-\gamma},\quad \forall\;(t,y)\in[0,T]\times(0,\infty);$$

(ii) *polynomial growth of* $u\circ I$ *:*

$$\exists\;\;\gamma>0\quad\textit{such that}\quad u(t,I(t,y))\ge-\gamma-y^{\gamma},\;\; \forall\;(t,y)\in[0,T]\times(0,\infty);$$

(iii) *Hölder continuity of* I *: for every* $y_0>0$ *there are constants* $\varepsilon(y_0)>0$, $k(y_0)>0$ *and* $\rho(y_0)\in(0,1)$ *such that*

$$|I(t,y)-I(t,y_0)|\;\le k(y_0)|y-y_0|^{\rho(y_0)},\quad \forall\;t\in[0,T],$$
$$\forall\;y\in(0,\infty)\cap(y_0-\varepsilon(y_0),y_0+\varepsilon(y_0));$$

(iv) $\forall\;t\in[0,T]$, *there is a set* $N\subset(0,\infty)$ *of positive Lebesgue measure such that* $I'(t,y)\triangleq\frac{\partial}{\partial y}I(t,y)$ *is well-defined and strictly negative* $\forall\;y\in N$.

Remark 3.3. *Assumption 3.2(i), (ii), together with (2.17) and the strict decrease of $I(t,\cdot)$, yields that*

$$\exists\ \gamma > 0 \ \text{ such that } |u(t, I(t,y))| \ \le \gamma + y^{\gamma} + y^{-\gamma}, \quad \forall\ (t,y) \in [0,T] \times (0,\infty).$$

Moreover, it is easy to see that $u \circ I$ inherits from I the property of being Hölder continuous as stated in Assumption 3.2(iii). In fact, if $y_0 > 0$ and $\varepsilon(y_0)$, $k(y_0)$, and $\rho(y_0)$ are taken as in Assumption 3.2(iii), then an application of the mean value theorem leads to

$$\begin{aligned} |u(t, I(t,y)) - u(t, I(t,y_0))| &\le u'(t, \iota(t))|I(t,y) - I(t,y_0)| \\ &\le Qk(y_0)|y - y_0|^{\rho(y_0)} \end{aligned}$$

for each $y \in (0,\infty) \cap (y_0 - \varepsilon(y_0), y_0 + \varepsilon(y_0))$, some function $\iota(t)$ with values between $I(t,y)$ and $I(t,y_0)$ and a bound Q on the continuous map $u'(t, I(t,\eta))$ with (t,η) ranging over the set $[0,T] \times [(0,\infty) \cap (y_0 - \varepsilon(y_0), y_0 + \varepsilon(y_0))]$.

Remark 3.4. *Notice that Assumptions 3.1 and 3.2(i), (ii), in conjunction with Remark 3.3, guarantee the validity of Assumption 2.7 in the preceding chapter; compare also with Remark 2.16.*

3.2 Feedback Formulae

For each $(t,y) \in [0,T] \times (0,\infty)$ and $t \le s \le T$, we consider the stochastic processes

$$Z^t(s) \triangleq \exp\left\{ -\int_t^s \vartheta^*(v)dW(v) - \frac{1}{2}\int_t^s \|\vartheta(v)\|^2 dv \right\}, \tag{3.4}$$

$$\begin{aligned} H^t(s) &\triangleq \exp\left\{ -\int_t^s r(v)dv \right\} Z^t(s) \\ &= \exp\Bigg\{ -\int_t^s r(v)dv - \int_t^s \vartheta^*(v)dW_0(v) \\ &\qquad\qquad + \frac{1}{2}\int_t^s \|\vartheta(v)\|^2 dv \Bigg\}. \end{aligned} \tag{3.5}$$

These extend the processes of (2.6) and (2.8), respectively, to initial times other than zero. In accordance with (2.37), we shall also consider the extended "adjusted" state-price density process

$$\begin{aligned}\Gamma^t(s) &\triangleq H^t(s) + \delta(s) \cdot E_s\left(\int_s^T e^{\int_s^\theta (\delta(v)-\alpha(v))dv} H^t(\theta)d\theta\right) \qquad (3.6)\\ &= H^t(s)\left[1 + \delta(s) \cdot E_s\left(\int_s^T e^{\int_s^\theta (\delta(v)-\alpha(v))dv} H^s(\theta)d\theta\right)\right]\\ &= H^t(s)\left[1 + \delta(s)\int_s^T e^{\int_s^\theta (-r(v)+\delta(v)-\alpha(v))dv} d\theta\right],\\ &= H^t(s)\mu(s), \qquad t \le s \le T. \qquad (3.7)\end{aligned}$$

We have invoked here the martingale property of $Z(\cdot)$, and have set

$$\mu(t) \triangleq 1 + \delta(t)w(t), \qquad t \in [0,T], \tag{3.8}$$

with

$$w(t) \triangleq \int_t^T e^{\int_t^s (-r(v)+\delta(v)-\alpha(v))dv} ds, \qquad t \in [0,T] \tag{3.9}$$

and

$$\begin{aligned} w'(t) \triangleq \frac{d}{dt}w(t) &= \big[r(t) + \alpha(t) - \delta(t)\big]w(t) - 1\\ &= \big[r(t) + \alpha(t)\big]w(t) - \mu(t), \qquad t \in [0,T]. \end{aligned} \tag{3.10}$$

Note that $w(\cdot)$, $\mu(\cdot)$ are deterministic, and that the former is the Markovian reduction of $\mathcal{W}(\cdot)$ in (2.62); i.e., $\mathcal{W}(\cdot) \equiv w(\cdot)$ within the context of the current chapter.

Furthermore, we define the diffusion process

$$Y^{(t,y)}(s) \triangleq y\Gamma^t(s), \qquad t \le s \le T, \tag{3.11}$$

which satisfies the linear stochastic differential equation

$$dY^{(t,y)}(s) = Y^{(t,y)}(s)\left[\left(\frac{\mu'(s)}{\mu(s)} - r(s)\right)ds - \vartheta^*(s)dW(s)\right], \tag{3.12}$$

or equivalently

$$dY^{(t,y)}(s) = Y^{(t,y)}(s)\left[\left(\frac{\mu'(s)}{\mu(s)} - r(s) + \|\vartheta(s)\|^2\right)ds - \vartheta^*(s)dW_0(s)\right], \tag{3.13}$$

and $Y^{(t,y)}(t) = y\mu(t)$, $Y^{(t,y)}(s) = yY^{(t,1)}(s) = yH(s)\mu(s)/H(t)$. Invoking the "Bayes rule" for conditional expectations, a computation akin to the one presented in the proof of Theorem 2.12 shows that the optimal wealth/standard of living process $\big(X_0(\cdot), z_0(\cdot)\big)$ of (2.52), (2.56), satisfies

$$\begin{aligned} X_0(t) - w(t)z_0(t) &= \frac{1}{\xi}\, E_t\left[\int_t^T Y^{(t,\xi)}(s)I(s, Y^{(0,\xi)}(s))ds\right] \\ &= E_t^0\left[\int_t^T e^{-\int_t^s r(v)dv}\mu(s)I(s, Y^{(0,\xi)}(s))ds\right] \\ &= \mathcal{X}\left(t, \frac{Y^{(0,\xi)}(t)}{\mu(t)}\right), \quad 0 \le t \le T \end{aligned} \tag{3.14}$$

with $\xi = \mathcal{Y}(x - wz)$. We have used here the definition (3.11), the Markov property of $Y^{(0,\xi)}(\cdot)$ under P^0 from (3.13), and introduced the function $\mathcal{X} : [0,T] \times (0,\infty) \to (0,\infty)$ defined as

$$\mathcal{X}(t,y) \triangleq E^0\left[\int_t^T e^{-\int_t^s r(v)dv}\mu(s)I(s, yY^{(t,1)}(s))ds\right]. \tag{3.15}$$

Since the process $Y^{(t,y)}(\cdot)$ is Markovian also under the original probability measure P (cf. (3.12)), the conditional expectation

$$E_t\left[\int_t^T Y^{(t,y)}(s)I(s, Y^{(t,y)}(s))ds\right]$$

is a function of $Y^{(t,y)}(t) = y\mu(t)$, i.e., is deterministic. Therefore, we get the representation

$$\begin{aligned} \mathcal{X}(t,y) &= E^0\left\{E_t^0\left[\int_t^T e^{-\int_t^s r(v)dv}\mu(s)I(s, Y^{(t,y)}(s))ds\right]\right\} \\ &= E\left\{\frac{Z(T)}{Z(t)}E_t\left[\int_t^T e^{-\int_t^s r(v)dv}Z(s)\mu(s)I(s, Y^{(t,y)}(s))ds\right]\right\} \end{aligned}$$

$$= \frac{1}{y} E \left\{ Z(T) E_t \left[\int_t^T Y^{(t,y)}(s) I(s, Y^{(t,y)}(s)) ds \right] \right\}$$
$$= \frac{1}{y} E \left[\int_t^T Y^{(t,y)}(s) I(s, Y^{(t,y)}(s)) ds \right], \tag{3.16}$$

a generalization of (2.48).

Lemma 3.5. *Suppose both Assumptions 3.1, 3.2 hold. Then, the function* $\mathcal{X}$ *of (3.15) belongs to the class* $C([0,T] \times (0,\infty)) \cap C^{1,2}([0,T) \times (0,\infty))$ *and solves the Cauchy problem*

$$\mathcal{X}_t(t,y) + \frac{1}{2}\|\vartheta(t)\|^2 y^2 \mathcal{X}_{yy}(t,y) + \big(\|\vartheta(t)\|^2 - r(t)\big) y \mathcal{X}_y(t,y) - r(t)\mathcal{X}(t,y)$$
$$= -\mu(t) I(t, y\mu(t)) \quad \text{on } [0,T) \times (0,\infty), \tag{3.17}$$
$$\mathcal{X}(T,y) = 0 \quad \text{on } (0,\infty). \tag{3.18}$$

Additionally, for every $t \in [0,T)$, $\mathcal{X}(t,\cdot)$ *is strictly decreasing with* $\mathcal{X}(t,0^+) = \infty$ *and* $\mathcal{X}(t,\infty) = 0$, *and so it has a strictly decreasing inverse function* $\mathcal{Y}(t,\cdot) : (0,\infty) \xrightarrow{onto} (0,\infty)$, *namely*

$$\mathcal{X}(t, \mathcal{Y}(t,x)) = x, \qquad \forall\, x \in (0,\infty). \tag{3.19}$$

The function $\mathcal{Y}$ *is of class* $C^{1,2}([0,T) \times (0,\infty))$ *as well.*

Proof: We shall show that

$$\mathcal{X}(t,y) = v(t, \log(y\mu(t))) \tag{3.20}$$

for $(t,y) \in [0,T] \times (0,\infty)$, where $v : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies a Cauchy problem for a non-degenerate, second-order partial differential equation, namely

$$v_t(t,\eta) + \frac{1}{2}\|\vartheta(t)\|^2 v_{\eta\eta}(t,\eta) + \left(\frac{\mu'(t)}{\mu(t)} - r(t) + \frac{1}{2}\|\vartheta(t)\|^2 \right) v_\eta(t,\eta)$$
$$-r(t)v(t,\eta) = -\mu(t) I(t, e^\eta), \qquad 0 \le t < T, \quad \eta \in \mathbb{R}, \tag{3.21}$$
$$v(T,\eta) = 0, \qquad \eta \in \mathbb{R}. \tag{3.22}$$

Standard results of partial differential equations theory (cf. Friedman (1964), Sections 1.7-1.9) show that there exists a unique solution in $C([0,T] \times (0,\infty)) \cap$

$C^{1,2}([0,T) \times (0,\infty))$ for this problem. Furthermore, for every $\varepsilon > 0$, there exists a positive constant $K(\varepsilon)$ such that

$$|v(t,\eta)| \leq K(\varepsilon) e^{\varepsilon \eta^2}, \qquad \forall\, \eta \in \mathbb{R}. \tag{3.23}$$

Given a pair $(t,y) \in [0,T) \times (0,\infty)$, an application of Itô's formula, in combination with (3.13) and (3.21), implies

$$\begin{aligned} d &\left[e^{-\int_t^s r(v)dv} v(s, \log Y^{(t,y)}(s)) \right] \\ &\qquad = -e^{-\int_t^s r(v)dv} \mu(s) I(s, Y^{(t,y)}(s)) ds \\ &\qquad\qquad - e^{-\int_t^s r(v)dv} v_\eta(s, \log Y^{(t,y)}(s)) \vartheta^*(s) dW_0(s). \end{aligned} \tag{3.24}$$

Let $n \in \mathbb{N}$ and define the stopping time

$$\tau_n \triangleq \left(T - \frac{1}{n}\right) \wedge \inf\left\{ s \in [t,T] : \ |\log Y^{(t,y)}(s)| \geq n \right\},$$

so that the function $v_\eta(s, \log Y^{(t,y)}(s))$ is uniformly bounded on $[t,\tau_n] \times \Omega$ and accordingly, the P^0-stochastic integral in (3.24) is actually a martingale. Thus, integrating the latter relationship over $[t,\tau_n]$ and taking expectations, we arrive at

$$\begin{aligned} v(t, \log(y\mu(t))) =& E^0 \int_t^{\tau_n} e^{-\int_t^s r(v)dv} \mu(s) I(s, Y^{(t,y)}(s)) ds \\ &+ E^0 e^{-\int_t^{\tau_n} r(v)dv} v(\tau_n, \log Y^{(t,y)}(\tau_n)), \end{aligned}$$

whence

$$v(t, \log(y\mu(t))) = \mathcal{X}(t,y) + \lim_{n\to\infty} E^0 \left(e^{-\int_t^{\tau_n} r(v)dv} v(\tau_n, \log Y^{(t,y)}(\tau_n)) \right), \tag{3.25}$$

by the monotone convergence theorem and (3.15). Moreover, the initial condition (3.22) gives

$$\lim_{n\to\infty} e^{-\int_t^{\tau_n} r(v)dv} v(\tau_n, \log Y^{(t,y)}(\tau_n)) = 0, \qquad \text{a.s.} \tag{3.26}$$

and our next task will be to pass the limit inside the expectation in (3.25) using domination arguments. Once this is done, we shall have our representation (3.20).

