Hedging and Optimization under Transaction Costs

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ABSTRACT

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In this paper we address problems of partial-hedging and utility maximization, in a general continuous-time multi-currency market with proportional transaction costs. After setting up the market model and describing its basic properties in Chapter 2, we study in Chapter 3 two partial-hedging problems for a cash-settled contingent claim: the problem of minimizing expected shortfall, and the problem of maximizing probability of perfect hedge. With the help of tools from non-smooth convex analysis, we establish the existence of optimal trading strategies and describe them in terms of appropriate dual optimization problems.

In Chapter 4, we consider a utility maximization problem under the same market model. Unlike the existing literature in mathematical finance, our utility function is the so-called "direct utility" depending on terminal consumption, rather than the "indirect utility" that depends on terminal wealth. We analyze the consumer's choice at the terminal time and compare our direct utility approach with the traditional indirect utility approach. We show that in a market with transaction costs, direct utility is much easier to handle than indirect utility, and that the standard tools from convex analysis and the smooth calculus of variations still suffice to establish the existence result.

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Chapter 1 Introduction

As is well-known, F. Black and M. Scholes presented in the celebrated paper [6] their famous formula for pricing options via partial differential equations (PDE). The modern martingale approach in mathematical finance was originated by yet another celebrated paper [19] by J. M. Harrison and S. Pliska. Among the fundamental assumptions on the financial markets, as studied in these pioneering papers and in most of the ensuing literature, is the assumption that the market is frictionless; in particular, there exist no transaction costs such as sales taxes, brokerage fees, telecommunication costs and membership fees in the exchange. In such a market, an investor has to pay (respectively, can receive the full amount of) only the quoted price of the asset when buying (respectively, selling) the asset.

The attempts to relax this assumption of no-transaction costs, have also taken either the PDE approach or the martingale approach. Since the optimization problems under transaction costs take the form of the so-called *singular stochastic control problem*, the PDE approach involves a free-boundary problem which is to be satisfied by the optimal value function. One of the important works in this spirit was done by M. H. A. Davis and A. R. Norman; in their paper [15] they obtained, with the help of the dynamic programming principle, an explicit solution to a utility maximization problem under an infinite time-horizon market with a single risky asset, whose price-process follows a geometric Brownian motion. Another important work in this period was the paper [32] by S. E. Shreve, H. M. Soner and J. Cvitanić, who considered the super-hedging problem of a call-option under the same setting as [15]; there authors proved that the upper-hedging price of a call-option is equal to the initial stock price, and that the optimal trading strategy is the simple "buy-and-hold" strategy. This result was generalized by [12] to the case of more general European contingent claims, and by [3] to the multi-asset case. The paper [35] has an extensive survey on the PDE approach to the transaction costs model with a single risky asset.

On the other hand, the martingale approach to continuous-time markets with transaction costs began to appear in the second half of the 90's. The fundamental work in this direction is the paper [9] by J. Cvitanić and I. Karatzas, who showed that in a single-risky-asset, Itô-process market model with proportional transaction costs, one can identify a pair of positive martingales ("conjugate processes"), satisfying certain additional conditions, which play the same role as the likelihood-ratio process of an equivalent martingale measure in an incomplete market without transaction costs. This fact enables us to take the convex-duality approach to hedging and portfolio optimization problems under transaction costs, exactly in the same manner as in an incomplete market. The results of that work were recently generalized by the papers [20] and [21] to the case of markets with several risky assets. This present thesis is on this recent spirit and framework of the martingale approach, and considers hedging and optimization problems in a general multi-asset, financial market under proportional transaction costs.

We organize this thesis as follows. First, in Chapter 2, we set up our model for the financial market, essentially in the spirit of [21]. In our model, there exist one risk-free asset, which will be called the *domestic currency*, and

d risky assets, which will called *foreign currencies*. The currency prices, i.e., the foreign/domestic exchange rates, are continuous semi-martingales. The basic structure of transaction costs, is that whenever we want to transfer as much as one domestic-currency-unit from the i-th currency account to the j-th currency account, then we actually need to withdraw as much $(1 + \lambda^{ij})$ domestic-currency-units from the i-th currency account because of the transaction cost of rate λ^{ij}. The set K of (2.3) below, which will be called the *solvency region*, is of fundamental importance in such a setting. We review the basic properties of the solvency region and, give some simple but useful results which have not been pointed out so far; see Propositions 2.2.4 and 2.2.5. We also review, in Section 2.3, the fundamental results about super-hedging of contingent claims. The basic situation in hedging-problems is the following. Suppose that we under-write a contingent claim at time $t = 0$. Then at maturity $(t = T)$, we are going to have to pay some random amount of domestic or foreign currency to the holder of the claim. The question then, is to find an initial endowment and a trading strategy which will provide terminal portfolio holdings vector in such a way, that we do not suffer any shortfall after covering the liability, *under any circumstances*; in other words, the probability of shortfall is zero. Theorem 2.3.7, obtained by [21], gives a description in terms of the conjugate processes about the minimum amount of initial endowment that is necessary for such riskless super-hedging.

We start our original contribution with Chapter 3. There, we consider two typical problems of partial-hedging. Suppose again that we are under-writing a contingent claim. This time, however, we also suppose that we are given some initial endowment which is strictly *less* than what is required for riskless super-hedging of the contingent claim. In such a situation, we cannot completely eliminate the risk of possible shortfall at maturity. As already indicated by the papers [32], [12] and [3] cited above, under proportional transaction costs, it is more the rule than the exception that we may not have initial endowment enough to find a riskless super-hedging trading strategy, simply because the minimum amount of initial endowment that is necessary for such riskless super-hedging is too high. Thus, it is frequently a crucial matter to find an alternative hedging scheme. The fundamental question of partial-hedging is to find a trading strategy that minimizes that risk. Obviously, such a question does not make sense unless we specify some risk-measure. We shall consider two different risk measures and show that for each of them, there exists a certain terminal portfolio holdings vector and the super-hedging of this terminal holdings is optimal for the original problem of partial-hedging.

In Chapter 4, we consider utility maximization problems. Our main contribution in this chapter is that we deal with a market with several consumption goods by using a so-called *direct utility function*. In mathematical finance, it is of common practice to specify a "utility" function of terminal wealth and/or of inter-temporal consumption, *measured in monetary-units*. In other words, utility depends on consumption only through the total expenditure allocated for consumption. Such a utility function is called an *indirect utility function*. A direct utility function is a more fundamental object, and depends on how much *physical amount* of consumption good is actually consumed. In a complete market without transaction costs, it was proved by P. Lakner in his doctoral dissertation that a *separation principle* holds, which relates direct and indirect utility functions. According to Lemma 3.3 of [27], in order to solve a utility maximization problem with a direct utility function, we may first consider a utility maximization problem over the total expenditure process with some suitable indirect utility function; then, with the optimal total expenditure process specified, we may consider an *allocation problem*, namely about how much portion of the total expenditure to allocate to each consumption good. In our model, Proposition 4.3.5 below corresponds to such a separation principle, and states that a utility maximization problem with a direct utility function is equivalent to that with a suitable indirect utility function. However, in our case, the resulting indirect utility function turns out to be rather hard to deal with. This is because in our market model, **(i)** there exist transaction costs, and **(ii)** several different currencies are available for purchasing consumption goods. We show in Section 4.4 that with direct utility function, we can prove the existence result by standard tools from convex analysis and the smooth calculus of variations.

Finally, in the Appendix, we provide some auxiliary results and ramifications. In Section A.1, we review the bipolar theorem (Theorem A.1.1 below) on the space of all non-negative random variables which are not necessarily integrable. We prove some related results which we have used in Section 3.4. Sections A.2 and B.1 contain the proofs for ramifications of our results in Chapters 3 and 4, respectively.

Chapter 2 The Model

2.1 Trading Strategies and Portfolio Holdings

Throughout this thesis, we fix a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and on it a filtration $\mathbb{F} \triangleq \{\mathscr{F}(t);\, t \in [0,T]\}$ satisfying the usual conditions. Also, whenever we deal with a martingale, we shall always take the RCLL (right-continuous with finite left-hand limits) modification; the usual conditions enable us to take this convention. The time-horizon $T \in (0, \infty)$ is a fixed constant. Our financial market consists of d-currencies with prices given by the components of an $\mathbb{F}$-semimartingale $S(\cdot) \triangleq \{(S^1(t), \dots, S^d(t));\, t \in [0,T]\}$, with values in $(0,\infty)^d$ and *continuous* paths. We assume that the first component $S^1(\cdot)$, denoting the "domestic currency", is normalized to be 1. We also assume the following.

Assumption 2.1.1. There exists at least one *equivalent martingale measure* $\mathbb{Q}$ for the process $S(\cdot)$; that is, $\mathbb{Q}$ is a probability measure, equivalent to $\mathbb{P}$, under which the process $S(\cdot)$ is a martingale.

The reason that we regard $S(\cdot)$ as a vector of currencies is that it is more convenient to take such a viewpoint for the utility maximization problem in Chapter 4. For the partial-hedging problems in Chapter 3, however, there is no such convenience in viewing the components of $S(\cdot)$ as prices of currencies; we may simply think of the first component $S^1(\cdot)$ as the risk-free asset or the bank-account, and of the remaining components $S^2(\cdot), \dots, S^d(\cdot)$ as risky assets such as equity stocks.

In our market, a transfer from the i-th currency account to the j-th currency account is subject to a proportional transaction cost at the rate λ^{ij}, where each λ^{ij} is a nonnegative constant such that $\lambda^{ii} = 0$, $\forall 1 \le i \le d$. This means that in order to transfer 1-unit of the i-th currency to the j-th currency, an investor needs to withdraw $(1+\lambda^{ij})$-units of the i-th currency and pay λ^{ij}-units to the exchange (or the broker) as a transaction cost. A *trading strategy* is an $\mathbb{R}_+^{d\times d}$-valued, $\mathbb{F}$-adapted process $L(\cdot) \triangleq (L^{ij}(\cdot))_{1\le i,j\le d}$ defined on $[0,T)$, with each component $L^{ij}(\cdot)$ having right-continuous, non-decreasing paths with $L^{ij}(T-) \triangleq \lim_{t\uparrow T} L^{ij}(t) < \infty$ a.s. The random variable $L^{ij}(t)$ denotes the cumulative amount of the i-th currency transferred to the j-th currency over the period $[0,t]$. Since $L(\cdot)$ is assumed to be right-continuous, $L(t)$ can depend on $S(t)$ at each time t; thus $dL(t)$ should be understood as the transfer made immediately after the price $S(t)$ has been observed. We do not require $L(0)$ to be equal to 0, which means a transfer at time 0 is allowed. We adopt the convention $L(0-) \equiv 0$, so that $L(0) - L(0-)$ is equal to the transfer made at time 0.

