The Numéraire Portfolio and Arbitrage in Semimartingale Models of Financial Markets

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To my parents, for bringing me up.

To my sister, who still shares the same room in my life.

To my friends, who make me laugh and cry.

I would like to thank my advisor Ioannis Karatzas
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ABSTRACT

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We study the existence of the *numéraire portfolio* under predictable convex constraints in a general semimartingale financial model. The numéraire portfolio generates a wealth process which makes the relative wealth processes of all other portfolios with respect to it supermartingales. Necessary and sufficient conditions for the existence of the numéraire portfolio are obtained in terms of the triplet of predictable characteristics of the asset price process. This characterization is then used to obtain further necessary and sufficient conditions, in terms of an arbitrage-type notion. In particular, the full strength of the "No Free Lunch with Vanishing Risk" (NFLVR) is not needed, only the weaker "No Unbounded Profit with Bounded Risk" (NUPBR) condition that involves the boundedness in probability of the terminal values of wealth processes. We show that this notion is the minimal a-priori assumption required, in order to proceed with utility optimization. The fact that it is expressed entirely in terms of predictable characteristics makes it easy to check, something that the stronger (NFLVR) condition lacks.

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0. Introduction

0.1. **Background and Discussion of Results.** The branch of Probability Theory that goes by the name "Stochastic Finance" is concerned (among other problems) with finding adequate descriptions of the way financial markets work. There exists a huge literature of models by now, and we do not attempt to give a history or summary of all the work that has been done. There is, however, a broad class of these models that has been used extensively: those for which the price processes of certain financial instruments[1] are considered to evolve as semimartingales. The concept of semimartingale is a very intuitive one: it connotes a process that can be decomposed into a *finite variation* term that represents the "signal" or "drift" and a *local martingale* term that represents the "noise" or "uncertainty". Discrete-time models can be embedded in this class, as can processes with independent increments and many other Markov processes, such as solutions to stochastic differential equations. Models that are not included are, for example, those where price processes are driven by fractional Brownian motion.

There are at least two good reasons for the choice of semimartingale asset price processes in modeling financial markets. The first is that semimartingales constitute the largest class of stochastic processes which can be used as integrators, in a theory that resembles as closely as possible the ordinary Lebesgue integration. In economic terms, integration with respect to a price process represents the wealth of an investment in the market, the integrand being the strategy that an investor uses. To be more precise, let us denote the price process of a certain tradeable asset by $S = (S_t)_{t \in \mathbb{R}_+}$; for the time being, S could

[1] These can be stocks, indices, currencies, etc.

be any random process. An investor wants to invest in this asset. As long as simple "buy-and-hold strategies" are being used, which in mathematical terms are captured by an *elementary integrand* θ, the "stochastic integral" of the strategy θ with respect to S is obviously defined: it is the sum of net gains or losses resulting from the use of the buy-and-hold strategy. Nevertheless, the need to consider strategies that are not of that simple and specific structure, but can change continuously in time[2] arises. If one wishes to extend the definition of the integral to this case, keeping the previous intuitive property for the case of simple strategies and requiring a very mild "dominated convergence" property, the Bichteler-Dellacherie theorem[3] states that S *has to be* a semimartingale.

A second reason why semimartingale models are ubiquitous, is the pioneering work on no-arbitrage criteria that has been ongoing during the last decades. Culminating with the papers [8] and [12] of Delbaen and Schachermayer, the connection has been established between the economic notion of no arbitrage — which found its ultimate incarnation in the "No Free Lunch with Vanishing Risk" (NFLVR) condition — and the mathematical notion of existence of equivalent probability measures, under which asset prices have some sort of martingale property. In [8] it was shown that if we want to restrict ourselves to the realm of locally bounded stock prices, and agree that we should banish arbitrage by use of simple strategies, the price process again has to be a semimartingale.

In this work, we consider a general semimartingale model and make no further mathematical assumptions. On the economic side, it is part of the assumptions

[2]An example of this is the hedging strategy of a European call option in the Black-Scholes model.

[3]See for example the book [5] by Bichteler himself.

that the asset prices are exogenously determined – in some sense they "fall from the sky", and an investor's behavior has no effect whatsoever on their movement. The usual practice is to assume that we are dealing with small investors and whistle away all the criticism, as we shall do. We also assume a frictionless market, in the sense that transaction costs for trading are non-existent or negligible.

Our main concern will be a problem which can be cast in the mold of dynamic stochastic optimization, though of a highly static and deterministic nature, since the optimization is being done in a path-by-path, pointwise manner. We explore a specific strategy whose wealth appears "better" when compared to the wealth generated by any other strategy, in the sense that the ratio of the two processes is a supermartingale. If such a strategy exists, it is essentially unique and we call it *the numéraire portfolio.*

We derive necessary and sufficient conditions for the numéraire portfolio to exist, in terms of the *predictable characteristics* of the stock-price process[4]. Since we are working in a more general setting where jumps are also allowed, there is the need to introduce a characteristic that measures the intensity of these jumps. Sufficient conditions have already been established in the paper [15] of Goll and Kallsen, where the focus is on the equivalent problem of maximizing expected logarithmic utility. These authors went on to show that their conditions are also necessary when the following assumptions hold: the problem of maximizing the expected log-utility has a finite value, no constraints are enforced on the strategies, and the (NFLVR) condition is satisfied. Further,

[4]These are the analogues (and generalizations) of the drift and volatility coëfficients in Itô process models.

Becherer [4] also discussed how under these assumptions the numéraire portfolio exists; as is somewhat well-known, it coincides with the log-optimal one. In both of these papers, deep results from Kramkov and Schachermayer [25] on utility maximization from terminal wealth had to be used, to obtain necessary and sufficient conditions.

Here we follow a bare-hands approach, which makes possible some improvements. First, the assumption of finite expected log-utility is dropped completely — there should be no reason for this to appear anyhow, since we are not working on the log-optimal problem. Secondly, we can enforce any type of closed convex constraints on portfolio choice, as long as these arrive in a predictable manner. Thirdly, and perhaps most controversially, we drop the (NFLVR) assumption and impose *no* normative assumption on the model. It turns out that the numéraire portfolio can exist even when the classical *No Arbitrage* (NA) condition fails.

In the context of stochastic portfolio theory, we feel there is no need for no-arbitrage assumptions to begin with: if there is arbitrage in the market, the role of optimization should be actually to find and utilize these opportunities, rather than ban the model. It is actually possible that the optimal strategy of an investor is *not* the arbitrage (see Examples 3.7 and 3.19). The usual practice of assuming that we can invest unconditionally on the arbitrage breaks down because of credit limit constraints: arbitrages are sure to give more capital than initially invested in at a fixed future date, but can do pretty badly meanwhile, and this imposes an upper bound on the money the investor can bet on it. If the previous reason for not banning arbitrage does not satisfy the reader, here is a more severe problem: in very general semimartingale financial markets *there does not seem to exist any computationally feasible way of deciding whether* *arbitrages exist or not.* This goes hand-in-hand with the fact that the existence of equivalent martingale measures — its incredible theoretical importance notwithstanding — is a purely normative assumption and *not easy to check*, at least by looking directly at the dynamics of the stock-price process.

Our second main result comes hopefully to shed some light on this situation. Having assumed nothing about the model when initially trying to decide whether the numéraire portfolio exists, we now take a step backwards and in the opposite-than-usual direction: *we ask ourselves what the existence of the numéraire portfolio can tell us about arbitrage-like opportunities in the market.* Here, the necessary and sufficient condition for existence is the boundedness in probability of the collection of terminal wealths attainable by trading. Readers acquainted with arbitrage notions will recognize that this boundedness in probability is one of the two conditions that comprise (NFLVR); what remains of course is the (NA) condition. One can go on further, and ask how severe this assumption (of boundedness in probability for the set of terminal wealths) really is. The answer is simple: when this condition fails, one cannot do utility optimization for *any* utility function; conversely if this assumption holds, one can proceed as usual with utility maximization.

The obvious advantage of not assuming the full (NFLVR) condition is that there is a direct way of checking whether the weaker condition of boundedness in probability holds, in terms of the predictable characteristics of the price process, i.e., in terms of the dynamics of the stock-price process. Furthermore, our result can be used to understand the gap between the concepts of (NA) and the stronger (NFLVR); the existence of the numéraire portfolio is exactly the bridge needed to go from (NA) to (NFLVR). This has already been understood for the continuous-path process case in the paper [9]; here we do it for the general case.

0.2. **Synopsis.** We offer here an overview of what is to come, so the reader does not get lost in the technical details and little detours.

Chapter 1 introduces the financial model, the ways that a financial agent can invest in this market, and the constraints that an investor faces. In Chapter 2 we introduce the numéraire portfolio. We discuss how it relates to other notions, and conclude with our first main result (Theorem 2.20) that provides necessary and sufficient conditions for the existence of the numéraire portfolio in terms of the predictable characteristics of the stock-price process.

Chapter 3 begins by recalling of some arbitrage notions and their interrelationships. We proceed to discuss the second main result, which establishes the equivalence of the existence of the numéraire portfolio with an arbitrage notion. Some applications are presented, namely in arbitrage equivalences for exponential Lévy markets and in utility optimization.

In Chapters 2 and 3 some of the proofs are not given, since they tend to be quite long; this is the content of the next three sections. In Chapter 4 we describe necessary and sufficient conditions for existence of wealth processes that are increasing and not constant. Chapter 5 deals with a deterministic, static case of the problem, where prices are modeled by exponential Lévy processes. After the case of exponential Lévy models, we proceed in Chapter 6 to general semimartingales.

Finally, we include an Appendix. In an effort to keep the text as self-contained as possible, we included there some topics that might not be as widely known as we would wish, and some results which, were they to be presented in the main text, would have interfered with its natural flow.

0.3. **General notation.** A vector p of the d-dimensional real Euclidean space $\mathbb{R}^d$ is understood as a $d \times 1$ (column) matrix. The transpose of p is denoted by $\mathrm{p}^\top$, and the usual Euclidean norm is $\|\mathrm{p}\| := \sqrt{\mathrm{p}^\top \mathrm{p}}$. We use superscripts to denote coordinates: $\mathrm{p} = (\mathrm{p}^1, \dots \mathrm{p}^d)^\top$. By $\mathbb{R}_+$ we denote the positive real half-line $[0, \infty)$.

The symbol "$\wedge$" denotes minimum: $f \wedge g = \min(f, g)$; for a real-valued function f its *negative part* is $f^- := -(f \wedge 0)$ and its *positive part* is $f^+ := \max(f, 0) = f + f^-$.

The *indicator function* of a set A is denoted by $\mathbf{1}_A$. To ease notation and the task of reading, subsets of $\mathbb{R}^d$ such as $\left\{x \in \mathbb{R}^d \mid \|x\| \leq 1\right\}$ are "schematically" denoted by $\{\|x\| \leq 1\}$; for the corresponding indicator function we write $\mathbf{1}_{\{\|x\| \leq 1\}}$.

A measure ν on $\mathbb{R}^d$ (Euclidean spaces are always supposed to be endowed with the Borel σ-algebra) is called a *Lévy measure*, if $\nu(\{0\}) = 0$ and $\int (1 \wedge \|x\|^2) \nu(\mathrm{d}x) < +\infty$. A *Lévy triplet* (b, c, ν) consists of a vector $b \in \mathbb{R}^d$, a $d \times d$ symmetric, non-negative definite matrix c, and a Lévy measure ν on $\mathbb{R}^d$. Once we have defined the price processes, the elements c and ν of the Lévy triplet will correspond to the instantaneous covariation rate of the continuous part and to the instantaneous jump intensity of the process, respectively. Also, b can be thought as an instantaneous drift rate, although one has to be careful with this interpretation, since b does not take into consideration the drift coming from large jumps of the process.

Suppose we have two measurable spaces $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$, a measure μ_1 on $(\Omega_1, \mathcal{F}_1)$, and a *transition measure* $\mu_2 : \Omega_1 \times \mathcal{F}_2 \mapsto \mathbb{R}_+$; i.e., for fixed $\omega_1 \in \Omega_1$, the set function $\mu_2(\omega_1, \cdot)$ is a measure on $(\Omega_2, \mathcal{F}_2)$, and for fixed $A \in \mathcal{F}_2$ the function $\mu_2(\cdot, A)$ is $\mathcal{F}_1$-measurable. We shall denote by $\mu_1 \otimes \mu_2$ the measure on the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ defined for $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$ as

$$(\mu_1 \otimes \mu_2)(E) := \int \left(\int \mathbf{1}_E(\omega_1, \omega_2) \mu_2(\omega_1, \mathrm{d}\omega_2) \right) \mu_1(\mathrm{d}\omega_1). \tag{0.1}$$

0.4. **Remarks of probabilistic nature.** For results concerning the general theory of stochastic processes described below, we refer the interested reader to the book [17] of Jacod and Shiryaev, especially the first two chapters.

We are given a *stochastic basis* $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where the *filtration* $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is assumed to satisfy the *usual hypotheses* of right continuity and augmentation by the $\mathbb{P}$-null sets. Without loss of generality we can assume that $\mathcal{F}_0$ is $\mathbb{P}$-trivial and that $\mathcal{F} = \mathcal{F}_\infty := \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_t$. The *probability measure* $\mathbb{P}$ will be fixed throughout and will receive no special mention. Every formula, relationship, etc. is supposed to be valid $\mathbb{P}$-a.s. (again, no special mention will be made). The *expectation* of random variables defined on the measure space $(\Omega, \mathcal{F}, \mathbb{P})$ will be denoted by $\mathbb{E}$.

The set $\Omega \times \mathbb{R}_+$ is the *base space*; a generic element will be denoted by (ω, t). Every process on the stochastic basis can be seen as a function from $\Omega \times \mathbb{R}_+$ with values in $\mathbb{R}^d$ for some $d \in \mathbb{N}$. The *predictable σ-algebra* on $\Omega \times \mathbb{R}_+$ is generated by all the adapted, left-continuous processes; we denote it by $\mathcal{P}$. Also, for any adapted, right-continuous process Y that admits left-hand limits, its *left-continuous version* Y_- is defined by setting $Y_-(0) := Y(0)$ and $Y_-(t) := \lim_{s \uparrow t} Y(s)$ for $t > 0$; this process is obviously predictable. We also define the *jump process* $\Delta Y := Y - Y_-$.

For a d-dimensional semimartingale X and a d-dimensional predictable process H, we shall denote by $H \cdot X$ the *stochastic integral* process, whenever this makes sense, in which case we shall be referring to H as being *X-integrable*[5]. Let us

[5] When we say that H is X-integrable we shall assume tacitly that it is predictable.

note that we are assuming *vector stochastic integration*. A good account of this can be found in [17] as well as in the paper [6] by Cherny and Shiryaev. Also, for two real-valued semimartingales X and Y, we define their *quadratic covariation* process by

$$[X, Y] := XY - X_0 Y_0 - X_- \cdot Y - Y_- \cdot X.$$

Finally, by $\mathcal{E}(Y)$ we shall be denoting the *stochastic exponential* of the linear semimartingale Y; we send the reader to Appendix A for more information.

1. The Market, Investments, and Constraints

1.1. **The stock prices model.** On the given stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ we shall consider a $(d+1)$-dimensional semimartingale $S \equiv (S^0, S^1, \ldots, S^d)$ that models the prices of $d+1$ assets. The vector $(S^1, \ldots, S^d)$ represents what we shall casually refer to as *stocks* and S^0 is the *money market* (or bank account). The only difference between the stocks and the money market is that the latter plays the role of a "benchmark", in the sense that wealth processes are quoted in units of S^0 and not nominally. As we shall see (and as is common in Mathematical Finance), for our problem we can assume that $S^0 \equiv 1$; in economic language, the interest rate is zero.

Coupled with S, there exists another $(d+1)$-dimensional semimartingale $X \equiv (X^0, X^1, \ldots, X^d)$ with $X_0 = 0$ and $\Delta X^i > -1$ for $i = 0, 1, \ldots, d$; X is the *returns process* and generates the asset prices S in a multiplicative way: $S^i = S^i_0 \mathcal{E}(X^i)$, $i = 0, 1, \ldots, d$. The assumption of a money market satisfying $S^0 \equiv 1$ (that will eventually be made) gives rise to a returns process $X^0 \equiv 0$.

Observe that we work with the stochastic — as opposed to the usual — exponential; in financial terms, we consider *simple* - as opposed to *compound* - *interest*. Simple interest is easier to comprehend in financial terms; also, as it turns out, the stochastic exponential is mathematically much better-suited to work with when we are dealing with stochastic processes.

Remark 1.1. Under our model we have $S > 0$ and $S_- > 0$; one can argue that this is not the most general case of a semimartingale model, since it does not allow for negative prices — for example, prices of forward contracts can take negative values. The general model should be an *additive* one: $S = S_0 + X$, where now X^i represents the cumulative *gains* of S^i after time zero and can be *any* semimartingale (without having to satisfy $\Delta X^i > -1$ for $i = 1, \dots, d$), as long as at least the money market process S^0, as well as its left-continuous version remain strictly positive.

In our discussion we shall be using the returns process X, not the stock-price process S directly. All the work we shall do carries to the additive model almost vis-a-vis; whenever there is a small change we trust that the reader can spot it. We choose to work under the multiplicative model since it is somehow more intuitive and more applicable: almost every model used in practice is written in this way.

The predictable characteristics of the returns process X will be important in our discussion. To this end, we fix the *canonical truncation function*[6] $x \mapsto x\mathbf{1}_{\{\|x\| \leq 1\}}$. With respect to the canonical truncation and write the *canonical decomposition* of the semimartingale X as

$$X = X^{\mathsf{c}} + B + \left[x\mathbf{1}_{\{\|x\| \leq 1\}}\right] * (\mu - \eta) + \left[x\mathbf{1}_{\{\|x\| > 1\}}\right] * \mu. \tag{1.1}$$

Some remarks on this representation are in order. First, μ is the *jump measure* of X, i.e., the random counting measure on $\mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$ defined by

$$\mu([0,t] \times A) := \sum_{0 \leq s \leq t} \mathbf{1}_A(\Delta X_s), \quad \text{for } t \in \mathbb{R}_+ \text{ and } A \subseteq \mathbb{R}^d \setminus \{0\}\,. \tag{1.2}$$

With this in mind, the last process that appears in equation (1.1) is just

$$\left[x\mathbf{1}_{\{\|x\| > 1\}}\right] * \mu \equiv \textstyle\sum_{0 \leq s \leq \cdot} \Delta X_s \mathbf{1}_{\{\|\Delta X_s\| > 1\}}$$

the *sum of the "big" jumps of* X; throughout the text, the asterisk denotes integration with respect to random measures. Once this term is subtracted

[6] In principle one could use any bounded Borel function h such that $h(x) = x$ in a neighborhood of $x = 0$; the use of this specific choice will merely facilitate some calculations and notation.

from X, what remains is a semimartingale with bounded jumps, thus a *special* semimartingale with a unique decomposition into a *predictable finite variation* part, denoted by B in (1.1), and a *local martingale* part. Finally, this last local martingale part can be decomposed further, into its *continuous* part, denoted by X^{c}, and its *purely discontinuous* part, which can be identified as the local martingale $\left[x\mathbf{1}_{\{\|x\|\leq 1\}}\right] * (\mu - \eta)$. Here, η is the *predictable compensator* of μ, so the purely discontinuous part is just a *compensated sum of the "small" jumps* — the ones with less than unit magnitude.

We define $C := [X^{\mathsf{c}}, X^{\mathsf{c}}]$ to be the *quadratic covariation* process of X^{c}. Then, the triple (B, C, η) is called the *triplet of predictable characteristics* of X. We set $G := \sum_{i=0}^{d} (C^{i,i} + \mathrm{Var}(B^i) + [1 \wedge (x^i)^2] * \eta)$; then G is an predictable, linear, increasing process, and all three processes (B, C, η) are absolutely continuous with respect to it. It follows that one can write

$$B = b \cdot G, \ C = c \cdot G, \ \text{and } \eta = G \otimes \nu \tag{1.3}$$

where all b, c and ν are predictable, b is a vector process, c is a positive-definite matrix-valued process and ν is a process with values in the space of Lévy measures. For the product-measure notation $G \otimes \nu$ (see formula (0.1)) we consider the measure induced by G. Any process $\tilde{G}$ with $\mathrm{d}\tilde{G}_t \sim \mathrm{d}G_t$ can be used in place of G, and is many times more natural. The final choice of an increasing process G reflects also the idea of an *operational clock* (as opposed to the natural time flow, described by t), since it should roughly give an idea of how fast the market is moving.

We abuse terminology and also call (b, c, ν) the triplet of predictable characteristics of X; this depends on G, but everything that we shall be discussing are invariant to the choice of G.

Remark 1.2. In "purely continuous" models, known in the literature as *quasi-left-continuous* (meaning that the price process does not jump at predictable times), G can be chosen continuous. Nevertheless, if we want to include discrete-time models, we must allow for the possibility that G has positive jumps too. Since C is a continuous increasing process, and also since by (1.1) we obtain that $\mathbb{E}[\Delta X_\tau \mathbf{1}_{\{\|\Delta X_\tau\| \leq 1\}} \mid \mathcal{F}_{\tau-}] = \Delta B_\tau$ for every predictable time τ, we have

$$c = 0 \text{ and } b = \int x \mathbf{1}_{\{\|x\| \leq 1\}} \nu(\mathrm{d}x), \text{ on the predictable set } \{\Delta G > 0\}. \tag{1.4}$$

Remark 1.3. We make a small technical observation. The properties of c being a symmetric positive-definite predictable process and ν a predictable process taking values in the space of Lévy processes, in general hold $\mathbb{P} \otimes G$-a.e. We shall assume that they hold *everywhere*, i.e., for all $(\omega, t) \in \Omega \times \mathbb{R}_+$; we can always do this by changing them on a predictable set of $\mathbb{P} \otimes G$-measure zero to be $c \equiv 0$ and $\nu \equiv 0$.

Definition 1.4. Let X be *any*[7] semimartingale with canonical representation (1.1), and consider an operational clock G such that the relationships (1.3) hold. If $\int_{\{\|x\|>1\}} \|x\| \, \nu(\mathrm{d}x) < \infty$ for $\mathbb{P} \otimes G$-a.e. $(\omega, t) \in \Omega \times \mathbb{R}_+$, then the *drift rate* of X (with respect to G) is defined as the quantity

$$b + \int_{\{\|x\|>1\}} x \nu(\mathrm{d}x).$$

The concept of drift rates will be used throughout. Their existence does not depend on the choice of the operational clock G, although the drift rate itself does. Under the assumption of Definition 1.4, if the increasing process $\left[\|x\| \, \mathbf{1}_{\{\|x\|>1\}}\right] * \eta = \left(\int_{\{\|x\|>1\}} \|x\| \, \nu(\mathrm{d}x)\right) \cdot G$ is *finite* (this happens if and only

[7]By "any" we mean *not* necessarily the returns process.

if X is a *special* semimartingale), then the predictable process

$$B + \left[x\mathbf{1}_{\{\|x\|>1\}}\right] * \eta = \left(b + \int_{\{\|x\|>1\}} x\nu(\mathrm{d}x)\right) \cdot G$$

is called the *drift* of X. If drifts exist, drift rates exist too; the converse is not true. Semimartingales that are *not* special might have well-defined drift rates. For example a σ-martingale is exactly a semimartingale with vanishing drift rate; cf. Appendix D on σ-localization for further discussion.

1.2. **Wealth processes and strategies.** Given an initial capital $w \in \mathbb{R}_+$, one can invest in the assets described by the process S by choosing a predictable, d-dimensional and $(S^1, \dots, S^d)$-integrable process process θ, which we shall refer to as *strategy*. The number θ^i_t represents the number of shares from the i^{th} stock held by the investor at time t. Let us denote the wealth process from such a strategy by W. The total amount of money invested in stocks is $\sum_{i=1}^d \theta^i S^i_-$; in order for the wealth process to satisfy the *self-financing* condition at every point in time, it is necessary that the remaining wealth $W_- - \sum_{i=1}^d \theta^i S^i_-$ be invested in the money market, which will result in returns $(W_- - \sum_{i=1}^d \theta^i S^i_-) \cdot X^0$. In this case the value of the investment is described by the process

$$W := w + \left(W_- - \sum_{i=1}^d \theta^i S^i_-\right) \cdot X^0 + \theta \cdot S. \tag{1.5}$$

Now, W represents the *nominal* amount of money that the investor has. It is not hard to solve this last equation, because it is linear in W. One can check directly (or consult Lemma A.3 of Appendix A) that the solution of equation 1.5, given in terms of the *discounted wealth* $\widetilde{W} := W/S^0$ and the vector $\widetilde{S} = (S^1/S^0, \dots, S^d/S^0)$ of *discounted stock prices*, is

$$\widetilde{W} = w + \theta \cdot \widetilde{S}.$$

A peep ahead in Definition 2.1 reveals that the numéraire portfolio is defined in terms of ratios of wealth processes; ratios of discounted wealth processes are the same as ratios of the original wealth processes, so we might as well work in discounted terms. From now on we assume that $S^0 \equiv 1$ and that S and X are d-dimensional processes with only "stock" components. Starting with capital $w \in \mathbb{R}_+$ and investing according to the strategy θ, the investor's wealth process is $W := w + \theta \cdot S$. The local boundedness away from zero and infinity of S^0 makes S-integrability equivalent to $\widetilde{S}$-integrability, so we lose nothing in the class of strategies.

Some restrictions have to be enforced so that the investor cannot use so-called *doubling strategies*. The assumption prevailing in this context is that the wealth process should be uniformly bounded from below by some constant. This has the very clear financial interpretation of a *credit limit* that the particular investor has to face. We shall set this credit limit to be zero; one can regard this as just shifting the wealth process to some extend, and working with this relative credit line instead of the absolute one. So, for any $w \in \mathbb{R}_+$ and any predictable, S-integrable process θ, the value process $W = w + \theta \cdot S$ is called *admissible* if $W \geq 0$.

1.3. **Constraints on strategies.** We start with an example in order to motivate Definition 1.6 below.

Example 1.5. Suppose that the investor is prevented from selling stock short or borrowing from the bank. In terms of the strategy and wealth process, this will mean that $\theta^i \geq 0$ for all $i = 1, 2, \ldots, d$ and also $\theta^\top S_- \leq W_-$. By setting $\mathfrak{C} := \left\{ \mathrm{p} \in \mathbb{R}^d \mid \mathrm{p}^i \geq 0 \text{ and } \sum_{i=1}^d \mathrm{p}^i \leq 1 \right\}$, the prohibition of short sales and borrowing is translated into the requirement $\left(\theta^i S^i_-\right)_{1 \le i \le d} \in W_- \mathfrak{C}$, where this relationship holds in an $\Omega \times \mathbb{R}_+$-pointwise manner.

The example leads us to consider the class of all possible constraints that can be represented this way; although in this particular case the set $\mathfrak{C}$ was non-random, we might have situations where the constraints depend on both time and the path.

Definition 1.6. Consider an arbitrary set-valued process $\mathfrak{C} : \Omega \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}^d)$ with $0 \in \mathfrak{C}(\omega, t)$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$. The admissible wealth process $W = w + \theta \cdot S$ will be called $\mathfrak{C}$*-constrained*, if the vector $\left(\theta^i S^i_-\right)_{1 \le i \le d}$ belongs to the set $W_- \mathfrak{C} \mathbf{1}_{\{W_- > 0\}} + \check{\mathfrak{C}} \mathbf{1}_{\{W_- = 0\}}$ in a $\Omega \times \mathbb{R}_+$-pointwise sense. Here

$$\check{\mathfrak{C}} := \bigcap_{a > 0} a \mathfrak{C} \tag{1.6}$$

is the set of *cone points*[8] of $\mathfrak{C}$.

We denote by $\mathcal{W}$ the class of all admissible, $\mathfrak{C}$-constrained wealth processes. By $\mathcal{W}^o$ we shall be denoting the subclass of $\mathcal{W}$ that consists of wealth processes W which stay strictly positive, in the sense that $W > 0$ and $W_- > 0$.

The special treatment of constraints on the set $\{W_- = 0\}$ is purely for continuity reasons: if we are allowed to invest according to $\theta(\omega, t)$ however small our capital is, we might as well be allowed to invest in it even if our capital is zero, provided we keep ourselves with positive wealth. In any case, the reader can easily ignore this; soon we shall be considering only wealth processes with $W_- > 0$.

[8] Tyrell Rockafellar in [29] calls $\check{\mathfrak{C}}$ the *recession cone* of $\mathfrak{C}$.

Let us give another example of constraints of this type. They actually follow from the positivity constraints and will not constrain the wealth processes further, but the point is that we can always include them in our constraint set.