To do so, we observe that condition (3.23) leads to the following estimate:

$$
\begin{aligned}
&\left| e^{-\int_t^{\tau_n} r(v)dv} v\Big(\tau_n, \log Y^{(t,y)}(\tau_n)\Big)\right| \\
&\le K(\varepsilon) e^{\int_t^T |r(v)|dv} e^{\varepsilon(\log Y^{(t,y)}(\tau_n))^2} \\
&\le K(\varepsilon) e^{\int_t^T |r(v)|dv} \exp\left\{ \varepsilon \left[\left| \log\Big(y \max_{0\le s\le T} \mu(s)\Big)\right| \right.\right. \\
&\qquad \left.\left. + \int_t^T \left| -r(v) + \frac{1}{2}\|\vartheta(v)\|^2 \right| dv + \sup_{t\le s\le T} \left| \int_t^s \vartheta^*(v) dW_0(v) \right| \right]^2 \right\}.
\end{aligned}
$$

Consequently, everything boils down to showing that

$$
E^0 \left[\exp\left\{ \varepsilon \sup_{t\le s\le T} \left| \int_t^s \vartheta^*(v) dW_0(v) \right|^2 \right\} \right] < \infty. \tag{3.27}
$$

This inequality follows from an application of Fernique's Theorem (e.g. Fernique (1974)) for the running maximum of Brownian motion. For a straightforward argument the reader is referred to Karatzas and Shreve (1998), Lemma 3.8.4, according to which condition (3.27) is satisfied for any chosen $0 < \varepsilon < 1/(2\bar{\tau})$ with $\bar{\tau} \triangleq k_2^2(T-t)$. From (3.25), coupled with the dominated convergence theorem and (3.26), we obtain the representation (3.20). Straightforward computations yield that $\mathcal{X}$ is of class $C([0,T]\times(0,\infty)) \cap C^{1,2}([0,T)\times(0,\infty))$ and solves the Cauchy problem of (3.17), (3.18).

We may now prove that $\mathcal{X}_y(t,y) < 0$. For $t \in [0,T)$, $y > 0$ and $h > 0$, the (strict) decreasing monotonicity of $I(t,\cdot)$ and the inequality $\mu(\cdot) \ge 1$ imply that

$$
\begin{aligned}
&\frac{1}{h}[\mathcal{X}(t,y) - \mathcal{X}(t,y+h)] \\
&\ge E^0 \int_t^T e^{-\int_t^s r(v)dv} \frac{1}{h} \left[I\left(s, yY^{(t,1)}(s)\right) - I\left(s, (y+h)Y^{(t,1)}(s)\right)\right] ds.
\end{aligned}
$$

The martingale $\int_t^s \vartheta^*(v)dW_0(v) = B(\langle M\rangle(s-t))$, $t \le s$, has a normal distribution with mean zero and variance $q^2(s) \triangleq \langle M\rangle(s-t)$ with respect to P^0. Recalling the set N of Assumption 3.2(iv) and setting $m(s) \triangleq \int_t^s(-r(v) + \|\vartheta(v)\|^2/2)dv$, observe that the last expression is no less than

$$\frac{1}{\sqrt{2\pi}}\int_{N'}\left(\int_t^T e^{-\int_t^s r(v)dv}\frac{1}{h}\Big[I\left(s, y\mu(s)e^{m(s)-q(s)\eta}\right) \right.$$
$$\left. - I\left(s,(y+h)\mu(s)e^{m(s)-q(s)\eta}\right)\Big]ds\right)e^{-\eta^2/2}d\eta,$$

where

$$N' \triangleq \Big\{\eta\in\mathbb{R} : y\mu(s)e^{m(s)-q(s)\eta}\in N, \text{ for some } s\in[t,T]\Big\}.$$

As $h\downarrow 0$, Fatou's lemma yields

$$-\mathcal{X}_y(t,y) \geq -\frac{1}{\sqrt{2\pi}}\int_{N'}\left(\int_t^T e^{-\int_t^s r(v)dv}I'\left(s,y\mu(s)e^{m(s)-q(s)\eta}\right)ds\right)e^{-\eta^2/2}d\eta > 0.$$

Thus, the implicit function theorem ensures the existence of the function $\mathcal{Y}:[0,T)\times(0,\infty)\to(0,\infty)$ that possesses the same smoothness with $\mathcal{X}$ on its domain and verifies (3.19). Finally, the claimed marginal values $\mathcal{X}(t,0^+)$ and $\mathcal{X}(t,\infty)$ are justified by the monotone and dominated convergence theorems, respectively. $\Diamond$

Remark 3.6. *Use (3.24) and (3.20) along with the notation* $\beta(t)=e^{-\int_0^t r(v)dv}$ *of (2.7), to derive*

$$d\left(\beta(s)\mathcal{X}\left(s,\frac{Y^{(0,y)}(s)}{\mu(s)}\right)\right) = -\beta(s)\Big[\mu(s)I(s,Y^{(0,y)}(s))ds$$
$$+\frac{Y^{(0,y)}(s)}{\mu(s)}\mathcal{X}_y\left(s,\frac{Y^{(0,y)}(s)}{\mu(s)}\right)\vartheta^*(s)dW_0(s)\Big],$$

or in integral form

$$\beta(t)\mathcal{X}\left(t,\frac{Y^{(0,y)}(t)}{\mu(t)}\right)+\int_0^t\beta(s)\mu(s)I(s,Y^{(0,y)}(s))ds$$
$$=\mathcal{X}(0,y)-\int_0^t\beta(s)\frac{Y^{(0,y)}(s)}{\mu(s)}\mathcal{X}_y\left(s,\frac{Y^{(0,y)}(s)}{\mu(s)}\right)\vartheta^*(s)dW_0(s), \qquad (3.28)$$

for $(t,y)\in[0,T]\times(0,\infty)$. *This expression will be useful for representing the optimal investment and consumption in feedback form.*

Remark 3.7. *One should also notice that the inequality of (3.23) leads to a condition that ensures a* unique *solution to the Cauchy problem (3.17), (3.18). In particular, $\mathcal{X}$ is its sole solution of class $C([0,T]\times(0,\infty))\cap C^{1,2}([0,T)\times(0,\infty))$ over the set of functions f that satisfy the growth condition*

$$\big(\forall\ \varepsilon>0\big)\ \big(\exists\ K(\varepsilon)>0\big)\ \textit{such that}\ |f(t,y)|\le K(\varepsilon)\ e^{\varepsilon\left(\log(y\mu(t))\right)^2},\quad \forall\ (t,y)\in[0,T]\times(0,\infty). \tag{3.29}$$

Indeed, if f solves the Cauchy problem of (3.17), (3.18) and satisfies (3.29), then $v(t,\eta)\triangleq f(t,\frac{e^\eta}{\mu(t)})$ satisfies (3.21), (3.22) and (3.23) for each $\varepsilon>0$, and (3.20) indicates that f coincides with $\mathcal{X}$.

Next, we shall establish feedback formulae for the optimal portfolio and consumption processes. By analogy with Theorem 2.12, for each $t\in[0,T)$, the active range for the running optimal wealth $X_0(t)$ and for the associated standard of living $z_0(t)$ will be

$$\mathcal{D}_t\triangleq\big\{(x',z')\in(0,\infty)\times[0,\infty);\ x'>w(t)z'\big\}. \tag{3.30}$$

Theorem 3.8. *Impose the Assumptions 3.1, 3.2. Then, the feedback formulae (3.1) for the optimal consumption $c_0(\cdot)$ and investment $\pi_0(\cdot)$ of the maximization problem in Definition 2.3 are valid, with*

$$C(t,x,z)\triangleq z+I\big(t,\mu(t)\mathcal{Y}(t,x-w(t)z)\big), \tag{3.31}$$

$$\Pi(t,x,z)\triangleq-(\sigma^*(t))^{-1}\vartheta(t)\cdot\frac{\mathcal{Y}\big(t,x-w(t)z\big)}{x\mathcal{Y}_x\big(t,x-w(t)z\big)}\ , \tag{3.32}$$

for $t\in[0,T)$ and any pair $(x,z)\in\mathcal{D}_t$.

Proof: Recalling Assumption 2.2 for the initial endowment x and standard of living z, i.e. $(x,z)\in\mathcal{D}_0$, we may use (3.14) to extract

$$Y^{(0,\xi)}(t)\big|_{\xi=\mathcal{Y}(0,x-wz)}=\mu(t)\xi(t),\ \text{ where }\ \xi(t)\triangleq\mathcal{Y}\big(t,X_0(t)-w(t)z_0(t)\big).$$

Hence the optimal consumption process of (2.50), thanks to (2.46), (2.52) and (2.39), (2.40), admits the representation

$$c_0(t) = z_0(t) + I\big(t, \mu(t)\xi(t)\big)$$

for $0 \leq t < T$, and (3.31) follows. The substitution $y = \mathcal{Y}(0, x-wz)$ together with (3.14) transform (3.28) to the equivalent formula

$$\beta(t)\Big[X_0(t) - w(t)z_0(t)\Big] + \int_0^t \beta(s)\mu(s)\Big[c_0(s) - z_0(s)\Big]ds$$
$$= x - wz - \int_0^t \beta(s)\xi(s)\mathcal{X}_y\big(s, \xi(s)\big)\vartheta^*(s)dW_0(s).$$

Differentiating (3.19), we obtain $\mathcal{X}_y\big(t, \mathcal{Y}(t, x-w(t)z)\big) = 1/\mathcal{Y}_x(t, x-w(t)z)$ for every $(x,z) \in \mathcal{D}_t$; setting $\xi_x(t) \triangleq \mathcal{Y}_x\big(t, X_0(t) - w(t)z_0(t)\big)$ and using (3.8), the above equation becomes

$$\beta(t)X_0(t) + \int_0^t \beta(s)c_0(s)ds = x - \int_0^t \beta(s)\frac{\xi(s)}{\xi_x(s)}\vartheta^*(s)dW_0(s) + \beta(t)w(t)z_0(t)$$
$$- wz - \int_0^t \beta(s)\delta(s)w(s)\Big[c_0(s) - z_0(s)\Big]ds + \int_0^t \beta(s)z_0(s)ds. \tag{3.33}$$

On the other hand, use (2.26) and (3.10) to compute

$$\beta(t)w(t)z_0(t) - wz = \int_0^t d\Big(\beta(s)w(s)z_0(s)\Big)$$
$$= -\int_0^t \beta(s)r(s)w(s)z_0(s)ds + \int_0^t \beta(s)\Big[\Big(r(s) + \alpha(s) - \delta(s)\Big)w(s) - 1\Big]z_0(s)ds$$
$$+ \int_0^t \beta(s)w(s)\Big(\delta(s)c_0(s) - \alpha(s)z_0(s)\Big)ds$$
$$= \int_0^t \beta(s)\delta(s)w(s)\Big[c_0(s) - z_0(s)\Big]ds - \int_0^t \beta(s)z_0(s)ds, \tag{3.34}$$

and conclude that (3.33) reads

$$\beta(t)X_0(t) + \int_0^t \beta(s)c_0(s)ds = x - \int_0^t \beta(s)\frac{\xi(s)}{\xi_x(s)}\vartheta^*(s)dW_0(s).$$

Comparing this to the stochastic integral equation (2.11) for the wealth process, it follows that the optimal strategy satisfies

$$X_0(t)\pi_0^*(t)\sigma(t) = -\theta^*(t)\frac{\xi(t)}{\xi_x(t)} \,,$$

which implies (3.32). $\diamondsuit$

3.3 The Hamilton-Jacobi-Bellman Equation

We shall now proceed with our Markov-based approach, by investigating in detail the analytical behavior of the value function for the optimization problem (2.31) as a solution of a partial differential equation, widely referred to as the Hamilton-Jacobi-Bellman equation. In this vein, we find it useful to generalize the time-horizon of our asset market $\mathcal{M}_0$ by taking initial date $t \in [0,T]$ rather than zero. Hence, for a fixed commencement time $t \in [0,T]$ and any given capital wealth/initial standard of living pair $(x,z) \in \mathcal{D}_t$ (cf. (3.30)), the wealth process $X^{t,x,\pi,c}(\cdot)$, corresponding to a portfolio strategy $\pi(\cdot)$ and a consumption process $c(\cdot)$, satisfies the stochastic integral equation

$$\begin{aligned} X(s) = x + \int_t^s [r(v)X(v) - c(v)]dv \\ + \int_t^s X(v)\pi^*(v)\sigma(v)dW_0(v), \quad t \le s \le T, \end{aligned} \tag{3.35}$$

and the respective standard of living process $z(\cdot)$ is developed by

$$z(s) = ze^{-\int_t^s \alpha(\theta)d\theta} + \int_t^s \delta(v)e^{-\int_v^s \alpha(\theta)d\theta}c(v)dv, \quad t \le s \le T. \tag{3.36}$$

In this context, we shall call *admissible at the initial condition* (t,x), and denote their class by $\mathcal{A}_0(t,x)$, all portfolio/consumption pairs (π,c) such that $X^{t,x,\pi,c}(s) \ge 0$, $\forall\ s \in [t,T]$, almost surely. Each of these pairs satisfies the budget constrain

$$E^0\left[\int_t^T e^{-\int_t^s r(v)dv}c(s)ds\right] \le x. \tag{3.37}$$

Conversely, for every given consumption plan $c(\cdot)$ satisfying (3.37), there exists a portfolio process $\pi(\cdot)$ so that $(\pi, c) \in \mathcal{A}_0(t,x)$ [cf. Lemma 2.1]. Furthermore, we extend the definition of the dynamic maximization problem of Definition 2.3 as

$$V(t,x,z) \triangleq \sup_{(\pi,c)\in\mathcal{A}'_0(t,x,z)} E\left[\int_t^T u(s, c(s) - z(s))ds\right], \tag{3.38}$$

where

$$\mathcal{A}'_0(t,x,z) \triangleq \Big\{(\pi,c) \in \mathcal{A}_0(t,x);\quad E\left[\int_t^T u^-(s, c(s) - z(s))ds\right] < \infty\Big\},$$

and $V(0,\cdot,\cdot) = V(\cdot,\cdot)$. Assumptions 3.1 and 3.2 imply

$$V(t,x,z) = G\big(t, \mathcal{Y}(t, x - w(t)z)\big), \quad (x,z) \in \mathcal{D}_t, \quad t \in [0,T), \tag{3.39}$$

now with

$$G(t,y) \triangleq E\left[\int_t^T u\big(s, I(s, yY^{(t,1)}(s))\big)ds\right], \quad (t,y) \in [0,T] \times (0,\infty) \tag{3.40}$$

by analogy with (2.59) and (2.60), as $G(0,\cdot) = G(\cdot)$. Clearly

$$V(T,x,z) = 0, \quad \forall\ (x,z) \in \mathcal{D}; \tag{3.41}$$

in fact, for every $t \in [0,T)$, $(x,z) \in \mathcal{D}_t$, we have that $V(t,x,z) < \infty$, and

$$\lim_{(x,z)\to(\chi,\zeta)} V(t,x,z) = \int_t^T u(s, 0^+)ds, \quad \forall\ (\chi,\zeta) \in \partial\mathcal{D}_t \tag{3.42}$$

with $\partial\mathcal{D}_t = \Big\{(x',z') \in [0,\infty)^2;\ x' = w(t)z'\Big\}$ the boundary of $\mathcal{D}_t$ (cf. (2.68)).