For an initial endowment $x \in \mathbb{R}^d$ and a trading strategy $L(\cdot)$, we define the *portfolio holdings process* $X_x^L(\cdot) \triangleq (X_x^{1L}(\cdot), \dots, X_x^{dL}(\cdot))$ by

$$X_x^{iL}(t) = x^i + \int_0^t X_x^{iL}(u-)\frac{dS^i(u)}{S^i(u)} + \sum_{j=1}^d \left\{ L^{ji}(t) - (1+\lambda^{ij})L^{ij}(t) \right\}, \quad t \in [0,T),\ i = 1, \dots, d. \tag{2.1}$$

The i-th element $X_x^{iL}(t)$ of the random vector $X_x^L(t)$ thus denotes the amount of the i-th currency held at time t. Each $X_x^{iL}(t)$ is measured in terms of the domestic currency; that is, $X_x^{iL}(t)/S^i(t)$ denotes the number of i-th currency units held at time t. The equation (2.1) can be rewritten in terms of the "number of currency units" held, namely as

$$\frac{X_x^{iL}(t)}{S^i(t)} = \frac{x^i}{S^i(0)} + \int_{[0,t]} \frac{1}{S^i(u)} \sum_{j=1}^{d} \left\{ dL^{ji}(u) - (1+\lambda^{ij}) dL^{ij}(u) \right\}, \tag{2.2}$$

for every $t \in [0, T)$ and $i = 1, \ldots, d$. Note that since we have assumed the trading strategy process $L(\cdot)$ to be right-continuous, the portfolio holdings process $X_x^L(\cdot)$, which is also right-continuous, should be understood as the position right after the transfer at time t has taken place. In particular, we have $X(0) \neq x$ if $L(0) \neq 0$.

2.2 The Solvency Region

Following [20], we define the *solvency region* K by

(2.3)

$$K \triangleq \left\{ x \in \mathbb{R}^d \,\middle|\, \exists a \in \mathbb{M}_+^d : x^i + \sum_{j=1}^{d} \left[a^{ji} - (1+\lambda^{ij}) a^{ij} \right] \geq 0, \ \forall i = 1, \ldots, d \right\},$$

and the *positive polar* K^* of K by

$$K^* \triangleq \left\{ y \in \mathbb{R}^d \mid y \cdot x \geq 0, \ \forall x \in K \right\}. \tag{2.4}$$

Here and in the sequel, we denote by $\mathbb{M}^d$ the space of $(d \times d)$-matrices with real-valued components such that the diagonal elements are equal to 0, and by $\mathbb{M}_+^d$ the space of matrices that belong to $\mathbb{M}^d$ and have non-negative elements. The vector space $\mathbb{M}^d$ can be identified with $\mathbb{R}^{d(d-1)}$. Similarly, under this identification, $\mathbb{M}_+^d$ can be identified with the non-negative orthant $\mathbb{R}_+^{d(d-1)}$ of $\mathbb{R}^{d(d-1)}$.

The economic significance of the solvency region K in (2.3) is the following. Suppose that the portfolio holdings vector x belongs to K. Then there exists a so-called "transfer matrix" $a \in \mathbb{M}^d_+$ such that

$$x^i + \sum_{j=1}^{d} \left[a^{ji} - (1+\lambda^{ij})a^{ij}\right] \geq 0, \ \forall i = 1, \ldots, d.$$

In other words: if, for each i and j, we transfer as much as a^{ij} domestic currency units, from the i-th currency to the j-th currency, then after all such transfers have been completed and the appropriate transaction costs have been paid, we obtain a new vector of portfolio holdings with non-negative amounts in *each* currency.

Proposition 2.2.1. *The sets K and K^* are closed convex polyhedral cones of $\mathbb{R}^d$ satisfying $K^* \subseteq \mathbb{R}^d_+ \subseteq K$. Furthermore, we have $(K^*)^* = K$, that is*

$$K = \{x \in \mathbb{R}^d \mid y \cdot x \geq 0, \ \forall y \in K^*\}.$$

Proof. It is obvious that $\mathbb{R}^d_+ \subseteq K$ and thus $K^* \subseteq \mathbb{R}^d_+$. To show that K is a closed convex polyhedral cone, define the set A by

(2.5)

$$A \triangleq \left\{(x,a) \in \mathbb{R}^d \times \mathbb{M}^d_+ \;\middle|\; x^i + \sum_{j=1}^{d} \left[a^{ji} - (1+\lambda^{ij})a^{ij}\right] \geq 0, \ \forall i = 1, \ldots, d\right\}.$$

Then, as the intersection of finitely many closed linear half-spaces of $\mathbb{R}^d \times \mathbb{M}^d \simeq \mathbb{R}^{d^2}$, A is a closed convex polyhedral cone. Since K is the image of A under the projection from $\mathbb{R}^d \times \mathbb{M}^d$ onto $\mathbb{R}^d$, it follows from Theorem 19.3 of [29] that K is a closed convex polyhedral cone. As the positive polar of K, K^* is a closed convex polyhedral cone from Corollary 19.2.2 of [29]. $\square$

The following property states that the cone K^* of (2.4) consists of those vectors of portfolio holdings which are *parsimonious*; the holdings in each of the currencies are worth there more than in any other currency, after factoring-in the transaction costs for the transfers. Although this property can be easily checked by direct computation, we offer a proof that will also be informative for Propositions 2.2.4 and 2.2.5 below.

Proposition 2.2.2. *The positive polar cone* K^* *of* (2.4) *can also be expressed as*

$$K^* = \left\{ y \in \mathbb{R}^d_+ \mid y^j - (1+\lambda^{ij})y^i \leq 0, \ \forall i \neq j \right\}. \tag{2.6}$$

Proof. Let A^* be the positive polar of the set A defined in (2.5), namely

$$A^* \triangleq \{(y,b) \in \mathbb{R}^d \times \mathbb{M}^d \mid y \cdot x + \mathrm{vec}(b) \cdot \mathrm{vec}(a) \geq 0, \ \forall (x,a) \in A\}, \tag{2.7}$$

where $\mathrm{vec}(a)$ is the vector corresponding to the matrix $a \in \mathbb{M}^d$ under the identification $\mathbb{M}^d \simeq \mathbb{R}^{d(d-1)}$. Notice that for each $(x,a) \in \mathbb{R}^d \times \mathbb{M}^d$, $(x,a) \in A$ if and only if

$$\begin{gathered} e_i \cdot x + \mathrm{vec}(c_i) \cdot \mathrm{vec}(a) \geq 0, \ \forall i = 1, \ldots, d, \\ \mathrm{vec}(\delta_{ij}) \cdot \mathrm{vec}(a) \geq 0, \ \forall i \neq j. \end{gathered}$$

where e_i is the i-th unit vector of $\mathbb{R}^d$, δ_{ij} is the (i,j)-th unit matrix of $\mathbb{M}^d$ and c_i is a matrix in $\mathbb{M}^d$ with (j,k)-element given by

$$c_i^{jk} = \left\{ \begin{array}{cl} 1 & \text{if} \quad j = i \neq k \\ -(1+\lambda^{ij}) & \text{if} \quad k = i \neq j \\ 0 & \text{otherwise} \end{array} \right\}. \tag{2.8}$$

Thus, from p.122 of [29] and (2.5), we see that

$$\begin{aligned} A^* &= \mathrm{cone}\,\{(e_i, c_i), \ (0, \delta_{ij}); \ i \neq j\} \\ &= \mathrm{cone}\,\{(e_1, c_1), \ldots, (e_d, c_d)\} + \left(\{0\} \times \mathbb{M}^d_+\right). \end{aligned}$$

Therefore, $(y,b) \in A^*$ if and only if there exist $\alpha \in \mathbb{R}^d_+$ and $u \in \mathbb{M}^d_+$ such that

$$y = \sum_{k=1}^{d} \alpha^k e_k \quad \text{and} \quad b = \sum_{k=1}^{d} \alpha^k c_k + u.$$

By solving the first equation for α^k and substituting it into the second, we see that $(y,b) \in A^*$ if and only if $y \in \mathbb{R}^d_+$ and

$$b^{ij} \geq y^j - (1+\lambda^{ij}) y^i, \ \forall i \neq j. \tag{2.9}$$

Finally, notice that since K is the image of A under the projection onto $\mathbb{R}^d$, we have

$$K^* = \left\{ y \in \mathbb{R}^d \mid (y,0) \in A^* \right\},$$

which, in conjunction with (2.9), gives the assertion. □

Corollary 2.2.3. *Every non-zero vector in the cone K^* has only strictly positive components. In other words, we have $K^* \setminus \{0\} \subseteq (0,\infty)^d$.*

Proof. Let $y \in K^* \setminus \{0\}$. Without loss of generality, we may assume that $y^1 > 0$. Then from (2.6), we have

$$0 < y^1 \leq (1+\lambda^{i1}) y^i, \ \forall i = 2, \ldots, d,$$

which implies $y \in (0,\infty)^d$. □

The next proposition states that the inequality on the right-hand-side of (2.3) can be actually replaced by equality. This implies that whenever $x_2 - x_1 \in K$, we may find a transfer matrix a with which we can move from the position x_2 to the position x_1 without making any surplus.