Example 1.7. Natural Constraints. An admissible strategy generates a wealth process that starts positive and stays positive. Thus, if $W = w + \theta \cdot S$, then we have $\Delta W \geq -W_-$, or $\theta^\top \Delta S \geq -W_-$, or further that $\sum_{i=1}^d \theta^i S^i_- \Delta X^i \geq -W_-$. Remembering the definition or the random measure ν from (1.3). we see that this requirement is equivalent to $\nu[\sum_{i=1}^d \theta^i S^i_- x^i < -W_-] = 0$, $\mathbb{P} \otimes G$-almost everywhere. Define now the random set-valued process (randomness comes through ν)

$$\mathfrak{C}_0 := \left\{ \mathrm{p} \in \mathbb{R}^d \mid \nu[\mathrm{p}^\top x < -1] = 0 \right\}; \tag{1.7}$$

we shall call it the set-valued process of *natural constraints*. The requirement $\Delta W \geq -W_-$ is exactly what corresponds to θ being $\mathfrak{C}_0$-constrained. Note that $\mathfrak{C}_0$ is not deterministic in general. It is now clear that we are not considering random constraints just for the sake of generality, but because they arise naturally as part of the problem.

Remark 1.8. In our Definition 1.6 of $\mathfrak{C}$-constrained strategies, the multiplicative factor W_- before $\mathfrak{C}$ might seem ad-hoc, but will be crucial in our analysis. We have already seen how it comes up naturally in certain constraint considerations. In any case, more wealth should give one more freedom in choosing the number of stocks for investing. Finally, observe that if $\mathfrak{C}$ is a cone process, we have $W_- \mathfrak{C} = \mathfrak{C} = \check{\mathfrak{C}}$, so that the constraints do not depend on the wealth level.

Eventually (see section 2.3) we shall ask for more structure on the set-valued process $\mathfrak{C}$, namely convexity, closedness and predictability. The reader can check that the examples presented have these properties; the "predictability structure" should be clear from the definition of $\mathfrak{C}_0$, which involves the predictable process ν.

1.4. **Stochastic exponential representation of wealth processes.** Pick a wealth process $W \in \mathcal{W}^o$. Since the process $W = w + \theta \cdot S$ satisfies $W > 0$ and $W_- > 0$, we can write $W = w\mathcal{E}(\pi \cdot X)$, where π is the X-integrable process with components $\pi^i := \theta^i S^i_- / W_-$ (here we are using a property of the stochastic integral that goes by the name "Second Associativity Theorem" and appears as Theorem 4.7 in [6]). The equivalent of $W > 0$ and $W_- > 0$ is $\pi^\top \Delta X > -1$. The original constraints $(\theta^i S^i_-)_{1 \le i \le d} \in W_- \mathfrak{C}$ translate for the process π in the requirement that

$$\pi(\omega, t) \in \mathfrak{C}(\omega, t), \ \Omega \times \mathbb{R}_+\text{-pointwise.} \tag{1.8}$$

The converse also holds: start with a set-valued process $\mathfrak{C}$ that represents constraints on portfolios. For any X-integrable process π with $\pi^\top \Delta X > -1$ and $\pi(\omega, t) \in \mathfrak{C}(\omega, t)$, $\Omega \times \mathbb{R}_+$-pointwise, set $W := w\mathcal{E}(\pi \cdot X)$. Then, for the S-integrable process θ with $\theta^i := \pi^i W_- / S^i_-$ we have $W = 1 + \theta \cdot S$ for some S-integrable θ. Both W and W_- are strictly positive and the requirement (1.8) amounts to θ being $\mathfrak{C}$-constrained: W is an element of $\mathcal{W}^o$.

To summarize the preceding discussion: we have shown the class equality $\mathcal{W}^o = \{w\mathcal{E}(\pi \cdot X) \mid w > 0 \text{ and } \pi \in \Pi\}$, where

$$\Pi := \left\{ \pi \in \mathcal{P} \mid \pi \text{ is } X\text{-integrable}, \ \pi^\top \Delta X > -1, \text{ and (1.8) holds} \right\}.$$

The elements of Π will be called *portfolios*; we make this distinction with the corresponding notion of strategy, previously denoted by θ. A portfolio $\pi \in \Pi$ is understood to generate the *wealth process* $W^\pi := \mathcal{E}(\pi \cdot X)$ and the strategy

θ with $\theta^i = \pi^i W^\pi_- / S^i_-$. It is clear that π^i signifies the *proportion* of our current wealth invested in the stock S^i.

Example 1.9. We give some (rather trivial) examples of portfolios. Here, we assume no constraint other than admissibility, i.e., $\mathfrak{C} = \mathfrak{C}_0$. Denote by e_i the unit vector with all zero entries but for the i^{th} coördinate, which is unit. Since $\mathrm{e}_i^\top \Delta X = \Delta X^i > -1$ for all $i = 1, 2, \ldots, d$, we have that any unit vector e_i is a portfolio. Since the zero vector is always a portfolio and the class Π is *predictably convex*[9] it follows that any predictable process π with $\pi \in [0,1]^d$ and $\sum_{i=1}^d \pi^i \leq 1$ is a portfolio. The quantity $1 - \sum_{i=1}^d \pi^i$ is the percentage of wealth that is not invested in any stock.

A more interesting example is the *market portfolio* $\mathfrak{m}$, that is defined by

$$\mathfrak{m}^i := \frac{S^i_-}{\sum_{j=1}^d S^j_-};$$

we leave the reader the task to prove that

$$W^{\mathfrak{m}} = \left(\sum_{j=1}^d S^j_0 \right)^{-1} \sum_{j=1}^d S^j.$$

In this sense, the wealth generated by $\mathfrak{m}$ follows the total capitalization of the market (relative to the initial total capitalization, of course), hence the name "market portfolio".

1.5. **Time horizons.** We shall be working on an infinite time planning horizon. Of course, any finite time horizon can be easily contained in this case, but let us spend a few lines to explain this in some detail.

[9]Predictable convexity means that if π and $\tilde{\pi}$ are elements of Π and α is a $[0,1]$-valued predictable process then $\alpha\pi + (1-\alpha)\tilde{\pi}$ also belongs in Π

Let us first discuss the *range of integration* of the portfolios that we are considering. Up to now, we merely asked a portfolio $\pi \in \Pi$ to be X-integrable. Some authors[10] require also the existence of the limit of the corresponding wealth process at infinity. There is a notion of *global integrability*, which is stronger than mere integrability plus the existence of the limit at infinity. One can consult the book [5] of Bichteler; a discussion from a slightly different viewpoint is made in Cherny and Shiryaev [7]. A brief account, together with some results that we shall be using later on, is given in Appendix C. Let us denote by Π_∞ the class of portfolios $\pi \in \Pi$ which are globally X-integrable; we also define the class $\mathcal{W}_\infty$ of wealth processes as the elements of $\mathcal{W}$ that are semimartingales up to infinity, and the corresponding "strictly positive" elements $\mathcal{W}^o_\infty$ as the value processes $W \in \mathcal{W}_\infty$ for which $\inf_{t \in \mathbb{R}_+} W^\pi_t > 0$.

Let us discuss now how we can embed any time-horizon in our discussion. Pick any (possibly infinite-valued) stopping time τ. We shall say that a portfolio π is *X-integrable up to τ*, if π is X^τ-integrable up to infinity, where X^τ represents the *stopped process* defined by $X^\tau_t := X_{\tau \wedge t}$ for all $t \in \mathbb{R}_+$. Under this proviso, one can define the classes Π_τ and $\mathcal{W}_\tau$ as before, but requiring integrability up to τ in place of plain integrability. It is clear that if $\tau_1 \leq \tau_2$ are two stopping times then $\Pi_{\tau_2} \subseteq \Pi_{\tau_1}$, with the same relationship holding for the wealth process classes too. Note that if $\tau = \infty$, the class Π_τ is exactly Π_∞ defined in the previous paragraph.

Here is something more interesting: pick some (again, possibly infinite-valued) *predictable* time τ; so that there exists an increasing sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that $\tau_n \uparrow \tau$ and $\tau_n < \tau$ on $\{\tau > 0\}$. Say that a portfolio π is *X-integrable for all times before τ* if it is X-integrable up to time τ_n for all

[10] Notably Delbaen and Schachermayer in their paper [8].

$n \in \mathbb{N}$. We define $\Pi_{\tau-}$ and $\mathcal{W}_{\tau-}$ as before, by the requirement of integrability for all times before τ, i.e., $\Pi_{\tau-} = \bigcap_{n \in \mathbb{N}} \Pi_{\tau_n}$, and a similar relationship for the wealth processes class. One easily sees that this definition is independent of the announcing sequence $(\tau_n)_{n \in \mathbb{N}}$. Obviously, $\Pi_\tau \subseteq \Pi_{\tau-}$ and $\mathcal{W}_\tau \subseteq \mathcal{W}_{\tau-}$. For $\tau = \infty$, the difference in these classes is exactly the difference between the requirements of plain and global integrability, and the classes $\mathcal{W}_{\infty-}$ and $\Pi_{\infty-}$ are exactly $\mathcal{W}$ and Π.

Now, with a clear conscience, we can utter the usual sentence: Since everything can be deduced from the infinite-horizon case, we shall assume it from now on, and not bother with remarks of the preceding type anymore. If we ever refer to "integrability for all times", it will have the usual meaning of simple integrability.

2. The Numéraire Portfolio: Definitions, General Discussion and Predictable Characterization of its Existence

2.1. **The numéraire portfolio.** Here is the central notion of our work.

Definition 2.1. A portfolio $\rho \in \Pi$ will be called (global) *numéraire portfolio*, if for every wealth process $\hat{W} \in \mathcal{W}$ the *relative wealth process*, defined as $\hat{W}/W^\rho$, is a supermartingale, and $W^\rho_\infty < +\infty$.

Since $0 \in \Pi$, $1/W^\rho$ is a positive supermartingale and the limit W^ρ_∞ that appears in this definition exists and is strictly positive. We ask it further to be finite, because we want it to have a "global" property. The corresponding local notion, where one does not impose $W^\rho_\infty < +\infty$, might be called "the numéraire for all times before infinity", following the discussion of section 1.5. Definition 2.1 in this form first appears in Becherer [4], where we send the reader for the history of this concept. A simple observation from that paper shows that the wealth process generated by numéraire portfolios is unique[11]: indeed, if there are two numéraire portfolios ρ_1 and ρ_2 in Π, both W^{ρ_1}/W^{ρ_2} and W^{ρ_2}/W^{ρ_1} are supermartingales; an application of Jensen's inequality then gives

$$1 \geq \mathbb{E}\left[W^{\rho_1}_t / W^{\rho_2}_t\right] \geq \left(\mathbb{E}\left[W^{\rho_2}_t / W^{\rho_1}_t\right]\right)^{-1} \geq 1, \text{ for all } t \in \mathbb{R}_+.$$

For the positive random variable $U := W^{\rho_1}_t / W^{\rho_2}_t$ we have $\mathbb{E}[U] = \mathbb{E}[U^{-1}] = 1$, so it must be that $U = 1$, i.e., $W^{\rho_1}_t = W^{\rho_2}_t$ for any fixed $t \in \mathbb{R}_+$; since both processes are càdlàg we have the result holding simultaneously for all $t \in \mathbb{R}_+$ so that $W^{\rho_1} = W^{\rho_2}$. The uniqueness of the stochastic exponential gives $\rho_1 \cdot X = \rho_2 \cdot X$,

[11] This fact will also become clear later on.

thus $\rho_1 = \rho_2$, $\mathbb{P} \otimes G$-almost everywhere. In this sense, the numéraire portfolio is unique too.

This uniqueness property of the numéraire portfolio should explain the use of the definite article "the" in its definition; nevertheless, there is also a second reason for using the definite article, and this is linguistic. In general, by "numéraire" we mean any strictly positive semimartingale process Y with $Y_0 = 1$ (it may not even be generated by a portfolio) such that it acts as an "inverse deflator" for our wealth processes, i.e., we see our investment according to a portfolio π relatively to Y, giving us a wealth of W^π / Y. Of course, if ρ satisfies the requirements of Definition 2.1, W^ρ can act as a numéraire is the sense of what we are discussing here. Nevertheless, we agree to call W^ρ "the numéraire", since it is in a sense the best tradable benchmark: whatever anyone else does, it looks as a supermartingale[12] through the lens of relative wealth to W^ρ.

In accordance with the preceding paragraph, note an amusing fact: the property of being the numéraire portfolio is numéraire-independent! Indeed, for any numéraire Y the relative wealth of two wealth processes seen relatively to Y is exactly equal to the relative wealth of the two wealth processes, since Y cancels out. The reader can now see why in the first place we decided to work using the money market to discount the wealth processes, thus assuming that $S^0 \equiv 1$.

Remark 2.2. The numéraire portfolio is introduced in Definition 2.1 as the solution to some optimization-type problem. As we shall see, it has at least four more such optimality properties, which we mention here with hints on where they will appear again. If ρ is the numéraire portfolio, then

[12] Supermartingale are in amny senses the stochastic analogues of decreasing functions.

- it is *growth-optimal*, in the sense that it maximizes the growth rate over all portfolios (cf. section 2.5);
- it maximizes the *asymptotic growth* of the wealth process it generates over all portfolios (Proposition 2.18);
- it is also the solution of a *log-utility maximization* problem. In fact, if this problem is defined in relative (as opposed to absolute) terms, the two are equivalent. For more infomation, consult Proposition 3.16;
- it minimizes the *reverse relative entropy* among all *supermartingale deflators*. Consult Definition 3.6 on supermartingale deflators and Remark 3.8 for these notions and results.

We can state now our basic problem.

Problem 2.3. *Find necessary and sufficient conditions for the existence of the numéraire portfolio in terms of the triplet of predictable characteristics of the stock-price process S (equivalently, of the returns process X).*

Example 2.4. We can already give the first example of a numéraire portfolio. The numéraire portfolio clearly exists and is equal to zero, if and only if all elements $W \in \mathcal{W}$ are supermartingales under $\mathbb{P}$. This is a trivial example, but we shall make use of it when we study arbitrage in exponential Lévy markets.

Also, although not needed in the sequel, let us stay in accordance with our problem and remark that the corresponding predictable characterization of this is that

$$\pi^\top b + \int \pi^\top x \mathbf{1}_{\{\|x\|>1\}} \nu(\mathrm{d}x) \leq 0,$$

for all predictable processes π with $\nu[\pi^\top x < -1] = 0$.

In this Chapter we shall be concerned with the solution of Problem 2.3, which appears as Theorem 2.20. In the next Chapter we shall also consider Problem

3.5, which asks for an arbitrage characterization of the existence of the numéraire portfolio.

The following simple result shows that the existence of the numéraire has some implications for the class of wealth processes $\mathcal{W}$.

Proposition 2.5. *Suppose that the numéraire portfolio ρ exists. Then, all wealth processes of $\mathcal{W}$ are semimartingales up to infinity (i.e., $\mathcal{W} = \mathcal{W}_\infty$) and ρ is globally X-integrable ($\rho \in \Pi_\infty$, or equivalently $W^\rho \in \mathcal{W}^o_\infty$).*

Proof. Let us start with ρ. We already know that $(W^\rho)^{-1}$ being a positive supermartingale implies that $(W^\rho_\infty)^{-1} := \lim_{t\to\infty} (W^\rho_t)^{-1}$ exists and is finite. The assumption $W^\rho_\infty < +\infty$ implies that $(W^\rho_\infty)^{-1}$ is strictly positive. Lemma C.2 of Appendix C gives both that ρ is globally X-integrable and that W^ρ is a semimartingale up to infinity. Now, pick any other $W \in \mathcal{W}$; since W/W^ρ is a positive supermartingale, Lemma C.2 applied again gives that it is a semimartingale up to infinity, and so will be $W = W^\rho\,(W/W^\rho)$. □

2.2. **Preliminary necessary and sufficient conditions for existence of the numéraire portfolio.** In order to figure out whether a portfolio $\rho \in \Pi$ is the numéraire portfolio we should (at least) check that W^π/W^ρ is a supermartingale for all $\pi \in \Pi$. This is seemingly weaker than the requirement of Definition 2.1, but the two are actually equivalent; see the proof of Lemma 2.8. For the time being, let us derive a convenient expression for the ratio W^π/W^ρ.

Thus, let us consider a baseline portfolio $\rho \in \Pi$ that produces a wealth W^ρ, and any other portfolio $\pi \in \Pi$; their relative wealth process is given by the ratio $W^\pi/W^\rho = \mathcal{E}(\pi \cdot X)/\mathcal{E}(\rho \cdot X)$. With the help of Lemma A.2 of Appendix A we get

$$(\mathcal{E}(\rho \cdot X))^{-1} = \mathcal{E}\left(-\rho \cdot X + \left(\rho^\top c \rho\right) \cdot G + \left[\frac{(\rho^\top x)^2}{1+\rho^\top x}\right] * \mu\right),$$

where μ is the jump measure of X defined in (1.2) and G is the operational clock appearing in (1.3). The last equality coupled with use of Yor's formula[13] $\mathcal{E}(Y_1)\mathcal{E}(Y_2) = \mathcal{E}(Y_1 + Y_2 + [Y_1, Y_2])$ will give

$$\frac{W^\pi}{W^\rho} = \mathcal{E}(\pi \cdot X)\ \mathcal{E}\left(-\rho \cdot X + \left(\rho^\top c \rho\right) \cdot G + \left[\frac{(\rho^\top x)^2}{1+\rho^\top x}\right] * \mu\right) = \mathcal{E}\left((\pi - \rho) \cdot X^{(\rho)}\right)$$

(we have skipped some calculations), where

$$X^{(\rho)} := X - (c\rho) \cdot G - \left[\frac{\rho^\top x}{1+\rho^\top x} x\right] * \mu.$$

Remark 2.6. Let us call $\pi^{(\rho)} := \pi - \rho$; then, we have the following situation: for any portfolio π, the portfolio $\pi^{(\rho)}$ when invested in the market described by the returns process $X^{(\rho)}$ generates a value equal to W^π / W^ρ. We can see the relative wealth process as the usual wealth that would be generated by investing in an auxiliary market. Of course, $X^{(\rho)}$ depends only on ρ as it should, since we only consider the baseline fixed.

For ρ to be the numéraire portfolio, we want W^π / W^ρ to be a supermartingale. In conjunction with Propositions D.2 and D.3, since W^π / W^ρ is a strictly positive process, the supermartingale property is equivalent to the σ-supermartingale one, which is in turn equivalent to requiring that its drift rate is finite and negative (for drift rates look at Definition 1.4). For the reader not familiar with the σ-localization technique, Kallsen's paper [19] is a good reference; for an overview of what is needed here, see Appendix D.

[13]Lemma A.1 in Appendix A.

Since $W^\pi / W^\rho = \mathcal{E}\left((\pi - \rho) \cdot X^{(\rho)}\right)$, the condition of negativity on the drift rate of W^π / W^ρ is equivalent to saying that the drift rate of the semimartingale $(\pi - \rho) \cdot X^{(\rho)}$ is negative. Straightforward computations give that this drift rate (if it exists) is

(2.1)
$$\mathfrak{rel}(\pi \mid \rho) := (\pi-\rho)^\top b - (\pi-\rho)^\top c\rho + \int \left[\frac{(\pi-\rho)^\top x}{1+\rho^\top x} - (\pi-\rho)^\top x \mathbf{1}_{\{\|x\|\le 1\}} \right] \nu(\mathrm{d}x).$$

The point of the notation $\mathfrak{rel}(\pi \mid \rho)$ is to serve as a reminder that this quantity is the rate of return of the *relative* wealth process W^π / W^ρ. Observe that the integrand in (2.1), namely

$$\frac{1+\pi^\top x}{1+\rho^\top x} - 1 - (\pi-\rho)^\top x \mathbf{1}_{\{\|x\|\le 1\}},$$

is ν-bounded from below by -1 on the set $\{\|x\| > 1\}$, whereas on the set $\{\|x\| \le 1\}$ (near $x = 0$) it behaves like $(\rho - \pi)^\top x x^\top \rho$, which is comparable to $\|x\|^2$. It follows that the integral always makes sense, but can take the value $+\infty$, so that the drift rate of W^π / W^ρ either exists (i.e., is finite) or takes the value $+\infty$. In any case, the quantity $\mathfrak{rel}(\pi \mid \rho)$ of (2.1) is well-defined. Let us record what we just proved.

Lemma 2.7. *Let π and ρ be two portfolios. Then, W^π / W^ρ is a supermartingale if and only if $\mathfrak{rel}(\pi \mid \rho) \le 0$, $\mathbb{P} \otimes G$-almost everywhere.*

Using this Lemma 2.6 we get the preliminary, necessary and sufficient conditions needed to solve the numéraire problem. In a different, more general form, these have already appeared in the paper [15] by Goll and Kallsen. We just state them here as a consequence of our previous discussion.

Lemma 2.8. *Suppose that $\mathfrak{C}$ is enriched with the natural constraints ($\mathfrak{C} \subseteq \mathfrak{C}_0$), and consider a process ρ with $\rho(\omega, t) \in \mathfrak{C}(\omega, t)$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$. In order* *for ρ to be the numéraire portfolio in the class Π, it is necessary and sufficient that*

(1) $\mathfrak{rel}(\pi \mid \rho) \leq 0$, *holds* $\mathbb{P} \otimes G$*-a.e. for every predictable* π *with* $\pi(\omega, t) \in \mathfrak{C}(\omega, t)$.

(2) ρ *is predictable; and*

(3) ρ *is globally* X*-integrable.*

Proof. The necessity is trivial, but for the fact that we ask condition (1) to holds not only for all *portfolios*, but for *any* predictable process π (which might not even be X-integrable). Suppose it holds for all portfolios, and take a predictable process π with $\pi(\omega, t) \in \mathfrak{C}(\omega, t)$. Then, $\pi_n := \pi \mathbf{1}_{\{\|\pi\| \leq n\}} + \rho \mathbf{1}_{\{\|\pi\| > n\}}$ is a portfolio, so that $\mathfrak{rel}(\pi \mid \rho) \mathbf{1}_{\{\|\pi\| \leq n\}} = \mathfrak{rel}(\pi_n \mid \rho) \leq 0$, and finally $\mathfrak{rel}(\pi \mid \rho) \leq 0$.

The three conditions are also sufficient for ensuring that W^π / W^ρ is a supermartingale for all predictable π with $\pi(\omega, t) \in \mathfrak{C}(\omega, t)$; we have to show that the latter property continues to hold even if we replace W^π with any $W \in \mathcal{W}$.

Of course, we can assume that $W_0 = 1$. Now, pick $W = 1 + \theta \cdot S$ for some $\mathfrak{C}$-constrained strategy θ and define the sequence of stopping times $\tau_n := \inf \{t \in \mathbb{R}_+ \mid W_t \leq 1/n\}$; the process $1 + (\theta \mathbf{1}_{[\![0, \tau_n]\!]}) \cdot S$ can be written as W^{π_n} for some π_n with $\pi_n(\omega, t) \in \mathfrak{C}(\omega, t)$, so that W / W^ρ is a supermartingale on the stochastic interval $[\![0, \tau_n]\!]$. Using Fatou's lemma then, one shows that it is also a supermartingale on the "interval type" set $\Gamma := \bigcup_{n \in \mathbb{N}} [\![0, \tau_n]\!]$. We need only show that $(\theta \mathbf{1}_{(\Omega \times \mathbb{R}_+) \setminus \Gamma} \cdot S) / W^\rho$ is also a supermartingale. We shall show, in fact, that the process $(\theta \mathbf{1}_{(\Omega \times \mathbb{R}_+) \setminus \Gamma}) \cdot S$ is identically zero, which is a way of saying that zero is an absorbing state for W.

We claim that for all $n \in \mathbb{N}$, the process $W^{(n)} := 1 + (n \theta \mathbf{1}_{(\Omega \times \mathbb{R}_+) \setminus \Gamma}) \cdot S$ is an element of $\mathcal{W}$; this happens because $W^{(n)} \geq n W_- \mathbf{1}_{(\Omega \times \mathbb{R}_+) \setminus \Gamma}$ and $\theta \mathbf{1}_{(\Omega \times \mathbb{R}_+) \setminus \Gamma} \in W_- \mathbf{1}_{(\Omega \times \mathbb{R}_+) \setminus \Gamma} \mathfrak{C} \mathbf{1}_{\{W_- \neq 0\}}$. Now, since each $W^{(n)}$ is bounded from below by one, we have that $W^{(n)}/W^{\rho}$ is a supermartingale for each $n \in \mathbb{N}$. But $W^{(n)}/W^{\rho} = 1/W^{\rho} + nW\mathbf{1}_{(\Omega\times\mathbb{R}_+)\setminus\Gamma}/W^{\rho}$; and if $W\mathbf{1}_{(\Omega\times\mathbb{R}_+)}$ is non-zero all these processes cannot be supermartingales, since they will be unbounded in probability. We thus conclude that $(\theta\mathbf{1}_{(\Omega\times\mathbb{R}_+)\setminus\Gamma}) \cdot S \equiv 0$, which finishes the proof. □

In order to obtain necessary and sufficient conditions for the existence of the numéraire portfolio in terms of predictable characteristics, the three conditions of Lemma 2.8 will be tackled one by one. For condition (1), one has to solve pointwise (for each fixed $(\omega, t) \in \Omega \times \mathbb{R}_+$) a convex optimization problem over the set $\mathfrak{C}(\omega, t)$. It is obvious that if (1) above is to hold for $\mathfrak{C}$, then it must also hold for the closed convex hull of $\mathfrak{C}$, so we might as well assume that $\mathfrak{C}$ is closed and convex if we want to find the process ρ. For condition (2), in order to prove that the solution we get is predictable, the set-valued process $\mathfrak{C}$ must have some predictable structure. We describe how this is done in the next section. After this, a simple test in terms of predictable characteristics will give us condition (3), and we shall be able to provide the solution of Problem 2.3 in Theorem 2.20.

2.3. **The predictable, closed convex structure of constraints.** Let us start with a remark concerning *degeneracies* that might appear in the market. This has to do with linear dependence that some stocks might exhibit at some points of the base space, and will cause seemingly different portfolios to produce the exact same wealth processes; they should then be treated as equivalent. To formulate this notion, consider two portfolios π_1 and π_2 with $W^{\pi_1} = W^{\pi_2}$. The uniqueness of the stochastic exponential will imply that $\pi_1 \cdot X = \pi_2 \cdot X$, and so the predictable process $\zeta := \pi_2 - \pi_1$ will satisfy $\zeta \cdot X \equiv 0$, which is easily seen to be equivalent to $\zeta \cdot X^{\mathsf{c}} = 0$, $\zeta^\top \Delta X = 0$ and $\zeta \cdot B = 0$. This makes the following definition plausible.

Definition 2.9. For a Lévy triplet (b, c, ν) define the linear subspace of *null investments* $\mathfrak{N}$ to be the set of vectors

$$\mathfrak{N} := \left\{ \zeta \in \mathbb{R}^d \mid \zeta^\top c = 0,\ \nu[\zeta^\top x \neq 0] = 0 \text{ and } \zeta^\top b = 0 \right\} \tag{2.2}$$

for which nothing happens if one invests in them. Two portfolios π_1 and π_2 satisfy $\pi_2(\omega, t) - \pi_1(\omega, t) \in \mathfrak{N}(\omega, t)$ for $\mathbb{P} \otimes G$-almost every $(\omega, t) \in \Omega \times \mathbb{R}_+$ if and only if $W^{\pi_1} = W^{\pi_2}$; we consider such π_1 and π_2 to be the same.

Here are the predictability, closedness and convexity requirements for our constraints.

Definition 2.10. The $\mathbb{R}^d$-set-valued process $\mathfrak{C}$ will be said to impose *predictable closed convex constraints* if

1. (1) $\mathfrak{N}(\omega, t) \subseteq \mathfrak{C}(\omega, t)$ for *all* $(\omega, t) \in \Omega \times \mathbb{R}_+$,
2. (2) $\mathfrak{C}(\omega, t)$ is a closed convex set, for *all* $(\omega, t) \in \Omega \times \mathbb{R}_+$, and
3. (3) $\mathfrak{C}$ is predictably measurable, in the sense that for any closed $F \subseteq \mathbb{R}^d$, we have $\{\mathfrak{C} \cap F \neq \emptyset\} := \{(\omega, t) \in \Omega \times \mathbb{R}_+ \mid \mathfrak{C}(\omega, t) \cap F \neq \emptyset\} \in \mathcal{P}$.