Lemma 3.9. *Consider Assumptions 3.1 and 3.2. Then, the function G of (3.40) is of class $C([0,T]\times(0,\infty)) \cap C^{1,2}([0,T)\times(0,\infty))$; and within the set of functions that satisfy the growth condition (3.29), G constitutes the unique solution of the Cauchy problem*

$$G_t(t,y) + \frac{1}{2}\|\vartheta(t)\|^2 y^2 G_{yy}(t,y) - r(t)yG_y(t,y) = -u(t, I(t, y\mu(t))) \quad (3.43)$$
$$\textit{on } [0,T) \times (0,\infty),$$
$$G(T,y) = 0 \quad \textit{on } (0,\infty). \quad (3.44)$$

Additionally,

$$G(t,y) - G(t,h) = y\mathcal{X}(t,y) - h\mathcal{X}(t,h) - \int_h^y \mathcal{X}(t,\xi)d\xi, \quad 0 < h < y < \infty, \quad (3.45)$$
$$G_y(t,y) = y\mathcal{X}_y(t,y), \quad (3.46)$$
$$G_{yy}(t,y) = \mathcal{X}_y(t,y) + y\mathcal{X}_{yy}(t,y), \quad 0 \le t < T, \quad y > 0. \quad (3.47)$$

Proof: To prove (3.43) and (3.44) we follow the same reasoning that led to (3.17) and (3.18), whereas now Remark 3.3 is in use and $v : [0,T]\times\mathbb{R} \to \mathbb{R}$ is the $C([0,T]\times(0,\infty))\cap C^{1,2}([0,T)\times(0,\infty))$ solution of the Cauchy problem

$$v_t(t,\eta) + \frac{1}{2}\|\vartheta(t)\|^2 v_{\eta\eta}(t,\eta) + \left(\frac{\mu'(t)}{\mu(t)} - r(t) - \frac{1}{2}\|\vartheta(t)\|^2\right) v_\eta(t,\eta) \quad (3.48)$$
$$= -u(t, I(t, e^n)), \quad 0 \le t < T, \quad \eta \in \mathbb{R},$$
$$v(T,\eta) = 0, \quad \eta \in \mathbb{R}. \quad (3.49)$$

By analogy with (3.24), employ Itô's formula, (3.12) and (3.48) to obtain

$$dv(s, \log Y^{(t,y)}(s)) = -\, u(s, I(s, Y^{(t,y)}(s)))ds - v_\eta(s, \log Y^{(t,y)}(s))\vartheta^*(s)dW(s),$$

thus, as in (3.25) and (3.26),

$$v(t, \log(y\mu(t))) = E\int_t^T u\big(s, I(s, Y^{(t,y)}(s))\big)ds = G(t,y),$$

which turns out to satisfy the Cauchy problem (3.43), (3.44).

Repeat the computations in (2.77) concerning an initial time $t \neq 0$ to obtain (3.45). Clearly, differentiation of the latter yields (3.46) and (3.47). Concluding, we reason uniqueness like in Remark 3.7. $\Diamond$

Theorem 3.10. (Hamilton-Jacobi-Bellman Equation): *Under the Assumptions 3.1 and 3.2, the value function* $V(t,x,z)$ *of (3.39), (3.41) is continuous on the domain* $\big\{(t,x,z);\ t\in[0,T],\ (x,z)\in\mathcal{D}_t\big\}$ *and of class* $C^{1,2,2}$ *on the domain* $\big\{(t,x,z);\ t\in[0,T),\ (x,z)\in\mathcal{D}_t\big\}$. *It satisfies the boundary conditions (3.41), (3.42), as well as the following Hamilton-Jacobi-Bellman equation*

$$\begin{aligned} V_t(t,x,z) + \max_{\substack{0\le c<\infty\\ \pi\in\mathbb{R}^d}} \bigg\{ &\frac{1}{2}\|\sigma^*(t)\pi\|^2 x^2 V_{xx}(t,x,z) \\ &+ \Big[r(t)x - c + \pi^*\sigma(t)\vartheta(t)x\Big] V_x(t,x,z) \\ &+ \Big[\delta(t)c - \alpha(t)z\Big] V_z(t,x,z) + u(t,c-z)\bigg\} = 0 \end{aligned} \tag{3.50}$$

on the latter domain. In particular, the maximization in (3.50) is attained by the pair $(\Pi(t,x,z), C(t,x,z))$ *of (3.31), (3.32) .*

Proof: We may differentiate (3.19) and (3.39) and make use of (3.46), (3.47) to compute

$$\begin{aligned} \mathcal{X}_t\big(t,\mathcal{Y}(t,x-w(t)z)\big) + \mathcal{X}_y\big(t,\mathcal{Y}(t,x-w(t)z)\big)\mathcal{Y}_t(t,x-w(t)z) &= 0, \\ \mathcal{X}_y\big(t,\mathcal{Y}(t,x-w(t)z)\big)\mathcal{Y}_x(t,x-w(t)z) &= 1, \end{aligned}$$

as well as,

$$\begin{aligned} V_t(t,x,z) &= G_t\big(t,\mathcal{Y}(t,x-w(t)z)\big) \\ &\qquad + G_y\big(t,\mathcal{Y}(t,x-w(t)z)\big)\Big[\mathcal{Y}_t(t,x-w(t)z) \\ &\qquad\qquad - w'(t)z\mathcal{Y}_x(t,x-w(t)z)\Big], \\ V_x(t,x,z) &= \mathcal{Y}(t,x-w(t)z), \\ V_z(t,x,z) &= -w(t)\mathcal{Y}(t,x-w(t)z), \\ V_{xx}(t,x,z) &= \mathcal{Y}_x(t,x-w(t)z), \end{aligned}$$

over $\big\{(t,x,z);\ t\in[0,T),\ (x,z)\in\mathcal{D}_t\big\}$. Thanks to these formulas, and (3.8), the left-hand side of (3.50) becomes

$$\begin{aligned} &G_t\big(t,\mathcal{Y}(t,x-w(t)z)\big) \\ &+ G_y\big(t,\mathcal{Y}(t,x-w(t)z)\big)\Big[\mathcal{Y}_t(t,x-w(t)z) - w'(t)z\mathcal{Y}_x(t,x-w(t)z)\Big] \end{aligned}$$

$$
\begin{aligned}
&+ r(t)x\mathcal{Y}(t, x - w(t)z) + \alpha(t)w(t)z\mathcal{Y}(t, x - w(t)z) \\
&+ \max_{0 \le c < \infty} \Big[u(t, c - z) - c\,\mu(t)\mathcal{Y}(t, x - w(t)z)\Big] \\
&+ \max_{\pi \in \mathbb{R}^d} \left[\frac{1}{2}\|\sigma^*(t)\pi\|^2 x^2 \mathcal{Y}_x(t, x - w(t)z) + \pi^*\sigma(t)\vartheta(t)x\mathcal{Y}(t, x - w(t)z)\right].
\end{aligned}
$$

Since both representations subject to maximization are strictly concave, their derivatives vanish by the optimal values of π and c. Solving these two equations, we arrive at the feedback forms of (3.31) and (3.32). Substituting next the maximizers into the previous expression, we equivalently have

$$
\begin{aligned}
&G_t\big(t, \mathcal{Y}(t, x - w(t)z)\big) \\
&+ G_y\big(t, \mathcal{Y}(t, x - w(t)z)\big)\Big[\mathcal{Y}_t(t, x - w(t)z) - w'(t)z\mathcal{Y}_x(t, x - w(t)z)\Big] \\
&+ r(t)x\mathcal{Y}(t, x - w(t)z) + \alpha(t)w(t)z\mathcal{Y}(t, x - w(t)z) \\
&+ u\Big(t, I\big(t, \mu(t)\mathcal{Y}(t, x - w(t)z)\big)\Big) \\
&\qquad\qquad\qquad - \mu(t)\mathcal{Y}(t, x - w(t)z)\Big[z + I\big(t, \mu(t)\mathcal{Y}(t, x - w(t)z)\big)\Big] \\
&- \frac{1}{2}\|\vartheta(t)\|^2 \frac{\mathcal{Y}^2(t, x - w(t)z)}{\mathcal{Y}_x(t, x - w(t)z)}.
\end{aligned}
$$

Let us set $y = \mathcal{Y}(t, x - w(t)z)$, thus $x = \mathcal{X}(t, y) + w(t)z$, and employ (3.46), (3.47) and (3.43) in order to simplify this as

$$
\begin{aligned}
G_t(t, y) &- y\Big[\mathcal{X}_t(t, y) + w'(t)z\Big] + r(t)y\Big[\mathcal{X}(t, y) + w(t)z\Big] + \alpha(t)w(t)yz \\
&+ u\big(t, I(t, y\mu(t))\big) - \mu(t)y\Big[z + I(t, y\mu(t))\Big] - \frac{1}{2}\|\vartheta(t)\|^2 y^2 \mathcal{X}_y(t, y) \\
= &- \frac{1}{2}\|\vartheta(t)\|^2 y^2 G_{yy}(t, y) + r(t)yG_y(t, y) - y\mathcal{X}_t(t, y) \\
&+ r(t)y\mathcal{X}(t, y) - \mu(t)yI(t, y\mu(t)) - \frac{1}{2}\|\vartheta(t)\|^2 y^2 \mathcal{X}_y(t, y) \\
&- zy\Big[w'(t) - r(t)w(t) - \alpha(t)w(t) + \mu(t)\Big] \\
= &- y\Big[\mathcal{X}_t(t, y) + \frac{1}{2}\|\vartheta(t)\|^2 y^2 \mathcal{X}_{yy}(t, y) + \big(\|\vartheta(t)\|^2 - r(t)\big)y\mathcal{X}_y(t, y) \\
&\qquad - r(t)\mathcal{X}(t, y) + \mu(t)I(t, y\mu(t))\Big] = 0,
\end{aligned}
$$

where the last two equalities follow by (3.10) and Lemma 3.5. ◊

Remark 3.11. *Carrying out the maximization according to the proof of Theorem 3.10, the equation (3.50) takes the conventional form*

$$V_t(t,x,z) + \mathrm{H}\Big(V_{xx}(t,x,z), V_x(t,x,z), V_z(t,x,z), t, x, z\Big) = 0, \tag{3.51}$$

where

$$\begin{aligned}\mathrm{H}(\mathrm{A},p,q,t,x,z) \triangleq &- \|\vartheta(t)\|^2 \, \frac{p^2}{2\mathrm{A}} + \Big[r(t)x - z - I(t,p-\delta(t)q)\Big]p \\ &+ \Big[(\delta(t)-\alpha(t))z + \delta(t)I(t,p-\delta(t)q)\Big]q \\ &+ u\big(t, I(t,p-\delta(t)q)\big)\end{aligned}$$

for $\mathrm{A} < 0$, $p > 0$ *and* $q < 0$. *Notice that we have achieved a closed-form solution of the* strongly nonlinear *Hamilton-Jacobi-Bellman equation (3.51), by solving instead the two* linear *equations (3.17), (3.43) subject to the appropriate initial and growth conditions, and then performing the composition (3.39).*

As a consequence, (3.50) provides a *necessary condition* that must be satisfied by the value function V of (3.38). On the contrary, due to the absence of an appropriate growth condition for V as each component of $(x,z) \in \mathcal{D}_t$ increases to infinity, (3.50) fails to be also sufficient; in other words, we cannot claim directly that V is the $C^{1,2,2}$ unique solution of (3.50) on $\big\{(t,x,z);\ t \in [0,T),\ (x,z) \in \mathcal{D}_t\big\}$ with boundary conditions (3.41), (3.42). We decide though to treat this matter by establishing a *necessary and sufficient condition* for the *convex dual* of V, defined as

$$\widetilde{V}(t,y) \triangleq \sup_{(x,z)\in\mathcal{D}_t} \big\{V(t,x,z) - \big(x - w(t)z\big)y\big\}, \quad y \in \mathbb{R}, \tag{3.52}$$

by analogy with (2.67). Doing so, we evade investigating the solvability of the *nonlinear* partial differential equation (3.50), since it turns out that $\widetilde{V}$ is equivalently characterized as the unique solution of a *linear* parabolic partial differential equation (cf. (3.57)) and V can be easily recovered by inverting the above Legendre-Fenchel transformation to have

$$V(t,x,z) = \inf_{y\in\mathbb{R}} \left\{ \widetilde{V}(t,y) + \big(x - w(t)z\big)y \right\}, \quad (x,z) \in \mathcal{D}_t.$$

We formalize these considerations within the context of the subsequent theorem.

Theorem 3.12. (Convex Dual of $V(t,\cdot,\cdot)$): *Under the Assumptions 3.1, 3.2, and given any $t \in [0,T)$, the function $V(t,\cdot,\cdot)$ is a generalized utility function, as defined in Theorem 2.17, and*

$$V_x(t,x,z) = \mathcal{Y}(t, x - w(t)z), \quad \forall\ (x,z) \in \mathcal{D}_t, \tag{3.53}$$

$$V_z(t,x,z) = -w(t)\mathcal{Y}(t, x - w(t)z), \quad \forall\ (x,z) \in \mathcal{D}_t. \tag{3.54}$$

Furthermore, the convex dual $\widetilde{V}(t,\cdot)$ of $V(t,\cdot,\cdot)$ of (3.52) is represented as

$$\widetilde{V}(t,y) = G(t,y) - y\mathcal{X}(t,y) = E\int_t^T \tilde{u}(s, yY^{(t,1)}(s))ds, \tag{3.55}$$

and satisfies

$$\widetilde{V}_y(t,y) = -\mathcal{X}(t,y), \tag{3.56}$$

for every $(t,y) \in [0,T]\times(0,\infty)$. Moreover, $\widetilde{V}$ is continuous on $[0,T]\times(0,\infty)$, of class $C^{1,2}$ on $[0,T)\times(0,\infty)$, and solves the Cauchy problem

$$\widetilde{V}_t(t,y) + \frac{1}{2}\|\vartheta(t)\|^2 y^2 \widetilde{V}_{yy}(t,y) - r(t)y\widetilde{V}_y(t,y) \tag{3.57}$$

$$= -\tilde{u}(t, y\mu(t)) \quad \text{on } [0,T)\times(0,\infty),$$

$$\widetilde{V}(T,y) = 0 \quad \text{on } (0,\infty). \tag{3.58}$$

If $\tilde{v}$ is a function of class $C\left([0,T]\times(0,\infty)\right) \cap C^{1,2}\left([0,T)\times(0,\infty)\right)$ that satisfies (3.57), (3.58), if $\tilde{v}_y$ has the same order of smoothness as $\tilde{v}$, and if the growth condition (3.29) holds for both $\tilde{v}$ and $\tilde{v}_y$, then $\widetilde{V} = \tilde{v}$.

Proof: Setting claim (2.68) aside, the first two parts of this result represent the Markovian analogue of Theorem 2.17. Therefore, all the respective assertions, including (3.53)−(3.56), can be proved through a similar methodology.

The claimed degree of regularity for $\widetilde{V}$ comes from Lemmata 3.5, 3.9 and formula (3.55). Put now together equations (3.55), (3.43), (3.17) and (2.18) to derive (3.57). The boundary condition (3.58) follows by (3.55).

To argue uniqueness, we assume the existence of another function $\tilde{v}$ satisfying (3.57), (3.58), such that $\tilde{v}$ and its partial derivative $\tilde{v}_y$ have the same smoothness and satisfy the growth condition (3.29); we shall show that $\widetilde{V} = \tilde{v}$. More precisely, differentiate (3.57), (3.58) and use (2.19) to verify that $-\tilde{v}_y$ is a solution of (3.17), (3.18), and consider Remark 3.7 to derive that $-\tilde{v}_y = \mathcal{X}$. It is also easy to check that $\tilde{v} - y\tilde{v}_y$ satisfies (3.43) and (3.44), thus $G = \tilde{v} - y\tilde{v}_y$, according to Lemma 3.9. Finally, (3.55) implies that $\widetilde{V} = \tilde{v}$. $\Diamond$

We illustrate now the significance of Theorem 3.12 by computing the value function and the feedback formulas for the optimal consumption and portfolio plans, in several examples.