Proposition 2.2.4. *The solvency region K can also be written as*

$$K = \left\{ x \in \mathbb{R}^d \;\middle|\; \exists a \in \mathbb{M}^d_+ : x^i + \sum_{j=1}^{d} \left[a^{ji} - (1+\lambda^{ij}) a^{ij} \right] = 0, \ \forall i = 1, \ldots, d \right\}. \tag{2.10}$$

Proof. Let K_0 be the set appearing in the right-hand-side of (2.10) and K_0^* be its positive polar. Then, as in the previous two propositions, we can show that K_0 and K_0^* are closed convex polyhedral cones of $\mathbb{R}^d$ and that K_0^* is given by

$$K_0^* = \left\{ y \in \mathbb{R}^d \;\middle|\; y^j - (1+\lambda^{ij})y^i \leq 0, \ \forall i \neq j \right\}. \tag{2.11}$$

It is clear that $K^* \subseteq K_0^*$. We claim $K^* = K_0^*$. To see this, let $y \in K_0^*$. Then we have

$$y^j \leq (1+\lambda^{ij})y^i \quad \text{and} \quad y^i \leq (1+\lambda^{ji})y^j, \quad \forall i \neq j,$$

and thus

$$\left[1 + \lambda^{ij} - \frac{1}{1+\lambda^{ji}}\right] y^i \geq 0, \quad \forall i = 1, \dots, d.$$

Since $1 + \lambda^{ij} - 1/(1+\lambda^{ji}) > 0$, it follows $y^i \geq 0$. This proves $K^* = K_0^*$ as claimed. Therefore, we have $K = (K^*)^* = (K_0^*)^* = K_0$. □

From Proposition 2.2.4, for each $x \in K$, there exists a matrix $a \in \mathbb{M}_+^d$ such that

$$x^i + \sum_{j=1}^{d} \left[a^{ji} - (1+\lambda^{ij})a^{ij}\right] = 0, \ \forall i = 1, \dots, d. \tag{2.12}$$

For each $x \in \mathbb{R}^d$, let $A(x)$ be the set of matrices $a \in \mathbb{M}_+^d$ that satisfy (2.12). The map $x \mapsto A(x)$ is then a multifunction of K into $\mathbb{M}^d$.

Proposition 2.2.5. *The multifunction $A(\cdot)$ admits a continuous (and thus, in particular, Borel-measurable) selection, which we hereafter denote by $\alpha(\cdot)$.*

Proof. Define a linear map $T\colon \mathbb{M}^d \to \mathbb{R}^d$ by

$$T(a) \triangleq \begin{bmatrix} \sum_{j=1}^d [a^{j1} - (1+\lambda^{1j})a^{1j}] \\ \vdots \\ \sum_{j=1}^d [a^{jd} - (1+\lambda^{dj})a^{dj}] \end{bmatrix}, \quad a \in \mathbb{M}^d,$$

or equivalently,

$$T(a) = \begin{bmatrix} \mathrm{vec}(c_1)\cdot\mathrm{vec}(a) \\ \vdots \\ \mathrm{vec}(c_d)\cdot\mathrm{vec}(a) \end{bmatrix} = c\,\mathrm{vec}(a),$$

where

$$c \triangleq \begin{bmatrix} \mathrm{vec}(c_1)' \\ \vdots \\ \mathrm{vec}(c_d)' \end{bmatrix} \in \mathbb{R}^{d\times d(d-1)}$$

with c_i given by (2.8). It is easy to see that the vectors $\mathrm{vec}(c_i)$; $i = 1, \ldots, d$, are linearly independent, and hence $\mathrm{rank}(T) = d$. Noting that we have a vector space isomorphism $\mathbb{M}^d \simeq \mathbb{R}^{d(d-1)} \simeq \mathbb{R}^d \times \mathbb{R}^{d(d-2)}$, we see that there exists a linear map $S\colon \mathbb{M}^d \to \mathbb{R}^{d(d-2)}$ with $\mathrm{rank}(S) = d(d-2)$ such that the linear map

$$(T, S):\ \mathbb{M}^d \ni a \mapsto (T(a), S(a)) \in \mathbb{R}^d \times \mathbb{R}^{d(d-2)}$$

is non-singular.

Now, from Proposition 2.2.4, we have $K = -T(\mathbb{M}^d_+)$ and thus

$$K = -T(\mathbb{M}^d_+) = \mathrm{proj}_{\mathbb{R}^d}[-(T,S)(\mathbb{M}^d_+)] = \mathrm{proj}_{\mathbb{R}^d}[-T(\mathbb{M}^d_+) \times \{0\}],$$

where $\mathrm{proj}_{\mathbb{R}^d}$ is the projection of $\mathbb{R}^{d(d-1)}$ onto the space $\mathbb{R}^d$ of the first d components. In particular, $-(T,S)^{-1}$ maps $K \times \{0\}$ into $\mathbb{M}^d_+$. Furthermore, by the definition of the multifunction A, we have $A(x) = (-T^{-1}(x)) \cap \mathbb{M}^d_+$, $\forall x \in K$. Thus, the map

$$\alpha(x) \triangleq -(T,S)^{-1}(x, 0), \quad x \in K,$$

gives a continuous selection of $A(\cdot)$. $\square$

As a closed cone in $\mathbb{R}^d$, the solvency region K induces the partial pre-ordering $\succeq$ on $\mathbb{R}^d$ by

$$x_2 \succeq x_1 \overset{\text{def}}{\Longleftrightarrow} x_2 - x_1 \in K. \tag{2.13}$$

According to the economic significance of the solvency region K, which we explained in the first paragraph of this section, the inequality $x_2 \succeq x_1$ signifies that, starting with the portfolio holdings vector x_2, an investor can move to the new portfolio holdings vector x_1, by making appropriate transfer. In particular, for each portfolio holdings vector $x \in \mathbb{R}^d$, the quantity

$$\ell(x) \triangleq \sup\{\xi \in \mathbb{R} \mid x \succeq (\xi, 0, \ldots, 0)\} \tag{2.14}$$

denotes the maximal amount of the domestic currency that can be obtained through liquidating the portfolio holdings vector x. We call the function $\ell(\cdot)$ the *liquidation function*; it is real-valued, as we explain right after Lemma 2.2.6 below.

For our purposes, it is sometimes convenient to "normalize" the vectors in the positive polar cone K^* so that the first component is equal to 1. We thus introduce the set

$$\Lambda \triangleq K^* \cap (\{1\} \times \mathbb{R}^{d-1}). \tag{2.15}$$

The set Λ of (2.15) has the following properties. The proofs can also be found in Lemma 5.1 of [3] and Section 3.3 of [4].

Lemma 2.2.6. **(i)** *Λ is non-empty and compact in $\mathbb{R}^d$;*

(ii) *$x_2 \succeq x_1$ if and only if $y \cdot (x_2 - x_1) \geq 0$, $\forall y \in \Lambda$.*

Proof. **(i)** From (2.6), it is clear that $(1, \ldots, 1) \in \Lambda$ and thus $\Lambda \neq \emptyset$. Since K^* is closed in $\mathbb{R}^e$, so also is Λ. Let $y \in \Lambda$. Since $y^1 = 1$, the equation (2.6) gives

$$0 \le y^j \le (1+\lambda^{1j}), \quad \forall j = 2, \dots, d,$$

which implies that the set Λ is bounded and thus compact.

(ii) Suppose that $y \cdot (x_2 - x_1) \ge 0$, $\forall y \in \Lambda$. Let $z \in K^*$ be arbitrary. If $z \ne 0$, we have $z^1 > 0$ from Corollary 2.2.3, and thus,

$$z \cdot (x_2 - x_1) = z^1 \left(\frac{z}{z^1}\right) \cdot (x_2 - x_1) \ge 0,$$

since $z/z^1 \in \Lambda$. For $z = 0$, we trivially have $z \cdot (x_2 - x_1) = 0$. It follows that $z \cdot (x_2 - x_1) \ge 0$, $\forall z \in K^*$, which implies $x_2 - x_1 \in K$ because $K = (K^*)^*$, and thus $x_2 \succeq x_1$. This proves the "if" part. The "only if" part is obvious from (2.13) and (2.3). $\square$

Part **(ii)** of Lemma 2.2.6 gives a "dual representation" of the partial preordering $\succeq$ of (2.13). As a consequence, we have the following dual representation

$$\ell(x) = \inf \{ y \cdot x \mid y \in \Lambda \}, \quad \forall x \in \mathbb{R}^d \tag{2.16}$$

for the liquidation function $\ell(\cdot)$. From Proposition 2.2.1, we have then

$$\ell(x) \ge 0 \iff x \in K.$$

Note also that from **(i)** of Lemma 2.2.6, the infimum in (2.16), and thus, by duality, the supremum in (2.14), is always attained.

2.3 Hedging

In this section, we summarize results from [21] which will be used frequently in our later analysis. We denote by $\mathscr{M}_0$ the set of all $\mathbb{P}$-martingales $Z(\cdot)$

with values in $(0,\infty)^d$ such that $\mathrm{diag}[S(t)]^{-1}Z(t) \in K^*$, $\forall t \in [0,T]$. Here and in the sequel, we denote by $\mathrm{diag}[v]$ the $(d \times d)$-diagonal matrix with diagonal elements $v^1, \dots, v^d$, for any vector $v = (v_1, \dots, v^d) \in \mathbb{R}^d$. For each $z \in \mathrm{diag}[S(0)]K^* \setminus \{0\}$, we denote by $\mathscr{D}_0(z)$ the set of terminal values of such martingales starting from z, i.e.,

$$\mathscr{D}_0(z) \triangleq \{Z(T) \mid Z(\cdot) \in \mathscr{M}_0,\ Z(0) = z\}. \tag{2.17}$$

Recall now Assumption 2.1.1, take an arbitrary equivalent martingale measure $\mathbb{Q}$ for $S(\cdot)$, and let $\rho(\cdot)$ be the (RCLL modification of the) strictly positive $\mathbb{P}$-martingale

$$\rho(t) \triangleq \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathscr{F}(t)\right], \quad t \in [0,T]. \tag{2.18}$$

Then, for each $z \in \mathrm{diag}[S(0)]K^* \setminus \{0\}$, the process $\rho(\cdot)\,\mathrm{diag}[S(\cdot)]\,\mathrm{diag}[S(0)]^{-1}z$ belongs to $\mathscr{D}_0(z)$ and in particular, the class $\mathscr{D}_0(z)$ is non-empty. Indeed, since $S(\cdot)$ is a martingale under $\mathbb{Q}$, Bayes' rule (Lemma 3.5.3 of [25]) implies that $\rho(\cdot)\,\mathrm{diag}[S(\cdot)]\,\mathrm{diag}[S(0)]^{-1}z$ is a martingale under $\mathbb{P}$.

In order for the hedging results of [21] (Theorems 2.3.6 and 2.3.7 below) to be valid, we need the following assumption on "efficient friction".

Assumption 2.3.1. The cone K^* has nonempty interior.

We see easily that $(1, \dots, 1) \in \mathrm{int}(K^*)$ from (2.6), if $\lambda^{ij} > 0$, $\forall i \neq j$. Also, it is clear that this assumption does not hold if $\lambda^{ij} = 0$, $\forall i$ and $\forall j$. This implies that our model does not include the market without transaction costs as a special case, and most of our results are particular to markets with transaction costs; see also the F-condition of [21] and [20]. From now on, we shall always assume that Assumption 2.3.1 holds.