In this definition we must assume that $\mathfrak{C}$ is closed and convex and contains every null vector for *every* $(\omega, t) \in \Omega \times \mathbb{R}_+$, not just in an "almost every" sense.

The first requirement in this definition can be construed as saying that we are giving investors *at least* the freedom to do nothing; that is, if an investment is to lead to absolutely no profit or loss, one should be free to make it. In the non-degenerate case this just becomes $0 \in \mathfrak{C}(\omega, t)$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$.

One can refer to Appendix B for more information about the measurability requirement $\{\mathfrak{C} \cap F \neq \emptyset\} \in \mathcal{P}$ for all closed $F \subseteq \mathbb{R}^d$, where the equivalence with other definitions of measurability is discussed.

In any case, we should include the following result, which is there to convince the reader that all the constraints in mind belong in this class.

Lemma 2.11. *Let $(H, \mathcal{H})$ be an auxiliary measure space and $\phi : H \times \mathbb{R}^d \mapsto \mathbb{R}$ a function that satisfies the following:*

(1) $\phi(h, 0) \leq 0$ *for all* $h \in H$,

(2) $\phi(h, \cdot)$ *is a convex function for fixed* $h \in H$.

(3) $\phi(\cdot, \mathrm{p})$ *is* $\mathcal{H}$*-measurable for fixed* $\mathrm{p} \in \mathbb{R}^d$,

Also, let $z : \Omega \times \mathbb{R}_+ \mapsto H$ be a predictable process such that $\phi(z(\omega, t), \mathfrak{N}(\omega, t)) = \{\phi(z(\omega, t), 0)\}$ (i.e., $\phi(z(\omega, t), \cdot)$ is constant-valued on $\mathfrak{N}(\omega, t)$) for all $(\omega, t) \in \Omega \times \mathbb{R}_+$. Then, the set

$$\mathfrak{C}(\omega, t) := \left\{ \mathrm{p} \in \mathbb{R}^d \mid \phi(z(\omega, t), \mathrm{p}) \leq 0 \right\}$$

is a predictable closed convex set-valued process.

Also, countable intersections of sets like this (where all $(H, \mathcal{H})$, ϕ and z can depend on $n \in \mathbb{N}$) will be predictable closed convex constrains.

Proof. We briefly explain why this is true, sending the reader to section B for all the details.

First of all, for the last sentence, the intersection of closed convex sets is again closed and convex, so one only has to consult Lemma B.3 to find out that the intersection of a sequence of measurable closed random sets is measurable too.

Now, the set $\left\{ \mathrm{p} \in \mathbb{R}^d \mid \phi(z, \mathrm{p}) \leq 0 \right\}$ is closed and convex (trivial from the convexity in p of the functions ϕ) and it also contains $\mathfrak{N}$ because $\phi(z, \mathfrak{N}) =$

$\{\phi(z,0)\}$ and $\phi(h,0) \leq 0$. Only the predictablity has to be discussed. Since $\phi(z(\omega,t),\mathrm{p})$ is a Carathéodory function, i.e, predictably measurable in $(\omega,t) \in \Omega \times \mathbb{R}_+$ for fixed $\mathrm{p} \in \mathbb{R}^d$ and continuous in $\mathrm{p} \in \mathbb{R}^d$ with $(\omega,t) \in \Omega \times \mathbb{R}_+$ fixed, the result follows from Lemma B.5. □

The *natural constraints* $\mathfrak{C}_0$ of (1.7) can be easily seen to satisfy the requirements of Definition 2.10; for the proof of the predictability requirement, which is very plausible from the definition, see Corollary B.6 of Appendix B. In view of this, we can always assume $\mathfrak{C} \subseteq \mathfrak{C}_0$, since otherwise we can replace $\mathfrak{C}$ by $\mathfrak{C} \cap \mathfrak{C}_0$ (and use the fact that intersections of closed predictable set-valued processes are also predictable — see Lemma B.3 of Appendix B).

Example 2.12. Let us give an example of "local volatility" constraints. Here, we assume that S (and thus X) is continuous. Risk-averse investors like to have as few fluctuations of their wealth as possible. One possible local measure of randomness is given by the square root of the quadratic variation rate of the log-wealth process $\log(W^\pi)$, which is equal to $(\pi^\top c \pi)^{1/2}$: this is the local volatility and is exactly equal to the quadratic variation rate of $\pi \cdot X$. Pick now a predictable positive process ϵ of some "small" numbers and require that $(\pi^\top c \pi)^{1/2} \leq \epsilon$. This is a predictable convex constraint (observe that for any $\pi \in \mathfrak{N}$ we have $\pi^\top c \pi = 0$).

We now give a little twist to the previous situation and turn it into the problem of following closely a baseline[14] portfolio. We shall use the notation and ideas of Remark 2.6, which explains how we can see relative wealth processes as usual wealth processes when investing in an auxiliary market. Suppose that we want to invest in such a way as to not deviate a lot from a baseline portfolio

[14] This could be any index, or the whole market.

ξ — for example, ξ could be the market portfolio $\mathfrak{m}$, in which case we would like to follow an index closely, but not exactly. Here is what can be done: instead of working under the original market, we work under the marker that is generated by the process $X^{(\xi)}$ as described in Remark 2.6, and we try to keep the local volatility in this new market low. In this sense, the constraint on the portfolio $\pi^{(\xi)}$ we can use would be $\big((\pi^{(\xi)})^\top c \pi^{(\xi)}\big)^{1/2} \leq \epsilon$. After the optimal portfolio $\pi_*^{(\xi)}$ has been found (by any kind of optimization in this auxiliary market), we can turn back to the original market and use $\pi_* := \xi + \pi_*^{(\xi)}$ as the optimal portfolio.

2.4. **Unbounded Increasing Profit.** We proceed with an effort to obtain a better understanding of condition (1) in Lemma 2.8. In this section we state a sufficient predictable condition of its failure; in the next section, when we state our first main theorem about the predictable characterization for the existence of the numéraire portfolio, we shall see that this condition is also necessary. The failure of that condition is intimately related to the existence of wealth processes that start with no capital at all, manage to escape from zero, and are furthermore increasing. The existence of such a possibility in a financial market amounts to the most egregious form of arbitrage.

Definition 2.13. A wealth process $W \in \mathcal{W}$ of the form $W = \theta \cdot S$ (in particular, with $W_0 = 0$), is called *unbounded increasing profit*, if $\theta \in \check{\mathfrak{C}}$ and W is an increasing process, not identically equal to zero. If no such wealth process exists we say that the *No Unbounded Increasing Profit* (NUIP) condition holds.

The qualifier "unbounded" reflects the fact that, since $\theta \in \check{\mathfrak{C}}$, one can invest as much as one wishes according to the strategy θ; by doing so, the investor's wealth will be multiplied accordingly. Of course, the numéraire cannot exist if such strategies exist (this was actually shown implicitly in the proof of Lemma

2.8). To obtain the connection with predictable characteristics, we also give the definition of the immediate arbitrage opportunity vectors in terms of the Lévy triplet.

Definition 2.14. Let (b, c, ν) be any Lévy triplet. Define the set $\mathfrak{I}$ of *immediate arbitrage opportunities* to be the set of vectors $\xi \in \mathbb{R}^d \setminus \mathfrak{N}$ such that the following three conditions hold:

(1) $\xi^\top c = 0$,
(2) $\nu[\xi^\top x < 0] = 0$,
(3) $\xi^\top b - \int \xi^\top x \mathbf{1}_{\{\|x\| \leq 1\}} \nu(\mathrm{d}x) \geq 0$.

Vectors of the set $\mathfrak{N}$ in (2.2) satisfy these three conditions, but cannot be considered "arbitrage opportunities" since they have zero returns. The set $\mathfrak{I}$ is a cone, with the entire "face" $\mathfrak{N}$ removed. When we want to make explicit the dependence of the set $\mathfrak{I}$ on the chosen Lévy triplet (b, c, ν), we write $\mathfrak{I}(b, c, \nu)$.

Assume for simplicity that X is a Lévy process[15] and that we can find a vector $\xi \in \mathfrak{I}$. In terms of the process $\xi \cdot X$, condition (1) of Definition 2.11 implies that there is no diffusion part, and condition (2) that there are no negative jumps; condition (iii) implies that $\xi \cdot X$ has finite first variation, though this is not as obvious. Using also the fact that $\xi \notin \mathfrak{N}$, we get that $\xi \cdot X$ is actually non-zero and increasing, and the same will hold for $W^\xi = \mathcal{E}(\xi \cdot X)$. We refer the reader to Chapter 4 for a thorough discussion.

Proposition 2.15. *The* (NUIP) *condition of Definition 2.13 is equivalent to the predictable set* $\left\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\right\}$ *being* $\mathbb{P} \otimes G$*-null. Here* $\check{\mathfrak{C}} := \bigcap_{a \in \mathbb{R}_+} a\mathfrak{C}$ *is the*

[15]Of course this is not necessary, but it eases the presentation.

set of cone points of $\mathfrak{C}$*, and*

$$\left\{ \mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset \right\} := \left\{ (\omega, t) \in \Omega \times \mathbb{R}_+ \mid \mathfrak{I}\left(b(\omega, t), c(\omega, t), \nu(\omega, t)\right) \cap \check{\mathfrak{C}}(\omega, t) \neq \emptyset \right\} .$$

The proof of this result is given in Chapter 4. As remarked before, one should at least read the first part of Chapter 4, which contains the one side of the argument: if there exists an unbounded increasing profit, the set $\left\{ \mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset \right\}$ *cannot* be $\mathbb{P} \otimes G$-null. The other direction, though it follows the same idea, has a "measurable selection" flavor and the reader might wish to skip or skim it.

Remark 2.16. One might wonder what connection the previous result has with our original problem. We attempt here a quick answer, showing that if the condition $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$ fails (i.e., if the set of cone points of our constraints $\mathfrak{C}$ exposes some immediate arbitrage opportunities), then one cannot find a $\rho \in \mathfrak{C}$ such that $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \mathfrak{C}$. To this end, pick a vector $\xi \in \mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$, and suppose that ρ satisfied $\mathfrak{rel}(\pi \mid \rho) \leq 0$, for all $\pi \in \mathfrak{C}$. Since $\xi \in \check{\mathfrak{C}}$, we have $n\xi \in \mathfrak{C}$ for all $n \in \mathbb{N}$ and the convex combination $(1 - n^{-1})\rho + \xi \in \mathfrak{C}$ too; but $\mathfrak{C}$ is closed, and so $\rho + \xi \in \mathfrak{C}$. Now

$$\mathfrak{rel}(\rho + \xi \mid \rho) \; = \; \ldots \; = \; \xi^\top b - \int \xi^\top x \mathbf{1}_{\{\|x\| \leq 1\}} \nu(\mathrm{d}x) + \int \frac{\xi^\top x}{1 + \rho^\top x} \nu(\mathrm{d}x)$$

is strictly positive from the definition of ξ, a contradiction. It follows that there cannot exist any ρ satisfying $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \mathfrak{C}$.

We already mentioned that the converse holds as well; namely if $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$, then one can find a ρ that satisfies the previous requirement. But the proof of this part is longer; it is carried out in Chapter 5.

2.5. **The growth-optimal portfolio and connection with the numéraire portfolio.** We hinted in Remark 2.16 that if $\left\{ \mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset \right\}$ is $\mathbb{P} \otimes G$-null, then one can find a process $\rho \in \mathfrak{C} \cap \mathfrak{C}_0$ such that $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \Pi$. It is actually also important to have an algorithmic way of computing ρ.

For a portfolio $\pi \in \Pi$, its *growth rate* is defined as the drift rate of the log-wealth process $\log W^{\pi}$. One can use the stochastic exponential formula (A.1) and formally (since it will not always exist) compute the growth rate of π to be

$$\mathfrak{g}(\pi) := \pi^\top b - \frac{1}{2}\pi^\top c \pi + \int \left[\log(1 + \pi^\top x) - \pi^\top x \mathbf{1}_{\{\|x\| \leq 1\}}\right] \nu(\mathrm{d}x). \tag{2.3}$$

It is well-understood by now that the numéraire portfolio and the portfolio that maximizes in an (ω, t)-pointwise sense the growth rate over all portfolios in Π are essentially the same. Let us describe this in some more detail, being a bit informal for now. A vector $\rho \in \mathfrak{C}$ maximizes this concave function $\mathfrak{g}$ if and only if the directional derivative of $\mathfrak{g}$ at the point ρ in the direction of $\pi - \rho$ is negative for any $\pi \in \Pi$. One can compute this directional derivative to be

$$(\nabla \mathfrak{g})_\rho (\pi - \rho) = (\pi - \rho)^\top b - (\pi - \rho)^\top c \rho + \int \left[\frac{(\pi - \rho)^\top x}{1 + \rho^\top x} - (\pi - \rho)^\top x \mathbf{1}_{\{\|x\| \leq 1\}} \right] \nu(\mathrm{d}x),$$

and this turns out to be exactly $\mathfrak{rel}(\pi \mid \rho)$.

Let us now be more formal. We do not know if we can differentiate under the integral appearing in equation 2.3; even worse, we do not know a priori whether the integral is well-defined. Both its positive and negative parts could lead to infinite results. The non-integrability of the negative part is not too severe, since one wants to *maximize* $\mathfrak{g}$: if a portfolio π leads to the negative part of the integrand integrate to infinity, all vectors $a\pi$ for $a \in [0, 1)$ will lead to a finite result. More annoying is the fact that the *positive* part can integrate to infinity, especially if one notices that if this happens for at least one vector $\pi \in \mathfrak{C}$, concavity will imply that it happens for *many* vectors — actually for *all* vectors in the relative interior of $\mathfrak{C}$, with the possible exception of those of the form $-a\pi$ for $a > 0$. This problem is exactly the "pathwise" analogue of the situation where the expected log-utility is infinite, and both can be resolved in the exact same way: by looking at *relative*, instead of absolute, quantities. This will become very clear when we reach section 3.3.

In the spirit of the above discussion, let us describe a class of Lévy measures for which the concave growth rate function $\mathfrak{g}$ of (2.3) *is* well-defined.

Definition 2.17. (i) A Lévy measure ν will be said to *integrate the log*, if we have

$$\int_{\{\|x\|>1\}} \log(1 + \|x\|)\nu(\mathrm{d}x) < \infty.$$

(ii) Consider any Lévy measure ν. A sequence $(\nu_n)_{n\in\mathbb{N}}$ of Lévy measures with $\nu_n \sim \nu$ that integrate the log, and with densities $f_n := \mathrm{d}\nu_n/\mathrm{d}\nu$ satisfying

$$0 < f_n \leq 1, \quad f_n(x) = 1 \text{ for } \|x\| \leq 1, \quad \text{and} \lim_{n\to\infty} \uparrow f_n = \mathbf{1}$$

will be called an *approximating sequence*.

There are many ways to choose the sequence $(\nu_n)_{n\in\mathbb{N}}$, or equivalently the densities $(f_n)_{n\in\mathbb{N}}$; for example one could take $f_n(x) = \mathbf{1}_{\{\|x\|\leq 1\}} + \|x\|^{-1/n} \mathbf{1}_{\{\|x\|>1\}}$. Observe also that the sets $\mathfrak{C}_0$, $\mathfrak{N}$ and $\mathfrak{I}$ remain unchanged if we move from the original triplet to any of the approximating triplets, which will be useful since it gives us that $\mathfrak{I}(b, c, \nu) \cap \check{\mathfrak{C}} = \emptyset$ if and only if $\mathfrak{I}(b, c, \nu_n) \cap \check{\mathfrak{C}} = \emptyset$ for all $n \in \mathbb{N}$.

The problem of the positive infinite value for the integral appearing in equation (2.3) disappears when the Lévy measure ν integrates the log. In the general case, where the growth-optimal problem takes an infinite value, our strategy will be the following: we shall solve the optimization problem concerning $\mathfrak{g}$ for a sequence of problems using the approximation described in Definition 2.17, and then show that the corresponding solutions converge to the solution of the original problem.

2.6. **An asymptotic optimality property of the numéraire portfolio.** Before we proceed, let us also give an "asymptotic growth optimality" property of the numéraire portfolio. Let us first note that, since by definition W^π / W^ρ is a positive supermartingale for every $\pi \in \Pi$, the $\lim_{t \to \infty} (W^\pi_t / W^\rho_t)$ exists and is $[0, +\infty)$-valued. As a consequence, for *any* increasing process H with $H_\infty = +\infty$ (H does not even have to be adapted!),

$$\limsup_{t \to \infty} \left(\frac{1}{H_t} \cdot \log \frac{W^\pi_t}{W^\rho_t} \right) \ \leq \ 0\,. \tag{2.4}$$

This version of "asymptotic growth optimality" was first observed and proved (for $H_t \equiv t$, but this is not too essential) in Algoet and Cover [1] for the discrete-time case; see also Karatzas and Shreve [23] and Goll and Kallsen [15] for a discussion of (2.4) for the continuous-path and the general semimartingale case respectively.

In Proposition 2.18 below, we separate the cases when $\lim_{t \to \infty} (W^\pi_t / W^\rho_t)$ is $(0, \infty)$-valued and when it is zero, by finding a predictable characterization of this dichotomy. Also, in the case of convergence to zero, we quantify how fast does this takes place. Note that Proposition 2.18 mostly makes sense if the numéraire portfolio exists for all times before infinity but not globally, i.e., if $W^\rho_\infty = +\infty$.

Proposition 2.18. *Assume that the numéraire portfolio ρ exists (not necessarily globally). For any other $\pi \in \Pi$, define the positive, predictable process*

$$\lambda^\pi := -\mathfrak{rel}(\pi \mid \rho) + \frac{1}{2}(\pi - \rho)^\top c (\pi - \rho) + \int h_a \left(\frac{1 + \pi^\top x}{1 + \rho^\top x} \right) \nu(\mathrm{d}x),$$

where $h_a(y) := [-\log a + (1 - a^{-1})y]\, \mathbf{1}_{[0,a)}(y) + [y - 1 - \log y]\, \mathbf{1}_{[a,+\infty)}(y)$ *for some* $a \in (0,1)$ *is a positive, convex function. Consider the increasing, predictable process* $\Lambda^\pi := \lambda^\pi \cdot G$. *Then*

$$\text{on } \{\Lambda^\pi_\infty < +\infty\}, \quad \text{we have} \quad \lim_{t \to \infty} \frac{W^\pi_t}{W^\rho_t} \in (0, +\infty), \text{ whereas}$$
$$\text{on } \{\Lambda^\pi_\infty = +\infty\}, \quad \text{we have} \quad \limsup_{t \to \infty} \left(\frac{1}{\Lambda^\pi_t} \cdot \log \frac{W^\pi_t}{W^\rho_t} \right) \leq -1 .$$

Remark 2.19. Some remarks are in order. Let us begin with the "strange looking" function h_a, that depends also on the (cut-off point) parameter $a \in (0,1)$. Ideally we would like to define $h_0(y) = y - 1 - \log y$ for all $y > 0$; but the problem is that the positive, predictable process $\int h_0 \left(\frac{1+\pi^\top x}{1+\rho^\top x} \right) \nu(\mathrm{d}x)$ may fail to be G-integrable, because the function $h_0(y)$ explodes to $+\infty$ as $y \downarrow 0$. For this reason, we define $h_a(y)$ to be equal to $h_0(y)$ for all $y \geq a$, and for $y \in [0,a)$ we define it in a linear way so that at the "gluing" point a, h_a is continuously differentiable. The functions $h_a(\cdot)$ all are finite-valued at $y = 0$ and satisfy $h_a \uparrow h_0$ as $a \downarrow 0$.

Now, let us focus on λ and Λ. Observe that λ^π is *predictably convex* in π, namely, if π_1 and π_2 are two portfolios and ϑ is a $[0,1]$-valued predictable process, then $\lambda^{\vartheta\pi_1 + (1-\vartheta)\pi_2} \leq \vartheta\lambda^{\pi_1} + (1-\vartheta)\lambda^{\pi_2}$. This, together with the fact that $\lambda^\pi = 0$ if and only if $\pi - \rho$ is a null investment, implies that λ^π can be seen as a measure of instantaneous deviation of π from ρ; by the same token, Λ^π_∞ can be seen as the total (cumulative) deviation of π from ρ. With these remarks in mind, Proposition 2.18 says in effect that, if an investment deviates a lot from the numéraire portfolio (i.e., if $\Lambda^\pi_\infty = +\infty$), its performance will lag considerably behind that of the numéraire portfolio. Only if an investment follows very closely the numéraire portfolio the whole time (i.e., if $\Lambda^\pi_\infty < +\infty$)

will the two wealth processes have comparable growths over the entire time period. Also, in connection with the previous paragraph, letting $a \downarrow 0$ in the definition of Λ we get equivalent measures of distance of a portfolio π from the numéraire portfolio, in the sense that the event $\{\Lambda^{\pi}_{\infty} = +\infty\}$ does not depend on the choice of a; nevertheless we get ever sharper results, since λ^{π} is increasing for decreasing $a \in (0, 1)$.

Proof. The proof of Proposition 2.18 is immediate from its abstract version, which is Lemma A.7 in the Appendix. One only has to notice that W^{π}/W^{ρ} is a positive supermartingale and to identify A of Lemma A.7 with $\mathfrak{rel}(\pi \mid \rho) \cdot G$, $\hat{C}$ with $\left((\pi - \rho)^{\top} c(\pi - \rho)\right) \cdot G$, and $h(1 + x) * \hat{\eta}$ with $\left(\int h\left(\frac{1+\pi^{\top}x}{1+\rho^{\top}x}\right)\nu(\mathrm{d}x)\right) \cdot G$, respectively. $\square$

2.7. **The first main result.** We are now ready to state the main result which describes the predictable characterization for the existence of the numéraire portfolio. We already discussed about condition (1) of Lemma 2.8 and its predictable characterization: *there exists a predictable process ρ with $\rho(\omega, t) \in \mathfrak{C}(\omega, t)$ such that $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \Pi$, if and only if $\left\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\right\}$ has zero $\mathbb{P} \otimes G$-measure.* If this holds we construct such a predictable process ρ, and the only thing that might keep ρ from being the numéraire portfolio is failure of global X-integrability. To cover this, for a given predictable process ρ define

$$\phi^{\rho} := \nu\left[\rho^{\top}x > 1\right] + \left|\rho^{\top}b + \int \left[\rho^{\top}x\left(\mathbf{1}_{(\|x\|>1)} - \mathbf{1}_{(\rho^{\top}x>1)}\right)\right]\nu(\mathrm{d}x)\right|.$$

Here is the statement of the main theorem; its proof is given in Chapter 6.

Theorem 2.20. *Consider a financial model described by a semimartingale returns process X and predictable closed convex constraints $\mathfrak{C}$. Then we have the following:*

The set $\left\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\right\}$ *is predictable. If it has zero* $\mathbb{P} \otimes G$*-measure, there exists a unique predictable process* ρ *with* $\rho(\omega, t) \in \mathfrak{C} \cap \mathfrak{N}^{\perp}(\omega, t)$ *such that* $\mathfrak{rel}(\pi \mid \rho) \leq 0$ *for all* $\pi \in \Pi$*. On the predictable set* $\left\{\int_{\{\|x\|>1\}} \log(1 + \|x\|)\nu(\mathrm{d}x) < \infty\right\}$*, the process* ρ *is obtained as the unique solution of the concave optimization problem*

$$\rho = \arg \max_{\pi \in \mathfrak{C} \cap \mathfrak{N}^{\perp}} \mathfrak{g}(\pi);$$

in general, it can be obtained as the limit of the solutions to the corresponding problems where one replaces ν *by* ν_n *(an approximating sequence) in the definition of* $\mathfrak{g}$*. Finally, if the previous process* ρ *is such that* $\left(\phi^{\rho} \cdot G\right)_{\infty} < +\infty$*, then* $\rho \in \Pi_{\infty}$*, and it is the numéraire portfolio.*

Conversely, suppose that the numéraire portfolio ρ *exists for the class* Π*. Then, the predictable random set* $\left\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\right\}$ *has zero* $\mathbb{P} \otimes G$*-measure, and* ρ *satisfies* $\left(\phi^{\rho} \cdot G\right)_{\infty} < +\infty$ *and* $\mathfrak{rel}(\pi \mid \rho) \leq 0$ *for all* $\pi \in \Pi$*.*

Remark 2.21. Let us pause slightly to remark on the above condition, which amounts to the global G-integrability of both

(2.5)

$$\phi_1^{\rho} := \nu\left[\rho^{\top} x > 1\right] \text{ and } \phi_2^{\rho} := \rho^{\top} b + \int \left[\rho^{\top} x \left(\mathbf{1}_{\{\|x\|>1\}} - \mathbf{1}_{\left\{\rho^{\top} x > 1\right\}}\right)\right] \nu(\mathrm{d}x).$$

Now, the integrability of ϕ_1^{ρ} simply states that the process $\rho \cdot X$ cannot make an infinite number of large positive jumps; but this must obviously be the case if $\rho \cdot X$ is to have any limit at infinity, let alone be a semimartingale up to infinity. The second term ϕ_2^{ρ}, is exactly the drift rate of the part of $\rho \cdot X$ that remains when we subtract all large positive jumps (more than unit in magnitude). This part has to be a special semimartingale up to infinity, so its drift rate must be globally G-integrable, which is exactly the requirement $(|\phi_2^{\rho}| \cdot G)_{\infty} < \infty$.

Example 2.22. Let us consider the case where X is a continuous-path semimartingale. Since the jump-measure ν is then identically equal to zero, an immediate arbitrage opportunity is a vector $\xi \in \mathbb{R}^d$ with $c\xi = 0$ and $\xi^\top b > 0$. It follows that there are no immediate arbitrage opportunities if and only if b lies in the range of c, i.e., if there exists a d-dimensional process ρ with $b = c\rho$; of course, if c is non-singular this always holds and $\rho = c^{-1}b$. It is easy to see that this "no immediate arbitrage opportunities" condition is equivalent to $\mathrm{d}B_t \ll \mathrm{d}[X, X]_t$. We refer the reader to [22], to Chapter 1 of [23], and to [9] for more thorough discussion.

Consider now the unconstrained case $\mathfrak{C} = \mathbb{R}^d$. The derivative of the growth rate is $(\nabla \mathfrak{g})_\pi = b - c\pi = c\rho - c\pi$, which is trivially zero for $\pi = \rho$, and so ρ will be the numéraire portfolio as long as $\big((\rho^\top c\rho) \cdot G\big)_\infty < \infty$, or, in the case where c^{-1} exists, when $\big((b^\top c^{-1} b) \cdot G\big)_\infty < \infty$.

Remark 2.23. Let us write $X = A + M$ for the unique decomposition of a special semimartingale X into a predictable finite variation part A and a local martingale M, which we further assume is locally square-integrable. By calling $\langle M, M \rangle$ the predictable compensator of $[M, M]$, the condition for absence of immediate arbitrage opportunities in continuous-path models is the very simple $\mathrm{d}A_t \ll \mathrm{d}\left\langle M, M \right\rangle_t$. This should be compared with the more complicated way we have defined this notion for general markets in Definition 2.14. One wonders if we can have the simple criterion of continuous-path models in more general situations. It is easy to see that $\mathrm{d}A_t \ll \mathrm{d}\left\langle M, M \right\rangle_t$ is necessary for absence of immediate arbitrage opportunities; nevertheless, it is not sufficient — it is too weak. Take for example X to be standard linear Poisson process. In the absence of constraints on portfolio choice, any positive portfolio is an immediate arbitrage opportunity. Nevertheless, $A_t = t$ and $M_t = X_t - t$ with $\langle M, M \rangle_t = t = A_t$, so that $\mathrm{d}A_t \ll \mathrm{d}\langle M, M \rangle_t$ holds trivially.

3. Arbitrage Characterization of the Numéraire Portfolio, and Applications in Mathematical Finance

3.1. **Arbitrage-type definitions.** There are two widely-known conditions relating to arbitrage in financial markets: the classical “No Arbitrage” and its stronger version “No Free Lunch with Vanishing Risk”. We recall them below, together with yet another notion; this is exactly what one needs to bridge the gap between the previous two, and it will actually be the most important for our discussion.

Definition 3.1. For the following three definitions we consider our financial model with constrains $\mathfrak{C}$ on the set of portfolios Π.

(1) A portfolio $\pi \in \Pi_\infty$ is said to generate an *arbitrage opportunity*, if it satisfies $\mathbb{P}[W^\pi_\infty \geq 1] = 1$ and $\mathbb{P}[W^\pi_\infty > 1] > 0$. If no such portfolio exists we say that the market satisfies the *no arbitrage* (NA) condition.