Example 3.13. *(Logarithmic utility).* Let $u(t,x) = \log x$ for $(t,x) \in [0,T] \times (0,\infty)$. We have $I(t,y) = 1/y$ and $\tilde{u}(t,y) = -\log y - 1$ for $(t,y) \in [0,T] \times (0,\infty)$. Motivated by the non-homogeneous term of (3.57), we seek appropriate functions $\nu, m : [0,T] \to \mathbb{R}$ such that

$$\tilde{v}(t,y) \triangleq -\nu(t)\log(y\mu(t)) - m(t) \tag{3.59}$$

satisfies (3.57), (3.58). Indeed, this is the case if and only if

$$\nu(t) = T - t, \tag{3.60}$$

$$m(t) = \int_t^T \left[1 - (T-s)\left(\frac{1}{2}\|\vartheta(s)\|^2 + r(s) - \frac{\mu'(s)}{\mu(s)}\right)\right] ds, \tag{3.61}$$

for $0 \le t \le T$. It follows that both $\tilde{v}$ given by (3.59) $-$ (3.61) and $\tilde{v}_y(t,y) = -\nu(t)/y$ are of class $C([0,T]\times(0,\infty)) \cap C^{1,2}([0,T)\times(0,\infty))$, and satisfy the growth condition (3.29). From Theorem 3.12, $\tilde{v}$ is the unique solution of the Cauchy problem (3.57), (3.58), thus $\widetilde{V} \equiv \tilde{v}$, and

$$\mathcal{X}(t,y) = \frac{\nu(t)}{y}, \;\; G(t,y) = \nu(t)\Big[1 - \log(y\mu(t))\Big] - m(t), \quad (t,y) \in [0,T]\times(0,\infty).$$

Therefore,

$$\mathcal{Y}(t,x) = \frac{\nu(t)}{x}, \quad x \in (0,\infty),$$
$$V(t,x,z) = \nu(t)\log\left(\frac{x - w(t)z}{\nu(t)\mu(t)}\right) + \nu(t) - m(t), \quad (x,z) \in \mathcal{D}_t,$$

and the feedback formulae (3.31), (3.32) for the optimal consumption and portfolio are

$$C(t,x,z) = z + \frac{x - w(t)z}{\nu(t)\mu(t)}, \quad (x,z) \in \mathcal{D}_t,$$
$$\Pi(t,x,z) = (\sigma^*(t))^{-1}\vartheta(t)\frac{x - w(t)z}{x}, \quad (x,z) \in \mathcal{D}_t,$$

for every $0 \le t < T$.

Example 3.14. *(Power utility).* For $p \in (-\infty,1) \setminus \{0\}$, let $u(t,x) = x^p/p$, $\forall\ (t,x) \in [0,T] \times (0,\infty)$. Now, $I(t,y) = y^{1/(p-1)}$ and $\tilde{u}(t,y) = \frac{1-p}{p}\, y^{p/(p-1)}$ for $(t,y) \in [0,T] \times (0,\infty)$. We seek for a function $\nu : [0,T] \to \mathbb{R}$ such that

$$\tilde{v}(t,y) = \frac{1-p}{p}\nu(t)\big(y\mu(t)\big)^{p/(p-1)}$$

satisfies (3.57), (3.58). This happens if and only if $\nu(\cdot)$ solves the ordinary differential equation

$$\nu'(t) + q(t)\nu(t) = -1, \quad t \in [0,T),$$
$$\nu(T) = 0$$

with

$$q(t) \triangleq \frac{p}{(1-p)^2}\left[\frac{1}{2}\|\vartheta(t)\|^2 + (1-p)\left(r(t) - \frac{\mu'(t)}{\mu(t)}\right)\right];$$

thus,

$$\nu(t) = \int_t^T e^{\int_t^s q(\theta)d\theta} ds.$$

Also, $\tilde{v}$ and

$$\tilde{v}_y(t,y) = -\frac{1}{y}\nu(t)\big(y\mu(t)\big)^{p/(p-1)}$$

have the smoothness claimed in Theorem 3.12, and satisfy condition (3.29). It turns out that $\widetilde{V} \equiv \tilde{v}$ and

$$\mathcal{X}(t,y) = \nu(t) y^{1/(p-1)} (\mu(t))^{p/(p-1)}, \quad G(t,y) = \frac{1}{p}\nu(t)\big(y\mu(t)\big)^{p/(p-1)},$$

$\forall\ (t,y) \in [0,T] \times (0,\infty)$. Moreover, for $0 \le t < T$, we have

$$\begin{aligned}
\mathcal{Y}(t,x) &= \frac{x^{p-1}}{(\nu(t))^{p-1}(\mu(t))^p}, \quad x \in (0,\infty),\\
V(t,x,z) &= \frac{1}{p}\nu(t)\left(\frac{x - w(t)z}{\nu(t)\mu(t)}\right)^p, \quad (x,z) \in \mathcal{D}_t,\\
C(t,x,z) &= z + \frac{x - w(t)z}{\nu(t)\mu(t)}, \quad (x,z) \in \mathcal{D}_t,\\
\Pi(t,x,z) &= \left(\sigma^*(t)\right)^{-1} \vartheta(t) \frac{x - w(t)z}{(1-p)x}, \quad (x,z) \in \mathcal{D}_t.
\end{aligned}$$

Remark 3.15. *Within the context of this chapter Detemple and Zapatero (1992) obtain a closed form representation for the optimal portfolio, by application of the Clark (1970) formula; this reduces to "feedback form" only for logarithmic and power-form utility functions. These feedback formulas now become a special case of (3.32) (cf. Examples 3.13 and 3.14) that was established in Theorem 3.8 for any arbitrary utility function.*

4 The Role of Stochastic Partial Differential Equations In Utility Optimization Under Habit Formation

4.1 Optimal Portfolio-Consumption Decisions in a Dynamic Framework

In Chapter 3 we specialized the results of Section 2.5 to the case of deterministic model coefficients. Employing carefully the interplay of partial differential equation theory with the Feynman-Kac results, we arrived at an initial-boundary value problem for the value function V of (3.38), also referred to as the Hamilton-Jacobi-Bellman equation (cf. Theorem 3.10). We were also able to obtain feedback-form expressions for the optimal policies. In this chapter we return to the stochastic control problem introduced in Definition 2.3, and prepare the ground for a systematic analysis based on the ideas of *dynamic programming*. Our motivation goes back to Theorem 2.12, which reveals the *dynamic* nature of the optimal wealth/standard of living pair $\big(X_0(\cdot), z_0(\cdot)\big)$ in terms of a stochastically developing range. Since the market model is not Markovian anymore, our analysis will be based on the recently developed theory of backward *stochastic* partial differential equations and their interrelation with appropriate *adapted versions of stochastic* Feynman-Kac formulas. This interplay will be based on the generalized Itô-Kunita-Wentzell formula, and will eventually permit us to show that the value function of problem (2.31) satisfies a nonlinear, backward *stochastic* Hamilton-Jacobi-Bellman partial differential equation of parabolic type.

By analogy with (3.1), we shall establish *stochastic* "feedback formulas" representing explicitly the optimal portfolio $\pi_0(t)$ and consumption $c_0(t)$ in closed forms, in terms of the current wealth $X_0(t)$ and standard of living $z_0(t)$. In particular, we shall get hold of suitable random fields $\mathfrak{C} : [0,T) \times (0,\infty) \times [0,\infty) \times \Omega \to (0,\infty)$ and $\mathfrak{P} : [0,T) \times (0,\infty) \times [0,\infty) \times \Omega \to \mathbb{R}^d$, for which

$$c_0(t) = \mathfrak{C}(t, X_0(t), z_0(t)) \quad \text{and} \quad \pi_0(t) = \mathfrak{P}(t, X_0(t), z_0(t)), \quad 0 \le t < T \quad (4.1)$$

hold almost surely. On the other hand, it will be clear from the randomness of $\mathfrak{C}$ and $\mathfrak{P}$ that the current level of wealth/standard of living will no longer constitute a sufficient statistic for the maximization problem of (2.31).

The conditions listed below will allow us to present the main concepts of our dynamic approach, with a minimum of technical fuss.

Assumption 4.1. *The assumptions made on the model coefficients $r(\cdot)$, $b(\cdot)$, $\vartheta(\cdot)$, $\sigma(\cdot)$, $\alpha(\cdot)$ and $\delta(\cdot)$ in Chapter 2 are in force. Moreover, these coefficients are continuous, $\delta(\cdot)$ is differentiable and the relative risk process $\vartheta(\cdot)$ is universally bounded, satisfying assumption (3.2). Finally, it will be assumed throughout that the expression $r(\cdot) - \delta(\cdot) + \alpha(\cdot)$ is non-random.*

This last assumption on $r(\cdot)-\delta(\cdot)+\alpha(\cdot)$ is rather severe, and can actually be omitted. However, it will be crucial in our effort to keep the required analysis and notation at manageable levels, without obscuring by technicalities the essential ideas of our reasoning.

In accordance with the comment following Assumption 3.1, the process $W_0(\cdot)$ of (2.9) is a d-dimensional Brownian motion relative to the filtration $\mathbb{F}$ under the equivalent martingale measure P^0, as constructed in (3.3).

Assumption 4.2. *Conditions (i), (ii) of Assumption 3.2 remain active, in conjunction with the following assumptions:*

(iii) *for each $t \in [0,T]$, the mappings $y \mapsto u(t,y)$ and $y \mapsto I(t,y)$ are of class $C^4((0,\infty))$;*

(iv) *the function $I'(t,y) = \frac{\partial}{\partial y} I(t,y)$ is strictly negative for every $(t,y) \in [0,T] \times (0,\infty)$;*

(v) *for every $t \in [0,T]$, the mapping $y \mapsto g(t,y) \triangleq yI'(t,y)$ is increasing and concave.*

Remark 4.3. *The first part of Remark 3.3 still holds, and the composite function $u(t, I(t,\cdot))$ has the order of smoothness posited in Assumption 4.2(iii) for its components, for every $t \in [0,T]$.*

Preparing the ground of our approach, we state the following useful implication of the generalized Itô-Kunita-Wentzell formula (e.g. Kunita (1990), Section 3.3, pp 92-93). This will enable us to carry out computations in a stochastically modulated dynamic framework.

Proposition 4.4. *Suppose that the random field* $\mathbf{F} : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ *is of class* $C^{0,2}([0,T] \times \mathbb{R}^n)$ *and satisfies*

$$\mathbf{F}(t,\mathrm{x}) = \mathbf{F}(0,\mathrm{x}) + \int_0^t \mathbf{f}(s,\mathrm{x})ds + \int_0^t \mathbf{g}^*(s,\mathrm{x})dW(s), \quad \forall\ (t,\mathrm{x}) \in [0,T] \times \mathbb{R}^n,$$

almost surely. Here $\mathbf{g} = (\mathbf{g}^{(1)}, \ldots, \mathbf{g}^{(d)})$*,* $\mathbf{g}^{(j)} : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}$*,* $j = 1, \ldots, d$ *are* $C^{0,2}([0,T] \times \mathbb{R}^n)$*,* $\mathbb{F}$*-adapted random fields, and* $\mathbf{f} : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ *is a* $C^{0,1}([0,T] \times \mathbb{R}^n)$ *random field. Furthermore, let* $\mathbf{X} = (\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(n)})$ *be a vector of continuous semimartingales with decompositions*

$$\mathbf{X}^{(i)}(t) = \mathbf{X}^{(i)}(0) + \int_0^t \mathbf{b}^{(i)}(s)ds + \int_0^t (\mathbf{h}^{(i)}(s))^* dW(s); \quad i = 1, \ldots, n,$$

where $\mathbf{h}^{(i)} = (\mathbf{h}^{(i,1)}, \ldots, \mathbf{h}^{(i,d)})$ *is an* $\mathbb{F}$*-progressively measurable, almost surely square integrable vector process, and* $\mathbf{b}^{(i)}(\cdot)$ *is an almost surely integrable process. Then* $\mathbf{F}(\cdot, \mathbf{X}(\cdot))$ *is also a continuous semimartingale, with decomposition*

$$\begin{aligned}
\mathbf{F}\big(t, \mathbf{X}(t)\big) &= \mathbf{F}\big(0, \mathbf{X}(0)\big) + \int_0^t \mathbf{f}\big(s, \mathbf{X}(s)\big)ds + \int_0^t \mathbf{g}^*\big(s, \mathbf{X}(s)\big)dW(s) \\
&+ \sum_{i=1}^n \int_0^t \frac{\partial}{\partial \mathbf{x}_i}\mathbf{F}\big(s, \mathbf{X}(s)\big)\mathbf{b}^{(i)}(s)ds + \sum_{i=1}^n \int_0^t \frac{\partial}{\partial \mathbf{x}_i}\mathbf{F}\big(s, \mathbf{X}(s)\big)\big(\mathbf{h}^{(i)}(s)\big)^* dW(s) \\
&+ \sum_{j=1}^d \sum_{i=1}^n \int_0^t \frac{\partial}{\partial \mathbf{x}_i}\mathbf{g}^{(j)}\big(s, \mathbf{X}(s)\big)\mathbf{h}^{(i,j)}(s)ds \qquad (4.2) \\
&+ \frac{1}{2}\sum_{\ell=1}^d \sum_{i=1}^n \sum_{k=1}^n \int_0^t \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_k}\mathbf{F}\big(s, \mathbf{X}(s)\big)\mathbf{h}^{(i,\ell)}(s)\mathbf{h}^{(k,\ell)}(s)ds
\end{aligned}$$

for every $0 \le t \le T$*.*

The following notation will also be in use throughout the chapter.

Notation 4.5. For any integer $k \geq 0$, let $C^k(\mathbb{R}^n, \mathbb{R}^d)$ denote the set of functions from $\mathbb{R}^n$ to $\mathbb{R}^d$ that are continuously differentiable up to order k. In addition, for any $1 \leq p \leq \infty$, any Banach space $\mathbb{X}$ with norm $\|\cdot\|_{\mathbb{X}}$, and any sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, let

- $\mathbb{L}^p_{\mathcal{G}}(\Omega, \mathbb{X})$ denote the set of all $\mathbb{X}$-valued, $\mathcal{G}$-measurable random variables $\mathbf{X}$ such that $E\|\mathbf{X}\|^p_{\mathbb{X}} < \infty$;
- $\mathbb{L}^p_{\mathbb{F}}(0, T; \mathbb{X})$ denote the set of all $\mathbb{F}$-progressively measurable, $\mathbb{X}$-valued processes $\mathbf{X} : [0, T] \times \Omega \to \mathbb{X}$ such that $\int_0^T \|\mathbf{X}(t)\|^p_{\mathbb{X}} dt < \infty,$ a.s.;
- $\mathbb{L}^p_{\mathbb{F}}(0, T; \mathbb{L}^p(\Omega; \mathbb{X}))$ denote the set of all $\mathbb{F}$-progressively measurable, $\mathbb{X}$-valued processes $\mathbf{X} : [0, T] \times \Omega \to \mathbb{X}$ such that $\int_0^T E\|\mathbf{X}(t)\|^p_{\mathbb{X}} dt < \infty$;
- $C_{\mathbb{F}}([0, T]; \mathbb{X})$ denote the set of all continuous, $\mathbb{F}$-adapted processes $\mathbf{X}(\cdot, \omega) : [0, T] \to \mathbb{X}$ for P-a.e. $\omega \in \Omega$.