An $\mathbb{R}^d$-valued stochastic process $X(\cdot)$ will be called *S-bounded from below* if there exists a real constant $\kappa \geq 0$ such that

$$X(t) \succeq -\kappa S(t), \quad \forall t \in [0, T) \tag{2.19}$$

holds almost surely. For an arbitrary initial portfolio holdings vector $x \in \mathbb{R}^d$, a trading strategy $L(\cdot)$ and a martingale $Z(\cdot) \in \mathscr{M}_0$, Itô's formula gives

$$\begin{aligned}
&Z(t) \cdot \mathrm{diag}[S(t)]^{-1} X_x^L(t) \\
&\qquad = Z(0) \cdot \mathrm{diag}[S(0)]^{-1} x + \int_{[0,t]} \mathrm{diag}[S(u)]^{-1} X_x^L(u-) \cdot dZ(u) \\
&\qquad\qquad + \sum_{i=1}^{d} \int_{[0,t]} \frac{Z^i(u)}{S^i(u)} \sum_{j=1}^{d} \{dL^{ji}(u) - (1+\lambda^{ij}) dL^{ij}(u)\} \\
&\qquad = Z(0) \cdot \mathrm{diag}[S(0)]^{-1} x + \int_{[0,t]} \mathrm{diag}[S(u)]^{-1} X_x^L(u-) \cdot dZ(u) \\
&\qquad\qquad + \sum_{i=1,j}^{d} \int_{[0,t]} \left\{ \frac{Z^j(u)}{S^j(u)} - (1+\lambda^{ij}) \frac{Z^i(u)}{S^i(u)} \right\} dL^{ij}(u).
\end{aligned}$$

On the right-most-side, the stochastic integral is a local martingale, and the last Lebesgue-Stieltjes integral is non-increasing in t because of (2.6); thus, the process $Z(\cdot) \cdot \mathrm{diag}[S(\cdot)]^{-1} X_x^L(\cdot)$ is a local supermartingale. Part **(i)** of the next lemma states that it is actually a supermartingale if $X_x^L(\cdot)$ is S-bounded from below. Part **(ii)** implies that the trading strategy $L(\cdot)$ is "admissible" in the sense of Definition 2.3.3 below, if and only if $X_x^L(\cdot)$ is S-bounded from below and $X_x^L(T-) \in K$ holds almost surely; see also Remark 2.3.5 below.

Lemma 2.3.2. *Let $x \in \mathbb{R}^d$ be an initial endowment vector and $L(\cdot)$ be a trading strategy. Suppose that the portfolio holdings process $X_x^L(\cdot)$ is S-bounded from below. Then the following assertions hold.*

- **(i)** *For every $Z(\cdot) \in \mathscr{M}_0$, the process $Z(\cdot) \cdot \mathrm{diag}[S(\cdot)]^{-1} X_x^L(\cdot)$ is a supermartingale;*

(ii) *If* $X_x^L(T-) \in K$ *a.s., then* $X_x^L(t) \in K$ *a.s. for every* $t \in [0,T)$.

To see **(i)**, it suffices to observe that the local supermartingale $Z(\cdot) \cdot \operatorname{diag}[S(\cdot)]^{-1} X_x^L(\cdot)$ is bounded from below by a martingale if $X_x^L(\cdot)$ is S-bounded from below. Part **(ii)** is an easy consequence of the supermartingale property, and of the fact that $\rho(\cdot) \operatorname{diag}[S(\cdot)] \operatorname{diag}[S(0)]^{-1} z \in \mathscr{M}_0$, $\forall z \in \operatorname{diag}[S(0)] K^* \setminus \{0\}$ with $\rho(\cdot)$ given by (2.18) for some equivalent martingale measure $\mathbb{Q}$. For a more complete proof, see Lemmas 3.1 and 3.3 there in [21]. For an argument similar to **(ii)**, see also the proof of the "sufficiency" part of Lemma 4.5.1 in Section 4.5.

Now, let $x \in \mathbb{R}^d$ be an initial portfolio holdings vector and $L(\cdot)$ be a trading strategy. Note that a component $X_x^{iL}(\cdot)$ of the random vector $X_x^L(\cdot)$ of (2.1) can take negative values, which means that an investor can take a short position for some currencies. We do not allow, however, the "total value" of the portfolio holdings vector $X_x^L(t)$ to be negative at any time t. More precisely, we make the following definition of admissibility.

Definition 2.3.3 (Admissible Trading Strategies). Let $x \in K$ be an initial endowment vector. A trading strategy $L(\cdot)$ is called *admissible* for x, if the following "no-bankruptcy" condition

$$X_x^L(t) \in K, \quad \forall t \in [0,T) \tag{2.20}$$

holds almost surely. We denote by $\mathscr{A}(x)$ the set of all admissible trading strategies.

Note that the condition $X_x^L(t) \in K$ is equivalent to $\ell(X_x^L(t)) \geq 0$, in terms of the liquidation function of (2.14).

Definition 2.3.4. A *contingent claim* is a K-valued, $\mathscr{F}(T)$-measurable random vector $G = (G^1, \dots, G^d)$. The i-th component denotes the amount of the i-th currency that the claim holder receives at time T from the issuer of the claim, measured in terms of the domestic currency; in other words, the number of i-th currency units received is $G^i/S^i(T)$.

The purpose of the issuer of the claim is to invest some initial endowment vector $x \in K$ over the time period $[0,T)$, by taking some admissible trading strategy $L(\cdot) \in \mathscr{A}(x)$, so that the terminal holdings vector $X_x^L(T-)$ "covers" the liability G in the sense that

$$X_x^L(T-) - G \in K, \quad \text{a.s.} \tag{2.21}$$

In other words, by appropriate transfers of his holdings at time $t = T$, the issuer of the claim can move his holdings $X_x^L(T-)$ to the target-position G, that he has to deliver to the holder (recall Proposition 2.2.4 and the discussion preceding it).

If such a trading strategy $L(\cdot) \in \mathscr{A}(x)$ exists, we say that the contingent claim G is *hedgeable* with the initial endowment x and the trading strategy $L(\cdot)$ *hedges* the contingent claim G. We denote by $\mathbb{A}_x$ the set of all contingent claims that are hedgeable with initial endowment x, i.e.,

$$\mathbb{A}_x \triangleq \{G \in \mathbb{L}^0(K) \mid \exists L(\cdot) \in \mathscr{A}(x) : X_x^L(T-) \succeq G \text{ a.s.}\} \tag{2.22}$$

for $x \in K$.

Here and in the sequel, we denote by $\mathbb{L}^0(E)$ the set of all E-valued, $\mathscr{F}(T)$-measurable random variables, for any Borel set E of some Euclidean space $\mathbb{R}^n$. Similarly, for $1 \le p \le \infty$, we denote by $\mathbb{L}^p(E)$ the set of p-th integrable (essentially bounded, if $p = \infty$) random vectors in $\mathbb{L}^0(E)$. We omit E and write $\mathbb{L}^0$ or $\mathbb{L}^p$, when $E = \mathbb{R}$.

If a contingent claim G is hedgeable with the amount x, and the holder agrees to pay the amount x to the issuer at time $t = 0$, as the price of the right to receive the amount G at time $t = T$, then the issuer can safely assume the liability associated with the contingent claim G without taking any risk of shortfall.

Remark 2.3.5. The paper [21] defines admissibility of a trading strategy $L(\cdot)$ to be S-boundedness from below of the portfolio holding process $X_x^L(\cdot)$. For our purpose, however, the only interesting trading strategies will be those with terminal holdings $X_x^L(T-) \in K$. By virtue of **(ii)** of Lemma 2.3.2, if $X_x^L(\cdot)$ is S-bounded from below and $X_x^L(T-) \in K$, then we automatically have $X_x^L(t) \in K$, $\forall t \in [0, T)$ almost surely. Similarly, while a contingent claim is defined in the paper [21] to be an arbitrary $\mathbb{R}^d$-valued, $\mathscr{F}(T)$-measurable random vector G such that $G \succeq -\kappa S(T)$ a.s. for some constant $\kappa \geq 0$, we need to consider, for our purposes, only K-valued contingent claims.

The next theorem is an easy consequence of Lemma 3.5 of [21] about the more general, so-called "S-Fatou" closedness of the set $\mathbb{A}_x$.

Theorem 2.3.6. *Under Assumption 2.3.1, the set $\mathbb{A}_x$ of (2.22) is closed under a.s.-convergence; i.e., if $\{G_n\}_{n \in \mathbb{N}} \subseteq \mathbb{A}_x$ is a sequence converging almost surely to some $\mathscr{F}(T)$-measurable random vector G, then we have $G \in \mathbb{A}_x$.*

If G is a hedgeable contingent claim with initial endowment $x \in K$, then there exists an admissible trading strategy $L(\cdot) \in \mathscr{A}(x)$ such that $X_x^L(T-) \succeq G$. It then follows from **(i)** of Lemma 2.3.2, the continuity of the paths of $S(\cdot)$, the uniform integrability of the martingale $\{Z(t); t \in [0, T]\}$ and Fatou's lemma that

$$
\begin{aligned}
\mathbb{E}\left[Z(T)\cdot\operatorname{diag}[S(T)]^{-1}G\right] &\le \mathbb{E}\left[Z(T)\cdot\operatorname{diag}[S(T)]^{-1}X_x^L(T-)\right] \\
&= \mathbb{E}\left[Z(T)\cdot\operatorname{diag}[S(T-)]^{-1}X_x^L(T-)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[Z(T)\cdot\operatorname{diag}[S(T-)]^{-1}X_x^L(T-) \mid \mathscr{F}(T-)\right]\right] \\
&= \mathbb{E}\left[Z(T-)\cdot\operatorname{diag}[S(T-)]^{-1}X_x^L(T-)\right] \\
&\le \varliminf_{t\uparrow T}\mathbb{E}\left[Z(t)\cdot\operatorname{diag}[S(t)]^{-1}X_x^L(t)\right] \\
&\le Z(0)\cdot\operatorname{diag}[S(0)]^{-1}x.
\end{aligned}
$$

The next Theorem 2.3.7 states that this condition is actually necessary and sufficient for the hedgeability of G. The theorem can be derived from the corresponding Theorem 3.2 of [21] (with slightly different terminology) by using **(ii)** of Lemma 2.3.2 above.