(2) A sequence of portfolios $(\pi_n)_{n\in\mathbb{N}}$ that are elements of Π_∞ is said to generate an *unbounded profit with bounded risk*, if the collection of positive random variables $(W^{\pi_n}_\infty)_{n\in\mathbb{N}}$ is unbounded in probability, i.e., if

$$\lim_{m\to\infty} \downarrow \left(\sup_{n\in\mathbb{N}} \mathbb{P}[W^{\pi_n}_\infty > m] \right) > 0.$$

If no such sequence exists, we say that the market satisfies the *no unbounded profit with bounded risk* (NUPBR) condition.

(3) A sequence $(\pi_n)_{n\in\mathbb{N}}$ of elements in Π_∞ is said to be a *free lunch with vanishing risk*, if there exist an $\epsilon > 0$ and an increasing sequence $(\delta_n)_{n\in\mathbb{N}}$ of real numbers with $0 \leq \delta_n \uparrow 1$, such that $\mathbb{P}[W^{\pi_n}_\infty \geq \delta_n] = 1$ as well as

$\mathbb{P}[W_\infty^{\pi_n} \geq 1 + \epsilon] \geq \epsilon$. If no such sequence exists we say that the market satisfies the *no free lunch with vanishing risk* (NFLVR) condition.

We have to use elements of Π_∞ in the previous definitions, in order to make sure that the limits exist. If there exists unbounded profit with bounded risk, one can find a sequence of wealth processes, each starting with less and less capital (converging to zero) and such that the terminal wealths are unbounded with a fixed probability. Thus, "Unbounded Profit with Bounded Risk" can be translated as "the *possibility* of making (a considerable) something out of almost nothing"; it should be contrasted with the classical notion of arbitrage, which can be translated as "the *certainty* of making something more out of something".

None of the two conditions (NA) and (NUPBR) implies the other, and they are not mutually exclusive. It is easy to see that they are both weaker than (NFLVR). In fact, we have the following result that gives the exact relationship between these notions under the case of cone constraints. Its proof can be found in [8] for the unconstrained case; we include it here for completeness.

Proposition 3.2. *Suppose that $\mathfrak{C}$ enforces predictable closed convex cone constraints. Then,* (NFLVR) *holds if and only if both* (NA) *and* (NUPBR) *hold.*

Proof. It is obvious that if either of the conditions (NA) or (NUPBR) fail, then (NFLVR) fails too.

Conversely, suppose that (NFLVR) fails. If (NA) fails there is nothing more to say, so suppose that (NA) holds and let $(\pi_n)_{n \in \mathbb{N}}$ generate a free lunch with vanishing risk. Since we have no arbitrage, the assumption $\mathbb{P}[W_\infty^{\pi_n} \geq \delta_n] = 1$ results in the stronger $\mathbb{P}[\inf_{t \in \mathbb{R}_+} W_t^{\pi_n} \geq \delta_n] = 1$. Construct a new sequence of portfolio $(\tilde{\pi}_n)_{n \in \mathbb{N}}$ by requiring $W^{\tilde{\pi}_n} = 1 + (1 - \delta_n)^{-1}(W^{\pi_n} - 1)$. The reader can readily check that $W^{\tilde{\pi}_n} \geq 0$ and that $\tilde{\pi}_n(\omega, t) \in \mathfrak{C}(\omega, t)$, $\Omega \times \mathbb{R}_+$-pointwise (here it is essential that $\mathfrak{C}$ be a cone). Further, $\mathbb{P}[W^{\pi_n}_\infty \geq 1+\epsilon] \geq \epsilon$ translates to $\mathbb{P}[W^{\tilde{\pi}_n}_\infty \geq 1+(1-\delta_n)^{-1}\epsilon] \geq \epsilon$, which means that $(\tilde{\pi}_n)_{n\in\mathbb{N}}$ generates an unbounded profit with bounded risk so that (NUPBR) fails and the proof is over. □

The (NFLVR) condition has proven very fruitful in understanding cases when we can change the original measure $\mathbb{P}$ to some other equivalent probability measure such that the stock-price processes (or, at least the wealth processes) has some kind of martingale (or maybe only supermartingale) property under $\mathbb{Q}$. The following definition puts us in the proper context for the statement of Theorem 3.4.

Definition 3.3. Consider a financial market model described by a semimartingale returns X and predictable closed convex cone constraints $\mathfrak{C}$. A probability measure $\mathbb{Q}$ will be called an *equivalent supermartingale measure*, if $\mathbb{Q} \sim \mathbb{P}$, and every $W \in \mathcal{W}$ is a $\mathbb{Q}$-supermartingales. If such a measure exists, we denote this fact by (ESMM).

Similarly define the concept of an *equivalent local martingale measure* (ELMM) $\mathbb{Q}$ by requiring $\mathbb{Q} \sim \mathbb{P}$ and that every $W \in \mathcal{W}$ is a $\mathbb{Q}$-local martingale.

In this definition we assume that $\mathfrak{C}$ are cone constraints. The reason is that if (ESMM) or (ELMM) holds for the market with convex constraints $\mathfrak{C}$, the same holds for the constraints $\overline{\mathsf{cone}(\mathfrak{C})}$, the closure of the *cone generated by* $\mathfrak{C}$.

The following theorem is one of the most well-known in mathematical finance; we give the "constrained" version.

Theorem 3.4. *For a financial market model with stock-price process S and predictable closed convex cone constraints $\mathfrak{C}$,* (NFLVR) *and* (ESMM) *are equivalent.*

In the unconstrained case, one can prove further that there exists a $\mathbb{Q} \sim \mathbb{P}$ such that the stock prices S become σ-martingales under $\mathbb{Q}$ — this version of the theorem is what is called the "Fundamental Theorem of Asset Pricing". The concept of σ-martingale is just the σ-localized equivalent of local martingales; the reader should consult Appendix D for more information. Since we are assuming *positive* stock price processes, and a positive σ-martingale is a local martingale, we conclude that S is a vector of local martingales under $\mathbb{Q}$. Nevertheless, it should be pointed out that Theorem 3.4 holds for *any* stock-price process, and then the local martingale concept is not sufficient. In any case, we know in particular that $\mathbb{Q}$ will be an equivalent local martingale measure according to Definition 3.3.

As a contrast to the preceding paragraph, let us note that because we are working under constraints, we cannot hope in general for anything better than an equivalent *supermartingale* measure in the statement of Theorem 3.4. One can see this easily in the case where X is a single-jump process which jumps at a stopping time τ with $\Delta X_\tau \in (-1, 0)$ and we are constrained in the cone of positive strategies. Under any measure $\mathbb{Q} \sim \mathbb{P}$, the process $S = \mathcal{E}(X)$, an element of $\mathcal{W}$, will be non-increasing and not identically equal zero, which prevents it from being a local martingale.

The implication (ESMM) $\Rightarrow$ (NFLVR) is easy; the reverse implication is considerably harder for the general semimartingale model. There is a saga of papers devoted in proving some version of it. A proof of Theorem 3.4 in the generality we assume, appears in the paper [18] by Kabanov, although all the crucial work has been done by Delbaen and Schachermayer in [8] and the theorem is certainly due to them. To make sure that Theorem 3.4 can be derived from Kabanov's result, one has to observe that the class of wealth processes $\mathcal{W}$ is convex and closed in the semimartingale[16] topology. A careful inspection of the proof in Mémin's work [28], of the fact that the class of all stochastic integrals with respect to the d-dimensional semimartingale X is closed under this topology, will convince the reader that one can actually pick the limiting semimartingale from a convergent sequence in $\mathcal{W}$ to be again an element of $\mathcal{W}$.

3.2. **The numéraire portfolio and arbitrage.** We now discuss the relationship of the numéraire portfolio with the arbitrage notions previously defined.

Problem 3.5. *Find necessary and sufficient conditions for the existence of the numéraire portfolio in terms of arbitrage notions (as the ones in Definition 3.1).*

The solution to this problem is our second main result, Theorem 3.12.

Let us start by assuming that everything is nice and the numéraire W^ρ exists globally. By way of definition, the process $(W^\rho)^{-1}$ acts as a "deflator", under which all wealth processes $W \in \mathcal{W}$ become supermartingales. Of course, $(W^\rho)^{-1}$ does not have to be the only process with this property.

Definition 3.6. The class of *supermartingale deflators* $\mathcal{D}$ is defined as

$$\mathcal{D} := \{D \geq 0 \mid D_0 = 1,\ DW \text{ is a supermartingale for all } W \in \mathcal{W}\}.$$

By $\mathcal{D}^o$ we denote the class of *equivalent supermartingale deflators*, that is of elements $D \in \mathcal{D}$ for which we further have $D_\infty > 0$.

[16] Sometimes this is called the *Émery* topology.

An equivalent supermartingale measure $\mathbb{Q}$ generates an equivalent supermartingale deflator, in terms of the positive martingale

$$D_t := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t}, \quad \text{for } t \in \mathbb{R}_+,$$

so that (ESMM) $\Rightarrow \mathcal{D}^o \neq \emptyset$. In general, the elements of $\mathcal{D}$ are supermartingales, not martingales, and the reverse implication $\mathcal{D}^o \neq \emptyset \Rightarrow$ (ESMM) does not hold, as can ne seen from the following simple example.

Example 3.7. THE 3-DIMENSIONAL BESSEL PROCESS. Consider a one-stock market, where the price process satisfies the stochastic differential equation $\mathrm{d}S_t = S_t^{-1}\mathrm{d}t + \mathrm{d}\beta_t$, $S_0 = 1$. Here, β is a standard, linear Brownian motion, so S is the *3-dimensional Bessel process*. We work on the finite time horizon $[0, 1]$.

Clearly, $b = S^{-2}$ and $c = S^{-2}$, so the numéraire portfolio for the unconstrained case exists and is $\rho = c^{-1}b = 1$. Thus $\mathcal{D}^o \neq \emptyset$. Although the numéraire portfolio exists, this market admits arbitrage. To wit, with the notation

$$\varphi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \quad \Phi(x) = \int_{-\infty}^{x} \varphi(u)\mathrm{d}u, \quad \text{for } x \in \mathbb{R},$$

consider the function $\kappa(x) := x\varphi(x)/\Phi(x)$, $x > 0$, which is easily seen to satisfy $\kappa(0+) = 0$ and $\kappa(+\infty) = 0$. It follows that κ is bounded, so that the the predictable process $\pi_t = \kappa\left(S_t/\sqrt{1-t}\right)$, for $t \in [0, 1]$ is a well-defined portfolio. In the next paragraph we shall see that simple use of Itô's formula shows that the wealth process generated by using the portfolio π is given by $W_t^\pi = c\Phi\left(S_t/\sqrt{1-t}\right)$, where c is chosen so that $W_0^\pi = 1$, i.e., $c := 1/\Phi(1)$. Since $S_1 > 0$, it follows that $W_1^\pi = 1/\Phi(1) > 1$, i.e., there exists arbitrage, and of course (NA), and thus (ESMM), fails.

To prove the preceding claim, set $Y_t := \Phi\left(\frac{S_t}{\sqrt{1-t}}\right)$ and observe that $Y > 0$. Then, one has to prove that

$$\frac{\mathrm{d}Y_t}{Y_t} = \kappa\left(\frac{S_t}{\sqrt{1-t}}\right)\frac{\mathrm{d}S_t}{S_t},$$

which is the same as

$$\mathrm{d}Y_t = \frac{1}{\sqrt{1-t}}\varphi\left(\frac{S_t}{\sqrt{1-t}}\right)\mathrm{d}S_t. \tag{3.1}$$

Let us write $\hat{S}_t = S_t/\sqrt{1-t}$ to easy the task of reading. In order to prove (3.1), first observe that

$$\mathrm{d}\hat{S}_t = \frac{1}{\sqrt{1-t}}\mathrm{d}S_t + \frac{S_t}{2(1-t)^{3/2}}\mathrm{d}t = \frac{1}{\sqrt{1-t}}\mathrm{d}S_t + \frac{\hat{S}_t}{2(1-t)}\mathrm{d}t,$$

so that further use of Itô's formula gives

$$\begin{aligned}
\mathrm{d}\Phi(\hat{S}_t) &= \varphi(\hat{S}_t)\mathrm{d}\hat{S}_t + \frac{1}{2(1-t)}\varphi'(\hat{S}_t)\mathrm{d}t \\
&= \frac{1}{\sqrt{1-t}}\varphi(\hat{S}_t)\mathrm{d}S_t + \frac{1}{2(1-t)}\left(\varphi'(\hat{S}_t) + \hat{S}_t\varphi(\hat{S}_t)\right)\mathrm{d}t \\
&= \frac{1}{\sqrt{1-t}}\varphi(\hat{S}_t)\mathrm{d}S_t,
\end{aligned}$$

where the last equality follows from $\varphi'(x) = -x\varphi(x)$. The claim is proved.

There exists also an indirect way to show that arbitrage exists, proposed by Delbaen and Schachermayer [11]. For this, one has to assume that the filtration $\mathbf{F}$ is the one generated by S, or equivalently by β. It is well-known that S^{-1} is a *strict local martingale*, i.e., $\mathbb{E}[S_t^{-1}] < 1$ for all $t > 0$. Furthermore, using the strong martingale representation property of β it can be seen that S_1^{-1} is the *only* candidate for an equivalent local martingale measure density. Since it fails to integrate to one, (ESMM) fails. Theorem 3.4 implies that (NFLVR) fails; the fact that it is actually (NA) which fails, will become clear after Theorem 3.12.

We note that *this is one of the rare occasions, when one can compute the arbitrage portfolio concretely.* We were successful in this, because of the very special structure of the 3-dimensional Bessel process; every model has to be attacked in a different way and there is no general theory that will spot the arbitrage. Nevertheless, we refer the reader to Fernolz, Karatzas and Kardaras [14] and Fernolz and Karatzas [13] for many examples of arbitrage relatively to the market portfolio, under easy-to-check, descriptive (as opposed to normative) conditions on market structure.

Remark 3.8. In order to keep the discussion complete, let us mention that, if the numéraire portfolio ρ exists, the supermartingale property of DW^ρ for all $D \in \mathcal{D}$ leads to the property

$$\mathbb{E}\left[\log W^\rho_\infty\right] = \inf_{D \in \mathcal{D}} \mathbb{E}\left[\log\left(D^{-1}_\infty\right)\right] \tag{3.2}$$

of the supermartingale deflator $(W^\rho_\infty)^{-1}$.

Indeed, to prove this, first observe that since every $D \in \mathcal{D}$ is itself is a supermartingale (take $\pi = 0$), the inequality $\log^- x \leq x^{-1}$ for all $x > 0$ shows that $\mathbb{E}\log^-(D^{-1}_\infty) \leq \mathbb{E}D_\infty \leq 1$, so that $\mathbb{E}\log(D^{-1}_\infty)$ always makes sense and can take the value $+\infty$. Now, if $\mathbb{E}\log(D^{-1}_\infty) = \infty$ for all $D \in \mathcal{D}$ there is nothing to prove. So, pick a $D \in \mathcal{D}$ with $\mathbb{E}\log(D^{-1}_\infty) < \infty$. Using the fact that DW^ρ is a supermartingale, that $\log(D^{-1}_\infty)$ is integrable and Jensen's inequality for the logarithmic function we get $\mathbb{E}\log W^\rho_\infty \leq \mathbb{E}\log(D^{-1}_\infty) < \infty$.

Equation (3.2) can be seen as an optimal property of the numéraire portfolio, dual to log-optimality (this is discussed in section 3.3). Also, we can consider it as a *minimal reverse relative entropy* property of $(W^\rho)^{-1}$ in the set $\mathcal{D}$. Let us explain: when an element $D \in \mathcal{D}$ is actually a probability measure $\mathbb{Q}$, i.e., $\mathrm{d}\mathbb{Q} = D_\infty \mathrm{d}\mathbb{P}$, then $H(\mathbb{P} \mid \mathbb{Q}) := \mathbb{E}[\log(D^{-1}_\infty)] = \mathbb{E}^\mathbb{Q}[D^{-1}_\infty \log(D^{-1}_\infty)]$ is the relative entropy of $\mathbb{P}$ with respect to $\mathbb{Q}$. In general (even when D is not a probability density), we could regard $\mathbb{E}[\log (D_\infty^{-1})]$ as the relative entropy of $\mathbb{P}$ with respect to D (whatever this might mean). The qualifier "reverse" comes from the fact that one usually considers minimizing the entropy of *another* equivalent probability measure $\mathbb{Q}$ with respect to the *original* $\mathbb{P}$ (what is called the *minimal entropy measure*). We refer the reader to Example 7.1 of Karatzas and Kou [21] for further discussion.

Let us now describe *what goes wrong if the numéraire portfolio fails to exist.* This can happen in two ways. First, the set $\left\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\right\}$ may not have zero $\mathbb{P} \otimes G$-measure; in this case, Proposition 2.15 shows that one can construct an unbounded increasing profit, the most egregious form of arbitrage. Secondly, in case the $\mathbb{P} \otimes G$-measure of $\left\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\right\}$ is zero, the constructed predictable process ρ can fail to be globally X-integrable. The next definition prepares the ground for the statement of Proposition 3.10, which describes what happens in this latter case.

Definition 3.9. Consider a sequence $(f_n)_{n \in \mathbb{N}}$ of random variables. Its *superior limit in the probability sense*, $\mathbb{P}\text{-}\limsup_{n \to \infty} f_n$, is defined as the essential infimum of the collection $\{g \in \mathcal{F} \mid \lim_{n \to \infty} \mathbb{P}[f_n \leq g] = 1\}$.

It is obvious that the sequence $(f_n)_{n \in \mathbb{N}}$ of random variables is unbounded in probability if and only if $\mathbb{P}\text{-}\limsup_{n \to \infty} |f_n| = +\infty$ with positive probability.

Of course, the $\mathbb{P}\text{-}\liminf$ can be defined analogously, and one can readily check that $(f_n)_{n \in \mathbb{N}}$ converges in probability if and only if its $\mathbb{P}\text{-}\liminf$ and $\mathbb{P}\text{-}\limsup$ coincide, but these last facts will not be used below.

Proposition 3.10. *Assume that the predictable set* $\left\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\right\}$ *has zero* $\mathbb{P} \otimes G$*-measure, and let* ρ *be the predictable process constructed as in Theorem 2.20.*

Pick any sequence $(\theta_n)_{n \in \mathbb{N}}$ *of* $[0,1]$*-valued predictable processes with* $\lim_{n \to \infty} \theta_n = \mathbf{1}$ *holding* $\mathbb{P} \otimes G$*-almost everywhere, and such that* $\rho_n := \theta_n \rho$ *is globally* X*-integrable for all* $n \in \mathbb{N}$*. Then,* $\overline{W}^{\rho}_{\infty} := \mathbb{P}\text{-}\limsup_{n \to \infty} W^{\rho_n}_{\infty}$ *is a* $(0, +\infty]$*-valued random variable, and does not depend on the choice of the sequence* $(\theta_n)_{n \in \mathbb{N}}$*. On the event* $\{(\phi^{\rho} \cdot G)_{\infty} < +\infty\}$ *the random variable* $\overline{W}^{\rho}_{\infty}$ *is an actual limit in probability, and*

$$\{\overline{W}^{\rho}_{\infty} = +\infty\} = \{(\phi^{\rho} \cdot G)_{\infty} = +\infty\}\,;$$

in particular, $\mathbb{P}[\overline{W}^{\rho}_{\infty} = +\infty] > 0$ *if and only if* ρ *fails to be globally* X*-integrable.*

The proof is the content of the second part of Chapter 6. The above result says, in effect, that *closely following the numéraire portfolio, when it is not globally* X*-integrable, one can make arbitrarily large gains with fixed, positive probability.* There are many ways to choose the sequence $(\theta_n)_{n \in \mathbb{N}}$. One particular example is $\theta_n := \mathbf{1}_{\Sigma_n}$ with $\Sigma_n := \{(\omega, t) \in \Omega \times \mathbb{R}_+ \mid t \in [0, n] \text{ and } \|\rho(\omega, t)\| \leq n\}$.

Remark 3.11. The failure of ρ to be globally X-integrable can happen in two distinct ways. Let us define the stopping time $\tau := \inf\{t \in \mathbb{R}_+ \mid (\phi^{\rho} \cdot G)_t = +\infty\}$ (this can be possibly infinite). In a similar fashion define a whole sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times by $\tau_n := \inf\{t \in \mathbb{R}_+ \mid (\phi^{\rho} \cdot G)_t \geq n\}$. We consider two cases.

First, let us suppose that $\tau > 0$ and $(\phi^{\rho} \cdot G)_{\tau} = +\infty$; then $\tau_n < \tau$ for all $n \in \mathbb{N}$ and $\tau_n \uparrow \tau$. By using the sequence $\rho_n := \rho \mathbf{1}_{[0, \tau_n]\!]}$ it is easy to see that $\lim_{n \to \infty} W^{\rho_n}_{\infty} = +\infty$ almost surely — this is a consequence of the supermartingale property of $\{(W^{\rho}_t)^{-1}, 0 \leq t < \tau\}$. An example of a situation when this happens in finite time (say, in $[0,1]$) is when the linear price-generating process X satisfies $\mathrm{d}X_t = (1-t)^{-1/2}\mathrm{d}t + \mathrm{d}\beta_t$, where β is a standard linear

Brownian motion. Then, $\rho_t = (1-t)^{-1/2}$ and thus $(\phi^\rho \cdot G)_t = \int_0^t (1-u)^{-1} \mathrm{d}u$, which of course gives us $\tau \equiv 1$.

Nevertheless, this is not the end of the story. With the notation set-up above we will give an example with $(\phi^\rho \cdot G)_\tau < +\infty$. Actually, we shall only time-reverse the example we gave before and show that in this case $\tau \equiv 0$. So, take the stock-generating process now to be $\mathrm{d}X_t = t^{-1/2}\mathrm{d}t + \mathrm{d}\beta_t$; then, $\rho_t = t^{-1/2}$ and $(\phi^\rho \cdot G)_t = \int_0^t u^{-1}\mathrm{d}u = +\infty$ for all $t \in \mathbb{R}_+$. We then get that $\tau = 0$. In this case we cannot invest in ρ as before in a "forward" manner, because it has a "singularity" at $t = 0$ and we cannot take full advantage of it. This is basically what makes the proof of Proposition 3.10 non-trivial.

Let us remark further that in the latter case, and for a continuous-path process X with no constraints (as the one described in this example), Delbaen and Schachermayer in their paper [9], as well as Levental and Skorohod in [27], show that one can actually create "instant arbitrage", which is a wealth process that never falls below its initial capital and is also not constant[17]. For the case of jumps it is an open question whether one can still construct this instant arbitrage.

Here is our second main result, that puts the numéraire portfolio in the context of arbitrage.

Theorem 3.12. *For a financial model described by the stock-price process S and the predictable closed convex constraints $\mathfrak{C}$, the following are equivalent:*

(1) *The numéraire portfolio exists.*

(2) *The set of equivalent supermartingale deflators $\mathcal{D}^o$ is non-empty.*

[17]This is *almost* the definition of an increasing unbounded profit, but weaker since the wealth process is not assumed to be increasing.

(3) *The* (NUPBR) *condition holds.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial: $(W^\rho)^{-1}$ is an element of $\mathcal{D}^o$.

Now, for the implication (2) $\Rightarrow$ (3), start by assuming that $\mathcal{D}^o \neq \emptyset$ and pick an element $D \in \mathcal{D}^o$. We wish to show that the set of terminal values of wealth processes in $\mathcal{W}_\infty$ that start with unit capital is bounded in probability. Since $D_\infty > 0$, this is equivalent to showing that the set $\{D_\infty W_\infty \mid W \in \mathcal{W}_\infty \text{ and } W_0 = 1\}$ is bounded in probability. But this is obvious, since every process DW for $W \in \mathcal{W}_\infty$ is a positive supermartingale and so, for all $a > 0$,

$$\mathbb{P}[D_\infty W_\infty > a] \leq \frac{\mathbb{E}[D_\infty W_\infty]}{a} \leq \frac{\mathbb{E}[D_0 W_0]}{a} = \frac{1}{a}.$$

Finally, for the implication (3) $\Rightarrow$ (1), suppose that the numéraire portfolio fails to exist. Then, according to the discussion that precedes Proposition 3.10, either we have opportunities for unbounded increasing profit, in which case (NUPBR) certainly fails, or ρ exists but is not globally X-integrable, in which case Definition 3.9, gives that (NUPBR) fails again. □

Remark 3.13. Conditions (2) and (3) of this last theorem remain invariant by an equivalent change of probability measure. Thus, the existence of the numéraire portfolio remains unaffected also, although the numéraire portfolio itself of course will change. Notice that although this would have been a pretty reasonable conjecture to have made from the outset, it does not follow directly from the definition of the numéraire portfolio by any trivial considerations.

Note that the discussion of the previous paragraph does not remain valid if we only consider *absolutely continuous* changes of measure (unless the price process is continuous). Even though one would rush to say that (NUPBR) would hold, let us remark that non-equivalent changes of measure might change the structure of admissible wealth processes, since now it will be easier for wealth processes to satisfy the positivity condition: in effect, the natural constraints set $\mathfrak{C}_0$ can be larger. Consider, for example, a finite time-horizon case where, under $\mathbb{P}$, X is a driftless compound Poisson process with $\nu(\{-1/2\}) = \nu(\{1/2\}) > 0$. It is obvious that $\mathfrak{C}_0 = [-2, 2]$ and X itself is a martingale. Now, the simple absolutely continuous change of measure that transforms the jump measure to $\nu_1(\mathrm{d}x) := \mathbf{1}_{\{x>0\}}\nu(\mathrm{d}x)$ charges only the point $x = 1/2$, $\mathfrak{C}_0 = (-2, \infty]$ and of course (NUIP) fails.

3.3. **Application to Utility Optimization.** A central problem of mathematical finance is the maximization of expected utility of an economic agent who can invest in the market. The point of this section is to convince the reader that for solving this problem, the full power of (NFLVR) is not necessary. Rather, the weaker (NUPBR) is the *minimal* "arbitrage" notion needed to proceed in the solution for any utility maximization problem. Here we shall try to convey that failure of the classical (NA) property — as described in item (1) of Definition 3.1 — will not prevent the investor from finding an optimal investment strategy, and this in many cases will *not* be an arbitrage. In a loose sense to become precise below, the problem of maximizing expected utility from terminal wealth for a rather large class of utility functions that has been considered in the literature, is solvable *if and only if* the special case of the logarithmic utility problem has a solution — which is exactly in the case when (NUPBR) holds.

To start, let us formalize preference structures. We assume that an investor is equipped with a *utility function*: this is defined as a concave and strictly increasing function $U : (0, \infty) \mapsto \mathbb{R}$. We also define $U(0)$ by continuity. Starting with initial capital $w > 0$, the objective of the investor is to find a portfolio

$\hat{\rho} \equiv \hat{\rho}(w) \in \Pi_\infty$ such that

$$\mathbb{E}\left[U(wW_\infty^{\hat{\rho}})\right] = \sup_{\pi \in \Pi_\infty} \mathbb{E}\left[U(wW_\infty^{\pi})\right] =: u(w). \tag{3.3}$$

Remark 3.14. The optimization problem (3.3) makes sense only if its value function u is finite. Due to the concavity of U, if $u(w) < +\infty$ for *some* $w > 0$, then $u(w) < +\infty$ for *all* $w > 0$ and u is continuous, concave and increasing.

When we have $u(w) = \infty$ for some (equivalently, all) $w > 0$, there are two cases. Either the supremum in (3.3) is not attained, so there is no solution; or, in case there exists a portfolio with infinite expected utility, the concavity of U will imply that there will be infinitely many of them.

Probably the most important example of a utility function is the logarithmic $U(w) = \log w$. Due to its special structure, the optimal portfolio (when it exists) does not depend on the initial capital, and is myopic, i.e., does not depend on the given time-horizon. The relationship between the numéraire and the log-optimal portfolio is well-known and established by now. In fact, under a suitable reformulation of log-optimality, we can show an equivalence between the two notions.

Definition 3.15. A portfolio $\rho \in \Pi_\infty$ will be called *relatively log-optimal*, if

$$\mathbb{E}\left[\log\left(\frac{W_\infty^{\pi}}{W_\infty^{\rho}}\right)\right] \leq 0, \text{ for every } \pi \in \Pi_\infty.$$

Of course, if a portfolio is log-optimal, then it is also relatively log-optimal. The two notions coincide if the value function of the log-optimal problem is finite. Nevertheless, if this fails, we can have existence of an essentially *unique* relatively log-optimal portfolio, when there will be infinitely many log-optimal portfolios.