Define similarly the set $C_{\mathbb{F}}([0, T]; \mathbb{L}^p(\Omega; \mathbb{X}))$, and let $\mathbb{R}^+$ stand for the positive real numbers.

4.2 A Stochastic Version of the Feedback Formulae

Recall the stochastic processes introduced in (3.4), (3.5), (3.6), and (3.11) with the dynamics of (3.12), (3.13). Thanks to Assumption 4.1, the computations leading to (3.7), in terms of the definitions (3.8), (3.9) and (3.10), are also valid in the present setting where the model coefficients are random in general. Clearly, $w(\cdot)$ remains deterministic as well. However, computations similar to those leading to (3.14) lead now to the generalized formula

$$X_0(t) = w(t)z_0(t) + \mathfrak{X}\left(t, \frac{Y^{(0,\xi)}(t)}{\mu(t)}\right), \qquad 0 \leq t \leq T \tag{4.3}$$

with $\xi = \mathcal{Y}(x - wz)$, where the random field $\mathfrak{X} : [0, T] \times \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ is given by

$$\mathfrak{X}(t, y) \triangleq E^0_t \left[\int_t^T e^{-\int_t^s r(v)dv} \mu(s) I(s, yY^{(t,1)}(s)) ds \right]. \tag{4.4}$$

This reduces to $\mathcal{X}$ of (3.15), i.e. $\mathfrak{X}(\cdot,\cdot) \equiv \mathcal{X}(\cdot,\cdot)$, in the Markovian framework discussed in Chapter 3. Moreover, a comparison of (2.48), (3.15), (3.16) and (4.4) divulges the dynamic and stochastic evolution of the function $\mathcal{X}(\cdot)$ as a random field in the sense that $\mathcal{X}(\cdot) = \mathcal{X}(0,\cdot) = \mathfrak{X}(0,\cdot)$.

We proceed with the derivation of the random fields $\mathfrak{C}$ and $\mathfrak{P}$ in (4.1) by formulating first a semimartingale decomposition for the random field $\mathfrak{X}$ of (4.4). A significant role in this program will be played by an appropriate backward stochastic partial differential equation, whose unique adapted solution will lead, via the generalized Itô-Kunita-Wentzell rule, to a stochastic Feynman-Kac formula and consequently to the desired decomposition for $\mathfrak{X}$.

Let us start by looking at the the Cauchy problem for the parabolic *Backward Stochastic PDE* (BSPDE)

$$
\begin{aligned}
-d\mathcal{U}(t,\eta) = \Bigg[&\frac{1}{2}\|\vartheta(t)\|^2\mathcal{U}_{\eta\eta}(t,\eta) + \left(\frac{\mu'(t)}{\mu(t)} - r(t) + \frac{1}{2}\|\vartheta(t)\|^2\right)\mathcal{U}_\eta(t,\eta) \\
&- r(t)\mathcal{U}(t,\eta) - \vartheta^*(t)\Psi_\eta(t,\eta) + \mu(t)I(t,e^\eta)\Bigg]dt - \Psi^*(t,\eta)dW_0(t), \\
&\eta \in \mathbb{R}, \quad 0 \le t < T, \qquad (4.5) \\
\mathcal{U}(T,\eta) = 0,& \quad \eta \in \mathbb{R} \qquad (4.6)
\end{aligned}
$$

for the pair of $\mathbb{F}$-adapted random fields $\mathcal{U}$ and Ψ. According to Assumptions 4.1, 4.2, and the study of parabolic backward stochastic partial differential equations by Ma and Yong (1997), the problem (4.5), (4.6) admits a unique solution pair $(\mathcal{U},\Psi) \in C_{\mathbb{F}}\big([0,T];\mathbb{L}^2(\Omega;C^3(\mathbb{R}^+))\big)\times\mathbb{L}^2_{\mathbb{F}}\big(0,T;\mathbb{L}^2(\Omega;C^2(\mathbb{R}^+;\mathbb{R}^d))\big)$. Apply the generalized Itô-Kunita-Wentzell formula (cf. Proposition 4.4) for a fixed pair $(t,y) \in [0,T) \times \mathbb{R}^+$, in conjunction with the dynamics of (3.13) and the equation of (4.5), to get

$$
\begin{aligned}
d&\left[e^{-\int_t^s r(v)dv}\mathcal{U}(s,\log Y^{(t,y)}(s))\right] \\
&= -e^{-\int_t^s r(v)dv}\mu(s)I(s,Y^{(t,y)}(s))ds \qquad (4.7) \\
&\quad - e^{-\int_t^s r(v)dv}\left[\vartheta(s)\mathcal{U}_\eta(s,\log Y^{(t,y)}(s)) - \Psi(s,\log Y^{(t,y)}(s))\right]^* dW_0(s),
\end{aligned}
$$

almost surely. Adopting the proof of Corollary 6.2 in the above citation (p. 76), integrate over $[t,T]$, take conditional expectations with respect to the martingale measure P^0, and make use of (4.4), (4.6) to end up with

$$\mathfrak{X}(t,y) = \mathcal{U}(t, \log(y\mu(t))) \tag{4.8}$$

for every $(t,y) \in [0,T] \times \mathbb{R}^+$. We may define, accordingly, the random field $\Psi^{\mathfrak{X}} : [0,T] \times \mathbb{R}^+ \times \Omega \to \mathbb{R}^d$ by

$$\Psi^{\mathfrak{X}}(t,y) \triangleq \Psi(t, \log(y\mu(t))), \tag{4.9}$$

and state the subsequent result.

Lemma 4.6. *Considering Assumptions 4.1 and 4.2, the pair of random fields* $(\mathfrak{X}, \Psi^{\mathfrak{X}})$*, where* $\mathfrak{X}$ *is provided by (4.4) and* $\Psi^{\mathfrak{X}}$ *by (4.9), is the unique* $C_{\mathbb{F}}\big([0,T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}^+))\big) \times \mathbb{L}^2_{\mathbb{F}}\big(0,T; \mathbb{L}^2(\Omega; C^2(\mathbb{R}^+;\mathbb{R}^d))\big)$ *solution of the Cauchy problem*

$$\begin{aligned} -d\mathfrak{X}(t,y) = \Bigg[&\frac{1}{2}\|\vartheta(t)\|^2 y^2 \mathfrak{X}_{yy}(t,y) + \big(\|\vartheta(t)\|^2 - r(t)\big)\, y\mathfrak{X}_y(t,y) - r(t)\mathfrak{X}(t,y) \\ &- \vartheta^*(t) y \Psi^{\mathfrak{X}}_y(t,y) + \mu(t) I(t, y\mu(t)) \Bigg] dt - \big(\Psi^{\mathfrak{X}}(t,y)\big)^* \, dW_0(t) \\ \text{on } [0,T) \times \mathbb{R}^+, & \end{aligned} \tag{4.10}$$

$$\mathfrak{X}(T,y) = 0 \quad \text{on } \ \mathbb{R}^+, \tag{4.11}$$

almost surely. Furthermore, for each $t \in [0,T)$*, we have that* $\mathfrak{X}(t,0^+) = \infty$*,* $\mathfrak{X}(t,\infty) = 0$ *and* $\mathfrak{X}(t,\cdot)$ *is strictly decreasing, establishing the existence of a strictly decreasing inverse random field* $\Upsilon(t,\cdot,\cdot) : \mathbb{R}^+ \times \Omega \xrightarrow{onto} \mathbb{R}^+$*, such as*

$$\mathfrak{X}(t, \Upsilon(t,x)) = x, \qquad \textit{for all } x \in \mathbb{R}^+, \tag{4.12}$$

almost surely. The random field Υ *is of class* $C_{\mathbb{F}}\big([0,T); C^3(\mathbb{R}^+)\big)$.

Proof: From (4.5), (4.6), (4.8) and (4.9), it is verified directly that the pair of random fields $(\mathfrak{X}, \Psi^{\mathfrak{X}})$ possesses the desired regularity and constitutes the unique solution of the Cauchy problem (4.5), (4.6), almost surely.

Next, we shall verify that $\mathfrak{X}_y(t,y)$ is strictly negative, almost surely. To this end, let $(t,y) \in [0,T) \times \mathbb{R}^+$, $h > 0$, and invoke the (strict) decrease of

$I(t,\cdot)$, coupled with (2.3), to certify that

$$\frac{1}{h}\left[\mathfrak{X}(t,y)-\mathfrak{X}(t,y+h)\right]$$
$$\geq e^{-\varrho}\ E_t^0\left[\int_t^T \frac{1}{h}\Big\{I(s,yY^{(t,1)}(s))-I\big(s,(y+h)Y^{(t,1)}(s)\big)\Big\}ds\right].$$

By the mean-value theorem, there is a real number $y_h\in[y,y+h]$ such that

$$I(s,yY^{(t,1)}(s))-I\big(s,(y+h)Y^{(t,1)}(s)\big)=-hY^{(t,1)}(s)I'\big(s,y_hY^{(t,1)}(s)\big),$$

and conditions (2.3), (2.27), (3.2) imply the inequality $Y^{(t,1)}(s)\leq\phi(s)Z_0^t(s)$, in terms of the deterministic function

$$\phi(t)\triangleq(1+\Delta w(t))e^{\varrho+\kappa_2^2(T-t)}$$

and the P^0-martingale

$$Z_0^t(s)\triangleq\exp\left\{-\int_t^s\vartheta^*(v)dW_0(v)-\frac{1}{2}\int_t^s\|\vartheta(v)\|^2dv\right\},\quad t\leq s\leq T. \tag{4.13}$$

Due to Assumption 4.2(v), the right-hand side of the former inequality achieves the lower bounds

$$\begin{aligned}&-e^{-\varrho}\ E_t^0\left[\int_t^T\frac{1}{y_h}\ g\big(s,y_h\phi(s)Z_0^t(s)\big)ds\right]\\&\geq -e^{-\varrho}\int_t^T\frac{1}{y_h}\ g\Big(s,y_h\phi(s)E_t^0\big(Z_0^t(s)\big)\Big)ds\\&=-e^{-\varrho}\int_t^T I'(s,y_h\phi(s))\phi(s)ds,\end{aligned}$$

where we have also used Jensen's inequality. Passing to the limit as $h\downarrow 0$, we obtain from Fatou's lemma

$$-\mathfrak{X}_y(t,y)\geq -e^{-\varrho}\int_t^T I'(s,y\phi(s))\phi(s)ds>0.$$

According to the implicit function theorem, the inverse random field Υ : $[0,T)\times\mathbb{R}^+\times\Omega\xrightarrow{onto}\mathbb{R}^+$ of $\mathfrak{X}$ exists almost surely, in the context of (4.12);

in fact, the two random fields enjoy the same order of regularity on their respective domains. Concluding, the claimed values of $\mathfrak{X}(t, 0^+)$ and $\mathfrak{X}(t, \infty)$ are easily confirmed, respectively, by the monotone and dominated convergence theorems. $\diamondsuit$

Remark 4.7. *At this point, we should note that Lemma 4.6 assigns to the pair of random fields* $(\mathfrak{X}, \Psi^{\mathfrak{X}})$ *an additional order of smoothness than is required in order to solve the stochastic partial differential equation (4.10), (4.11). Nevertheless, this extra smoothness allows us to apply the Itô-Kunita-Wentzell formula, as we did already in (4.7).*

Furthermore, the above lemma yields the representation

$$\mathfrak{X}(t,y) = \int_t^T \left[\frac{1}{2}\|\vartheta(s)\|^2 y^2 \mathfrak{X}_{yy}(s,y) + \left(\|\vartheta(s)\|^2 - r(s)\right) y \mathfrak{X}_y(s,y) - r(s)\mathfrak{X}(s,y) - \vartheta^*(s) y \Psi^{\mathfrak{X}}_y(s,y) + \mu(s) I(s, y\mu(s))\right] ds - \int_t^T \left(\Psi^{\mathfrak{X}}(s,y)\right)^* dW_0(s)$$

for the pair $(\mathfrak{X}, \Psi^{\mathfrak{X}})$, *namely, the semimartingale decomposition of the stochastic processes* $\mathfrak{X}(\cdot, y)$ *defined in (4.4) for each* $y \in \mathbb{R}^+$.

The random field Υ *represents the random dynamic extension of the function* $\mathcal{Y}$, *established in Lemma 3.5. In particular,* $\Upsilon(0,\cdot) \equiv \mathcal{Y}(0,\cdot)$.

Remark 4.8. *Combining (2.7), (4.7), (4.8) and (4.9), we obtain the dynamics*

$$d\left[\beta(s)\mathfrak{X}\left(s, \frac{Y^{(0,y)}(s)}{\mu(s)}\right)\right] = -\beta(s)\Bigg\{\mu(s) I(s, Y^{(0,y)}(s)) ds + \left[\vartheta(s)\frac{Y^{(0,y)}(s)}{\mu(s)}\mathfrak{X}_y\left(s, \frac{Y^{(0,y)}(s)}{\mu(s)}\right) - \Psi^{\mathfrak{X}}\left(s, \frac{Y^{(0,y)}(s)}{\mu(s)}\right)\right]^* dW_0(s)\Bigg\}.$$

Therefore, via integration, we arrive at the relationship

$$\beta(t)\mathfrak{X}\left(t, \frac{Y^{(0,y)}(t)}{\mu(t)}\right) + \int_0^t \beta(s)\mu(s)I(s, Y^{(0,y)}(s))ds \tag{4.14}$$
$$= \mathfrak{X}(0,y) - \int_0^t \beta(s)\left[\vartheta(s)\frac{Y^{(0,y)}(s)}{\mu(s)}\mathfrak{X}_y\left(s, \frac{Y^{(0,y)}(s)}{\mu(s)}\right) - \Psi^{\mathfrak{X}}\left(s, \frac{Y^{(0,y)}(s)}{\mu(s)}\right)\right]^* dW_0(s)$$

for every $(t,y) \in [0,T] \times \mathbb{R}^+$, *almost surely.*

We are in position now to develop the stochastic feedback formulae for the optimal investment and consumption processes. In view of (4.3), the effective state space of the optimal wealth/standard of living pair $\big(X_0(\cdot), z_0(\cdot)\big)$ is given by (3.30) as well. This results from the Assumption 4.1 about the quantity $r(\cdot) - \delta(\cdot) + \alpha(\cdot)$, which conditions $w(\cdot)$ to remains non-random in our context, as it was in the setting of Section 3.