Theorem 2.3.7. *Let Assumption 2.3.1 hold. Let G be a contingent claim, and let $x \in K$ be an initial endowment. Then G is hedgeable with x, if and only if*

$$
\mathbb{E}\left[Z(T)\cdot\operatorname{diag}[S(T)]^{-1}G\right] \le Z(0)\cdot\operatorname{diag}[S(0)]^{-1}x, \quad \forall Z(\cdot)\in\mathscr{M}_0, \tag{2.23}
$$

or equivalently,

$$
\mathbb{E}\left[Z\cdot\operatorname{diag}[S(T)]^{-1}G\right] \le z\cdot\operatorname{diag}[S(0)]^{-1}x, \qquad \forall Z\in\mathscr{D}_0(z) \text{ and } \forall z\in\operatorname{diag}[S(0)]K^*\setminus\{0\}. \tag{2.24}
$$

Chapter 3 Partial-Hedging of Cash-Settled Contingent Claims

3.1 Introduction

In Section 2.3, we briefly described the results about hedging a contingent claim. In particular, Theorem 2.3.7 gave a necessary and sufficient condition for hedgeability in terms of duality. However, it has been pointed out that hedging a contingent claim often requires a large amount of initial capital. For example, in the case of simple European call-options, the minimum initial cash required for hedging is just the initial stock price, which is a trivial upper bound on the value of the option; and the optimal strategy for hedging a call-option is the so-called *buy-and-hold* strategy; see [32] or [28]. Therefore, unless the option writer is willing to take some risks of possible shortfall, writing a call-option is meaningless, and hence, for an option writer, taking and controlling the risk of shortfall is essentially a decision problem.

With such a situation in mind, a natural question arises: *What is the best strategy to follow when the initial capital is less than the upper hedging price?* This is the main problem which we consider in this chapter. First of all, in order to evaluate the performance of the trading strategy, we need to specify a certain criterion. We consider two different criteria; the first is *minimization of expected shortfall*, and the second is *maximization of the probability of perfect hedge*. We also consider a modified version of the second criterion, namely, *maximization of the expected success-ratio*. These risk-measures are considered by [17] and [18] in the case of a complete market, by means of the Neyman-Pearson fundamental lemma; the authors also discussed several interesting cases, including the incomplete market and the volatility jump stock process. The paper [24] discussed the connection between Monge-Ampère-type equations and the optimal value function of the problem of maximizing the probability of perfect hedge in markets with partial information. The result obtained by [24] was generalized by the paper [33]. The paper [33] also considered the case where wealth processes have nonlinear drift. As for the problem of minimizing expected shortfall, the paper [10] considered the case of a complete market with a margin requirement, and also took into account the uncertainty of the knowledge about the real-world measure. The paper [8] gave an explicit formula for the optimal target wealth to attain the minimum expected shortfall in terms of the dual optimization problem for the case of incomplete and constrained markets, which is similar to the formula which we obtain in this chapter.

In this thesis, we consider partial-hedging problems for cash-settled contingent claims and with initial endowment in cash, that is, the payoff of the contingent claim and the initial endowment allocated for the effort to hedge the contingent claim arise in the domestic currency. Since the contingent claims and initial endowments appearing in this chapter are all one-dimensional, we omit the super-script to denote them. We denote by $\mathbb{L}^0_+$ the set of $\mathscr{F}(T)$-measurable random variables with values in $[0, \infty)$, and by $\mathbb{L}^1_+$ the set of integrable random variables in $\mathbb{L}^0_+$.

3.2 Cash-Settled Contingent Claims

A *cash-settled contingent claim* is simply an $\mathscr{F}(T)$-measurable, non-negative random variable G; then the random vector $(G, 0, \dots, 0)$ is a contingent claim, as defined in Section 2.3. Given an initial endowment $x \geq 0$ in cash (i.e., domestic currency), we denote by $\mathbb{A}_x^1$ the set of all cash-settled contingent claims G that are hedgeable with initial endowment x, i.e.,

$$\mathbb{A}_x^1 \triangleq \{G \in \mathbb{L}_+^0 \mid (G, 0, \dots, 0) \in \mathbb{A}_{(x,0,\dots,0)}\}. \tag{3.1}$$

Using the liquidation function ℓ of (2.14), we can write the set $\mathbb{A}_x^1$ as

(3.2)
$$\mathbb{A}_x^1 = \left\{G \in \mathbb{L}_+^0 \mid \exists L \in \mathscr{A}((x, 0, \dots, 0)) : \ell(X^L_{(x,0,\dots,0)}(T-)) \geq G, \quad \text{a.s.}\right\}.$$

For each $z > 0$, we define a "cash-settled version" $\mathscr{D}_0^1(z)$ of the set $\mathscr{D}_0(\cdot)$ of (2.17) by

(3.3)
$$\mathscr{D}_0^1(z) \triangleq \left\{ Z \in \mathbb{L}_+^0 \,\middle|\, \begin{array}{l} \exists (Z^2, \dots, Z^d) \in \mathbb{L}^0(\mathbb{R}_+^{d-1}), \exists (z^2, \dots, z^d) \in \mathbb{R}_+^{d-1} \\ \text{such that} \quad (Z, Z^2, \dots, Z^d) \in \mathscr{D}_0((z, z^2, \dots, z^d)) \end{array} \right\}.$$

Noting $\mathscr{D}_0^1(z) = z\mathscr{D}_0^1(1)$ and our assumption $S^1(\cdot) \equiv 1$, we can see the following "cash-settled version" of Theorem 2.3.7.

Theorem 3.2.1. *Let G be a cash-settled contingent claim and let $x \geq 0$ be an initial endowment in cash. Then G is hedgeable with x, i.e., $G \in \mathbb{A}_x^1$ (equivalently, $(G, 0, \dots, 0)$ is hedgeable with $(x, 0, \dots, 0)$), if and only if*

$$G(0) \triangleq \sup_{Z \in \mathscr{D}_0^1(1)} E[ZG] \leq x.$$

In other words,

$$\mathbb{A}_x^1 = \{G \in \mathbb{L}_+^0 \mid \mathbb{E}[ZG] \leq x, \quad \forall Z \in \mathscr{D}_0^1(1)\}. \tag{3.4}$$

The number $G(0)$ is called the *upper-hedging price* of G. It is equal to the minimal amount of cash required to hedge the cash-settled contingent claim G.

For our purposes, we need to enlarge slightly the space $\mathscr{D}_0^1(z)$. As a dual of the space $\mathbb{A}_x^1$ in (3.1), we choose the space $\mathscr{D}^1(z)$ defined by

$$\mathscr{D}^1(z) \triangleq \left\{H \in \mathbb{L}_+^0 \mid \mathbb{E}[HG] \le zx,\ \forall x \ge 0 \text{ and } \forall G \in \mathbb{A}_x^1\right\} \tag{3.5}$$

for every $z \ge 0$. From (3.4), we can easily see that $\mathscr{D}_0^1(z) \subseteq \mathscr{D}^1(z)$.

For later use, we summarize some of the basic properties of the set $\mathbb{A}_x^1$ of (3.2) in Proposition 3.2.2 below. Given a subset A of $\mathbb{L}_+^0$, we define the *polar* A° of A by

$$A^\circ \triangleq \{g \in \mathbb{L}_+^0 \mid \mathbb{E}[fg] \le 1, \quad \forall f \in A\}. \tag{3.6}$$

Also, we say the set A is *solid* in $\mathbb{L}_+^0$ if $f \in A$, $g \in \mathbb{L}_+^0$ and $g \le f$ a.s. imply $g \in A$.

Proposition 3.2.2. **(i)** *The set $\mathbb{A}_1^1$ of (3.2) is convex, solid and closed in $\mathbb{L}_+^0$ under the topology of convergence in probability;*

(ii) *The constant function 1 belongs to $\mathbb{A}_1^1$;*

(iii) $\mathbb{A}_x^1 = x\mathbb{A}_1^1$ *for every* $x \ge 0$.

Proof. **(i)** Convexity and solidity are obvious. We show the closedness. First, let $\{G_n\}_{n\in\mathbb{N}} \subseteq \mathbb{A}_1^1$ be a sequence which converges almost surely to some $G \in \mathbb{L}_+^0$. Then, the sequence $\{(G_n, 0, \dots, 0)\}_{n\in\mathbb{N}} \subseteq \mathbb{A}_{(1,0,\dots,0)}$ clearly converges to $(G, 0, \dots, 0)$, and thus $(G, 0, \dots, 0) \in \mathbb{A}_{(1,0,\dots,0)}$ from Theorem 2.3.6. This implies that $G \in \mathbb{A}_1^1$.

Now, given a sequence $\{G_n\}_{n=1}^{\infty} \subseteq \mathbb{A}_1^1$ that converges to $G \in \mathbb{L}_+^0$ in probability as $n \to \infty$, we can extract a subsequence $\{G_{n_k}\}_{k=1}^{\infty}$ that converges to G as $k \to \infty$, almost surely. Thus, $G \in \mathbb{A}_1^1$.

(ii) By taking $L(\cdot) \equiv 0$, we see $X_{(1,0,\dots,0)}^0(\cdot) \equiv (1,0,\dots,0) \succeq 0$, and thus $(1,0,\dots,0) \in \mathbb{A}_1$. It follows from (3.1) that $1 \in \mathbb{A}_1^1$.

(iii) Clearly, $\mathbb{A}_0^1 = \{0\}$. Let $x > 0$. Given $G \in \mathbb{A}_1^1$, take an admissible trading strategy $L(\cdot) \in \mathscr{A}((1,0,\dots,0))$ such that $X_{(1,0,\dots,0)}^L(T-) \succeq (G,0,\dots,0)$ a.s. Consider the trading strategy $xL(\cdot)$. From (2.2), we have

$$X_{(x,0,\dots,0)}^{xL}(\cdot) = xX_{(1,0,\dots,0)}^L(\cdot).$$

This shows that $xL(\cdot) \in \mathscr{A}((x,0,\dots,0))$ and

$$X_{(x,0,\dots,0)}^{xL}(T-) = xX_{(1,0,\dots,0)}^L(T-) \succeq x(G,0,\dots,0).$$

Thus $xG \in \mathbb{A}_x^1$. Conversely, let $G \in \mathbb{A}_x^1$ and take an admissible trading strategy $L(\cdot) \in \mathscr{A}((x,0,\dots,0))$ such that $X_{(x,0,\dots,0)}^L(T-) \succeq (G,0,\dots,0)$. By the same argument, we can show that the trading strategy $L(\cdot)/x$ belongs to $\mathscr{A}((1,0,\dots,0))$ and $X_{(1,0,\dots,0)}^{L/x}(T-) \succeq (G/x,0,\dots,0)$. This shows $G/x \in \mathbb{A}_1^1$ and thus $G = x(G/x) \in x\mathbb{A}_1^1$. $\square$

There is a similar characterization for the sets of (3.5).