Proposition 3.16. *A numéraire portfolio exists if and only if a relatively log-optimal problem portfolio exists, in which case the two are the same.*

Proof. Suppose ρ is the numéraire portfolio. Then, for any other $\pi \in \Pi_\infty$, we have $\mathbb{E}[W^\pi_\infty / W^\rho_\infty] \leq 1$, and Jensen's inequality gives $\mathbb{E}[\log(W^\pi_\infty / W^\rho_\infty)] \leq 0$.

Let us now assume that the numéraire portfolio does not exist; we shall show that a relative log-optimal portfolio does not exist either. By way of contradiction, suppose that $\hat{\rho}$ was a relatively log-optimal portfolio.

First, we observe that $\left\{ \mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset \right\}$ must have zero $\mathbb{P} \otimes G$-measure. To see why, suppose the contrary. Then, by Proposition 2.15, we could select a portfolio $\xi \in \Pi_\infty$ that produces an unbounded increasing profit. According to Remark 2.16, we would have that $\hat{\rho} + \xi \in \Pi_\infty$ and $\mathfrak{rel}(\hat{\rho} \mid \hat{\rho} + \xi) \leq 0$ with strict inequality holding on a predictable set of positive $\mathbb{P} \otimes G$-measure; this would mean that the process $W^{\hat{\rho}} / W^{\hat{\rho}+\xi}$ is a non-constant positive supermartingale, so that Jensen's inequality again would give $\mathbb{E}[\log(W^{\hat{\rho}}_\infty / W^{\hat{\rho}+\xi}_\infty)] < 0$, contradicting the relative log-optimality of $\hat{\rho}$.

Continuing, since the numéraire portfolio does not exist and we already showed that $\left\{ \mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset \right\}$ has full $\mathbb{P} \otimes G$-measure, we must have that ρ (the candidate for the numéraire portfolio) is not globally X-integrable. In particular, the predictable set $\{\hat{\rho} \neq \rho\}$ must have non-zero $\mathbb{P} \otimes G$-measure. But then we can find a predictable set $\Sigma \subseteq \{\hat{\rho} \neq \rho\}$ such that Σ has non-zero $\mathbb{P} \otimes G$-measure and such that $\rho \mathbf{1}_\Sigma \in \Pi_\infty$. This implies $\hat{\rho} \mathbf{1}_{(\Omega \times \mathbb{R}_+) \setminus \Sigma} + \rho \mathbf{1}_\Sigma \in \Pi_\infty$, and since $\mathfrak{rel}(\hat{\rho} \mid \hat{\rho} \mathbf{1}_{(\Omega \times \mathbb{R}_+) \setminus \Sigma} + \rho \mathbf{1}_\Sigma) = \mathfrak{rel}(\hat{\rho} \mid \rho) \mathbf{1}_\Sigma \leq 0$, with strict inequality on Σ, the same discussion as in the end of the preceding paragraph shows that $\hat{\rho}$ cannot be the relatively log-optimal portfolio. □

Corollary 3.17. *A relatively log-optimal portfolio exists if and only if the condition* (NUPBR) *holds.*

Example 3.18. Take a one-stock market model with $S_t = \exp(\beta_{\tau \wedge t})$, where β is a standard, linear Brownian motion and τ is an almost surely finite stopping time with $\mathbb{E}[\beta_\tau^+] = +\infty$. For the logarithmic utility $U(w) = \log w$ we have $u(w) \geq \mathbb{E}[\log(wS_\infty)] = +\infty$, so that the log-utility optimization problem does not have a unique solution. In this case, $\rho = 1/2$ is both the numéraire portfolio and the relative log-optimal portfolio. We shall also give the more "natural" Example 3.24 that involves finite time-horizon.

Example 3.19. Here, we continue Example 3.7 on the 3-dimensional Bessel process market. Since the numéraire portfolio exists, (NUPBR) holds. We have also seen that (NA) fails — there exists arbitrage in the market, as described in Example 3.7. An investor with logarithmic utility will choose the portfolio $\rho = 1$ as his optimal investment; in this case we have finite expected utility, since even $\mathbb{E}[S_T] < \infty$. *Even though arbitrage opportunities exist in the market, the investor's optimal choice is clearly* not *an arbitrage.*

Remark 3.20. In the case where $\Pi \neq \Pi_\infty$, i.e., when we are working in a "truly infinite" time horizon, one can define analogously a portfolio $\rho \in \Pi$ to be relatively log-optimal if for all other $\pi \in \Pi$ we have

$$\mathbb{E}\left[\limsup_{t \to \infty} \left(\log\left(\frac{W_t^\pi}{W_t^\rho}\right)\right)\right] \leq 0.$$

Almost the exact same proof as the one in Proposition 3.16 will show that a relatively log-optimal portfolio exists if and only if the numéraire portfolio exists for all times before infinity, and that the two portfolios must coincide. We also refer the reader to Karatzas [20] and the references cited there, for this and related results.

We return now to the general case of a general utility U and show that if the relative log-optimal problem fails to have a solution, then none of the other utility problems has a solution either.

Proposition 3.21. *Assume that* (NUPBR) *fails. Then, for any utility function U, the corresponding utility maximization problem either does not have a solution or has infinitely many of them. More precisely: if $U(\infty) = +\infty$, then $u(w) = +\infty$ for all $w > 0$, so we either have no solution (in the case where the supremum is not attained) or an infinite number of them (in the case where the supremum is attained); whereas if $U(\infty) < +\infty$ there is no solution.*

Proof. Since (NUPBR) fails, pick an $\epsilon > 0$ and a sequence $(\pi_n)_{n\in\mathbb{N}}$ of elements of Π_∞ such that, with $A_n := \{W_\infty^{\pi_n} \geq n\}$, we have $\mathbb{P}[A_n] \geq \epsilon$, for all $n \in \mathbb{N}$.

If we suppose that $U(\infty) = +\infty$, then it is obvious that, for all $w > 0$ and $n \in \mathbb{N}$ we have $u(w) \geq \mathbb{E}[U(wW_\infty^{\pi_n})] \geq \epsilon U(wn)$; so that $u(w) = +\infty$ and we have the result stated in the proposition in view of Remark 3.14.

Now, suppose that $U(\infty) < \infty$; then of course $U(w) \leq u(w) \leq U(\infty) < \infty$ for all $w > 0$. Furthermore, u is also concave, thus continuous. Pick any $w > 0$, suppose that $\pi \in \Pi_\infty$ is optimal for U with initial capital w, and observe

$$u(w + n^{-1}) \geq \mathbb{E}[U(wW_\infty^\pi + n^{-1}W_\infty^{\pi_n})] \geq \mathbb{E}[U(wW_\infty^\pi + \mathbf{1}_{A_n})].$$

Pick $M > 0$ large enough so that $\mathbb{P}[wW_\infty^\pi > M] \leq \epsilon/2$; since U is concave we know that for any $y \in (0, M]$ we have $U(y + 1) - U(y) \geq U(M + 1) - U(M)$. Set $a := (U(M+1) - U(M))\epsilon/2$ - this is a strictly positive because U is strictly increasing. Then, $\mathbb{E}[U(wW^\pi_\infty + \mathbf{1}_{A_n})] \geq \mathbb{E}[U(wW^\pi_\infty) + a] = u(w) + a$; this implies $u(w + n^{-1}) \geq u(w) + a$ for all $n \in \mathbb{N}$, and contradicts the continuity of u. □

Having resolved the situation when (NUPBR) fails, let us now assume that it holds. We shall have to put a little bit more structure on the utility functions that we consider, so let us suppose that they are continuously differentiable and that they satisfy the *Inada conditions* $U'(0) = +\infty$ and $U'(+\infty) = 0$. We also assume that we are in the unconstrained case ($\mathfrak{C} = \mathbb{R}^d$).

The (NUPBR) condition is equivalent to the existence of the numéraire portfolio ρ. Since all wealth processes when divided by W^ρ become supermartingales, we conclude that the change of numéraire which utilizes W^ρ as a benchmark produces a market for which the *original* measure $\mathbb{P}$ is a supermartingale measure (see Delbaen and Schachermayer [10] for this "change of numéraire" technique). In particular, (NFLVR) holds and the duality results of the paper [12] allows us to write down the *superhedging duality*, valid for any positive, $\mathcal{F}$-measurable random variable H:

$$\inf \{w > 0 \mid \exists\, \pi \in \Pi_\infty \text{ with } wW^\pi_\infty \geq H\} = \sup_{D \in \mathcal{D}} \mathbb{E}[D_\infty H].$$

This relationship allows one to show that the utility optimization problems admit a solution (when their value is finite). We shall not go into the details, but send the reader to the papers [25, 26] of Kramkov and Schachermayer for more information.

3.4. **Arbitrage equivalences for exponential Lévy financial models.** In this section we present a complete characterization of the arbitrage situation in exponential Lévy financial models. By "exponential Lévy", we mean that the returns process X is a $\mathbf{F}$-Lévy process, i.e., for all $0 \leq s < t$, the increment

$X_t - X_s$ is independent of the σ-algebra $\mathcal{F}_s$ and has a distribution that only depends on the difference $t - s$. Then, X has a deterministic triplet of characteristics (b, c, ν) with respect to the canonical truncation function and the natural time flow $G(t) = t$. For this section we assume that the constraints $\mathfrak{C}$ are deterministic. We shall consider both the finite and the infinite horizon case, since they turn out to be radically different. In the finite-horizon case, if there is any kind of arbitrage, it is of the worst kind: an unbounded increasing profit. In the infinite-horizon case we basically always have arbitrage, unless the original measure $\mathbb{P}$ is a supermartingale measure (which makes sense, given that we have an infinite amount of time to do it and things are evolving constantly). We use the numéraire portfolio to prove the latter fact; we have not been able to find a proof in the existing literature.

Theorem 3.22. *Consider an exponential Lévy model. Suppose that $\mathfrak{C}$ is a non-random fixed closed convex cone of $\mathbb{R}^d$ with $\mathfrak{N} \subseteq \mathfrak{C}$ which enforces constraints on portfolios. On a finite financial planning horizon $[0, T]$, the following are equivalent:*

(1) *There exists a $\mathbb{Q} \sim \mathbb{P}$ under which $\mathrm{p}^\top X$ is a Lévy supermartingale for every $\mathrm{p} \in \mathfrak{C}$.*
(2) *The* (ESMM) *condition holds;*
(3) *The* (NFLVR) *condition holds;*
(4) *The* (NA) *condition holds;*
(5) *The* (NUIP) *condition holds;*
(6) *The* (NUPBR) *condition holds;*
(7) *The numéraire portfolio exists;*
(8) $\mathfrak{I} \cap \mathfrak{C} = \emptyset$.

If $\mathfrak{C} = \mathbb{R}^d$ *one can replace (1) and (2) by the stronger:*

(1′) *There exists a* $\mathbb{Q} \sim \mathbb{P}$ *under which* X *is a Lévy martingale.*

(2′) *The* (ELMM) *condition holds;*

Proof. For the implication (1) ⇒ (2), observe from (2.1) that under $\mathbb{Q}$ we have $\mathfrak{rel}_{\mathbb{Q}}(0 \mid \pi) \leq 0$ for all $\pi \in \Pi$, so $\mathbb{Q}$ is an equivalent supermartingale measure. All the implications (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (8) are trivial or follow from Propositions 2.15 and 3.2. Also, Theorem 3.12 implies that (6) ⇔ (7) ⇔ (8).

It remains to show that (8) ⇒ (1) and we are done. We rush through the steps of the proof of this fact that relies on the *Esscher transform*, since the technique is pretty well known; see for example Rogers [24].

First of all, by an equivalent change of measure that preserves the Lévy property, we can assume that for all $\mathrm{p} \in \mathbb{R}^d$ one has $\mathbb{E}[\exp(\mathrm{p}^\top X_T)] < \infty$. With this understanding, set $f(\mathrm{p}) := \mathbb{E}[\exp(-\mathrm{p}^\top X_T)]$ for $\mathrm{p} \in \mathfrak{C}$; the function f is convex and differentiable. Also, let $f_* := \inf_{\mathrm{p} \in \mathfrak{C}} f(\mathrm{p})$; nothing changes if we restrict this infimum on $\mathfrak{C} \cap \mathfrak{N}^\perp$.

The infimum f_* must be achieved by a point in $\mathfrak{C}$; otherwise, if there exists a minimizing sequence of elements of $\mathfrak{C} \cap \mathfrak{N}^\perp$ which is a divergent in norm, we can show[18] that we can construct a unit vector $\zeta \in \mathfrak{C} \cap \mathfrak{N}^\perp$ that is an immediate arbitrage opportunity, i.e., the assumption $\mathfrak{I} \cap \mathfrak{C} = \emptyset$ is violated.

We know that the infimum is attained at some point $\mathrm{p}_* \in \mathfrak{C}$; by differentiating f in each direction $\mathrm{p}_* + \mathrm{p}$ for all other $\mathrm{p} \in \mathfrak{C}$ we get $\mathbb{E}[-\mathrm{p}^\top X_T \exp(-\mathrm{p}_*^\top X_T)] \geq 0$. It then easily follows that if we set $Z_T := \exp(-\mathrm{p}_*^\top X_T)/f(\mathrm{p}_*)$, the strictly positive random variable Z_T defines a probability measure $\mathbb{Q}(A) := \mathbb{E}[Z_T \mathbf{1}_A]$ such

[18]The reader can check Chapter 5 to see how this is done.

that all $(\mathrm{p}^\top X_t)_{t\in[0,T]}$ for $\mathrm{p} \in \mathfrak{C}$ are $\mathbb{Q}$-Lévy processes[19] and since $\mathbb{E}^{\mathbb{Q}}[\mathrm{p}^\top X_T] \leq 0$ for all $\mathrm{p} \in \mathfrak{C}$, we get that $(\mathrm{p}^\top X_t)_{t\in[0,T]}$ are $\mathbb{Q}$-supermartingales. □

The following example shows that *the deflator corresponding to the numéraire portfolio in an exponential Lévy market need not be a martingale, and in fact is a strict supermartingale.* This is why we had to go through the Esscher transform in the proof of Theorem 3.22. This should be contrasted to the continuous-path case where, in the absence of constraints, $(W^\rho)^{-1}$ is at least a local martingale. The example also shows that we cannot expect to be able in general to compute the numéraire portfolio just by naively trying to solve $\nabla\mathfrak{g}(\rho) = \mathfrak{rel}(0 \mid \rho) = 0$, because sometimes this equation simply fails to have a solution.

Example 3.23. Consider a one-dimensional Lévy process with $b \in \mathbb{R}$, $c = 0$ and $\nu(\mathrm{d}x) = (1+x)\mathbf{1}_{(-1,1]}(x)\mathrm{d}x$, where $\mathrm{d}x$ is the usual Lebesgue measure. We have $\mathfrak{C}_0 = [-1, 1]$ and, for any $\pi \in (-1, 1)$,

$$\mathfrak{g}'(\pi) = b + \int_{-1}^{1} \left[\frac{x(1+x)}{1+\pi x} - (x+1)x\mathbf{1}_{\{\|x\|\leq 1\}} \right] \mathrm{d}x = b - \frac{2}{3} + \int_{-1}^{1} \frac{x(1+x)}{1+\pi x}\mathrm{d}x;$$

it is easy to see that $\mathfrak{g}'$ is decreasing in $\pi \in (-1, 1)$, that $\mathfrak{g}'(-1) = +\infty$ and $\mathfrak{g}'(1) = b - 2/3$. We can infer that if $b > 2/3$, there is no solution to the equation $\mathfrak{g}'(\pi) = 0$. In that case the numéraire portfolio is $\pi = 1$ and $(W^\rho)^{-1}$ is a strict Lévy supermartingale, since $\mathfrak{rel}(0 \mid 1) = -\mathfrak{g}'(1) < 0$.

In this example it seems that the drift is very favorable to the investor, who is inclined to invest more than $\pi = 1$ in order to get more growth, but cannot do that because negative wealth is not allowed.

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[19] If not familiar with this fact, the reader can check that the Lévy structure is retained under these measure transformations.

Example 3.24. This is another example where the log-utility optimization problem does not have a unique solution, in the spirit of Example 3.18. Again, consider a one-dimensional Lévy process with $b \in \mathbb{R}$, $c = 0$ and $\nu(\mathrm{d}x) = \big(\mathbf{1}_{(-1,1]}(x) + x^{-1}\left(\log(1+x)\right)^{-2}\mathbf{1}_{[1,\infty)}(x)\big)\,\mathrm{d}x$. We now have that $\mathfrak{C}_0 = [0,1]$. Observe that for all $\pi \in (0,1)$ the process $\log W_\pi \equiv \log \mathcal{E}(\pi^\top X)$ is a Lévy process; a moment's reflection shows that its jump measure behaves like $y^{-2}\mathrm{d}y$ as $y \to +\infty$ and like $e^y \mathrm{d}y$ as $y \to -\infty$, which means that $\log W_\pi$ has infinite expectation. We see that the problem of maximizing expected log-utility does not have unique solution. Of course, the numéraire exists and will be unique.

For $\pi \in (0,1)$, the simple calculation

$$\mathfrak{g}'(\pi) = \int_{-1}^{1} \frac{x}{1+\pi x}\mathrm{d}x + \int_{1}^{\infty} \frac{1}{(1+\pi x)\left(\log(1+x)\right)^2}\mathrm{d}x$$

will give us $\mathfrak{g}'(0+) = \int_1^\infty \left(\log(1+x)\right)^{-2}\mathrm{d}x = +\infty$ and likewise $g'(1-) = -\infty$, which means that the numéraire portfolio ρ belongs to the *open* interval $(0,1)$. Let us note — just for fun — that numerical results show that $\rho \cong .916$. Although the expected log-utility is infinite, the numéraire portfolio does not put all the weight on the stock.

Here is the infinite time-horizon result.

Proposition 3.25. *Consider an exponential Lévy financial model with infinite financial planning horizon and deterministic, fixed closed convex constraints $\mathfrak{C}$. Then, the following are equivalent:*

(1) *The original probability $\mathbb{P}$ is a supermartingale measure;*
(2) *The* (ESMM) *condition holds;*
(3) *The* (NFLVR) *condition holds;*
(4) *The* (NUPBR) *condition holds;*

(5) *The* (NA) *condition holds.*

Remark 3.26. The predictable characterization for the original measure $\mathbb{P}$ to be a supermartingale measure is very easy. We just have to check that for every $\pi \in \mathfrak{C}$ such that $\nu\left[\pi^\top x < -1\right] = 0$, we have $\pi^\top b + \int \pi^\top x \mathbf{1}_{\{\|x\|>1\}} \nu(\mathrm{d}x) \leq 0$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) are all trivial; we only prove (4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1). We show that if $\mathbb{P}$ is not a supermartingale measure, then both (NUPBR) and (NA) fail.

So, assume that $\mathbb{P}$ is not a supermartingale measure. If $\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$, then (NUIP) fails and so both (NUPBR) and (NA) will fail. On the other hand, if $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$, the numéraire portfolio exists for all times before infinity: it is a constant portfolio ρ that gives rise to a positive supermartingale $(W^\rho)^{-1}$. We claim that $(W^\rho_\infty)^{-1} = 0$; for otherwise the process $\mathcal{L}((W^\rho)^{-1})$, which is the Lévy process $-\rho \cdot X^{(\rho)}$ in the notation of section 2.2, would be a semimartingale up to infinity according to Lemma C.2. But a Lévy process cannot have a limit to infinity unless it is identically constant, which means that we should have $-\rho \cdot X^{(\rho)} \equiv 0$ and so $W^\rho = \mathcal{E}(-\rho \cdot X^{(\rho)}) \equiv 1$, or $\rho \in \mathfrak{N}$. But this cannot happen unless $\mathbb{P}$ is a supermartingale measure, and we are working under the assumption that it is not. Now, the fact $W^\rho_\infty = \infty$ allows us to construct portfolios $\pi_n \in \Pi$ by requiring $\pi_n := \rho \mathbf{1}_{[0,\tau_n]}$, where τ_n is the finite stopping time $\tau_n := \inf\left\{t \in \mathbb{R}_+ \mid W^\rho_t \geq n\right\}$. We deduce that $W^{\pi_n}_\infty \geq n$, which shows that both conditions (NUPBR) and (NA) fail. $\square$

4. The "No Unbounded Increasing Profit" Condition

This Chapter is devoted to the proof of Proposition 2.15; at this point the reader should be reminded of the context of that proposition that was given in section 2.4 and the discussion therein. We split the proof in three steps.

- **If** $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ $\mathbb{P} \otimes G$**-null, then (NUIP) holds:** Let us suppose that π is a portfolio that creates unbounded increasing profit; we shall show that $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ is not $\mathbb{P} \otimes G$-null. By way of definition we have that $\{\pi \in \check{\mathfrak{C}}\}$ has full $\mathbb{P} \otimes G$-measure, so we wish to prove that $\{\pi \in \mathfrak{I}\}$ has strictly positive $\mathbb{P} \otimes G$-measure.

Now, W^π has to be a non-decreasing process, which means that the same will hold for $\pi \cdot X$. We also shall have $\pi \cdot X \neq 0$ with positive probability. This means that the predictable set $\{\pi \notin \mathfrak{N}\}$ has strictly positive $\mathbb{P} \otimes G$-measure, and it will suffice to show that properties (1), (2) and (3) of Definition 2.14 hold $\mathbb{P} \otimes G$-a.e.

Since $\pi \cdot X$ is a process with finite variation we must have that $\pi \cdot X^{\mathsf{c}} = 0$, which translates into $\pi^\top c = 0$, $\mathbb{P} \otimes G$-a.e.; because $\pi \cdot X$ is increasing, one has $\mathbf{1}_{\{\pi^\top x < 0\}} * \mu = 0$, so that $\nu[\pi^\top x < 0] = 0$, $\mathbb{P} \otimes G$-a.e.

Finally, since $\pi \cdot X$ of finite variation, one can decompose

$$\pi \cdot X = \pi \cdot B - [\pi^\top x \mathbf{1}_{\{\|x\| \leq 1\}}] * \eta + [\pi^\top x] * \mu. \tag{4.1}$$

The last term $[\pi^\top x] * \mu$ is a pure-jump increasing process, while for the sum of the first two we have

$$\Delta\Big(\pi \cdot B - [\pi^\top x \mathbf{1}_{\{\|x\| \leq 1\}}] * \eta\Big) = \big(\pi^\top b - \textstyle\int \pi^\top x \mathbf{1}_{\{\|x\| \leq 1\}} \nu(\mathrm{d}x)\big)\, \Delta G = 0\,,$$

where the last equality follows from (1.4). It follows that the sum of the first two terms on the right-hand side of equation (4.1) is the continuous part of $\pi \cdot X$ (when seen as a finite variation process) and thus has to be increasing; this translates to the requirement $\pi^\top b - \int \pi^\top x \mathbf{1}_{\{\|x\| \leq 1\}} \nu(\mathrm{d}x) \geq 0$, $\mathbb{P} \otimes G$-a.e., and ends the proof.

• **The set-valued process $\mathfrak{I}$ is predictable:** In order to prove the other half of Proposition 2.15, we need to select a predictable process from the set $\left\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\right\}$. For this, we shall have to prove that $\mathfrak{I}$ is a predictable set-valued process. Nevertheless $\mathfrak{I}$ is not closed, and it is usually helpful to work with closed sets when trying to apply selection results.

For a Lévy triplet (b, c, ν) and every $a > 0$, define $\mathfrak{I}^a$ to be the set of vectors of $\mathbb{R}^d$ such that (1) to (3) of Definition 2.14 hold, and where we also require that

$$\xi^\top b + \int \frac{\xi^\top x}{1 + \xi^\top x} \mathbf{1}_{\{\|x\| \geq 1\}} \nu(\mathrm{d}x) \geq a^{-1}. \tag{4.2}$$

The following lemma contains the properties of these sets that will be useful.

Lemma 4.1. *With the previous definition we have:*

(1) *The sets $\mathfrak{I}^a$ are increasing in $a > 0$; we have $\mathfrak{I}^a \subseteq \mathfrak{I}$ and also $\mathfrak{I} = \bigcup_{a>0} \mathfrak{I}^a$. In particular, $\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$ if and only if $\mathfrak{I}^a \cap \check{\mathfrak{C}} \neq \emptyset$ for all large enough $a > 0$.*
(2) *For all $a > 0$, the set $\mathfrak{I}^a$ is closed and convex.*

Proof. Let us first note that because of conditions (1) to (3) of Definition 2.14, we have that the left-hand-side of (4.2) is well-defined (the integrant is positive since $\nu[\xi^\top x < 0] = 0$) and has to be positive. In fact, if $\xi \in \mathfrak{I}$, it has to be *strictly* positive, otherwise we would have $\xi \in \mathfrak{N}$. The fact that $\mathfrak{I}^a$ is increasing for $a > 0$ is trivial, and part (1) of this lemma follows immediately.

For the second part, we first show that $\mathfrak{I}^a$ is closed. It is obvious that the subset of $\mathbb{R}^d$ consisting of vectors ξ such that $\xi^\top c = 0$ and $\nu[\xi^\top x < 0] = 0$ is closed. On this last set, the functions $\xi^\top x$ are ν-positive. For a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathfrak{I}^a$ with $\xi_n \to \xi$, Fatou's lemma will give that

$$\int \xi^\top x \mathbf{1}_{\{\|x\|\leq 1\}} \nu(\mathrm{d}x) \leq \liminf_{n\to\infty} \int \xi_n^\top x \mathbf{1}_{\{\|x\|\leq 1\}} \nu(\mathrm{d}x) \leq \liminf_{n\to\infty} \xi_n^\top b = \xi^\top b,$$

so that ξ satisfies requirement (3) of Definition 2.14 also. The "large jumps" measure $\mathbf{1}_{\{\|x\|>1\}}\nu(\mathrm{d}x)$ is finite, and we can use the bounded convergence theorem to get

$$\xi^\top b + \int \frac{\xi^\top x}{1+\xi^\top x} \mathbf{1}_{\{\|x\|\geq 1\}} \nu(\mathrm{d}x) = \lim_{n\to\infty} \left\{ \xi_n^\top b + \int \frac{\xi_n^\top x}{1+\xi_n^\top x} \mathbf{1}_{\{\|x\|\geq 1\}} \nu(\mathrm{d}x) \right\} \geq a^{-1}$$

and the fact that $\mathfrak{I}^a$ is closed is established. Finally, convexity is trivial as soon as one uses the fact that the function $x \mapsto x/(1+x)$ is concave for $x > 0$. □

For the remainder of this Chapter, we denote by $\mathfrak{I}$ the $\mathbb{R}^d$-set-valued process $\mathfrak{I}\left(b(\omega,t), c(\omega,t), \nu(\omega,t)\right)$; same for $\mathfrak{I}^a$. From $\mathfrak{I} = \bigcup_{n\in\mathbb{N}} \mathfrak{I}^n$ and Lemma B.3, in order to prove predictability of $\mathfrak{I}$ we only have to prove predictability of $\mathfrak{I}^a$.

To this end, define the following $\mathbb{R}$-valued functions, with arguments from $(\Omega \times \mathbb{R}_+) \times \mathbb{R}^d$, hiding the dependence in the argument $(\omega, t) \in \Omega \times \mathbb{R}_+$:

$$\begin{aligned}
f_1(\mathrm{p}) &= \mathrm{p}^\top c, \qquad f_2(\mathrm{p}) = \int \frac{((\mathrm{p}^\top x)^-)^2}{1+((\mathrm{p}^\top x)^-)^2} \nu(\mathrm{d}x), \\
f_3^n(\mathrm{p}) &= \mathrm{p}^\top b - \int \mathrm{p}^\top x \mathbf{1}_{\{n^{-1}<\|x\|\leq 1\}} \nu(\mathrm{d}x), \text{ for all } n \in \mathbb{N}, \text{ and} \\
f_4(\mathrm{p}) &= \mathrm{p}^\top b + \int \frac{\mathrm{p}^\top x}{1+\mathrm{p}^\top x} \mathbf{1}_{\{\|x\|\geq 1\}} \nu(\mathrm{d}x).
\end{aligned}$$

Observe that all these functions are predictably measurable in $(\omega, t) \in \Omega \times \mathbb{R}_+$ and continuous in p (follows from applications of the dominated convergence theorem).

In a limiting sense, also define $f_3(\mathrm{p}) \equiv f_3^{\infty}(\mathrm{p}) = \mathrm{p}^\top b - \int \mathrm{p}^\top x \mathbf{1}_{\{\|x\| \leq 1\}} \nu(\mathrm{d}x)$; but observe that this function might not even be well-defined since both the positive and negative parts of the integrand might have infinite ν-integral.