Theorem 4.9. *Under the Assumptions 4.1 and 4.2, the optimal consumption* $c_0(\cdot)$ *and the optimal trading strategy* $\pi_0(\cdot)$ *of the dynamic optimization problem (2.31) admit the stochastic feedback forms of (4.1), determined by the random fields*

$$\mathfrak{C}(t,x,z) \triangleq z + I\big(t, \mu(t)\Upsilon(t, x - w(t)z)\big), \tag{4.15}$$
$$\mathfrak{P}(t,x,z) \triangleq -\frac{1}{x}(\sigma^*(t))^{-1}\left[\vartheta(t)\frac{\Upsilon\big(t, x-w(t)z\big)}{\Upsilon_x\big(t, x-w(t)z\big)} - \Psi^{\mathfrak{X}}\Big(t, \Upsilon\big(t, x - w(t)z\big)\Big)\right], \tag{4.16}$$

for $t \in [0,T)$ *and any pair* $(x,z) \in \mathcal{D}_t$.

Proof: For any initial wealth x and standard of living z such that $(x,z) \in \mathcal{D}_0$ of (3.30), we may rewrite (4.3) as

$$Y^{(0,\mathfrak{J})}(t)\big|_{\mathfrak{J}=\Upsilon(0,x-wz)} = \mu(t)\mathfrak{J}(t)$$

with $\mathfrak{J}(t) \triangleq \Upsilon\big(t, X_0(t) - w(t)z_0(t)\big)$. From (2.46) and (2.52), it develops that the optimal consumption process of (2.50) is expressed by

$$c_0(t) = z_0(t) + I\big(t, \mu(t)\mathfrak{J}(t)\big)$$

for $0 \le t < T$, and (4.15) is proved. Considering (4.14) for $y = \Upsilon(0, x - wz)$, in connection with (4.3), we obtain

$$\beta(t)\Big[X_0(t) - w(t)z_0(t)\Big] + \int_0^t \beta(s)\mu(s)\Big[c_0(s) - z_0(s)\Big]ds$$
$$= x - wz - \int_0^t \beta(s)\Big[\vartheta(s)\mathfrak{I}(s)\mathfrak{X}_y\left(s, \mathfrak{I}(s)\right) - \Psi^{\mathfrak{X}}(s, \mathfrak{I}(s))\Big]^* dW_0(s).$$

Or equivalently,

$$\beta(t)X_0(t) + \int_0^t \beta(s)c_0(s)ds$$
$$= x - \int_0^t \beta(s)\left[\vartheta(s)\frac{\mathfrak{I}(s)}{\mathfrak{I}_x(s)} - \Psi^{\mathfrak{X}}(s, \mathfrak{I}(s))\right]^* dW_0(s) + \beta(t)w(t)z_0(t)$$
$$- wz - \int_0^t \beta(s)\delta(s)w(s)\Big[c_0(s) - z_0(s)\Big]ds + \int_0^t \beta(s)z_0(s)ds, \quad (4.17)$$

where for any $(x, z) \in \mathcal{D}_t$, we have differentiated (4.12) arriving at

$$\mathfrak{X}_y\big(t, \Upsilon(t, x - w(t)z)\big) = 1/\Upsilon_x(t, x - w(t)z),$$

and have defined

$$\mathfrak{I}_x(t) \triangleq \Upsilon_x\big(t, X_0(t) - w(t)z_0(t)\big).$$

Returning to (3.34), the formula (4.17) reduces to

$$\beta(t)X_0(t) + \int_0^t \beta(s)c_0(s)ds$$
$$= x - \int_0^t \beta(s)\left[\vartheta(s)\frac{\mathfrak{I}(s)}{\mathfrak{I}_x(s)} - \Psi^{\mathfrak{X}}(s, \mathfrak{I}(s))\right]^* dW_0(s),$$

almost surely. A comparison of the later with the integral expression (2.11) implies that

$$X_0(t)\pi_0^*(t)\sigma(t) = -\left[\vartheta(t)\frac{\mathfrak{I}(t)}{\mathfrak{I}_x(t)} - \Psi^{\mathfrak{X}}(t, \mathfrak{I}(t))\right]^*,$$

which indicates (4.16). $\Diamond$

Remark 4.10. *The feedback formulas (4.15), (4.16) amount to an adapted, stochastic version of those in (3.31), (3.32). In fact, it is not hard to see that (4.15) coincides with (3.31) and (4.16) coincides with (3.32) under the assumption of deterministic model coefficients. However, in the present chapter where the above assumption is not imposed, the random fields (4.15), (4.16) cease to be independent of the past information associated with the market filtration* $\mathbb{F}$. *As an immediate result, the vector of current level of wealth and standard of living ceases to be a sufficient statistic for the utility maximization problem of Definition 2.3. In other words, an economic agent trying to form an efficient financial investment according to the deployed theory, is required successively to track the whole trading history of the market over the time-horizon* $[0, T]$.

4.3 The Stochastic Hamilton-Jacobi-Bellman Equation

We devote this section to an additional characterization of the value function (2.31), as solution of a stochastic Hamilton-Jacobi-Bellman equation. By analogy with the method of dynamic porgramming, we return to the time-horizon generalization employed in Section 3.3; namely, we consider portfolio plans $\pi(\cdot)$ and consumption policies $c(\cdot)$, for which the dynamics of the corresponding wealth $X^{t,x,\pi,c}(\cdot)$ and standard of living $z(\cdot)$ are governed by (3.35), (3.36) for every $t \in [0, T]$ and $(x, z) \in \mathcal{D}_t$. Here though, any admissible portfolio/consumption pair $(\pi, c) \in \mathcal{A}_0(t, x)$ at the initial condition (t, x) satisfies the extended (cf. (3.37)) budget constraint

$$E_t \left[\int_t^T H^t(s) c(s) ds \right] \leq x, \tag{4.18}$$

almost surely. Furthermore, a variant of Lemma 2.1, subject to an initial date t that is not necessarily zero, shows that if (4.18) stands for a consumption process $c(\cdot)$, then we can always fashion a portfolio strategy $\pi(\cdot)$ such that $(\pi, c) \in \mathcal{A}_0(t, x)$. In addition, the optimization problem of Definition 2.3 is extended by the random field

$$\mathbf{V}(t, x, z) \triangleq \operatorname*{ess\,sup}_{(\pi,c) \in \mathfrak{A}_0'(t,x,z)} E_t \left[\int_t^T u\left(s, c(s) - z(s)\right) ds \right], \tag{4.19}$$

where

$$\mathfrak{A}_0'(t,x,z) \triangleq \left\{ (\pi,c) \in \mathcal{A}_0(t,x);\ E_t\left[\int_t^T u^-\left(s, c(s)-z(s)\right) ds\right] < \infty,\ \text{a.s.} \right\},$$

and $\mathbf{V}(0,\cdot,\cdot) = V(\cdot,\cdot)$. Summoning Assumptions 4.1 and 4.2, we obtain

$$\mathbf{V}(t,x,z) = \mathfrak{G}\big(t, \Upsilon(t, x - w(t)z)\big), \quad (x,z) \in \mathcal{D}_t, \quad t \in [0,T), \tag{4.20}$$

almost surely, where we have also introduced the random field $\mathfrak{G} : [0,T] \times \mathbb{R}^+ \times \Omega \mapsto \mathbb{R}$ through

$$\mathfrak{G}(t,y) \triangleq E_t\left[\int_t^T u\big(s, I(s, yY^{(t,1)}(s))\big) ds\right]. \tag{4.21}$$

One observes immediately that the random fields (4.19) and (4.21) constitute the dynamic, probabilistic analogues of those in (3.38) and (3.40) respectively, since in the setting of non-random coefficients it follows that $\mathbf{V}(\cdot,\cdot,\cdot) \equiv V(\cdot,\cdot,\cdot)$ and $\mathfrak{G}(\cdot,\cdot) \equiv G(\cdot,\cdot)$. Of course, $V(\cdot,\cdot) = V(0,\cdot,\cdot) = \mathbf{V}(0,\cdot,\cdot)$ and $G(\cdot) = G(0,\cdot) = \mathfrak{G}(0,\cdot)$ hold as well [cf. (2.31), (2.60)], in compliance with the temporal and stochastic evolution of the function $\mathcal{X}(\cdot)$ described in the previous section. A direct implication of definition (4.19) is

$$\mathbf{V}(T,x,z) = 0, \quad \forall\ (x,z) \in \mathcal{D}, \tag{4.22}$$

and actually, $\mathbf{V}(t,x,z) < \infty$ for every $(x,z) \in \mathcal{D}_t$. Moreover, for all $(x,z) \in \mathcal{D}_t$, $t \in [0,T)$, we have that

$$\lim_{(x,z)\to(\chi,\zeta)} \mathbf{V}(t,x,z) = \int_t^T u(s,0^+)ds, \quad \forall\ (\chi,\zeta) \in \partial\mathcal{D}_t \tag{4.23}$$

by analogy with (3.42).

We shall next derive a semimartingale decomposition for the random field $\mathfrak{G}$ of (4.21). Recalling Assumptions 4.1, 4.2, and making use of the methodology developed in the proof of (4.10), (4.11), we consider the unique solution $(\mathcal{V}, \Phi) \in C_{\mathbb{F}}\big([0,T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}^+))\big) \times \mathbb{L}^2_{\mathbb{F}}\big(0,T; \mathbb{L}^2(\Omega; C^2(\mathbb{R}^+;\mathbb{R}^d))\big)$ of the Cauchy problem

$$-d\mathcal{V}(t,\eta) = \Bigg[\frac{1}{2}\|\vartheta(t)\|^2\mathcal{V}_{\eta\eta}(t,\eta) + \left(\frac{\mu'(t)}{\mu(t)} - r(t) - \frac{1}{2}\|\vartheta(t)\|^2\right)\mathcal{V}_{\eta}(t,\eta)$$
$$- \vartheta^*(t)\Phi_\eta(t,\eta) + u(t, I(t,e^\eta))\Bigg]dt - \Phi^*(t,\eta)dW(t),$$
$$\eta \in \mathbb{R}, \quad 0 \le t < T, \tag{4.24}$$
$$\mathcal{V}(T,\eta) = 0, \quad \eta \in \mathbb{R}, \tag{4.25}$$

almost surely. As in (4.7), an application of Itô-Kunita-Wentzell formula, in conjunction with (3.12) and (4.24), yields

$$d\mathcal{V}(s, \log Y^{(t,y)}(s)) = -\, u\left(s, I(s, Y^{(t,y)}(s))\right) ds$$
$$- \Big[\mathcal{V}_\eta(s, \log Y^{(t,y)}(s))\vartheta(s) - \Phi(s, \log Y^{(t,y)}(s))\Big]^* dW(s),$$

and by analogy with (4.8), leads to

$$\mathcal{V}(t, \log(y\mu(t))) = E_t\left[\int_t^T u\big(s, I(s, Y^{(t,y)}(s))\big)ds\right] = \mathfrak{G}(t,y). \tag{4.26}$$

We also introduce the random field $\Phi^{\mathfrak{G}} : [0,T] \times \mathbb{R}^+ \times \Omega \to \mathbb{R}^d$ via

$$\Phi^{\mathfrak{G}}(t,y) \triangleq \Phi(t, \log(y\mu(t))), \tag{4.27}$$

and we have the following result.

Lemma 4.11. *Adopting Assumptions 4.1 and 4.2, the pair of random fields* $(\mathfrak{G}, \Phi^{\mathfrak{G}})$*, where* $\mathfrak{G}$ *is given by (4.21) and* $\Phi^{\mathfrak{G}}$ *by (4.27), belongs to the class* $C_{\mathbb{F}}\big([0,T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}^+))\big) \times \mathbb{L}^2_{\mathbb{F}}\big(0,T; \mathbb{L}^2(\Omega; C^2(\mathbb{R}^+; \mathbb{R}^d))\big)$ *and is the unique solution of the Cauchy problem*

$$-d\mathfrak{G}(t,y) = \Bigg[\frac{1}{2}\|\vartheta(t)\|^2 y^2\mathfrak{G}_{yy}(t,y) - r(t)y\mathfrak{G}_y(t,y)$$
$$- \vartheta^*(t)y\Phi^{\mathfrak{G}}_y(t,y) + u(t, I(t, y\mu(t)))\Bigg]dt - \big(\Phi^{\mathfrak{G}}(t,y)\big)^* dW(t)$$

$$on\ [0,T)\times\mathbb{R}^+, \tag{4.28}$$

$$\mathfrak{G}(T,y)=0 \quad on\ \mathbb{R}^+, \tag{4.29}$$

almost surely. Moreover, for every $(t,y)\in[0,T)\times\mathbb{R}^+$ *we have*

$$\mathfrak{G}(t,y)-\mathfrak{G}(t,h)=y\mathfrak{X}(t,y)-h\mathfrak{X}(t,h) - \int_h^y \mathfrak{X}(t,\xi)d\xi, \quad 0<h<y<\infty, \tag{4.30}$$

$$\mathfrak{G}_y(t,y)=y\mathfrak{X}_y(t,y), \tag{4.31}$$

$$\mathfrak{G}_{yy}(t,y)=\mathfrak{X}_y(t,y)+y\mathfrak{X}_{yy}(t,y), \tag{4.32}$$

almost surely.

Once again (cf. Remark 4.7), the additional smoothness of $(\mathfrak{G},\Phi^{\mathfrak{G}})$ will be essential in the formalization of explicit calculations, and the semimartingale decomposition of the process $\mathfrak{G}(\cdot,y)$, $y\in\mathbb{R}^+$, is realized by

$$\mathfrak{G}(t,y)=\int_t^T\left[\frac{1}{2}\|\vartheta(s)\|^2y^2\mathfrak{G}_{yy}(s,y)-r(s)y\mathfrak{G}_y(s,y) - \vartheta^*(s)y\Phi_y^{\mathfrak{G}}(s,y)+u(s,I(s,y\mu(s)))\right]ds-\int_t^T\left(\Phi^{\mathfrak{G}}(s,y)\right)^*dW(s).$$

Proof of Lemma 4.11: Use (4.24), (4.25), (4.26) and (4.27) to check that the pair of random fields $(\mathfrak{G},\Phi^{\mathfrak{G}})$ has the asserted order of regularity and is the unique solution of the Cauchy problem (4.28), (4.29), almost surely.

The proof of the remaining formulas (4.30), (4.31) and (4.32) is like the proof of (3.45), (3.46) and (3.47) in Lemma 3.9, except now we are dealing with conditional expectations. ◊

We carry on our analysis with the subsequent lemma, which copes with the semimartingale decomposition of the random field Υ, defined in Lemma 4.6.