Proposition 3.2.3. **(i)** *The set $\mathscr{D}^1(1)$ of (3.5) is convex, solid and closed in $\mathbb{L}_+^0$ under the topology of convergence in probability. More precisely, it is the smallest such subset of $\mathbb{L}_+^0$ that contains the set $\mathscr{D}_0^1(1)$ of (3.3);*

(ii) *$\mathscr{D}^1(1)$ is bounded in $\mathbb{L}^1$;*

(iii) *$\mathscr{D}^1(z) = z\mathscr{D}^1(1)$ for every $z \geq 0$.*

Proof. **(i)** From (3.5) with $z = 1$, (3.6) and **(iii)** of Proposition 3.2.2, we have

$$\mathscr{D}^1(1) = (\mathbb{A}^1_1)^\circ.$$

On the other hand, (3.4) of Theorem 3.2.1 gives

$$\mathbb{A}^1_1 = (\mathscr{D}^1_0(1))^\circ.$$

Combining these two, we get

$$\mathscr{D}^1(1) = (\mathscr{D}^1_0(1))^{\circ\circ}.$$

But then, from the bipolar theorem of [5], $\mathscr{D}^1(1)$ is equal to the smallest convex, solid subset of $\mathbb{L}^0_+$, closed under the topology of convergence in probability $\mathbb{P}$, that contains $\mathscr{D}^1_0(1)$; for a brief synopsis of the bipolar theorem, see Appendix A.1.

(ii) Since the constant function 1 belongs to $\mathbb{A}^1_1$, we have $\mathbb{E}[H] \leq 1$ for every $H \in \mathscr{D}^1(1)$.

(iii) This follows immediately from (3.5). □

The next proposition indicates that the enlargement of the dual space $\mathscr{D}^1_0$ to $\mathscr{D}^1$ is "minimal", in the sense that it preserves the dual characterization of the upper-hedging price.

Proposition 3.2.4. *For every $G \in \mathbb{L}^0_+$, we have*

$$G(0) = \sup_{Z \in \mathscr{D}^1_0(1)} \mathbb{E}[ZG] = \sup_{H \in \mathscr{D}^1(1)} \mathbb{E}[HG]. \tag{3.7}$$

Proof. Since $\mathscr{D}^1_0(1) \subseteq \mathscr{D}^1(1)$, it is obvious that

$$\sup_{Z \in \mathscr{D}^1_0(1)} \mathbb{E}[ZG] \leq \sup_{H \in \mathscr{D}^1(1)} \mathbb{E}[HG].$$

To prove the reverse inequality, we first claim that for each fixed $G \in \mathbb{A}^1_x$, the map $J_G \colon \mathbb{L}^0_+ \to [0,\infty]$ defined by

$$J_G(H) \triangleq \mathbb{E}[HG] \tag{3.8}$$

is lower semi-continuous under the topology of convergence in probability. But, since this topology is also given by the metric

$$\varrho(H,K) \triangleq \mathbb{E}[1 \wedge |H-K|], \quad H,\, K \in \mathbb{L}^0_+,$$

we only need show that

$$\mathbb{E}[HG] \leq \lim_{n\to\infty} E[H_n G] \tag{3.9}$$

for any sequence $\{H_n\}_{n=1}^\infty \subseteq \mathbb{L}^0_+$, convergent to $H \in \mathbb{L}^0_+$ in probability, such that $\lim_{n\to\infty} E[H_n G]$ exists in $[0,\infty]$. Given such a sequence $\{H_n\}_{n=1}^\infty$, we may take a subsequence $\{H_{n_k}\}_{k=1}^\infty$ which converges to H almost surely; see, for example, Exercise 4.2.7 of [7]. It then follows from Fatou's lemma that

$$\mathbb{E}[HG] \leq \lim_{k\to\infty} \mathbb{E}[H_{n_k} G] = \lim_{n\to\infty} \mathbb{E}[H_n G],$$

which gives (3.9).

We now proceed to prove the assertion of the proposition. Let $H \in \mathscr{D}^1(1)$ be arbitrary. We first consider the case where $\mathbb{E}[HG] < \infty$. In this case, from the lower semi-continuity of the map $J_G(\cdot)$ of (3.8), for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathbb{E}[HG] < \mathbb{E}[KG] + \varepsilon$$

holds for every $K \in \mathbb{L}^0_+$ satisfying $\varrho(H,K) = \mathbb{E}[1 \wedge |H-K|] < \delta$. But since $\mathscr{D}^1(1)$ is the smallest convex, solid, closed set that contains $\mathscr{D}^1_0(1)$, there exists a convex combination of the form $\sum_{j=1}^m \alpha_j W_j$ such that $\varrho(H, \sum_{j=1}^m \alpha_j W_j) < \delta$, where $0 \leq \alpha_j \leq 1$, for every $j = 1, \ldots, m$, $\sum_{j=1}^m \alpha_j = 1$, $W_j \in \mathbb{L}^0_+$ and

$W_j \le Z_j$ for every $j = 1, \dots, m$ for some random variables $Z_j \in \mathscr{D}_0^1(1)$; see Proposition A.1.2 in Appendix A.1 for a proof of this claim. It then follows that

$$\begin{aligned}\mathbb{E}[HG] &< \mathbb{E}\left[\left(\sum_{j=1}^{m} \alpha_j W_j\right) G\right] + \varepsilon = \sum_{j=1}^{m} \alpha_j \mathbb{E}[W_j G] + \varepsilon \\ &\le \sum_{j=1}^{m} \alpha_j \mathbb{E}[Z_j G] + \varepsilon \le \left(\max_{1\le j\le m} \mathbb{E}[Z_j G]\right)\left(\sum_{j=1}^{m} \alpha_j\right) + \varepsilon \\ &\le \max_{1\le j\le m} \mathbb{E}[Z_j G] + \varepsilon \le \sup_{Z\in\mathscr{D}_0^1(1)} \mathbb{E}[ZG] + \varepsilon.\end{aligned}$$

Let $\varepsilon \downarrow 0$ to obtain

$$\mathbb{E}[HG] \le \sup_{Z\in\mathscr{D}_0^1(1)} \mathbb{E}[ZG].$$

Next, consider the case where $\mathbb{E}[HG] = \infty$. Then, the lower semi-continuity of the map $J_G(\cdot)$ of (3.8) implies that for every $M > 0$ there exists a number $\delta > 0$ such that $M < \mathbb{E}[KG]$ for every $K \in \mathscr{D}^1(1)$ with $\varrho(H, K) < \delta$. As in the above, we can show that

$$\sup_{Z\in\mathscr{D}_0^1(1)} \mathbb{E}[ZG] = \mathbb{E}[ZG] = \infty.$$

In either case, we have

$$\mathbb{E}[HG] \le \sup_{Z\in\mathscr{D}_0^1(1)} \mathbb{E}[ZG].$$

By taking the supremum over $H \in \mathscr{D}^1(1)$, we obtain the assertion. □

3.3 Minimizing Expected Shortfall

Suppose that G is a cash-settled contingent claim such that

$$G(0) \triangleq \sup_{Z\in\mathscr{D}_0^1(1)} \mathbb{E}[ZG] < \infty. \tag{3.10}$$

According to Theorem 3.2.1, with initial endowment $x \geq G(0)$, the claim issuer can hedge the contingent claim G without risk. However, as we mentioned in the introduction of this chapter, the upper-hedging price $G(0)$ is often too high, for the holder of the claim to agree to pay it at time $t = 0$. Now, given $x < G(0)$, there exists no hedging trading strategy for G, and the claim issuer must take some risks of shortfall. The natural question is then "what is the optimal trading strategy that minimizes the risk?". This is the partial-hedging problem we shall consider in the rest of this chapter. Clearly, what we need to do first, is to give a precise meaning to the concept "risk". In this section and the next, we choose the *expected shortfall* $\mathbb{E}(G - \ell(X^L_{(x,0,\ldots,0)}(T-)))^+$ as the risk measure and consider the minimization problem of it. The value function of this problem is then given by

$$V(x) \triangleq \inf_{L(\cdot)\in\mathscr{A}((x,0,\ldots,0))} \mathbb{E}\left(G - \ell(X^L_{(x,0,\ldots,0)}(T-))\right)^+, \quad x \geq 0. \tag{3.11}$$

To exclude uninteresting case, we assume, in addition to (3.10), that

$$\mathbb{E}[G] < \infty. \tag{3.12}$$

If the process $S(\cdot)$ is a martingale under the original probability measure $\mathbb{P}$, then we have $1 \in \mathscr{D}^1_0(1)$, and thus the inequality (3.12) automatically follows from (3.10). Finally, note that the value function $V(\cdot)$ of (3.11) can also be expressed as

$$V(x) = \inf_{\xi\in\mathbb{A}^1_x} \mathbb{E}(G - \xi)^+, \tag{3.13}$$

where the set $\mathbb{A}^1_x$ is defined by (3.1).

We adopt the *convex-duality approach*. In this approach, the role of "dual variables" will be played by the elements of the spaces $\mathscr{D}^1(z)$, $z > 0$ in (3.5). And as the "conjugate" of the convex objective function $R(u) \triangleq$

$(g-u)^+$, $u \geq 0$, for any given fixed $g \geq 0$, we consider the concave function $\tilde{R}\colon [0,\infty) \to \mathbb{R}$ defined as

$$\tilde{R}(v) \triangleq \inf_{u \geq 0}[(g-u)^+ + vu] = (1 \wedge v)g, \quad v \geq 0. \tag{3.14}$$

By definition, we have

$$(g-u)^+ \geq (1 \wedge v)g - vu, \quad \forall u \geq 0,\ \forall v \geq 0, \tag{3.15}$$

with equality if

$$u \in I(v). \tag{3.16}$$

Here I is the multifunction

$$I(v) \triangleq \left\{ \begin{array}{lll} \{g\} & \text{if} & 0 \leq v < 1 \\ [0,g] & \text{if} & v = 1 \\ \{0\} & \text{if} & v > 1 \end{array} \right\}. \tag{3.17}$$

The function $v \mapsto -\tilde{R}(-u)$ is the so-called Legendre-Fenchel transform of the map $u \mapsto (g-u)^+$, which is by now a well-known standard tool for optimization problems in mathematical finance.