Also, consider the sequence of set-valued processes, where $n \in \mathbb{N}$:

$$\mathfrak{A}_n^a := \left\{ \mathrm{p} \in \mathbb{R}^d \mid f_1(\mathrm{p}) = 0,\ f_2(\mathrm{p}) = 0,\ f_3^n(\mathrm{p}) \geq 0,\ f_4(\mathrm{p}) \geq a^{-1} \right\}$$

of which the "infinite" version coincides with $\mathfrak{I}^a$:

$$\mathfrak{I}^a \equiv \mathfrak{A}_\infty^a := \left\{ \mathrm{p} \in \mathbb{R}^d \mid f_1(\mathrm{p}) = 0,\ f_2(\mathrm{p}) = 0,\ f_3(\mathrm{p}) \geq 0,\ f_4(\mathrm{p}) \geq a^{-1} \right\}.$$

Because of the requirement $f_2(\mathrm{p}) = 0$, the function f_3 can be considered well-defined (but not finite, since it can take the value $-\infty$). In any case, it is easy to see that for any element p of the set $\left\{ \mathrm{p} \in \mathbb{R}^d \mid f_2(\mathrm{p}) = 0 \right\}$, we have that $f_3^n(\mathrm{p}) \downarrow f_3(\mathrm{p})$, so that the sequence $(\mathfrak{A}_n^a)_{n \in \mathbb{N}}$ is decreasing and that $\mathfrak{A}_n^a \downarrow \mathfrak{I}^a$. But every $\mathfrak{A}_n^a$ is closed and predictable (refer to Lemmata B.3 and B.5) and thus so is $\mathfrak{I}^a$.

Remark 4.2. Since $\left\{ \mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset \right\} = \bigcup_{n \in \mathbb{N}} \left\{ \mathfrak{I}^n \cap \check{\mathfrak{C}} \neq \emptyset \right\}$ and all the random set-valued processes $\mathfrak{I}^n$ and $\check{\mathfrak{C}}$ are closed and predictable, Proposition B.3 gives us that the set $\left\{ \mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset \right\}$ is predictable.

• **(NUIP) implies that $\left\{ \mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset \right\}$ is $\mathbb{P} \otimes G$-null:** We are now ready to finish the proof of Proposition 2.15. Let us suppose that $\left\{ \mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset \right\}$ is not $\mathbb{P} \otimes G$-null; under this assumption, we shall construct an unbounded increasing profit. Start by observing that $\mathfrak{I} = \bigcup_{n \in \mathbb{N}} \left(\left\{ \mathrm{p} \in \mathbb{R}^d \mid \|\mathrm{p}\| \leq n \right\} \cap \mathfrak{I}^n \right)$, where $\mathfrak{I}^n$ is the set-valued process defined in the previous section. It follows that there exists some $n \in \mathbb{N}$ such that the *convex, closed* and *predictable* set-valued process $\mathcal{B}^n := \left\{ \mathrm{p} \in \mathbb{R}^d \mid \|\mathrm{p}\| \le n \right\} \cap \mathfrak{I}^n \cap \check{\mathfrak{C}}$ has $(\mathbb{P} \otimes G)(\{\mathcal{B}^n \neq \emptyset\}) > 0$. According to Theorem B.7, there exists a predictable process π with $\pi(\omega, t) \in \mathcal{B}^n(\omega, t)$ when $\mathcal{B}^n(\omega, t) \neq \emptyset$ and $\pi(\omega, t)$ if $\mathcal{B}^n(\omega, t) = \emptyset$. Now, π is bounded (by n), so $\pi \in \Pi$. Using the reverse reasoning of the one we used in the beggining of this Chapter (when we were proving that if $\left\{ \mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset \right\}$ $\mathbb{P} \otimes G$-null, then (NUIP) holds) we get that $\pi \cdot X$ is a non-decreasing process, and the same will be true of W^π. Now, we *must* have that $\mathbb{P}[W^\pi_\infty > 1] > 0$, otherwise we would have $\pi \cdot X \equiv 0$, which is impossible since $(\mathbb{P} \otimes G)(\{\pi \notin \mathfrak{N}\}) > 0$ by construction.

5. The Numéraire Portfolio for Exponential Lévy Markets

We saw in Lemma 2.8 that if the numéraire portfolio ρ exists, it has to satisfy $\mathfrak{rel}(\pi \mid \rho) \leq 0$ pointwise, $\mathbb{P} \otimes G$-a.e. In order to find necessary and sufficient conditions for the existence of a (predictable) process ρ to satisfy this inequality, it makes sense first to consider the corresponding static, deterministic problem. Since Lévy processes correspond to constant, deterministic triplets of characteristics with respect to the natural time flow $G(t) = t$, for the results in this Chapter the reader is welcome to regard X as a Lévy process with characteristic triplet (b, c, ν); this means that $B_t = bt$, $C_t = ct$ and $\eta(\mathrm{d}t, \mathrm{d}x) = \nu(\mathrm{d}x)\mathrm{d}t$. We also take $\mathfrak{C}$ to be a closed convex subset of $\mathbb{R}^d$; we remark that $\mathfrak{C}$ can be enriched as to accommodate the natural constraints $\mathfrak{C}_0 = \left\{\pi \in \mathbb{R}^d \mid \nu[\pi^\top x < -1] = 0\right\}$.

The following result, which will be the focus of this Chapter, is the deterministic analogue of Theorem 2.20.

Theorem 5.1. *Let (b, c, ν) be a Lévy triplet and $\mathfrak{C}$ a closed convex subset of $\mathbb{R}^d$. Then the following are equivalent:*

(1) $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$.

(2) *There exists a unique vector $\rho \in \mathfrak{C} \cap \mathfrak{N}^\perp$ with $\nu\left[\rho^\top x \leq -1\right] = 0$ such that $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \mathfrak{C}$. If the Lévy measure ν integrates the log, the vector ρ is characterized as $\rho = \arg\max_{\pi \in \mathfrak{C} \cap \mathfrak{N}^\perp} \mathfrak{g}(\pi)$. In general, ρ is the limit of a series of problems, in which ν is replaced by a sequence of approximating measures.*

We have already shown (consult Remark 2.16) that if (1) fails, than (2) fails as well. We need to show that if (1) holds, then (2) holds too. Chapter 5 is devoted to the proof of this fact.

Remark 5.2. Combining requirements (1) and (2) of Definition 2.10 we get that $\mathfrak{C} = \mathfrak{C} + \mathfrak{N}$: indeed, for any $\pi \in \mathfrak{C}$ and $\zeta \in \mathfrak{N} \subseteq \mathfrak{C}$ we have that $n\zeta \in \mathfrak{C}$ for any $n \in \mathbb{N}$ and the convex combination $(1 - n^{-1})\pi + \zeta$ belongs to $\mathfrak{C}$ as well; but $\mathfrak{C}$ is closed, and so $\pi + \zeta \in \mathfrak{C}$. Now, $\mathfrak{C}$ is closed and $\mathfrak{N}$ is a linear subspace; this means that $\mathsf{pr}_{\mathfrak{N}^\perp}\mathfrak{C} = \mathfrak{C} \cap \mathfrak{N}^\perp$ is also closed in the subspace $\mathfrak{N}^\perp$, where $\mathsf{pr}_{\mathfrak{N}^\perp}$ is the usual Euclidean projection on $\mathfrak{N}^\perp$, the orthogonal complement of $\mathfrak{N}$.

The vector ρ constructed as described in Theorem 5.1 will be the unique vector that satisfies $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \mathfrak{C}$ only in the case where $\mathfrak{N} = \{0\}$; in general the set of all the solutions will be $\rho + \mathfrak{N}$ (where ρ is the special solution lying on $\mathfrak{N}^\perp$). The reason is, of course, that for any $\zeta \in \mathfrak{N}$ and any vectors π and ρ of $\mathfrak{C}$ one has $\mathfrak{rel}(\pi + \zeta \mid \rho) = \mathfrak{rel}(\pi \mid \rho)$, as one can check immediately from (2.1), (2.2). With the same notation, we have furthermore $\mathfrak{g}(\pi + \zeta) = \mathfrak{g}(\pi)$.

The conclusion from the above discussion is that, with no loss of generality, we can restrict our attention to the set $\mathfrak{C} \cap \mathfrak{N}^\perp$ for the portfolios. Any degeneracy originally present in the market disappears on this set. We shall need to restrict our attention to that set, since we shall be using an approximation procedure for obtaining the solution and we want our corresponding approximating solutions to remain bounded (in order to have a limit). If we do not project them on $\mathfrak{N}^\perp$ we cannot make sure that these sequences do not escape to infinity: our portfolios might "get lost far away in the (sub)space $\mathfrak{N}$". Furthermore, the fact that we are choosing the solution in a unique way will rid us of "measurable selection" procedure later.

In the process of the proof we shall need the following simple characterization of the condition $\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$:

Lemma 5.3. *If* $\xi \in \check{\mathfrak{C}} \setminus \mathfrak{N}$*, then* $\xi \in \mathfrak{I}$ *if and only if* $\mathfrak{rel}(0 \mid a\xi) \leq 0$ *for all* $a \in \mathbb{R}_+$*.*

Proof. The fact that $\xi \in \mathfrak{I} \cap \check{\mathfrak{C}}$ implies $\mathfrak{rel}(0 \mid a\xi) \leq 0$ for all $a \in \mathbb{R}_+$ is trivial.

For the converse, let $\xi \in \check{\mathfrak{C}} \setminus \mathfrak{N}$ satisfy $\mathfrak{rel}(0 \mid a\xi) \leq 0$ for all $a \in \mathbb{R}_+$; we wish to show that $\xi \in \mathfrak{I}$. The second condition of Definition 2.14 is readily satisfied, since we assume that $\mathfrak{C}$ contains the natural constraints. Now, for all $a \in \mathbb{R}_+$, we have $-a^{-1}\mathfrak{rel}(0 \mid a\xi) \geq 0$; writing this down we get

$$\xi^\top b - a\xi^\top c\xi + \int \left[\frac{\xi^\top x}{1 + a\xi^\top x} - \xi^\top x \mathbf{1}_{\{\|x\| \leq 1\}} \right] \nu(\mathrm{d}x) \geq 0.$$

The integrand is ν-integrable and is pointwise decreasing in $a > 0$ (remember that $\nu[\xi^\top x < 0] = 0$), so we must have $\xi^\top c = 0$ (condition (1) of Definition 2.14), which now implies that

$$\xi^\top b + \int \left[\frac{\xi^\top x}{1 + a\xi^\top x} - \xi^\top x \mathbf{1}_{\{\|x\| \leq 1\}} \right] \nu(\mathrm{d}x) \geq 0.$$

Using the dominated convergence theorem and letting $a \uparrow \infty$ we get condition (3) of Definition 2.14, namely $\xi^\top b - \int \xi^\top x \mathbf{1}_{\{\|x\| \leq 1\}} \nu(\mathrm{d}x) \geq 0$. □

We make one final observation. On several occasions during the course of the proof we shall use Fatou's lemma in the following form: if we are given a *finite* measure κ and a sequence $(f_n)_{n \in \mathbb{N}}$ of measurable functions that are κ-uniformly bounded from below, then $\int \liminf_{n \to \infty} f_n(x) \kappa(\mathrm{d}x) \leq \liminf_{n \to \infty} \int f_n(x) \kappa(\mathrm{d}x)$. The finite measures κ that we shall consider will be of the form $(\|x\| \wedge k)^2 \, \nu(\mathrm{d}x)$, where $k \in \mathbb{R}_+$ and ν is our Lévy measure.

We can now proceed with the proof of the sufficiency of the condition $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$ in solving $\mathfrak{rel}(\pi \mid \rho) \leq 0$. We shall first do so for the case of a Lévy measure that integrates the log, then extend to the general case.

• **Proof of Theorem 5.1 for a Lévy measure that integrates the log.** We are trying to show (1) ⇒ (2) of Theorem 5.1, so let us assume $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$. For this section we also make the assumption $\int_{\{\|x\|>1\}} \log(1 + \|x\|)\nu(\mathrm{d}x) < \infty$.

Recall from section 2.5 the growth rate function

$$\mathfrak{g}(\pi) := \pi^\top b - \frac{1}{2}\pi^\top c\pi + \int \left[\log(1 + \pi^\top x) - \pi^\top x \mathbf{1}_{\{\|x\|\leq 1\}}\right] \nu(\mathrm{d}x)$$

of (2.3). This is a concave function on $\mathfrak{C}$ and is well-defined, in the sense that we always have $\mathfrak{g}(\pi) < +\infty$ (because ν integrates the log), but can take the value $-\infty$ on the boundary of $\mathfrak{C}$. Nevertheless, if we restrict our attention to

$$\mathfrak{C}^\diamond = \left\{\pi \in \mathfrak{C} \mid \nu[\pi^\top x \leq -u] = 0 \text{ for some } u < 1\right\}, \tag{5.1}$$

then we also have $g(\pi) > -\infty$.

Let us agree to call $\mathfrak{g}_* := \sup_{\pi\in\mathfrak{C}} \mathfrak{g}(\pi)$, and let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of vectors in $\mathfrak{C}$ with $\mathfrak{g}(\rho_n) \to \mathfrak{g}_*$. Since for any $\pi \in \mathfrak{C}$ and any $\zeta \in \mathfrak{N}$ we have $\mathfrak{g}(\pi + \zeta) = \mathfrak{g}(\pi)$, we can choose the sequence ρ_n to take values on the subspace $\mathfrak{N}^\perp$ (recall the discussion of Remark 5.2). We shall prove later that this sequence is bounded in $\mathbb{R}^d$; for the time being let us take this for granted and, without loss of generality, suppose that $(\rho_n)_{n\in\mathbb{N}}$ converges to a point $\rho \in \mathfrak{C}$ (otherwise, choose a convergent subsequence). The concavity of $\mathfrak{g}$ implies that $\mathfrak{g}_*$ is a finite number and it is obvious from continuity that $\mathfrak{g}(\rho) = \mathfrak{g}_*$. Of course, we have that $\nu\left[\rho^\top x \leq -1\right] = 0$, otherwise $\mathfrak{g}(\rho) = -\infty$.

Pick now any $\pi \in \mathfrak{C}^\diamond$ in the notation of (5.1); we then have that the mapping $[0, 1] \ni u \mapsto \mathfrak{g}(\rho + u(\pi - \rho))$ is well-defined (i.e., real-valued), concave and decreasing, so that the right-derivative at $u = 0$ should be negative; of course, this derivative is just $\mathfrak{rel}(\pi \mid \rho)$, so we have the result for $\pi \in \mathfrak{C}^\diamond$.

The extension of the inequality $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \mathfrak{C}$ now follows easily. Indeed, if $\pi \in \mathfrak{C}$, then for $0 \leq u < 1$ we have $u\pi \in \mathfrak{C}^\diamond$ and $\mathfrak{rel}(u\pi \mid \rho) \leq 0$; by using Fatou's lemma one can easily check that we also have $\mathfrak{rel}(\pi \mid \rho) \leq 0$. We do not do this now since anyway we shall have the chance to do this again three times in the sequel in a slightly more complicated manner.

We still have to show that the sequence $(\rho_n)_{n \in \mathbb{N}}$ of vectors of $\mathfrak{C} \cap \mathfrak{N}^\perp$ is bounded. Suppose that $(\rho_n)_{n \in \mathbb{N}}$ is unbounded, and without loss of generality also that the sequence of unit-length vectors $\xi_n := \rho_n / \|\rho_n\|$ converges to a unit-lenth vector $\xi \in \mathfrak{N}^\perp$ (picking a subsequence otherwise). We shall use Lemma 5.3 applied to the vector ξ and show that $\xi \in \mathfrak{I} \cap \check{\mathfrak{C}}$, contradicting condition (1) of Theorem 5.1.

Start by picking any $a \in \mathbb{R}_+$; for all large enough $n \in \mathbb{N}$ we have $a\xi_n \in \mathfrak{C}$, and since $\mathfrak{C}$ is closed we have $a\xi \in \mathfrak{C}$ as well, which implies $\xi \in \check{\mathfrak{C}}$ (since $a \in \mathbb{R}_+$ is arbitrary). We have $\xi \in \check{\mathfrak{C}} \setminus \mathfrak{N}$, and only need to show $\mathfrak{rel}(0 \mid a\xi) \leq 0$. For this, we can assume that the sequence $(\rho_n)_{n \in \mathbb{N}}$ is picked in such a way that the functions $[0, 1] \ni u \mapsto \mathfrak{g}(u\rho_n)$ are increasing; otherwise, replace ρ_n by the vector $u\rho_n$ for the choice of $u \in [0, 1]$ that maximizes $[0, 1] \ni u \mapsto \mathfrak{g}(u\rho_n)$. This would imply that eventually, for all large enough $n \in \mathbb{N}$ we have $\mathfrak{rel}(0 \mid a\xi_n) \leq 0$; this means

$$\int \left[\frac{-\xi_n^\top x}{1 + a\xi_n^\top x} + \xi_n^\top x \mathbf{1}_{\{\|x\| \leq 1\}} \right] \nu(\mathrm{d}x) \leq \xi_n^\top b - a\xi_n^\top c \xi_n.$$

If we can show that we can apply Fatou's lemma to the quantity on the left-hand-side of this inequality, we get the same inequality with ξ in place of ξ_n

and so $\mathfrak{rel}(0 \mid a\xi) \leq 0$; an application of Lemma 5.3 shows that $\xi \in \mathfrak{I} \cap \check{\mathfrak{C}}$, contradicting condition (1) of Theorem 5.1.

To show that we can actually apply Fatou's lemma, let us show that the integrand is bounded from below for the finite measure $(\|x\| \wedge k)^2 \nu(\mathrm{d}x)$ with $k := 1 \wedge (2a)^{-1}$. Since $\xi_n^\top x/(1 + a\xi_n^\top x) \leq a^{-1}$ and $\left|\xi_n^\top x\right| \leq \|x\|$, the integrand is uniformly bounded from below by $-(a^{-1} + 1)$. Thus, we only need consider what happens on the set $\{\|x\| \leq k\}$; but there, the integrand is equal to $-a(\xi_n^\top x)^2/(1+a\xi_n^\top x)$, which cannot be less than $-2a\|x\|^2$ and we are done. □

• **The extension to general Lévy measures.** We now have to extend the result of the previous section to the case where ν does not necessarily integrate the log. Recall from Definition 2.17 the use of the approximating triplets (b, c, ν_n), where for every $n \in \mathbb{N}$ we define the measure $\nu_n(\mathrm{d}x) := f_n(x)\nu(\mathrm{d}x)$; all these measures integrate the log. We assume throughout that $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$.

We remarked that the sets $\mathfrak{N}$ and $\mathfrak{I}$ remain invariant if we change the Lévy measure from ν to ν_n. Then, since we have $\mathfrak{I}(b, c, \nu_n) \cap \check{\mathfrak{C}} = \emptyset$, the discussion in the previous Chapter, gives us unique vectors $\rho_n \in \mathfrak{C} \cap \mathfrak{N}^\perp$ such that $\mathfrak{rel}_n(\pi \mid \rho_n) \leq 0$ for all $\pi \in \mathfrak{C}$, where $\mathfrak{rel}_n$ is associated with the triplet (b, c, ν_n).

As before, the constructed sequence $(\rho_n)_{n\in\mathbb{N}}$ is bounded. To prove it, we shall use Lemma 5.3 again, in the exact same way that we did for the case of a measure that integrates the log. Assume by way of contradiction that $(\rho_n)_{n\in\mathbb{N}}$ is not bounded. By picking a subsequence if necessary, assume without loss of generality that $\|\rho_n\|$ diverges to infinity. Now, call $\xi_n := \rho_n/\|\rho_n\|$. Again, by picking a further subsequence if the need arises, assume that $\xi_n \to \xi$, where ξ is a unit vector in $\mathfrak{N}^\perp$. Since $\rho_n \in \mathfrak{C}$ for all $n \in \mathbb{N}$ it follows that $a\xi \in \mathfrak{C}$ for all $a \in \mathbb{R}_+$, i.e., $\xi \in \check{\mathfrak{C}} \setminus \mathfrak{N}$. We know that for sufficiently large $n \in \mathbb{N}$, we have that

$\mathfrak{rel}_n(0 \mid a\xi_n) \leq 0$; equivalently

$$\int \left[\frac{-\xi_n^\top x}{1 + a\xi_n^\top x} f_n(x) + \xi_n^\top x \mathbf{1}_{\{\|x\| \leq 1\}} \right] \nu(\mathrm{d}x) \leq \xi_n^\top b - a\xi_n^\top c \xi_n.$$

The situation is exactly the same as in the proof in the case of a measure that integrates the log, but for the appearance of the density $f_n(x)$ which can only have a positive effect on any lower bounds that we have established there, since $0 < f_n \leq 1$. We show that the integrand is bounded from below for the finite measure $(\|x\| \wedge k)^2 \, \nu(\mathrm{d}x)$ with $k = 1 \wedge (2a)^{-1}$, thus we can apply Fatou's lemma to the left-hand-side of this inequality to get the same inequality with ξ in place of ξ_n, and so $\mathfrak{rel}(0 \mid a\xi) \leq 0$. Invoking Lemma 5.3, we arrive at a contradiction with the assumption $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$.

• Now that we know that $(\rho_n)_{n \in \mathbb{N}}$ is a bounded sequence, we can assume that it converges to a point $\rho \in \mathfrak{C} \cap \mathfrak{N}^\perp$, picking a subsequence if needed. We shall show now that ρ satisfies $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \mathfrak{C}$. Pick any $\pi \in \mathfrak{C}$; we know that we have

$$\int \left[\frac{(\pi - \rho_n)^\top x}{1 + \rho_n^\top x} f_n(x) - (\pi - \rho_n)^\top x \mathbf{1}_{\{\|x\| \leq 1\}} \right] \nu(\mathrm{d}x) \leq -(\pi - \rho_n)^\top b + (\pi - \rho_n)^\top c \rho_n$$

for all $n \in \mathbb{N}$. Yet once more, we shall use Fatou's lemma on the left-hand-side to get to the limit the inequality

$$\int \left[\frac{(\pi - \rho)^\top x}{1 + \rho^\top x} - (\pi - \rho)^\top x \mathbf{1}_{\{\|x\| \leq 1\}} \right] \nu(\mathrm{d}x) \leq -(\pi - \rho)^\top b + (\pi - \rho)^\top c \rho \,;$$

and so that $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \mathfrak{C}$.

To justify the use of Fatou's lemma, we shall show that the integrands are uniformly bounded from below for the measure $(\|x\| \wedge k)^2 \, \nu(\mathrm{d}x)$, where we set $k = 1 \wedge (2 \sup_{n \in \mathbb{N}} \|\rho_n\|)^{-1}$ which is a strictly positive number from the boundedness of $(\rho_n)_{n \in \mathbb{N}}$. First, observe that the integrands are uniformly bounded by

$-1-\sup_{n\in\mathbb{N}} \|\pi - \rho_n\|$, which is a finite number. Thus, we only need worry about the set $\{\|x\| \leq k\}$. There, the integrands are equal to $(\pi-\rho_n)^\top x(\rho_n^\top x)/(1+\rho_n^\top x)$; this cannot be less than $-2\sup_{n\in\mathbb{N}} (\|\pi - \rho_n\| \, \|\rho_n\|) \, \|x\|^2$, and Fatou's lemma can be used.

Up to now we have shown that $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \mathfrak{C}$ for the limit ρ of a subsequence of $(\rho_n)_{n\in\mathbb{N}}$. Nevertheless, carrying the previous steps we see that *every* subsequence of $(\rho_n)_{n\in\mathbb{N}}$ has a further convergent subsequence whose limit $\hat{\rho} \in \mathfrak{C} \cap \mathfrak{N}^\perp$ satisfies $\mathfrak{rel}(\pi \mid \hat{\rho}) \leq 0$ for all $\pi \in \mathfrak{C}$. The uniqueness of the vector $\rho \in \mathfrak{C} \cap \mathfrak{N}^\perp$ that satisfies $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \mathfrak{C}$ gives that $\hat{\rho} = \rho$, and we conclude that the whole sequence $(\rho_n)_{n\in\mathbb{N}}$ converges to ρ. □

6. The Numéraire Portfolio for General Semimartingales

The purpose of this Chapter is to provide the proofs of Theorem 2.20 and Proposition 3.10

• **Proof of Theorem 2.20.** We are ready now to prove our first main result. We start with a predictable characterization of global X-integrability that the predictable process ρ, candidate of being the numéraire portfolio, must satisfy.

Lemma 6.1. *Suppose that ρ is a predictable process with $\nu[\rho^\top x \leq -1] = 0$ and $\mathfrak{rel}(0 \mid \rho) \leq 0$. Then, ρ is globally X-integrable, if and only if we have $(\phi^\rho \cdot G)_\infty < \infty$ for the increasing, predictable process*

$$\phi^\rho := \nu[\rho^\top x > 1] + \left| \rho^\top b + \int \rho^\top x \left(\mathbf{1}_{\{\|x\|>1\}} - \mathbf{1}_{\{|\rho^\top x|>1\}} \right) \nu(\mathrm{d}x) \right| .$$

Proof. We have to show that ϕ_1^ρ and ϕ_2^ρ of (2.5) are globally G-integrable. According to Theorem C.3 of Appendix C, only the sufficiency has to be proved, since the necessity holds trivially (remember that $\nu[\rho^\top x \leq -1] = 0$). Furthermore, from the same theorem, the sufficiency will be shown if we can prove that the predictable processes $\psi_1^\rho := \rho^\top c \rho$ and $\psi_2^\rho := \int \left(1 \wedge \left(\rho^\top x\right)^2\right) \nu(\mathrm{d}x)$ are globally G-integrable (observe that ψ_3^ρ of that theorem is already covered by ϕ_2^ρ).

Dropping the "ρ" superscripts, we embark on proving the global G-integrability of ψ_1 and ψ_2, assuming the global G-integrability of ϕ_1 and ϕ_2. We have that ψ_2 will certainly be globally G-integrable, if one can show that the positive process

$$\psi_2' := \int \frac{(\rho^\top x)^2}{1 + \rho^\top x} \mathbf{1}_{\{|\rho^\top x| \leq 1\}} \nu(\mathrm{d}x) + \int \frac{\rho^\top x}{1 + \rho^\top x} \mathbf{1}_{\{\rho^\top x > 1\}} \nu(\mathrm{d}x)$$

is globally G-integrable. Using also the fact that both $-\mathfrak{rel}(0 \mid \rho)$ and $\psi_1 = \rho^\top c \rho$ are positive processes, we get that ψ_1 and ψ_2 will certainly be globally G-integrable if we can show that $\psi_1 + \psi_2' - \mathfrak{rel}(0 \mid \rho)$ is globally G-integrable. But one can compute this sum to be equal to

$$\rho^\top b + \int \rho^\top x \left(\mathbf{1}_{\{\|x\|>1\}} - \mathbf{1}_{\{|\rho^\top x|>1\}} \right) \nu(\mathrm{d}x) + 2 \int \frac{\rho^\top x}{1 + \rho^\top x} \mathbf{1}_{\{\rho^\top x > 1\}} \nu(\mathrm{d}x);$$

the first two terms equal exactly ϕ_2, which is globally G-integrable, and the last (third) term is globally G-integrable because ϕ_1 is. □

Continuing, let us remark that the last paragraph of Theorem 2.20 follows directly from Lemmata 2.8 and 6.1. It remains to prove all the claims of the second paragraph of Theorem 2.20.

The fact that $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ is predictable has been shown in Remark 4.2.

Now, suppose that $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ has zero $\mathbb{P} \otimes G$-measure. We first assume that ν integrates the log, $\mathbb{P} \otimes G$-almost everywhere. We set $\rho = 0$ on $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$. Now, on $\{\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset\}$, according to Theorem 5.1, there exists a (uniquely defined) process ρ with $\rho^\top \Delta X > -1$ that satisfies $\mathfrak{rel}(\pi \mid \rho) \leq 0$, and $\mathfrak{g}(\rho) = \max_{\pi \in \mathfrak{C} \cap \mathfrak{N}^\perp} \mathfrak{g}(\pi)$; by Theorem B.7, this process ρ is predictable and we are done.

Now, we drop the assumption that ν integrates the log. By considering an approximating sequence $(\nu_n)_{n \in \mathbb{N}}$ and keeping every ν_n predictable (this is trivial if all densities f_n are deterministic), we get a sequence of $(\rho_n)_{n \in \mathbb{N}}$ that take values in $\mathfrak{C} \cap \mathfrak{N}^\perp$ and solve the corresponding approximating problems. As was shown in Chapter 5 (in the last part of the proof, the case of a general Lévy measure), the sequence $(\rho_n)_{n \in \mathbb{N}}$ will converge pointwise to a process ρ, which will thus be predictable and for which we have $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \Pi$.