Lemma 4.12. *Consider the hypotheses of Lemma 4.6. Then, there exists a pair of random fields* $(\Theta, \Sigma) \in \mathbb{L}_{\mathbb{F}}\big(0, T^{'}; C^1(\mathbb{R}^+)\big) \times \mathbb{L}^2_{\mathbb{F}}\big(0, T^{'}; C^2(\mathbb{R}^+; \mathbb{R}^d)\big)$ *for each* $0 < T^{'} < T$*, such that*

$$-d\Upsilon(t,x) = \Theta(t,x)dt - \Sigma^*(t,x)dW_0(t) \tag{4.33}$$

holds almost surely, for every $(t,x) \in [0,T)\times\mathbb{R}^+$*. In particular, these random fields are uniquely determined by the relationships:*

$$\begin{aligned}
&\frac{1}{2}\Big[\|\Sigma(t,x)\|^2 - \|\vartheta(t)\|^2\Upsilon^2(t,x)\Big]\mathfrak{X}_{yy}(t,\Upsilon(t,x)) \\
&+ \Big[\big(r(t) - \|\vartheta(t)\|^2\big)\Upsilon(t,x) + \vartheta^*(t)\Sigma(t,x) - \Theta(t,x)\Big]\mathfrak{X}_y(t,\Upsilon(t,x)) + r(t)x \\
&+ \Big[\Sigma(t,x) + \vartheta(t)\Upsilon(t,x)\Big]^*\Psi^{\mathfrak{X}}_y(t,\Upsilon(t,x)) + \vartheta^*(t)\Psi^{\mathfrak{X}}(t,\Upsilon(t,x)) \\
&= \mu(t)I\big(t,\mu(t)\Upsilon(t,x)\big)
\end{aligned} \tag{4.34}$$

and

$$\mathfrak{X}_y(t,\Upsilon(t,x))\Sigma(t,x) + \Psi^{\mathfrak{X}}(t,\Upsilon(t,x)) = 0. \tag{4.35}$$

Proof: Let $(t,x) \in [0,T) \times \mathbb{R}^+$. Invoking equation (4.10) for $\mathfrak{X}$ and postulating the representation (4.33) for Υ, we may apply differentials and Proposition 4.4 on identity (4.12), and integrate over $[0,t]$, to compute

$$\begin{aligned}
&\int_0^t \bigg\{\frac{1}{2}\Big[\|\Sigma(s,x)\|^2 - \|\vartheta(s)\|^2\Upsilon^2(s,x)\Big]\mathfrak{X}_{yy}(s,\Upsilon(s,x)) \\
&\qquad + \Big[\big(r(s) - \|\vartheta(s)\|^2\big)\Upsilon(s,x) + \vartheta^*(s)\Sigma(s,x) - \Theta(s,x)\Big]\mathfrak{X}_y(s,\Upsilon(s,x)) \\
&\qquad + r(s)x + \Big[\Sigma(s,x) + \vartheta(s)\Upsilon(s,x)\Big]^*\Psi^{\mathfrak{X}}_y(s,\Upsilon(s,x)) + \vartheta^*(s)\Psi^{\mathfrak{X}}(s,\Upsilon(s,x)) \\
&\qquad - \mu(s)I\big(s,\mu(s)\Upsilon(s,x)\big)\bigg\}ds \\
&+ \int_0^t \Big\{\mathcal{X}_y(s,\mathcal{Y}(s,x))\Sigma(s,x) + \Psi^{\mathcal{X}}(s,\mathcal{Y}(s,x))\Big\}^* dW(s) = 0,
\end{aligned}$$

almost surely; (2.9) has also been used. Thus, the uniqueness for the decomposition of a continuous semimartingale [e.g. Karatzas and Shreve (1991), p 149] implies that both integrals of the above equation vanish. Differentiation of the Lebesgue integral implies (4.34), while the quadratic variation of the stochastic integral vanishes as well, leading to (4.35). The derived equations define uniquely the random fields Θ and Σ, assigning to them the claimed order of adaptivity, integrability and smoothness. $\diamondsuit$

Lemma 4.13. *Under the Assumptions 4.1, 4.2, the random fields $\Psi^{\mathfrak{X}}$ and $\Phi^{\mathfrak{G}}$ of (4.9) and (4.27) accordingly, satisfy almost surely the relationship*

$$\Phi^{\mathfrak{G}}_y(t,y) - y\Psi^{\mathfrak{X}}_y(t,y) = 0, \quad \forall\ (t,y) \in [0,T) \times \mathbb{R}^+. \tag{4.36}$$

Proof: Taking differentials and integrating (4.31) over $[z,y]$, $0 < z < y < \infty$, we get

$$d\mathfrak{G}(t,y) - d\mathfrak{G}(t,z) = y d\mathfrak{X}(t,y) - z d\mathfrak{X}(t,z) - \int_z^y d\mathfrak{X}(t,\lambda)d\lambda,$$

almost surely, for $0 \le t < T$. Perform next the substitutions signified by (4.10) and (4.28) in the above formula, and equalize the respective martingale parts (e.g. Karatzas and Shreve (1991), Problem 3.3.2) to end up at

$$\Phi^{\mathfrak{G}}(t,y) - \Phi^{\mathfrak{G}}(t,z) = y\Psi^{\mathfrak{X}}(t,y) - z\Psi^{\mathfrak{X}}(t,z) - \int_z^y \Psi^{\mathfrak{X}}(t,\lambda)d\lambda. \tag{4.37}$$

Of course, (4.37) is valid only if the interchange of Lebesgue and Itô integrals

$$\int_z^y \int_0^t \Psi^{\mathfrak{X}}(s,\lambda)dW(s)\ d\lambda = \int_0^t \int_z^y \Psi^{\mathfrak{X}}(s,\lambda)d\lambda\ dW(s)$$

holds almost surely, for each $t \in [0,T)$. But this is true, due to the observation that $L(t,\cdot) = \int_z^{\cdot} \Psi^{\mathfrak{X}}(t,\lambda)d\lambda$ is a C^2 random field on $[z,\infty)$, and Exercise 3.1.5 in Kunita (1990). Differentiating (4.37) we obtain (4.36). $\diamondsuit$

We are ready now to state the main result of this section.

Theorem 4.14. (Stochastic Hamilton-Jacobi-Bellman Equation): *Under Assumptions 4.1 and 4.2, the pair of random fields $(\mathbf{V},\Xi)$, here the value field $\mathbf{V}(t,x,z)$ is given by (4.20), (4.22), and*

$$\Xi(t,x,z) \triangleq \Phi^{\mathfrak{G}}\big(t,\Upsilon(t,x-w(t)z)\big) - \Upsilon(t,x-w(t)z)\Psi^{\mathfrak{X}}\big(t,\Upsilon(t,x-w(t)z)\big), \tag{4.38}$$

is of class

$$C_{\mathbb{F}}\big(\{t\in[0,T];\ \mathbf{V}(t,\cdot,\cdot)\in C^{3,3}(\mathcal{D}_t)\}\big)\times\mathbb{L}^2_{\mathbb{F}}\big(\{t\in[0,T);\ \Xi(t,\cdot,\cdot)\in C^{2,2}(\mathcal{D}_t;\mathbb{R}^d)\}\big).$$

Furthermore, this pair $(\mathbf{V},\Xi)$ *solves on* $\{(t,x,z);\ t\in[0,T),\ (x,z)\in\mathcal{D}_t\}$ *the stochastic Hamilton-Jacobi-Bellman dynamic programming partial differential equation*

$$\begin{aligned}-d\mathbf{V}(t,x,z)=\operatorname*{ess\,sup}_{\substack{0\le c<\infty\\ \pi\in\mathbb{R}^d}}\Big\{&\frac{1}{2}\|\sigma^*(t)\pi\|^2x^2\mathbf{V}_{xx}(t,x,z)\\&+\Big[r(t)x-c+\pi^*\sigma(t)\vartheta(t)x\Big]\mathbf{V}_x(t,x,z)\\&+\Big[\delta(t)c-\alpha(t)z\Big]\mathbf{V}_z(t,x,z)\\&+\pi^*\sigma(t)x\Xi_x(t,x,z)+u(t,c-z)\Big\}dt\\-\Xi(t,x,z)dW(t)\end{aligned}\tag{4.39}$$

with the boundary conditions (4.22) and (4.23), almost surely. Furthermore, the pair of random fields $(\mathfrak{P}(t,x,z),\mathfrak{C}(t,x,z))$ *of (4.15), (4.16) provides the optimal values for the maximization in (4.39).*

Proof: Differentiation of (4.12), (4.20) and (4.38), in combination with (4.31), (4.32) and (4.36), leads almost surely to

$$\begin{aligned}&\mathfrak{X}_y\big(t,\Upsilon(t,x-w(t)z)\big)\Upsilon_x(t,x-w(t)z)=1,\\&\mathbf{V}_x(t,x,z)=\Upsilon(t,x-w(t)z),\\&\mathbf{V}_z(t,x,z)=-w(t)\Upsilon(t,x-w(t)z),\\&\mathbf{V}_{xx}(t,x,z)=\Upsilon_x(t,x-w(t)z),\\&\Xi_x(t,x,z)=-\Upsilon_x(t,x-w(t)z)\Psi^{\mathfrak{X}}\big(t,\Upsilon(t,x-w(t)z)\big)\end{aligned}$$

for $(x,z)\in\mathcal{D}_t,\ 0\le t<T$. Using these formulae and (3.8), we may rewrite the right-hand side of (4.39) as

$$\Bigg[r(t)x\Upsilon(t,x-w(t)z)+\alpha(t)w(t)z\Upsilon(t,x-w(t)z)$$
$$+\operatorname*{ess\,sup}_{0\le c<\infty}\big\{u(t,c-z)-c\,\mu(t)\Upsilon(t,x-w(t)z)\big\}$$

$$+ \operatorname*{ess\,sup}_{\pi \in \mathbb{R}^d} \left\{ \frac{1}{2} \|\sigma^*(t)\pi\|^2 x^2 \Upsilon_x(t, x - w(t)z) + \pi^* \sigma(t) x \Big[\vartheta(t) \Upsilon(t, x - w(t)z) - \Upsilon_x(t, x - w(t)z) \Psi^{\mathfrak{X}}\big(t, \Upsilon(t, x - w(t)z)\big) \Big] \right\} \Bigg] dt$$

$$- \Big[\Phi^{\mathfrak{G}}\big(t, \Upsilon(t, x - w(t)z)\big) - \Upsilon(t, x - w(t)z) \Psi^{\mathfrak{X}}\big(t, \Upsilon(t, x - w(t)z)\big) \Big] dW(t).$$

The strict concavity of both expressions to be maximized allows us to differentiate and solve the resulting equations, in order to attain the optimal values of c and π. These values turn out to coincide with (4.15) and (4.16), respectively. Substituting them now into the later expression, we are driven to

$$\begin{aligned} &\Big[r(t) x \Upsilon(t, x - w(t)z) + \alpha(t) w(t) z \Upsilon(t, x - w(t)z) \\ &\quad + u\Big(t, I\big(t, \mu(t) \Upsilon(t, x - w(t)z)\big)\Big) \\ &\qquad - \mu(t) \Upsilon(t, x - w(t)z) \Big[z + I\big(t, \mu(t) \Upsilon(t, x - w(t)z)\big) \Big] \\ &\quad - \frac{1}{2\Upsilon_x(t, x - w(t)z)} \Big\| \vartheta(t) \Upsilon(t, x - w(t)z) \\ &\qquad - \Upsilon_x(t, x - w(t)z) \Psi^{\mathfrak{X}}\big(t, \Upsilon(t, x - w(t)z)\big) \Big\|^2 \Big] dt \\ &- \Big[\Phi^{\mathfrak{G}}\big(t, \Upsilon(t, x - w(t)z)\big) - \Upsilon(t, x - w(t)z) \Psi^{\mathfrak{X}}\big(t, \Upsilon(t, x - w(t)z)\big) \Big]^* dW(t). \end{aligned} \tag{4.40}$$

On the other hand, usage of differentials on (4.20) implies that

$$d\mathbf{V}(t, x, z) = d\mathfrak{G}\big(t, \Upsilon(t, x - w(t)z)\big) - w'(t) z \Upsilon_x(t, x - w(t)z) \mathfrak{G}_y\big(t, \Upsilon(t, x - w(t)z)\big),$$

and couple (4.33) with (2.9) to derive the alternative representation of Υ:

$$d\Upsilon(t, x) = \big[\vartheta^*(t) \Sigma(t, x) - \Theta(t, x) \big] dt + \Sigma^*(t, x) dW(t),$$

almost surely. Then, a straightforward application of Itô-Kunita-Wentzell formula, involving (4.28), yields that the left-hand side of (4.39) is equal to

$$
\begin{aligned}
-\Bigg[&\frac{1}{2}\Big[\|\Sigma(t,x-w(t)z)\|^2-\|\vartheta(t)\|^2\Upsilon^2(t,x-w(t)z)\Big]\mathfrak{G}_{yy}\big(t,\Upsilon(t,x-w(t)z)\big)\\
&+\Big[r(t)\Upsilon(t,x-w(t)z)+\vartheta^*(t)\Sigma(t,x-w(t)z)\\
&\qquad\qquad-\Theta(t,x-w(t)z)\Big]\mathfrak{G}_{y}\big(t,\Upsilon(t,x-w(t)z)\big)\\
&+\Big[\Sigma(t,x-w(t)z)+\vartheta(t)\Upsilon(t,x-w(t)z)\Big]^*\Phi^{\mathfrak{G}}_y\big(t,\Upsilon(t,x-w(t)z)\big)\\
&-u\Big(t,I\big(t,\mu(t)\Upsilon(t,x-w(t)z)\big)\Big)\Bigg]dt\\
-\Bigg[&\mathfrak{G}_y\big(t,\Upsilon(t,x-w(t)z)\big)\Sigma(t,x-w(t)z)+\Phi^{\mathfrak{G}}\big(t,\Upsilon(t,x-w(t)z)\big)\Bigg]^*dW(t),
\end{aligned}
$$

which via (4.31) and (4.32) becomes

$$
\begin{aligned}
-\Bigg[&\frac{1}{2}\|\Sigma(t,x-w(t)z)\|^2\mathfrak{X}_y\big(t,\Upsilon(t,x-w(t)z)\big)\\
&+\Upsilon(t,x-w(t)z)\Bigg\{\frac{1}{2}\Big[\|\Sigma(t,x-w(t)z)\|^2\\
&\qquad\qquad-\|\vartheta(t)\|^2\Upsilon^2(t,x-w(t)z)\Big]\mathfrak{X}_{yy}\big(t,\Upsilon(t,x-w(t)z)\big)\\
&\qquad+\Big[r(t)\Upsilon(t,x-w(t)z)-\frac{1}{2}\|\vartheta(t)\|^2\Upsilon(t,x-w(t)z)\\
&\qquad+\vartheta^*(t)\Sigma(t,x-w(t)z)-\Theta(t,x-w(t)z)\Big]\mathfrak{X}_y\big(t,\Upsilon(t,x-w(t)z)\big)\Bigg\}\\
&+\Big[\Sigma(t,x-w(t)z)+\vartheta(t)\Upsilon(t,x-w(t)z)\Big]^*\Phi^{\mathfrak{G}}_y\big(t,\Upsilon(t,x-w(t)z)\big)\\
&-u\Big(t,I\big(t,\mu(t)\Upsilon(t,x-w(t)z)\big)\Big)\Bigg]dt\\
-\Big[&\Phi^{\mathfrak{G}}\big(t,\Upsilon(t,x-w(t)z)\big)-\Upsilon(t,x-w(t)z)\Psi^{\mathfrak{X}}\big(t,\Upsilon(t,x-w(t)z)\big)\Big]^*dW(t).
\end{aligned}
$$