Now, for any $\xi \in \mathbb{A}^1_x$, $z \geq 0$ and $H \in \mathscr{D}^1(1)$, the equations (3.15) and (3.5) give

$$\mathbb{E}(G-\xi)^+ \geq \mathbb{E}[(1 \wedge zH)G] - z\mathbb{E}[H\xi] \geq \mathbb{E}[(1 \wedge zH)G] - zx, \tag{3.18}$$

with equality if

$$\xi \in I(zH) \quad \text{and} \quad \mathbb{E}[H\xi] = x. \tag{3.19}$$

Notice that the first condition is equivalent to the existence of an $\mathscr{F}(T)$-measurable random variable U such that $0 \leq U \leq G$ and

$$\xi = G1_{\{zH<1\}} + U1_{\{zH=1\}}. \tag{3.20}$$

From this observation, we see that, with suitable z, H and U, the random variable ξ given by (3.20) attains the infimum in (3.13) if it belongs to $\mathbb{A}^1_x$ and satisfies $\mathbb{E}[H\xi] = x$. In order to find such z, H and U, we consider the *dual optimization problem* given by

$$W(x) \triangleq \sup_{z \geq 0,\, H \in \mathscr{D}^1(1)} \{\mathbb{E}[(1 \wedge zH)G] - zx\} = \sup_{z \geq 0} \gamma(z; x) \leq V(x) \tag{3.21}$$

thanks to (3.18) and (3.13), where

$$\gamma(z; x) \triangleq \tilde{V}(z) - zx, \quad z \geq 0,\ x \geq 0 \tag{3.22}$$

$$\tilde{V}(z) \triangleq \sup_{H \in \mathscr{D}^1(1)} \mathbb{E}[(1 \wedge zH)G], \quad z \geq 0. \tag{3.23}$$

By virtue of **(iii)** of Proposition 3.2.3, the concave function $\tilde{V}(\cdot)$ can also be expressed as

$$\tilde{V}(z) = \sup_{K \in \mathscr{D}^1(z)} \mathbb{E}[(1 \wedge K)G], \quad \forall z \geq 0. \tag{3.24}$$

The main result is the following theorem.

Theorem 3.3.1. *Let G be a cash-settled contingent claim satisfying the condition* (3.10) *and* (3.12), *and let x be initial endowment such that $0 < x < G(0)$. Then:*

- **(i)** *There exist a number $\hat{z} > 0$ and a random variable $\hat{H} \in \mathscr{D}^1(1)$ that attain the first supremum in* (3.21)*;*

- **(ii)** *There exists a random variable $U \in \mathbb{L}^0_+$ with $U \leq G$ a.s. such that the random variable*

$$\hat{\xi} \triangleq G 1_{\{\hat{z}\hat{H} < 1\}} + U 1_{\{\hat{z}\hat{H} = 1\}},$$

given by the right-hand-side of (3.20) *with $(z, H) \equiv (\hat{z}, \hat{H})$, belongs to $\mathbb{A}^1_x$, satisfies $\mathbb{E}[\hat{H}\hat{\xi}] = x$ and thereby attains the infimum in* (3.13).

We shall prove this theorem in the next section. The next corollary easily follows once this theorem is proved.

Corollary 3.3.2. *There exists a trading strategy* $L(\cdot) \in \mathscr{A}((x, 0, \dots, 0))$ *that attains the infimum in* (3.11).

Proof of Corollary 3.3.2. From the part **(ii)** of Theorem 3.3.1, we know that the random variable $\hat{\xi} \in \mathbb{A}_x^1$ attains the infimum in (3.13). Since $\hat{\xi}$ belongs to $\mathbb{A}_x^1$, there exists a trading strategy $\hat{L}(\cdot) \in \mathscr{A}((x, 0, \dots, 0))$ such that $\ell(X^L_{(x,0,\dots,0)}(T-)) \geq \hat{\xi}$, a.s., and hence

$$V(x) = \mathbb{E}(G - \hat{\xi})^+ \geq \mathbb{E}\left(G - \ell(X^{\hat{L}}_{(x,0,\dots,0)}(T-))\right)^+ \geq V(x).$$

This shows that the trading strategy $\hat{L}(\cdot)$ attains the infimum in (3.11). $\square$

3.4 Existence via Convex Duality

In this section we shall construct an optimal $\hat{\xi} \in \mathbb{A}_x^1$ for the primal problem, in terms of the solution of the dual problem (3.21). This will be done with the help of Lemmas 3.4.1 – 3.4.3 below. We can see by now that, in spite of the presence of transaction costs, the basic convex-duality structure of the optimization problem is completely parallel to that of an incomplete market; see [8]. We may thus mainly follow Section 3 of [8]; a similar methodology for testing composite hypotheses versus composite alternatives in mathematical statistics, was followed in the article [11].

Lemma 3.4.1. *For each* $z > 0$ *there exists a random variable* $\hat{H} \equiv \hat{H}(z) \in \mathscr{D}^1(1)$ *that attains the supremum in* (3.23).

Proof. Take a sequence $\{H_n\}_{n=1}^{\infty} \subseteq \mathscr{D}^1(1)$ such that $\lim_{n\to\infty} \uparrow \mathbb{E}[(1 \wedge zH_n)G] = \tilde{V}(z)$. First, from Proposition 3.2.3, we know that $\mathscr{D}^1(1)$ is bounded in $\mathbb{L}^1$.

Thus, from Komlós' theorem (see, for example, [30]), there exists a subsequence $\{H_{n_k}\}_{k=1}^{\infty}$ of $\{H_n\}_{n=1}^{\infty}$ such that the sequence $\{\Theta_k\}_{k=1}^{\infty}$ defined by

$$\Theta_k \triangleq \frac{1}{k}\sum_{i=1}^{k} H_{n_i}, \quad k \in \mathbb{N}$$

converges, almost surely, to some random variable $\hat{H} \in \mathbb{L}^0_+$. Since

$$\mathbb{E}[\Theta_k \xi] = \frac{1}{k}\sum_{i=1}^{k} \mathbb{E}[H_{n_i}\xi] \le \sup_{i\in\mathbb{N}} \mathbb{E}[H_{n_i}\xi] \le 1, \quad \forall \xi \in \mathbb{A}^1_1,$$

we have $\Theta_k \in \mathscr{D}^1(1)$, $\forall k \in \mathbb{N}$, and thus $\hat{H} \in \mathscr{D}^1(1)$ from the closedness of the set $\mathscr{D}^1(1)$ in probability. Next, the fact that $\Theta_k \in \mathscr{D}^1(1)$ and the concavity of the functional $H \mapsto \mathbb{E}[(1 \wedge zH)G]$ imply that

$$\tilde{V}(z) \ge \mathbb{E}[(1 \wedge z\Theta_k)G] \ge \frac{1}{k}\sum_{i=1}^{k} \mathbb{E}[(1 \wedge zH_{n_i})G], \quad \forall k \in \mathbb{N}. \tag{3.25}$$

Let $k \to \infty$. Then, being the Cesàro average of the set of numbers $\{\mathbb{E}[(1 \wedge zH_{n_i})G]\}_{i=1}^{k}$, the right-hand-side of (3.25) converges to $\tilde{V}(z)$, which implies that

$$\lim_{k\to\infty} \mathbb{E}[(1 \wedge z\Theta_k)G] = \tilde{V}(z). \tag{3.26}$$

Finally, since $0 \le (1 \wedge z\Theta_k)G \le G \in \mathbb{L}^1$, $\forall k \in \mathbb{N}$, the dominated convergence theorem gives

$$\mathbb{E}[(1 \wedge z\hat{H})G] = \lim_{k\to\infty} \mathbb{E}[(1 \wedge z\Theta_k)G]. \tag{3.27}$$

The equations (3.26) and (3.26) now give

$$\tilde{V}(z) = \mathbb{E}[(1 \wedge z\hat{H})G].$$

□

We next show that the function $\gamma(\cdot\,;x)$ of (3.22) is concave and attains its supremum at some $0 < \hat{z} < \infty$, when $0 < x < G(0)$. In order to prove this, we need to investigate the behavior of the function $\gamma(\cdot\,;x)$.

Lemma 3.4.2. *The function $\gamma(\cdot\,;x)$ defined by (3.22), is continuous and concave on $[0,\infty)$, and satisfies*

$$\lim_{z\to 0} \frac{\gamma(z;x)}{z} = G(0) - x > 0, \tag{3.28}$$

as well as

$$\lim_{z\to\infty} \frac{\gamma(z;x)}{z} = -x. \tag{3.29}$$

Proof of the concavity of $\gamma(\cdot\,;x)$: It suffices to prove the concavity of $\tilde{V}(\cdot)$. Let $z_1 \geq 0$, $z_2 > 0$, $K_1 \in z_1\mathscr{D}^1(1)$, $K_2 \in z_2\mathscr{D}^1(1)$ and $0 < \alpha < 1$. We clearly have

$$\mathbb{E}[(\alpha K_1 + (1-\alpha)K_2)\xi] = \alpha\mathbb{E}[K_1\xi] + (1-\alpha)\mathbb{E}[K_2\xi] \leq \alpha z_1 + (1-\alpha)z_2, \quad \forall \xi \in \mathbb{A}_1^1, \tag{3.30}$$

which implies that $\alpha K_1 + (1-\alpha)K_2 \in \mathscr{D}^1(\alpha z_1 + (1-\alpha)z_2)$. It then follows from (3.24) and the concavity of the map $K \mapsto (1\wedge K)G$ that

$$\begin{aligned}\tilde{V}(\alpha z_1 + (1-\alpha)z_2) &\geq \mathbb{E}[(1\wedge(\alpha K_1 + (1-\alpha)K_2))G] \\ &\geq \alpha\mathbb{E}[(1\wedge K_1)G] + (1-\alpha)\mathbb{E}[(1\wedge K_2)G].\end{aligned}$$

Taking the supremum of the right-hand-side over $K_1 \in z_1\mathscr{D}^1(1)$ and $K_2 \in z_2\mathscr{D}^1(1)$, and using (3.24), we obtain

$$\tilde{V}\,(\alpha z_1 + (1-\alpha)z_2) \geq \alpha\tilde{V}(z_1) + (1-\alpha)\tilde{V}(z_2).$$

$\square$

Proof of the continuity of $\gamma(\cdot\,;x)$*:* It suffices to prove that $\lim_{z\downarrow 0}\tilde{V}(z)=0$. By definition, we have $0\le\tilde{V}(z)=\sup_{H\in\mathscr{D}^1(1)}\mathbb{E}[(1\wedge zH)G]\le z\sup_{H\in\mathscr{D}^1(1)}\mathbb{E}[HG]=zG(0)$. Since $G(0)<\infty$, it follows that $\lim_{z\downarrow 0}\tilde{V}(z)=0$. □