Now that we have the our predictable candidate-process ρ for the numéraire portfolio, we only have to check that it is globally X-integrable; according to

Lemma 6.1 this is exactly covered by the predictable criterion $(\phi^\rho \cdot G)_\infty < +\infty$. By Lemma 2.8, we are done. □

• **Proof of Proposition 3.10.** We embark on the proof by first defining $\Omega_0 := \{(\phi^\rho \cdot G)_\infty < \infty\}$ and $\Omega_A := \{(\phi^\rho \cdot G)_\infty = \infty\} = \Omega \setminus \Omega_0$.

First, we show the result for Ω_0. Assume that $\mathbb{P}[\Omega_0] > 0$, and call $\mathbb{P}_0$ the probability measure one gets by conditioning $\mathbb{P}$ on the set Ω_0. The process ρ of course remains predictable when viewed under the new measure; and because we are restricting ourselves on Ω_0, ρ is globally X-integrable under $\mathbb{P}_0$.

The dominated convergence theorems for Lebesgue, as well as for stochastic integrals, gives that all three sequences of processes $\rho_n \cdot X$, $[\rho_n \cdot X^{\mathsf{c}}, \rho_n \cdot X^{\mathsf{c}}]$ and $\sum_{s \leq \cdot} \left[\rho_n^\top \Delta X_s - \log(1 + \rho_n^\top \Delta X_s)\right]$ converge uniformly (in $t \in \mathbb{R}_+$) in $\mathbb{P}_0$-measure to three processes, not depending on the sequence $(\rho_n)_{n \in \mathbb{N}}$. Then, the stochastic exponential formula (A.1) gives that $W_\infty^{\rho_n}$ converges in $\mathbb{P}_0$-measure to a random variable, not depending on the sequence $(\rho_n)_{n \in \mathbb{N}}$. Since the limit of the sequence $(\mathbf{1}_{\Omega_0} W_\infty^{\rho_n})_{n \in \mathbb{N}}$ is the same under both the $\mathbb{P}$-measure and the $\mathbb{P}_0$-measure, we conclude that, on Ω_0, the sequence $(W_\infty^{\rho_n})_{n \in \mathbb{N}}$ converges in $\mathbb{P}$-measure to a real-valued random variable, independently of the choice of the sequence $(\rho_n)_{n \in \mathbb{N}}$.

Now we have to tackle the set Ω_A, which is more tricky. We shall have to further use a "helping sequence of portfolios". Suppose $\mathbb{P}[\Omega_A] > 0$, otherwise there is nothing to prove. Under this assumption, there exist a sequence of $[0,1]$-valued predictable processes $(h_n)_{n \in \mathbb{N}}$ such that each $\pi_n := h_n \rho$ is globally X-integrable and such that the sequence of terminal values $((\pi_n \cdot X)_\infty)_{n \in \mathbb{N}}$ is unbounded in probability[20]. Lemma A.4 shows that $(W_\infty^{\pi_n})_{n\in\mathbb{N}}$ is also unbounded in probability. Then, $\mathbb{P}[\limsup_{n\to\infty} W_\infty^{\pi_n} = +\infty] > 0$, where the lim sup is taken in probability and not almost surely (see Definition 3.9).

Let us return to our original sequence of portfolios $(\rho_n)_{n\in\mathbb{N}}$ with $\rho_n = \theta_n\rho$ and show that $\{\limsup_{n\to\infty} W_\infty^{\pi_n} = +\infty\} \subseteq \{\limsup_{n\to\infty} W_\infty^{\rho_n} = +\infty\}$. Both of these upper limits, and in fact all the lim sup that will appear until the end of the proof, are supposed to be in $\mathbb{P}$-measure. Since each θ_n is $[0,1]$-valued and $\lim_{n\to\infty}\theta_n = \mathbf{1}$, one can choose an increasing sequence $(k(n))_{n\in\mathbb{N}}$ of natural numbers such that the sequence $(W_\infty^{\theta_{k(n)}\pi_n})_{n\in\mathbb{N}}$ is unbounded in $\mathbb{P}$-measure on the set $\{\limsup_{n\to\infty} W_\infty^{\pi_n} = +\infty\}$. Now, each process $W^{\theta_{k(n)}\pi_n}/W^{\rho_{k(n)}}$ is a positive supermartingale, since $\mathfrak{rel}(\theta_{k(n)}\pi_n \mid \rho_{k(n)}) = \mathfrak{rel}(\theta_{k(n)}h_n\rho \mid h_n\rho) \leq 0$, the last inequality due to the fact that $[0,1] \ni u \mapsto \mathfrak{g}(u\rho)$ is increasing, and so the sequence of random variables $(W_\infty^{\theta_{k(n)}\pi_n}/W_\infty^{\rho_{k(n)}})_{n\in\mathbb{N}}$ is bounded in probability. ¿From the last two facts follows that the sequence of random variables $(W_\infty^{\rho_{k(n)}})_{n\in\mathbb{N}}$ is also unbounded in $\mathbb{P}$-measure on $\{\limsup_{n\to\infty} W_\infty^{\pi_n} = +\infty\}$.

Up to now we have shown that $\mathbb{P}[\limsup_{n\to\infty} W_\infty^{\rho_n} = +\infty] > 0$ and we also know that $\{\limsup_{n\to\infty} W_\infty^{\rho_n} = +\infty\} \subseteq \Omega_A$; the only things that remains is to show that the last set inclusion is actually an equality, mod $\mathbb{P}$. To do so, define $\Omega_B := \Omega_A \setminus \{\limsup_{n\to\infty} W_\infty^{\rho_n} = +\infty\}$ and assume that $\mathbb{P}[\Omega_B] > 0$. Working under the conditional measure on Ω_B (denote by $\mathbb{P}_B$), and following the exact same steps we carried out two paragraphs ago, we find predictable processes $(h_n)_{n\in\mathbb{N}}$ such that each $\pi_n := h_n\rho$ is globally X-integrable under $\mathbb{P}_B$ and such that the sequence of terminal values $((\pi_n \cdot X)_\infty)_{n\in\mathbb{N}}$ is unbounded in $\mathbb{P}_B$-probability; then

[20]Readers unfamiliar with this fact should consult a book like [5]: for example, the result that we are mentioning can be seen as a rather direct consequence of Corollary 3.6.10, in page 128 of that book.

$\mathbb{P}_B[\limsup_{n \to \infty} W_\infty^{\rho_n} = +\infty] > 0$, which contradicts the definition of Ω_B and we are done. $\square$

Appendix A. Stochastic Exponentials

If Y is a linear semimartingale, its *stochastic exponential* $\mathcal{E}(Y)$ is defined to be the unique solution Z to the stochastic integral equation $Z = 1 + Z_- \cdot Y$; it is given by the formula

$$\mathcal{E}(Y) = \exp\left\{Y - \frac{1}{2}[Y^{\mathsf{c}}, Y^{\mathsf{c}}]\right\} \prod_{s \leq \cdot} \left\{(1 + \Delta Y_s) \exp(-\Delta Y_s)\right\}, \tag{A.1}$$

where Y^{c} denotes the *continuous martingale part* of the semimartingale Y. The stochastic exponential $Z = \mathcal{E}(Y)$ satisfies $Z > 0$ and $Z_- > 0$ if and only if $\Delta Y > -1$; from (A.1) we have $\log \mathcal{E}(Y) \leq Y$. When we are given a process Z that satisfies $Z > 0$ and $Z_- > 0$ we can invert the stochastic exponential operator and get the *stochastic logarithm* $\mathcal{L}(Z)$, which is defined as the stochastic integral $\mathcal{L}(Z) := (1/Z_-) \cdot Z$. The stochastic logarithm will then satisfy $\Delta \mathcal{L}(Z) > -1$. In other words, we have a one to one correspondence between the class of semimartingales Y that satisfy $\Delta Y > -1$ and the class of semimartingales Z that satisfy $Z > 0$ and $Z_- > 0$ given by the stochastic exponential operator and having the stochastic logarithm as its inverse.

The following result is commonly known as Yor's formula.

Lemma A.1. *If Y and Z are two linear semimartingales, then*

$$\mathcal{E}(Y)\mathcal{E}(Z) = \mathcal{E}(Y + Z + [Y, Z]). \tag{A.2}$$

Proof. Since $\mathcal{E}(Y) = 1 + \mathcal{E}(Y)_- \cdot Y$ and $\mathcal{E}(Z) = 1 + \mathcal{E}(Z)_- \cdot Z$, integration by parts gives

$$\mathcal{E}(Y)\mathcal{E}(Z) = 1 + (\mathcal{E}(Y)_-\mathcal{E}(Z)_-) \cdot Y + (\mathcal{E}(Y)_-\mathcal{E}(Z)_-) \cdot Z + (\mathcal{E}(Y)_-\mathcal{E}(Z)_-) \cdot [Y, Z].$$

Now, collect terms and use the uniqueness of the stochastic exponential. □

Lemma A.2. *If R is a linear semimartingale with $\Delta R > -1$, then we have $\mathcal{E}(R)^{-1} = \mathcal{E}(Z)$, where*

$$Z = -R + [R^{\mathsf{c}}, R^{\mathsf{c}}] + \sum_{s \leq \cdot} \frac{(\Delta R_s)^2}{1 + \Delta R_s}. \tag{A.3}$$

Proof. First, we observe that since the process $\mathcal{E}(R)^{-1}$ is bounded away from zero, its stochastic logarithm exists; besides the process that is a candidate for being its stochastic logarithm is a well-defined semimartingale, since we know that $\sum_{s \leq \cdot} (\Delta R_s)^2 < \infty$. Since $\mathcal{E}(R)\mathcal{E}(Z) = 1$, with the help of Yor's formula (A.2) we get that $\mathcal{E}(0) = 1 = \mathcal{E}(R)\mathcal{E}(Z) = \mathcal{E}(R+Z+[R,Z])$, and the uniqueness of the stochastic exponential implies that $R + Z + [R, Z] = 0$. By splitting this last equation into its continuous and purely discontinuous parts, one can guess and then easily check that it is solved by Z of (A.3). □

As a simple application, here is the solution to the equation (1.5).

Lemma A.3. *If W satisfies the equation (1.5), then the discounted wealth process is given by $\widetilde{W} = w + \theta \cdot \widetilde{S}$.*

Proof. Remember that $S^0 = \mathcal{E}(X^0)$ and consider the semimartingale Z which solves $(S^0)^{-1} = \mathcal{E}(Z)$. The only thing that will be used from the previous lemma is that $X^0 + Z + [X^0, Z] = 0$; we use this fact in the equality of the second line in

$$\begin{aligned} \widetilde{W} = \mathcal{E}(Z)W &= w + (S^0_-)^{-1} \cdot W - ((S^0_-)^{-1} W_-) \cdot Z + (S^0_-)^{-1}[Z, W] \\ &= w + \theta \cdot \left[(S^0)^{-1} \cdot \left(S - S_- \cdot X^0 + [S - S_- \cdot X^0, Z] \right) \right], \end{aligned}$$

and leave the details to the reader. Setting $w = S^i_0$ and $\theta = \mathrm{e}_i$ (the unit vectors) in this equation we get that $(S^0)^{-1} \cdot (S - S_- \cdot X^0 + [S - S_- \cdot X^0, Z]) = \widetilde{S} - S_0$, in which case we can write $\widetilde{W} = w + \theta \cdot \widetilde{S}$. □

The following lemma will help us prove Proposition 3.10; although easy to believe, its proof (at least the one we were able to put together) is slightly tedious.

Lemma A.4. *Let $\mathcal{R}$ be a collection of linear semimartingales such that $R_0 = 0$, $\Delta R > -1$ and $\mathcal{E}(R)^{-1}$ is a (positive) supermartingale for all $R \in \mathcal{R}$ (in particular, $\mathcal{E}(R)_\infty$ exists and takes values in $(0, \infty]$). Then, the collection of processes $\mathcal{R}$ is unbounded in probability (see the remark below) if and only if the collection of positive random variables $\{\mathcal{E}(R)_\infty \mid R \in \mathcal{R}\}$ is unbounded in probability.*

Remark A.5. A class $\mathcal{R}$ of semimartingales will be called "unbounded in probability", if the collection of random variables $\{\sup_{t \in \mathbb{R}_+} |R_t| \mid R \in \mathcal{R}\}$ is unbounded in probability. Similar definitions will apply for (un)boundedness from above and below, taking one-sided suprema. Without further comment, we shall only consider boundedness notions "in probability" through the course of the proof.

Proof. Since $R \geq \log \mathcal{E}(R)$ for all $R \in \mathcal{R}$, one side of the equivalence is trivial, and we only have to prove that if $\mathcal{R}$ is unbounded then $\{\mathcal{E}(R)_\infty \mid R \in \mathcal{R}\}$ is unbounded. We split the proof of this into four steps.

As a first step, observe that since $\{\mathcal{E}(R)^{-1} \mid R \in \mathcal{R}\}$ is a collection of positive supermartingales, it is bounded from above, so that $\{\log \mathcal{E}(R) \mid R \in \mathcal{R}\}$ is bounded from below. Since $R \geq \log \mathcal{E}(R)$ for all $R \in \mathcal{R}$ and $\mathcal{R}$ is unbounded, it follows that it *must* be unbounded from above.

Let us now show that the collection of *random variables* $\{\mathcal{E}(R)_\infty \mid R \in \mathcal{R}\}$ is unbounded if and only if the collection of *semimartingales* $\{\mathcal{E}(R) \mid R \in \mathcal{R}\}$ is unbounded (from above, of course, since they are positive). One direction is trivial: if the semimartingale class is unbounded, the random variable class is unbounded too; we only need show the other direction. Unboundedness of $\{\mathcal{E}(R) \mid R \in \mathcal{R}\}$ means that we can pick an $\epsilon > 0$ so that, for any $n \in \mathbb{N}$, there exists a semimartingale $R^n \in \mathcal{R}$ such that for the stopping times $\tau_n := \inf\{t \in \mathbb{R}_+ \mid \mathcal{E}(R^n)_t \geq n\}$ we have $\mathbb{P}[\tau_n < \infty] \geq \epsilon$. Each $\mathcal{E}(R^n)^{-1}$ is a supermartingale, so

$$\mathbb{P}[\mathcal{E}(R^n)_\infty^{-1} \leq n^{-1/2}] \geq \mathbb{P}[\mathcal{E}(R^n)_\infty^{-1} \leq n^{-1/2} \mid \tau_n < \infty]\, \mathbb{P}[\tau_n < \infty] \geq \epsilon(1 - n^{-1/2}),$$

which shows that the sequence $(\mathcal{E}(R^n)_\infty)_{n\in\mathbb{N}}$ is unbounded and the claim of this paragraph is proved.

What we want to prove now is that if $\mathcal{R}$ is unbounded, then $\{\mathcal{E}(R) \mid R \in \mathcal{R}\}$ is unbounded too. Define the class $\mathcal{Z} := \{\mathcal{L}\left(\mathcal{E}(R)^{-1}\right) \mid R \in \mathcal{R}\}$; we have $Z_0 = 0$, $\Delta Z > -1$ and that Z is a local supermartingale for all $Z \in \mathcal{Z}$.

For our third step, we show that if the collection $\mathcal{Z}$ is bounded from below, then it is also bounded from above. To this end, pick any $\epsilon > 0$. We can find an $M \in \mathbb{R}_+$ such that the stopping times $\tau_Z := \inf\{t \in \mathbb{R}_+ \mid Z_t \leq -M + 1\}$ satisfy $\mathbb{P}[\tau_Z < \infty] \leq \epsilon/2$ for all $Z \in \mathcal{Z}$. Since $\Delta Z > -1$, we have $Z_{\tau_Z} \geq -M$ and so each stopped process Z^{τ_z} is a supermartingale (it is a local supermartingale bounded uniformly from below). Then, with $y_\epsilon := 2M/\epsilon$ we have that $\mathcal{Z}$ is bounded from above as well, because

$$\mathbb{P}[\sup_{t\in\mathbb{R}_+} Z_t > y_\epsilon] \leq \epsilon/2 + \mathbb{P}[\sup_{t\in\mathbb{R}_+} Z_t^{\tau_Z} > y_\epsilon] \leq \epsilon/2 + (1 + y_\epsilon/M)^{-1} \leq \epsilon\,.$$

We have now all the ingredients for the proof. Suppose that $\mathcal{R}$ is unbounded; we discussed that it has to be unbounded from above. According to Lemma A.2, every $Z \in \mathcal{Z}$ is of the form (A.3). When $\mathcal{Z}$ is unbounded from below, things are pretty simple, because $\log \mathcal{E}(Z) \leq Z$ for all $Z \in \mathcal{Z}$ so that $\{\log \mathcal{E}(\mathcal{Z}) \mid Z \in \mathcal{Z}\}$ is unbounded from below and thus $\{\mathcal{E}(R) \mid R \in \mathcal{R}\} = \{\exp(-\log \mathcal{E}(Z)) \mid Z \in \mathcal{Z}\}$ is unbounded from above.

It remains to see what happens if $\mathcal{Z}$ is bounded from below. The third step (two paragraphs ago) of this proof implies that $\mathcal{Z}$ is bounded from above as well. Then, because of equation (A.3) and the unboundedness from above of $\mathcal{R}$, this would mean that the collection $\{\ [R^{\mathsf{c}}, R^{\mathsf{c}}] + \sum_{s \leq \cdot} [(\Delta R_s)^2 / (1 + \Delta R_s)] \mid R \in \mathcal{R}\ \}$ of increasing processes is also unbounded. Now, for $Z \in \mathcal{Z}$ we have

$$\log \mathcal{E}(Z) = -\log \mathcal{E}(R) = -R + \frac{1}{2}[R^{\mathsf{c}}, R^{\mathsf{c}}] + \sum_{s \leq \cdot} [\Delta R - \log(1 + \Delta R)]$$

from (A.3) and the stochastic exponential formula, so that

$$Z - \log \mathcal{E}(Z) = \frac{1}{2}[R^{\mathsf{c}}, R^{\mathsf{c}}] + \sum_{s \leq \cdot} \left[\log(1 + \Delta R_s) - \frac{\Delta R_s}{1 + \Delta R_s} \right].$$

Since the collection $\{\ [R^{\mathsf{c}}, R^{\mathsf{c}}] + \sum_{s \leq \cdot} [(\Delta R_s)^2 / (1 + \Delta R_s)] \mid R \in \mathcal{R}\ \}$ is unbounded (as we discussed), it follows that the collection of increasing processes on the right-hand-side of the last equation is unbounded too. Since $\mathcal{Z}$ is bounded, this means that $\{\log \mathcal{E}(\mathcal{Z}) \mid Z \in \mathcal{Z}\}$ is unbounded from below, and we conclude again as before. □

Remark A.6. Without the assumption that $\{\mathcal{E}(R)^{-1} \mid R \in \mathcal{R}\}$ consists of supermartingales, this lemma is not longer true. In fact, take $\mathcal{R}$ to have only one element R with $R_t = at + \beta_t$, where $a \in (0, 1/2)$ and β is a standard linear Brownian motion. Then, R is bounded from below and unbounded from above, nevertheless $\log \mathcal{E}(R)_t = (a - 1/2)t + \beta_t$ is bounded from above, and unbounded from below.

The following result is the abstract version of Proposition 2.18.

Lemma A.7. *Let X be a local supermartingale with $\Delta X > -1$ and Doob-Meyer decomposition $X = M - A$, where A is an increasing, predictable process. With $\hat{C} := [X^{\mathsf{c}}, X^{\mathsf{c}}]$ being the quadratic covariation of the continuous local martingale part of X and $\hat{\eta}$ the predictable compensator of the jump measure $\hat{\mu}$, define the increasing, predictable process $\Lambda := A + \hat{C}/2 + h(1+x) * \hat{\eta}$, where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is the convex function defined for some $a \in (0,1)$ as*

$$h(y) := \left[-\log a + (1 - a^{-1})y\right] \mathbf{1}_{[0,a)}(y) + [y - 1 - \log y]\, \mathbf{1}_{[a,+\infty)}(y),$$

. Consider also the positive supermartingale $Y = \mathcal{E}(X)$. Then,

$$\begin{aligned} &\text{on } \{\Lambda_\infty < +\infty\}, \qquad \lim_{t\to\infty} Y_t \in (0, +\infty)\,; \\ &\text{on } \{\Lambda_\infty = +\infty\}, \qquad \limsup_{t\to\infty} \frac{\log Y_t}{\Lambda_t} \le -1\,. \end{aligned}$$

Remark A.8. In the course of the proof, we shall make heavy use of the following fact: for a locally square integrable martingale N with angle-bracket process $\langle N, N\rangle$, on the event $\{\langle N, N\rangle_\infty < +\infty\}$ the limit N_∞ exists and is finite, while on the event $\{\langle N, N\rangle_\infty = +\infty\}$ we have $\lim_{t\to\infty}(N_t / \langle N, N\rangle_t) = 0$.

Note also that if $N = v(x) * (\hat{\mu} - \hat{\eta})$, then $\langle N, N\rangle \le v(x)^2 * \hat{\eta}$ (equality holds if and only if N is quasi-left-continuous). Combining this with the previous remarks we get that on $\{(v(x)^2 * \hat{\eta})_\infty < +\infty\}$ the limit N_∞ exists and is finite, while on $\{(v(x)^2 * \hat{\eta})_\infty = +\infty\}$ we have $\lim_{t\to\infty}[\,N_t/(v(x)^2 * \hat{\eta})_t\,] = 0\,.$

Proof. For the supermartingale $Y = \mathcal{E}(X)$, the stochastic exponential formula (A.1) gives $\log Y = X - [X^{\mathsf{c}}, X^{\mathsf{c}}]/2 - \sum_{s \leq \cdot} [\Delta X_s - \log(1 + \Delta X_s)]$, or equivalently that

$$\text{(A.4)} \qquad \log Y = (M^{\mathsf{c}} - \hat{C}/2) - A + \big(x * (\hat{\mu} - \hat{\eta}) - [x - \log(1+x)] * \hat{\mu} \big).$$

Let us start with the continuous local martingale part. We use Remark A.8 twice: first, on the event $\{\hat{C}_\infty < +\infty\}$, M^{c}_∞ exists and is real-valued; secondly, on the event $\{\hat{C}_\infty = +\infty\}$, we have $\lim_{t \to \infty} (M^{\mathsf{c}}_t - \hat{C}_t/2)/(\hat{C}_t/2) = -1$.

Now we deal with the purely discontinuous local martingale part. Let us first define the two indicator functions $l := \mathbf{1}_{[-1,-1+a)}$ and $r := \mathbf{1}_{[-1+a,+\infty)}$, where l and r stand for mnemonics for *l*eft and *r*ight. Define the three semimartingales

$$\begin{aligned} E &:= [l(x) \log(1+x)] * \hat{\mu} - [l(x) x] * \hat{\eta}, \\ F &:= [r(x) \log(1+x)] * (\hat{\mu} - \hat{\eta}) + [r(x) h(1+x)] * \hat{\eta}, \\ D &:= x * (\hat{\mu} - \hat{\eta}) - [x - \log(1+x)] * \hat{\mu} \ = \ E \ + \ F. \end{aligned}$$

We claim that on $\{(h(1+x) * \hat{\eta})_\infty < +\infty\}$, both E_∞ and F_∞ exist and are real-valued. For E, this happens because $([l(x) h(1+x)] * \hat{\eta})_\infty < +\infty$ implies that there will only be a finite number of times when $\Delta X \in (-1, -1+a]$, so that both terms in the definition of E will have a limit at infinity. Turning to F, the second term in its definition is obviously finite-valued at infinity, while for the local martingale term $[r(x) \log(1+x)] * (\hat{\mu} - \hat{\eta})$ we need only observe that it has finite predictable quadratic variation (because of the set inclusion $\{([r(x) h(1+x)] * \hat{\eta})_\infty < +\infty\} \subseteq \big\{ \big([r(x) \log^2(1+x)] * \hat{\eta} \big)_\infty < +\infty \big\}$), then use Remark A.8.

Now we turn attention to the event $\{(h(1+x) * \hat{\eta})_\infty = +\infty\}$, on which at least one of $([l(x) h(1+x)] * \hat{\eta})_\infty$ and $([r(x) h(1+x)] * \hat{\eta})_\infty$ must be infinite. We deal with the event $\{([r(x)h(1+x)] * \hat{\eta})_\infty = \infty\}$ first; there, from the definition of F and Remark A.8, it is easy to see $\lim_{t\to\infty} F_t / ([r(x)h(1+x)] * \hat{\eta})_t = -1$.

Let us work next on the event $\{([l(x)h(1+x)] * \hat{\eta})_\infty = \infty\}$. We know that the inequality $\log y \leq y - 1 - h(y)$ holds for $y > 0$; using this last inequality in the first term of the definition of E, we obtain the comparison

$$E \leq [l(x)(x - h(1+x))] * (\hat{\mu} - \hat{\eta}) - [l(x)h(1+x)] * \hat{\eta}$$

which, along with Remark A.8, gives $\limsup_{t\to\infty} E_t / ([l(x)h(1+x)] * \hat{\eta})_t \leq -1$.

Let us summarize the last paragraphs for the purely discontinuous part. On $\{(h(1+x) * \hat{\eta})_\infty < +\infty\}$, the limit D_∞ exists and is finite; and on the event $\{(h(1+x) * \hat{\eta})_\infty = +\infty\}$, we have: $\limsup_{t\to\infty} \left(D_t / \left(h(1+x) * \hat{\eta}\right)_t\right) \leq -1$.

From the previous discussion on the continuous and the purely discontinuous local martingale parts of $\log Y$ and the definition of Λ, the result follows. □

Remark A.9. In the previous proposition, if we make the additional assumption $\Delta X \geq -1 + \varepsilon$ for some $\varepsilon > 0$, then by considering simply $h(y) = y - 1 - \log y$ in the definition of Λ, we get $\lim_{t\to\infty} \left(\log Y_t / \Lambda_t\right) = -1$ on the set $\{\Lambda_\infty = +\infty\}$, i.e., the exact speed of convergence of $\log Y$ to $-\infty$.

Appendix B. **Measurable Random Subsets**

Throughout this section we shall be working on a measurable space $(\tilde{\Omega}, \mathcal{P})$; although the results are general, for us $\tilde{\Omega}$ will be the base space $\Omega \times \mathbb{R}_+$ and $\mathcal{P}$ the *predictable* σ-algebra. The metric of the Euclidean space $\mathbb{R}^d$, its denoted by "dist" and its generic point by z. We shall be following Chapter 17 of [2] to which we send the reader for more information about this subject, treated there in much more generality.

A *random subset of* $\mathbb{R}^d$ is just a random variable taking values in $2^{\mathbb{R}^d}$, the powerset (class of all subsets) of $\mathbb{R}^d$. Thus, a random subset of $\mathbb{R}^d$ is a function $\mathfrak{A} : \tilde{\Omega} \mapsto 2^{\mathbb{R}^d}$. A random subset $\mathfrak{A}$ of $\mathbb{R}^d$ will be called *closed* (resp., *convex*) if the set $\mathfrak{A}(\tilde{\omega})$ is closed (resp., convex) for *every* $\tilde{\omega} \in \tilde{\Omega}$.

We have to require some measurability requirement for these types of processes, and thus we must ask for some measurable structure on the space $2^{\mathbb{R}^d}$. Thus, we endow $2^{\mathbb{R}^d}$ with the smallest σ-algebra that makes all the functions

$$2^{\mathbb{R}^d} \ni A \mapsto \mathrm{dist}(z, A) \in \mathbb{R}_+ \cup \{+\infty\}$$

(by way of definition, $\mathrm{dist}(z, \emptyset) = +\infty$) measurable for all $z \in \mathbb{R}^d$. This definition is very elegant, but to make some use of it we have to establish some equivalent formulations. The following result should give the reader more insight.

Proposition B.1. *The constructed σ-algebra on $2^{\mathbb{R}^d}$ is also the smallest σ-algebra that makes any of the following three classes of functions measurable.*

(1) $2^{\mathbb{R}^d} \in A \mapsto \mathbf{1}_{\{A \cap K \neq \emptyset\}}$, *for every compact* $K \subseteq \mathbb{R}^d$
(2) $2^{\mathbb{R}^d} \in A \mapsto \mathbf{1}_{\{A \cap F \neq \emptyset\}}$, *for every closed* $F \subseteq \mathbb{R}^d$
(3) $2^{\mathbb{R}^d} \in A \mapsto \mathbf{1}_{\{A \cap G \neq \emptyset\}}$, *for every open* $G \subseteq \mathbb{R}^d$

Proof. We first show that each $2^{\mathbb{R}^d} \in A \mapsto \mathbf{1}_{\{A \cap K \neq \emptyset\}}$, for compact $K \subseteq \mathbb{R}^d$ is measurable with respect to the σ-algebra on $2^{\mathbb{R}^d}$. Pick a countable dense subset D of K (this will be a recurring theme in the proofs to follow). We have that

$$\{A \cap K \neq \emptyset\} \Longleftrightarrow \min_{z \in K} \mathrm{dist}(z, A) = 0 \Longleftrightarrow \inf_{z \in D} \mathrm{dist}(z, A) = 0,$$

and the claim follows (for the second equivalence we *must* require compactness).