Finally, Lemmata 4.12 and 4.13 transform the latter to

$$
\begin{aligned}
-\Bigg[& -r(t)\big[x-w(t)z\big]\Upsilon(t,x-w(t)z) - u\Big(t, I\big(t,\mu(t)\Upsilon(t,x-w(t)z)\big)\Big) \\
& + \mu(t) I\big(t,\mu(t)\Upsilon(t,x-w(t)z)\big) + \frac{1}{2}\|\vartheta(t)\|^2 \frac{\Upsilon^2(t,x-w(t)z)}{\Upsilon_x(t,x-w(t)z)} \\
& - \vartheta^*(t)\Psi^{\mathfrak{X}}\big(t,\Upsilon(t,x-w(t)z)\big) \\
& + \frac{1}{2}\|\Psi^{\mathfrak{X}}\big(t,\Upsilon(t,x-w(t)z)\big)\|^2 \Upsilon_x(t,x-w(t)z)\Bigg] dt \\
- \Big[& \Phi^{\mathfrak{G}}\big(t,\Upsilon(t,x-w(t)z)\big) - \Upsilon(t,x-w(t)z)\Psi^{\mathfrak{X}}\big(t,\Upsilon(t,x-w(t)z)\big)\Big]^* dW(t).
\end{aligned}
$$

Expanding the norm in (4.40) and recalling (3.10), we conclude that both sides of (4.39) coincide almost surely. $\diamondsuit$

Remark 4.15. *Invoking Remark 3.11, we may rewrite equation (4.39) in the* nonlinear *form*

$$
\begin{aligned}
-d\mathbf{V}_t(t,x,z) = \mathcal{H}\Big(&\mathbf{V}_{xx}(t,x,z), \mathbf{V}_x(t,x,z), \mathbf{V}_z(t,x,z), \Xi_x(t,x,z), t, x, z\Big) dt \\
& - \Xi(t,x,z) dW(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}(\mathrm{A},p,q,\mathrm{B},t,x,z) \triangleq & -\frac{1}{2\mathrm{A}}\|\vartheta(t)p+\mathrm{B}\|^2 + \Big[r(t)x - z - I(t,p-\delta(t)q)\Big]p \\
& + \Big[(\delta(t)-\alpha(t))z + \delta(t) I(t,p-\delta(t)q)\Big] q \\
& + u\big(t, I(t,p-\delta(t)q)\big)
\end{aligned}
$$

for $\mathrm{A}<0,\ p>0,\ q<0$ *and* $\mathrm{B}\in\mathbb{R}$.

Remark 4.16. *Theorem 4.14 provides a rare illustration of the Peng (1992) approach to stochastic Hamilton-Jacobi-Bellman equations. More precisely, it formulates the nonlinear stochastic partial differential equation satisfied by the value random field of the stochastic optimal control problem (4.19).*

By analogy with Theorem 3.12, we conclude our program with the exposition of the *linear* parabolic backward stochastic partial differential equation (4.47), whose *unique* solution is represented by *the convex dual* of $\mathbf{V}$:

$$\widetilde{\mathbf{V}}(t,y) \triangleq \operatorname*{ess\,sup}_{(x,z)\in\mathcal{D}_t} \big\{\mathbf{V}(t,x,z) - \big(x - w(t)z\big)y\big\}, \quad y \in \mathbb{R}.$$

Resolving (4.47), we may invert the previous Legendre-Fenchel transform to re-cast the random field $\mathbf{V}$ as

$$\mathbf{V}(t,x,z) = \operatorname*{ess\,inf}_{y\in\mathbb{R}} \big\{\widetilde{\mathbf{V}}(t,y) + \big(x - w(t)z\big)y\big\}, \quad (x,z) \in \mathcal{D}_t,$$

almost surely.

Theorem 4.17. (Convex Dual of $\mathbf{V}(t,\cdot)$): *Considering Assumptions 4.1, 4.2, and a given* $t \in [0,T)$, $\mathbf{V}(t,\cdot,\cdot)$ *is a generalized utility function, as defined in Theorem 2.17, almost surely; also,*

$$\mathbf{V}_x(t,x,z) = \Upsilon(t, x - w(t)z), \quad \forall\ (x,z) \in \mathcal{D}_t, \tag{4.41}$$
$$\mathbf{V}_z(t,x,z) = -w(t)\Upsilon(t, x - w(t)z), \quad \forall\ (x,z) \in \mathcal{D}_t. \tag{4.42}$$

Furthermore, for $(t,y) \in [0,T] \times \mathbb{R}^+$, *we have*

$$\widetilde{\mathbf{V}}(t,y) = \mathfrak{G}(t,y) - y\mathfrak{X}(t,y) \tag{4.43}$$
$$= E_t\left[\int_t^T \tilde{u}(s, yY^{(t,1)}(s))ds\right], \tag{4.44}$$
$$\widetilde{\mathbf{V}}_y(t,y) = -\mathfrak{X}(t,y), \tag{4.45}$$

almost surely. Finally, the pair of random fields $(\widetilde{\mathbf{V}}, \Lambda)$, *where*

$$\Lambda(t,y) \triangleq \Phi^{\mathfrak{G}}(t,y) - y\Psi^{\mathfrak{X}}(t,y), \quad (t,y) \in [0,T] \times \mathbb{R}^+, \tag{4.46}$$

belongs to $C_{\mathbb{F}}\big([0,T];\mathbb{L}^2(\Omega; C^3(\mathbb{R}^+))\big) \times \mathbb{L}^2_{\mathbb{F}}\big(0,T;\mathbb{L}^2(\Omega; C^2(\mathbb{R}^+;\mathbb{R}^d))\big)$ *and is the unique solution of the following Cauchy problem for the linear BSPDE*

$$
\begin{aligned}
-d\widetilde{\mathbf{V}}(t,y) = \Bigg[&\frac{1}{2}\|\vartheta(t)\|^2 y^2 \widetilde{\mathbf{V}}_{yy}(t,y) - r(t)y\widetilde{\mathbf{V}}_y(t,y) - \vartheta^*(t)y\Lambda_y(t,y) \\
&+ \tilde{u}(t, y\mu(t))\Bigg]dt - \Lambda^*(t,y)dW(t) \qquad on\ [0,T)\times\mathbb{R}^+, \quad (4.47) \\
\widetilde{\mathbf{V}}(T,y) = 0 \quad &on\ \mathbb{R}^+. \qquad (4.48)
\end{aligned}
$$

Merging now (4.38) and (4.46), we notice that the random fields Ξ and Λ of the martingale parts of $\mathbf{V}$ and $\widetilde{\mathbf{V}}$, respectively, are related via the expression

$$\Xi(t,x,z) = \Lambda\big(t, \Upsilon(t, x - w(t)z)\big), \quad t \in [0,T), \quad (x,z) \in \mathcal{D}_t, \tag{4.49}$$

almost surely.

Proof of Theorem 4.17: As regards the first two parts of the theorem, it suffices to imitate the proof of their Markovian analogues in Theorem 3.12, keeping in mind the new feature of conditional expectation.

From Lemma 4.6, Lemma 4.11, identity (4.43) and definition (4.46), it is easy to verify the stated regularity for the pair $(\widetilde{\mathbf{V}}, \Lambda)$, while the equations (4.47) and (4.48) are direct implications of equations (4.43), (4.28), (2.18), (4.10) and (4.29) with (4.11). $\Diamond$

Remark 4.18. *In conclusion, we should emphasize that the stochastic partial differential equations of Lemmata 4.6, 4.11, and Theorems 4.14, 4.17 are stochastic versions of their Markovian counterparts studied in the previous chapter; cf. Lemmata 3.5, 3.9, and Theorems 3.10, 3.12, respectively. To our knowledge, this is the first concrete illustration of BSPDE's in a stochastic control context beyond the classical linear/quadratic regulator worked out in Peng (1992).*

Finally, we shall elaborate on how Example 3.13 can be modified in order to illustrate Theorem 4.17 as an alternative computational method for the value random field and the stochastic feedback formulas of the optimal portfolio/consumption pair.

Example 4.19. *(Logarithmic utility).* Take $u(t,x) = \log x$, $\forall\ (t,x) \in [0,T] \times \mathbb{R}^+$; thus, $I(t,y) = 1/y$, $\tilde{u}(t,y) = -\log y - 1$ for $(t,y) \in [0,T] \times \mathbb{R}^+$. Our goal is to find an $\mathbb{F}$-*adapted* pair of random fields that satisfies (4.47), (4.48). In particular, we recall Example 3.13 where the model coefficients are deterministic, the Cauchy problem (4.47), (4.48) reduces to (3.57), (3.58), and its solution is provided by (3.59) – (3.61). Accordingly, we introduce here the $\mathbb{F}$-adapted random field

$$\tilde{\mathbf{v}}(t,y) \triangleq -\nu(t)\log(y\mu(t)) - \mathbf{m}(t)$$

for $(t,y) \in [0,T] \times \mathbb{R}^+$, with

$$\nu(t) = T - t,$$
$$\mathbf{m}(t) = E_t\left[\int_t^T \left\{1 - (T-s)\left(\frac{1}{2}\|\vartheta(s)\|^2 + r(s) - \frac{\mu'(s)}{\mu(s)}\right)\right\} ds\right].$$

Moreover, the completeness of the market stipulates the existence of an $\mathbb{R}^d$-valued, $\mathbb{F}$-progressively measurable, square-integrable process $\ell(\cdot)$, such that the Brownian martingale

$$\mathbf{M}(t) = E_t\left[\int_0^T \left\{1 - (T-s)\left(\frac{1}{2}\|\vartheta(s)\|^2 + r(s) - \frac{\mu'(s)}{\mu(s)}\right)\right\} ds\right]$$

has the representation

$$\mathbf{M}(t) = \mathbf{M}(0) + \int_0^t \ell^*(s)dW(s), \quad 0 \le t \le T.$$

It is verified directly that the pair $(\tilde{\mathbf{v}}, \ell)$, where the random field $\tilde{\mathbf{v}}$ is of class $C_{\mathbb{F}}\big([0,T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}^+))\big)$, satisfies (4.47), (4.48). Therefore, Theorem 4.17 implies that $(\tilde{\mathbf{v}}, \ell)$ agrees with $(\widetilde{\mathbf{V}}, \Lambda)$, and

$$\mathfrak{X}(t,y) = \frac{\nu(t)}{y}, \quad \mathfrak{G}(t,y) = \nu(t)\big[1 - \log(y\mu(t))\big] - \mathbf{m}(t), \quad (t,y) \in [0,T] \times \mathbb{R}^+.$$

Consequently, for $0 \leq t < T$, it transpires that

$$\Upsilon(t,x) = \frac{\nu(t)}{x}, \quad x \in \mathbb{R}^+,$$
$$\mathbf{V}(t,x,z) = \nu(t)\log\left(\frac{x - w(t)z}{\nu(t)\mu(t)}\right) + \nu(t) - \mathbf{m}(t), \quad (x,z) \in \mathcal{D}_t.$$

For this special choice of utility preference, $\mathfrak{X}$ (and so Υ) is deterministic, and the feedback formulas (4.15), (4.16) for the optimal consumption and portfolio decisions are the same as those of Example 3.13.

Remark 4.20. *In the case $\delta(\cdot) = \alpha(\cdot) = 0$ and $z = 0$, namely, without habit formation in the market model, we have that $\mu(\cdot) = 1$ from (3.8), whence our analysis remains valid for a* random *interest rate process $r(\cdot)$ as well. Then, this chapter generalizes the interrelation of classical utility optimization problems with the principles of dynamic programming, as explored in Karatzas, Lehoczky and Shreve (1987) for the special case of deterministic coefficients.*

5 Conclusion And Open Problems

In this thesis we explored various aspects of portfolio-consumption utility optimization under the presence of addictive habits in complete financial markets. The effective state space of the optimal wealth and standard of living processes was identified as a random wedge, and the investor's value function was found to exhibit properties similar to those of a utility function. Of particular interest is the interplay between the dynamic programming principles and the (stochastic) partial differential equation theory that led to the characterization of the value function (random field) as a solution of a highly non-linear (stochastic) Hamilton-Jacobi-Bellman partial differential equation, respectively. In fact, the convex dual of the value function (random field) turned out to constitute the unique solution of a parabolic backward (stochastic) partial differential equation, respectively. A byproduct of this analysis was an additional representation for the optimal investment-consumption policies on the current level of the optimal wealth and standard of living processes.

Another aspect, still at question, that stems from our specification of preferences is the existence of an optimal portfolio/consumption pair in an incomplete market; that is, the number of stocks is strictly smaller than the dimension of the driving Brownian motion. Following the duality methodology deployed by He and Pearson (1991) and Karatzas, Lehoczky, Shreve and Xu (1991), one can fashion fictitious stocks in order to complete artificially the market, and thereby invoke the analysis presented in Chapter 2 to ensure existence of optimal policies. This pair remains optimal in the primal incomplete market only if the corresponding portfolio does not invest in the additional stocks at all. The fictitious completion is parametrized by a certain family of continuous exponential local martingales which includes $Z(\cdot)$ of (2.6), and gives rise to an analogous class of state-price density processes. As stated in Schroder and Skiadas (2002), an associated dual optimization problem can be defined in terms of the respective parametrized "adjusted" state-price density processes, such that a possible minimizer induces a null demand for the imaginary stocks. In contrast though to the former two papers, the dual functional fails now to be convex with respect to the dual parameter, excluding the usage of standard minimization techniques. Evidently, we are in need of new minimization considerations for the dual problem.

Returning to complete markets, our dynamic programming approach might also yield explicit solutions for the Hamilton-Jacobi-Bellman equation, deterministic or stochastic, related with a utility function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$, i.e., of exponential type. In this case the marginal utility at zero is finite and the "addiction" condition (2.25) is removed; i.e., consumption plans are allowed to drop bellow the contemporaneous standard of living, abolishing any further restrictions on the initial endowment x and standard of living z (cf. Assumption 2.2). In other words, an economic agent has the choice of reducing consumption to *decumulate* habits after periods of high consumption expenditures and standard of living buildup. At the same time, past consumption behavior continues to affect current consumption choices, leading to the development of "non-addictive" habits. Moreover, the natural constraint of a non-negative consumption plan remains intact in the model. Detemple and Karatzas (2003) resolved the respective consumption-portfolio problem in terms of an endogenously determined stopping time, after which the consumption non-negativity constraint cease to bind. The existence of an optimal pair was demonstrated and the optimal consumption process was provided in closed-form. Hence, given the dynamic programming approach presented in this thesis, an associated deterministic or stochastic partial differential equation might provide information about the optimal investment strategy as well.

In conclusion, one may attempt to investigate the latter optimization problem for more general preferences in which utility comes not just from net consumption $c(\cdot) - z(\cdot)$, but is a generalized utility function $u(t, c(t), z(t))$ of time t, current consumption $c(t)$ and standard of living $z(t)$ (cf. Theorems 2.17, 3.12). In this generalized setting, existence of optimal policies remains still an open question. From a preliminary study, one is formally led to a system of coupled *Forward-Backward Recursive Stochastic Equations,* which do not seem to be covered by the extant theory. Thus, progress on this front might also yield interesting results on this aspect of Stochastic Analysis.

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