Now, pick a closed $F \in 2^{\mathbb{R}^d}$ and let $(K_n)_{n \in \mathbb{N}}$ be the sequence of compact sets increasing to $\mathbb{R}^d$. Then, we have $\{A \cap F \neq \emptyset\} = \bigcup_{n \in \mathbb{N}} \{A \cap (F \cap K_n) \neq \emptyset\}$ and each $F \cap K_n$ is compact; thus any function of class (2) is measurable if every function of class (1) is measurable.

If G is an open set in $\mathbb{R}^d$, pick a sequence F_n of closed sets with $F_n \uparrow G$; then $\{A \cap G \neq \emptyset\} = \bigcup_{n \in \mathbb{N}} \{A \cap F_n \neq \emptyset\}$, so every function of class (3) is measurable if every function of class (2) is.

Finally, observe that for any $z \in \mathbb{R}^d$ and any $a > 0$ we have that $\mathrm{dist}(z, A) < a$ if and only if $\left\{y \in \mathbb{R}^d \mid \mathrm{dist}(y, z) < r\right\} \cap A \neq \emptyset$ for some rational $0 \leq r < a$. Thus, the functions $2^{\mathbb{R}^d} \in A \mapsto \mathrm{dist}(z, A)$ are measurable if all functions of class (3) are. □

A *Carathéodory function* is a mapping from $\tilde{\Omega} \times \mathbb{R}^d$ in some other topological space (that is also a measurable space with its Borel σ-algebra) that is measurable with respect to the first argument (keeping the second fixed) and continuous with respect to the second (keeping the first fixed). From the definition of the σ-algebra on $2^{\mathbb{R}^d}$, the random subset $\mathfrak{A}$ of $\mathbb{R}^d$ is measurable if and only if the function

$$\tilde{\Omega} \times \mathbb{R}^d \ni (\tilde{\omega}, z) \mapsto \mathrm{dist}(z, \mathfrak{A}(\tilde{\omega})) \in \mathbb{R}_+ \cup \{+\infty\}$$

is Carathéodory (continuity in z is evident from the triangle inequality).

By Proposition B.1, a random subset $\mathfrak{A}$ of $\mathbb{R}^d$ is measurable if for any compact $K \subseteq \mathbb{R}^d$, the set $\{\mathfrak{A} \cap K \neq \emptyset\} := \left\{\tilde{\omega} \in \tilde{\Omega} \mid \mathfrak{A}(\tilde{\omega}) \cap K \neq \emptyset\right\}$ is $\mathcal{P}$-measurable.

Remark B.2. Suppose that random subset $\mathfrak{A}$ is a *singleton* $\mathfrak{A}(\tilde{\omega}) = \{a(\tilde{\omega})\}$ for some $a : \tilde{\Omega} \mapsto \mathbb{R}^d$. Then, $\mathfrak{A}$ is measurable if and only if $\{a \in K\} \in \mathcal{P}$ for all closed $K \subseteq \mathbb{R}^d$, i.e., if and only if a is $\mathcal{P}$-measurable.

We now deal with unions and intersections of random subsets of $\mathbb{R}^d$.

Lemma B.3. *Suppose that $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ is a sequence of measurable random subsets of $\mathbb{R}^d$. Then, the union random subset $\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$ is also measurable. If furthermore each $\mathfrak{A}_n$ is closed, the intersection random subset $\bigcap_{n\in\mathbb{N}} \mathfrak{A}_n$ is measurable too.*

Proof. Let $\mathfrak{A} := \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$. For any subset $G \subseteq \mathbb{R}^d$ we have $\{\mathfrak{A} \cap G \neq \emptyset\} = \bigcup_{n\in\mathbb{N}} \{\mathfrak{A}_n \cap G \neq \emptyset\}$, so the first assertion follows,

For the second, call $\mathfrak{C} := \bigcap_{n\in\mathbb{N}} \mathfrak{A}_n$. We can assume that $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ is a decreasing sequence; otherwise, replace $\mathfrak{A}_n$ them with $\mathfrak{A}'_n := \bigcap_{1\leq k\leq n} \mathfrak{A}_k$; each $\mathfrak{A}'_n$ is measurable: for compact $K \subseteq \mathbb{R}^d$ and D a dense countable subset of K we have

$$\{\mathfrak{A}'_n \cap K \neq \emptyset\} = \left\{\min_{z\in K} \max_{1\leq k\leq n} \mathrm{dist}(z, \mathfrak{A}_k) = 0\right\} = \left\{\inf_{z\in D} \max_{1\leq k\leq n} \mathrm{dist}(z, \mathfrak{A}_k) = 0\right\},$$

which is an element of $\mathcal{P}$. Now, since K is a compact set we have $\{\mathfrak{C} \cap K \neq \emptyset\} = \bigcap_{n\in\mathbb{N}} \{\mathfrak{A}'_n \cap K \neq \emptyset\} \in \mathcal{P}$. □

Lemma B.4. *Let $\mathfrak{C}$ be a closed and convex measurable random set. If $x \in \mathbb{R}^d$, then the random element* p*, defined for $\tilde{\omega} \in \tilde{\Omega}$ as the (unique) projection of x onto $\mathfrak{C}(\tilde{\omega})$ is $\mathcal{P}$-measurable.*

Proof. Let $\mathfrak{A} := \{z \in \mathbb{R}^d \mid \operatorname{dist}(x,z) \leq \operatorname{dist}(x,\mathfrak{C})\}$. Since for every compact K we have $\{\mathfrak{A} \cap K \neq \emptyset\} = \{\operatorname{dist}(x,K) \leq \operatorname{dist}(x,\mathfrak{C})\} \in \mathcal{P}$, $\mathfrak{A}$ is a measurable random subset. Then, $\mathfrak{C} \cap \mathfrak{A} = \{\mathrm{p}\}$ is also measurable and we conclude in view of Remark B.2. $\square$

The following lemma gives a way to construct measurable, closed random subsets of $\mathbb{R}^d$. To state it, we shall need a slight generalization of the notion of a Carathéodory function. So, for a measurable closed random subset $\mathfrak{A}$ of $\mathbb{R}^d$, a function f that maps $\tilde{\Omega} \times \mathbb{R}^d$ to another topological space will be called *Carathéodory on* $\mathfrak{A}$, if it is measurable (with respect to the product σ-algebra on $\tilde{\Omega} \times \mathbb{R}^d$) and if $f(\tilde{\omega}, z)$ is a continuous function of $z \in \mathcal{A}(\tilde{\omega})$, for all $\tilde{\omega} \in \tilde{\Omega}$. Of course, if $\mathfrak{A} \equiv \mathbb{R}^d$, we recover the usual notion of a Carathéodory function.

Lemma B.5. *Let E be any topological space, $F \subseteq E$ a closed subset, and $\mathfrak{A}$ a closed and convex random subset of $\mathbb{R}^d$. If $f : \tilde{\Omega} \times \mathbb{R}^d \to E$ is a Carathéodory function on $\mathfrak{A}$, then $\mathfrak{C} := \{z \in \mathfrak{A} \mid f(\cdot, z) \in F\}$ is closed and measurable.*

Proof. The fact that $\mathfrak{C}$ is closed is obvious, since f is Carathéodory on $\mathfrak{A}$. Now, pick any compact $K \subseteq \mathbb{R}^d$; we wish to show that $\{\mathfrak{C} \cap K \neq \emptyset\} \in \mathcal{P}$.

First observe that one can find a sequence of $\mathcal{P}$-measurable, $\mathbb{R}^d$-valued processes $(\mathrm{p}_n)_{n \in \mathbb{N}}$ such that, for every $\tilde{\omega} \in \tilde{\Omega}$, the countable set $\{\mathrm{p}_1(\tilde{\omega}), \mathrm{p}_2(\tilde{\omega}), \ldots\}$ is dense in $\mathfrak{A}(\tilde{\omega}) \cap K$. Indeed, if D is a dense countable subset of K enumerated as $D = \{x_1, x_2, \ldots\}$, then the processes p_n, defined as the projections of x_n on $\mathfrak{A}$ are $\mathcal{F}$-measurable by Lemma B.4 and they are obviously dense in $\mathfrak{A} \cap K$. Now we can write $\{\mathfrak{C} \cap K \neq \emptyset\} = \{\inf_{n \in \mathbb{N}} \operatorname{dist}_E(f(\cdot, \mathrm{p}_n), F) = 0\}$, which is $\mathcal{P}$-measurable because each function $\tilde{\Omega} \in z \mapsto \operatorname{dist}_E(f(\cdot, \mathrm{p}_n), F) \in \mathbb{R}_+$ is. $\square$

Corollary B.6. *The set-valued process* $\mathfrak{C}_0 = \left\{ \mathrm{p} \in \mathbb{R}^d \mid \nu \left[\mathrm{p}^\top x < -1 \right] = 0 \right\}$ *of natural constraints in (1.7) is predictable.*

Proof. Just write $\mathfrak{C}_0 = \left\{ \mathrm{p} \in \mathbb{R}^d \mid \int \phi(1 + \mathrm{p}^\top x) \nu(\mathrm{d}x) \right\}$, where we have set $\phi(x) := (x^-)^2 / (1 + (x^-)^2)$, and use Lemma B.5; here one has to also remember Remark 1.3 on a nice version of the predictable characteristics. □

The last result focuses on the measurability of the "argument" process in random optimization problems.

Theorem B.7. *Suppose that* $\mathfrak{C}$ *is a closed and convex, measurable, non-empty random subset of* $\mathbb{R}^d$ *and* $f : \tilde{\Omega} \times \mathbb{R}^d \mapsto \mathbb{R} \cup \{-\infty\}$ *is a Carathéodory function on* $\mathfrak{C}$. *Consider the optimization problem* $f_*(\tilde{\omega}) = \max_{z \in \mathfrak{C}} f(\tilde{\omega}, z)$. *We have that:*

(1) *The value function* f_* *is* $\mathcal{P}$*-measurable.*
(2) *Suppose that for all* $\tilde{\omega}$, $f_*(\tilde{\omega})$ *is finite and there exists a unique* $z_*(\tilde{\omega}) \in \mathfrak{C}(\tilde{\omega})$ *satisfying* $f(\tilde{\omega}, z_*(\tilde{\omega})) = f_*(\tilde{\omega})$. *Then,* z_* *is* $\mathcal{P}$*-measurable.*

In particular, if $\mathfrak{C}$ *is a closed and convex, measurable, non-empty random subset of* $\mathbb{R}^d$, *we can find a* $\mathcal{P}$*-measurable* $h : \tilde{\Omega} \to \mathbb{R}^d$ *with* $h(\tilde{\omega}) \in \mathfrak{C}(\tilde{\omega})$ *for all* $\tilde{\omega} \in \tilde{\Omega}$.

Proof. For conclusion (1), just as in the proof of Lemma B.5, pick a dense, countable subset D of $\mathbb{R}^d$, and find a sequence of $\mathcal{P}$-measurable, $\mathbb{R}^d$-valued processes $(\mathrm{p}_n)_{n \in \mathbb{N}}$, such that for every $\tilde{\omega} \in \tilde{\Omega}$ the countable random subset $\{\mathrm{p}_1(\tilde{\omega}), \mathrm{p}_2(\tilde{\omega}), \ldots\}$ is dense in $\mathfrak{C}$. Then, $f_*(\tilde{\omega}) = \sup_{n \in \mathbb{N}} f(\tilde{\omega}, \mathrm{p}_n(\tilde{\omega}))$ is $\mathcal{P}$-measurable.

Now, for conclusion (2), Lemma B.5 gives us that the random subset of $\mathbb{R}^d$ defined as $\{z \in \mathfrak{C} \mid f(\tilde{\omega}, z) = f_*(\tilde{\omega})\}$ is $\mathcal{P}$-measurable; but this is exactly the singleton $\{z_*(\tilde{\omega})\}$, and once more Remark B.2 gives us that z_* is an $\mathcal{P}$-measurable process.

For the last statement, one can use for example the function $f(x) = -\|x\|$ and the result already obtained. □

In case the maximizer is not unique, one can still measurably select from the set of maximizers. This result is more difficult; in any case we shall not be using it.

Appendix C. Semimartingales up to Infinity and Global Stochastic Integration

We recall here a few important concepts from [7] and prove a few useful results.

Definition C.1. Let $X = (X_t)_{t \in \mathbb{R}_+}$ be a semimartingale, and assume that $X_\infty := \lim_{t \to \infty} X_t$ exists. Then X will be called a *semimartingale up to infinity* if the process $\tilde{X}$ defined on the time interval $[0,1]$ by $\tilde{X}_t = X_{\frac{t}{1-t}}$ (of course, $\tilde{X}_1 = X_\infty$) is a semimartingale relative to the filtration $\tilde{\mathbf{F}} = (\tilde{\mathcal{F}}_t)_{t \in [0,1]}$ defined by

$$\tilde{\mathcal{F}}_t := \begin{cases} \mathcal{F}_{\frac{t}{1-t}}, & \text{for } 0 \leq t < 1; \\ \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_t, & \text{for } t = 1. \end{cases}$$

We define similarly local martingales up to infinity, processes of finite variation up to infinity, etc., if the corresponding process $\tilde{X}$ has the property.

Until the end of this subsection, a "tilde" over a process, means that we are considering the process of the previous definition, with the new filtration $\tilde{\mathbf{F}}$.

To appreciate the difference between the concepts of (plain) semimartingale and semimartingale up to infinity, consider the simple example where X is the deterministic, continuous process $X_t := t^{-1} \sin t$; it is obvious that X is a semimartingale and that $X_\infty = 0$, but $\mathrm{Var}(X)_\infty = +\infty$ and thus X cannot be a semimartingale up to infinity (recall that a deterministic semimartingale must be of finite variation).

Every semimartingale up to infinity X can be written as the sum $X = A + M$, where A is a process of finite variation up to infinity (which simply means that $\mathrm{Var}(A)_\infty < \infty$) and M is a local martingale up to infinity (which means that there exists an increasing sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ with $\{\tau_n = +\infty\} \uparrow \Omega$ such that each of the stopped processes M^{τ_n} is a uniformly integrable martingale).

Here are examples of semimartingales up to infinity.

Lemma C.2. *If Z is a positive supermartingale, then it is a special semimartingale up to infinity. If furthermore $Z_\infty > 0$, then $\mathcal{L}(Z)$ is also a special semimartingale up to infinity, and both processes Z^{-1} and $\mathcal{L}(Z^{-1})$ are semimartingales up to infinity.*

Proof. We start with the Doob-Meyer decomposition $Z = M - A$, where M is a local martingale with $M_0 = Z_0$ and A is an increasing, predictable process. Since M is a positive local martingale, it is a supermartingale too, and we can infer that both limits Z_∞ and M_∞ exist and are integrable. This means that A_∞ exists and actually $\mathbb{E}[A_\infty] = \mathbb{E}[M_\infty] - \mathbb{E}[Z_\infty] < \infty$, so A is a predictable process of integrable variation up to infinity. It remains to show that M is a local martingale up to infinity. Set $\tau_n := \inf\{t \geq 0 \mid M_t \geq n\}$; this obviously satisfies $\{\tau_n = +\infty\} \uparrow \Omega$ (the supremum of a positive supermartingale is finite). Since $\sup_{0 \leq t \leq \tau_n} M_t \leq n + M_{\tau_n} \mathbf{1}_{\{\tau_n < \infty\}}$ and by the optional sampling theorem $\mathbb{E}[M_{\tau_n} \mathbf{1}_{\{\tau_n < \infty\}}] \leq \mathbb{E}[M_0] < \infty$, we get $\mathbb{E}[\sup_{0 \leq t \leq \tau_n} M_t] < \infty$. Thus, the local martingale M^{τ_n} is actually a uniformly integrable martingale and thus Z is a special semimartingale up to infinity.

Now assume that $Z_\infty > 0$. Since Z is a supermartingale, this will mean that both $\tilde{Z}$ and $\tilde{Z}_-$ are bounded away from zero. Since $\tilde{Z}_-^{-1}$ is locally bounded and $\tilde{Z}$ is a special semimartingale, $\mathcal{L}(\tilde{Z}) = \tilde{Z}_-^{-1} \cdot \tilde{Z}$ will be a special semimartingale as well, meaning that $\mathcal{L}(Z)$ is a special semimartingale up to infinity. Furthermore, Itô's formula applied to the inverse function $(0, \infty) \ni x \mapsto x^{-1}$ implies that $\tilde{Z}^{-1}$

is a semimartingale up to infinity and since $\tilde{Z}_-$ is locally bounded, $\mathcal{L}(\tilde{Z}^{-1}) = \tilde{Z}_- \cdot \tilde{Z}^{-1}$ is a semimartingale, which finishes the proof. □

Consider a d-dimensional semimartingale X. A predictable process H will be called *globally X-integrable* if it is X-integrable and the semimartingale $H \cdot X$ is a semimartingale up to infinity. The following result is proved in [7].

Theorem C.3. *Let X be a d-dimensional semimartingale with triplet (b, c, ν) with respect to the canonical truncation function and the operational clock G. A predictable process ρ is globally X-integrable, if and only if the predictable processes below are globally G-integrable:*

$$\psi_1^\rho := \rho^\top c \rho, \quad \psi_2^\rho := \int \left(1 \wedge \left(\rho^\top x\right)^2\right) \nu(\mathrm{d}x), \quad \textit{and}$$

$$\psi_3^\rho := \rho^\top b + \int \rho^\top x \left(\mathbf{1}_{\{\|x\|>1\}} - \mathbf{1}_{\{|\rho^\top x|>1\}}\right) \nu(\mathrm{d}x)$$

The process ψ_1^ρ controls the quadratic variation of the continuous martingale part of $\rho \cdot X$, whereas ψ_2^ρ controls the quadratic variation of the "small-jump" purely discontinuous martingale part of $\rho \cdot X$ and the intensity of the "large jumps". Also, ψ_3^ρ controls the drift term of $\rho \cdot X$ when the large jumps are subtracted; it is actually the drift rate of the bounded-jump part (that corresponds to "b" in the triplet of characteristics) in the canonical decomposition of $\rho \cdot X$.

This theorem is very general; we use it to prove Lemma 6.1, which gives a simple necessary and sufficient condition for global integrability of the candidate for the numéraire portfolio. The reader should contrast Theorem C.3 with Lemma 6.1, where one does not have to worry about the large negative jumps of $\rho \cdot X$, about the quadratic variation of its continuous martingale part, or about the quadratic variation of its small-jump purely discontinuous parts. This might look surprising, but follows because in Lemma 6.1 we assume $\nu[\rho^\top x \leq -1] = 0$ and $\mathfrak{rel}(0 \mid \rho) \leq 0$: there are not many negative jumps (none above unit magnitude), and the drift dominates the quadratic variation.

Appendix D. σ-Localization

A very good account of the concept of σ-localization is given in the paper [19] by Kallsen. Here we recall briefly what is needed.

For a semimartingale Z and a predictable set Σ, define $Z^{\Sigma} := Z_0 \mathbf{1}_{\Sigma}(0) + \mathbf{1}_{\Sigma} \cdot Z$.

Definition D.1. Let $\mathcal{Z}$ be a class of semimartingales. Then, the corresponding σ-localized class $\mathcal{Z}_{\sigma}$ is defined as the set of all semimartingales Z for which there exists an increasing sequence $(\Sigma_n)_{n \in \mathbb{N}}$ of predictable sets, such that $\Sigma_n \uparrow \Omega \times \mathbb{R}_+$ (up to an evanescent set) and $Z^{\Sigma_n} \in \mathcal{Z}$ for all $n \in \mathbb{N}$.

If the corresponding class $\mathcal{Z}$ has a name (like "supermartingales") we baptize the class $\mathcal{Z}_{\sigma}$ with the same name preceded by "σ-" (like "σ-supermartingales").

The concept of σ-localization is a natural extension of the well-known concept of localization along a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times, as one can easily see by considering the predictable sets $\Sigma_n \equiv [\![0, \tau_n]\!] := \{(\omega, t) \mid 0 \leq t \leq \tau_n(\omega) < \infty\}$.

Let us define the set $\mathcal{U}$ of semimartingales Z, such that the collection of random variables $\{Z_{\tau} \mid \tau \text{ is a stopping time}\}$ is uniformly integrable — also known in the literature as semimartingales of class (D). The elements of $\mathcal{U}$ admit the *Doob-Meyer decomposition* $Z = A + M$ into a predictable finite variation part A with $A_0 = 0$ and $\mathbb{E}[\mathrm{Var}(A)_{\infty}] < \infty$ and a uniformly integrable martingale M. It is then obvious that the localized class $\mathcal{U}_{\mathsf{loc}}$ corresponds to all special semimartingales; they are exactly the ones which admit a Doob-Meyer decomposition as before, but where now A is only a predictable, finite variation process with $A_0 = 0$ and M a local martingale. Let us remark that the local supermartingales (resp., local submartingales) correspond to these elements of $\mathcal{U}_{\mathsf{loc}}$ with A decreasing (resp., increasing). This last result can be found for example in [16].

One can have very intuitive interpretation of some σ-localized classes in terms of the predictable characteristics of Z.

Proposition D.2. *Consider a linear semimartingale Z, and let (b, c, ν) be the triplet of predictable characteristics of Z relative to the canonical truncation function and the operational clock G. Then,*

(1) *Z belongs to $\mathcal{U}_{\mathsf{loc}}$ if and only if the predictable process $\int |x| \mathbf{1}_{\{|x|>1\}} \nu(\mathrm{d}x)$ is G-integrable.*
(2) *Z belongs to $\mathcal{U}_\sigma$ if and only if $\int |x| \mathbf{1}_{\{|x|>1\}} \nu(\mathrm{d}x) < \infty$.*
(3) *Z is a σ-supermartingale if and only if $\int |x| \mathbf{1}_{\{|x|>1\}} \nu(\mathrm{d}x) < \infty$ and $b + \int x \mathbf{1}_{\{|x|>1\}} \nu(\mathrm{d}x) \leq 0$.*

Proof. The first statement follows from the fact that a linear semimartingale Z is a special semimartingale (i.e., a member of $\mathcal{U}_{\mathsf{loc}}$) if and only if $\left[|x| \mathbf{1}_{\{|x|>1\}}\right] * \hat{\eta}$ is a finite, increasing predictable process (one can consult [16] for this fact). The second statement follows easily from the first and σ-localization. Finally, the third follows for the fact that for a process in $\mathcal{U}_{\mathsf{loc}}$ the predictable finite variation part is given by the process $\left(b + \int \left[x \mathbf{1}_{\{|x|>1\}}\right] \nu(\mathrm{d}x)\right) \cdot G$ and using the last remark before the proposition, the first part of the proposition, and σ-localization. □

Results like the last proposition are very intuitive, because $b + \int x \mathbf{1}_{\{|x|>1\}} \nu(\mathrm{d}x)$ represents the infinitesimal drift rate (with respect to G) of the semimartingale Z; we expect this rate to be negative (resp., positive) in the case of σ-supermartingales (resp., σ-submartingales). The importance of σ-localization is that it allows us to talk directly about drift *rates* of processes, rather than about drifts. Sometimes drift rates exist, but cannot be integrated to give a drift process; this is when the usual localization technique fails, and the concept of σ-localization becomes useful.

What follows is a result giving sufficient conditions about the local supermartingale (or even plain supermartingale) property of a semimartingale, when all that is known is that it is a σ-supermartingale.

Proposition D.3. *Suppose that Z is a linear semimartingale with triplet of characteristics (b, c, ν) relative to the canonical truncation function and the operational clock G.*

(1) *Suppose that Z is a σ-supermartingale. Then, the following are equivalent:*
   (a) *Z is a local supermartingale.*
   (b) *The positive, predictable process $\int (-x)\mathbf{1}_{\{x<-1\}}\nu(\mathrm{d}x)$ is G-integrable.*

(2) *If Z is a σ-supermartingale (resp., σ-martingale) that is bounded from below by a constant, then it is a local supermartingale (resp., local martingale). If furthermore $\mathbb{E}[Z_0^+] < \infty$, it is a supermartingale.*

(3) *If Z is bounded from below by a constant, then it is a supermartingale if and only if $\mathbb{E}[Z_0^+] < \infty$ and $b + \int x\mathbf{1}_{\{|x|>1\}}\nu(\mathrm{d}x) \leq 0$.*

Proof. For the proof of (1), the implication (a) $\Rightarrow$ (b) follows from part (1) of Proposition D.2. For (b) $\Rightarrow$ (a), assume that $\int (-x)\mathbf{1}_{\{x<-1\}}\nu(\mathrm{d}x)$ is G-integrable. Since Z is a σ-supermartingale, it follows from part (3) of Proposition D.2 that $\int x\mathbf{1}_{\{x>1\}}\nu(\mathrm{d}x) \leq -b + \int (-x)\mathbf{1}_{\{x<-1\}}\nu(\mathrm{d}x)$. Now, this last inequality implies that $\int |x|\, \mathbf{1}_{\{|x|>1\}}\nu(\mathrm{d}x) \leq -b + 2\int (-x)\mathbf{1}_{\{x<-1\}}\nu(\mathrm{d}x)$; the last dominating process is G-integrable, thus $Z \in \mathcal{U}_{\mathsf{loc}}$ (again, part (1) of Proposition D.2). The special semimartingale Z has predictable finite variation part equal to $\left(b + \int x\mathbf{1}_{\{x>1\}}\nu(\mathrm{d}x)\right) \cdot G$, which is decreasing, so that Z is a local supermartingale.

For part (2), we can of course we can assume that Z is positive. We discuss the case of a σ-supermartingale; the σ-martingale case follows in the same way. According to part (1) of this proposition, we only need to show that $\int(-x)\mathbf{1}_{\{x<-1\}}\nu(\mathrm{d}x)$ is G-integrable. Since the negative jumps of Z are bounded in magnitude by Z_-, we have that $\int(-x)\mathbf{1}_{\{x<-1\}}\nu(\mathrm{d}x) \leq (Z_-)\nu\left[x < -1\right]$, which is G-integrable, because $\nu\left[x < -1\right]$ is G-integrable and Z_- is locally bounded. Now, if we further assume that $\mathbb{E}[Z_0] < \infty$, Fatou's lemma for conditional expectations gives us that the positive local supermartingale Z is a supermartingale.

Let us move on to part (3) and assume that Z is positive. First assume that Z is a supermartingale. Then, of course we have $\mathbb{E}[Z_0] < \infty$ and that Z is an element of $\mathcal{U}_\sigma$ (and even of $\mathcal{U}_{\mathsf{loc}}$) and part (3) of Proposition D.2 ensures that $b + \int x\mathbf{1}_{\{|x|>1\}}\nu(\mathrm{d}x) \leq 0$. Now, assume that Z is a positive semimartingale with $\mathbb{E}[Z_0] < \infty$ and that $b + \int x\mathbf{1}_{\{|x|>1\}}\nu(\mathrm{d}x) \leq 0$. Then, of course we have that $\int x\mathbf{1}_{\{x>1\}}\nu(\mathrm{d}x) < \infty$. Also, since Z is positive we always have that $\nu\left[x < -Z_-\right] = 0$ so that $\int(-x)\mathbf{1}_{\{x<-1\}}\nu(\mathrm{d}x) < \infty$ too. Part (2) of Proposition D.2 will give us that $Z \in \mathcal{U}_\sigma$, and part (3) of the same proposition that Z is a σ-supermartingale. Finally, part (2) of this proposition gives us that Z is a supermartingale. $\square$

Proposition D.3 has been known for some time and made its first appearance in the paper [3] of Ansel and Stricker. The authors did not deal directly with σ-martingales, but with semimartingales Z which are of the form $Z = Z_0 + H \cdot M$, where M is a martingale and H is M-integrable (a *martingale transform*). Of course, martingale transforms are σ-martingales and vice-versa. The corollary of

Proposition D.3 when the σ-martingale Z is bounded from below by a constant, is sometimes called "The Ansel-Stricker theorem". The case when Z is a σ-supermartingale bounded from below with $\mathbb{E}[Z_0^+] < \infty$ is proved in [19].

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