Proof of (3.28): Since

$$\frac{\tilde{V}(z)}{z}=\sup_{H\in\mathscr{D}^1(1)}\mathbb{E}\left[\left(\frac{1}{z}\wedge H\right)G\right]\le\sup_{H\in\mathscr{D}^1(1)}\mathbb{E}[HG]=G(0),$$

we have

$$\overline{\lim_{z\downarrow 0}}\frac{\gamma(z;x)}{z}=\overline{\lim_{z\downarrow 0}}\frac{\tilde{V}(z)}{z}-x\le G(0)-x. \tag{3.31}$$

To show the reverse inequality for the limit-inferior, we take, for each $\varepsilon>0$, a random variable $H_\varepsilon\in\mathscr{D}^1(1)$ such that

$$\mathbb{E}[H_\varepsilon G]>\sup_{H\in\mathscr{D}^1(1)}\mathbb{E}[HG]-\varepsilon=G(0)-\varepsilon.$$

Then, for each $z>0$, we have

$$\begin{aligned}\frac{\gamma(z;x)}{z}&=\sup_{H\in\mathscr{D}^1(1)}\mathbb{E}\left[HG1_{\{zH<1\}}+\frac{G}{z}1_{\{zH\ge 1\}}\right]-x\\&\ge\sup_{H\in\mathscr{D}^1(1)}\mathbb{E}[HG1_{\{zH<1\}}]-x\\&\ge\mathbb{E}[H_\varepsilon G1_{\{zH_\varepsilon<1\}}]-x.\end{aligned}$$

Letting $z\downarrow 0$, we obtain from Fatou's lemma

$$\underline{\lim_{z\downarrow 0}}\frac{\gamma(z;x)}{z}\ge\underline{\lim_{z\downarrow 0}}\mathbb{E}[H_\varepsilon G1_{\{zH_\varepsilon<1\}}]-x\ge\mathbb{E}[H_\varepsilon G]-x>G(0)-\varepsilon-x.$$

Since $\varepsilon>0$ is arbitrary, we conclude that

$$\underline{\lim_{z\downarrow 0}}\frac{\gamma(z;x)}{z}\ge G(0)-x. \tag{3.32}$$

In conjunction with (3.31), the equation (3.32) now gives (3.28). □

Proof of (3.29): Since $V(x) \geq \gamma(z;x)$ from (3.21), we have

$$\frac{V(x)}{z} \geq \frac{\gamma(z;x)}{z} = \frac{\tilde{V}(z)}{z} - x \geq -x, \quad \forall z > 0.$$

Then, since $0 \leq V(x) \leq \mathbb{E}[G] < \infty$, we may let $z \to \infty$ to obtain

$$0 \geq \overline{\lim_{z\to\infty}} \frac{\tilde{V}(z)}{z} - x \geq -x,$$

and thus

$$x \geq \overline{\lim_{z\to\infty}} \frac{\tilde{V}(z)}{z} \geq 0.$$

By letting $x \downarrow 0$, we obtain

$$\lim_{z\to\infty} \frac{\tilde{V}(z)}{z} = 0.$$

It follows that

$$\lim_{z\to\infty} \frac{\gamma(z;x)}{z} = \lim_{z\to\infty} \frac{\tilde{V}(z)}{z} - x = -x.$$

This completes the proof of the lemma. $\square$

Lemma 3.4.3. *The function $\gamma(\cdot\,;x)$ attains its supremum at some $0 < \hat{z} < \infty$.*

Proof. Because of (3.29), we cannot have $\sup_{0<\zeta<\infty} \gamma(\zeta;x) = \lim_{\zeta\uparrow\infty} \gamma(\zeta;x) > \gamma(z;x)$, $\forall z \in (0,\infty)$. Therefore, the concave function $\gamma(\cdot\,;x)$ either attains its supremum at some $0 < \hat{z} < \infty$, or else $\gamma(z;x) \leq \gamma(0;x) = 0$, $\forall z > 0$. Suppose the latter is true. Then $\gamma(z;x)/z \leq 0$, $\forall z > 0$; but this is again impossible, because of (3.28). Therefore, the function $\gamma(\cdot\,;x)$ must attain its supremum at some $0 < \hat{z} < \infty$. $\square$

With the help of Lemmas 3.4.1 – 3.4.3, we now proceed to prove Theorem 3.3.1.

Proof of Theorem 3.3.1: Let $0 < \hat{z} < \infty$ be the number given in Lemma 3.4.3, and denote by $\hat{H} \equiv \hat{H}(\hat{z}) \in \mathscr{D}^1(1)$ the random variable given in Lemma 3.4.1. Then it is clear that the pair $(\hat{z}, \hat{H})$ attains the first supremum in (3.21). This proves **(i)**.

To prove **(ii)**, we introduce the space

$$\mathbb{L} \triangleq \mathbb{R} \times \mathbb{L}^1 \tag{3.33}$$

equipped with the norm

$$\|(z,K)\| \triangleq |z| + \mathbb{E}|K|, \quad (z,K) \in \mathbb{L}, \tag{3.34}$$

and consider its subset

$$\begin{aligned} \mathscr{G}^1 &\triangleq \{(z,H) \in \mathbb{L} \mid z \geq 0,\ H \in \mathscr{D}^1(z)\} \\ &= \{(z,zH) \in \mathbb{L} \mid z \geq 0,\ H \in \mathscr{D}^1(1)\}. \end{aligned} \tag{3.35}$$

Then from Proposition 3.2.3, $\mathscr{G}$ is convex in $\mathbb{L}$. Moreover, $\mathscr{G}^1$ is closed in the norm topology of $\mathbb{L}$. To see this, let $\{(z_n, z_n H_n)\}_{n=1}^{\infty}$ be any sequence in $\mathscr{G}^1$ that converges to some $(z,K) \in \mathbb{L}$. Then $z_n \to z$ and $\mathbb{E}|z_n H_n - K| \to 0$ as $n \to \infty$. If $z = 0$, then $\mathbb{E}|z_n H_n| = z_n \mathbb{E}|H_n| \to 0$ as $n \to \infty$, because of the $\mathbb{L}^1$-boundedness of $\mathscr{D}^1(1)$, thus $K \equiv 0$ a.s., and we are done. If $z \neq 0$, then

$$\begin{aligned} \mathbb{E}\left|H_n - \frac{K}{z}\right| &\leq \frac{1}{z}\Big(\mathbb{E}|zH_n - z_n H_n| + \mathbb{E}|z_n H_n - K|\Big) \\ &= \frac{1}{z}\Big((z - z_n)\mathbb{E}|H_n| + \mathbb{E}|z_n H_n - K|\Big) \\ &\to 0, \quad \text{as} \quad z \to \infty, \end{aligned}$$

again by the $\mathbb{L}^1$-boundedness of $\mathscr{D}^1(1)$. Since $\mathscr{D}^1(1)$ is closed under the topology of convergence in probability, it is closed in $\mathbb{L}^1$ as well; it follows that $K/z \in \mathscr{D}^1(1)$, thus $(z,K) \in \mathscr{G}^1$.

On the space $\mathbb{L}$, we consider the functional $\tilde{U}\colon \mathbb{L} \to \mathbb{R}$ given by

$$\tilde{U}(z,K) \triangleq -\mathbb{E}[(1\wedge K)G] + zx, \quad (z,K)\in\mathbb{L}. \tag{3.36}$$

It is easy to see that this functional is convex and proper. By using the dominated convergence theorem, we can easily check that $\tilde{U}$ is lower semi-continuous under the norm topology of $\mathbb{L}$. Furthermore, from Lemma 3.4.1 and Lemma 3.4.3, we know that $\tilde{U}$ attains the infimum over $\mathscr{G}^1$ at $(\hat{z},\hat{z}\hat{H}) \in \mathscr{G}^1\backslash\{(0,0)\}$.

Therefore, from standard results on convex optimization (for example, Corollary 4.6.3 of [2]), it follows that there exists a pair $(\hat{y},\hat{Y})$ in the dual space $\mathbb{L}^* = \mathbb{R}\times\mathbb{L}^\infty$ that satisfies

$$-(\hat{y},\hat{Y}) \in \partial\tilde{U}(\hat{z},\hat{z}\hat{H}) \tag{3.37}$$

and

$$(\hat{y},\hat{Y}) \in N(\hat{z},\hat{z}\hat{H}). \tag{3.38}$$

Here, $\partial\tilde{U}(\hat{z},\hat{z}\hat{H})$ is the subdifferential of $\tilde{U}$, and $N(\hat{z},\hat{z}\hat{H})$ is the normal cone of $\mathscr{G}^1$, at the point $(\hat{z},\hat{z}\hat{H})$. These sets are given by

$$\partial\tilde{U}(\hat{z},\hat{z}\hat{H}) \triangleq \left\{ (y,Y)\in\mathbb{L}^* \;\middle|\; \begin{array}{c} \big[-\mathbb{E}[(1\wedge\hat{z}\hat{H})G] + zx\big] - \big[-\mathbb{E}[(1\wedge K)G] + \hat{z}x\big] \\ \leq \mathbb{E}[(\hat{z}\hat{H}-K)Y] + (\hat{z}-z)y, \quad \forall (z,K)\in\mathbb{L} \end{array} \right\}, \tag{3.39}$$

and

$$N(\hat{z},\hat{z}\hat{H}) \triangleq \left\{ (y,Y)\in\mathbb{L}^* \;\middle|\; z\mathbb{E}[HY] + zy \leq \hat{z}\mathbb{E}[\hat{H}Y] + \hat{z}y, \quad \forall (z,zH)\in\mathscr{G}^1 \right\}; \tag{3.40}$$

see, for example Propositions 4.4.4 and 4.3.3 of [2]. By definition, (3.37) and (3.38) are equivalent to

$$\begin{aligned}(3.41)\qquad & -\mathbb{E}[(1\wedge \hat{z}\hat{H})G]+\hat{z}x+\mathbb{E}[(1\wedge K)G]-zx\\ &\qquad \le -\mathbb{E}[(\hat{z}\hat{H}-K)\hat{Y}]-(\hat{z}-z)\hat{y},\quad \forall (z,K)\in\mathbb{L},\end{aligned}$$

and

$$(3.42)\qquad z\mathbb{E}[H\hat{Y}]+z\hat{y}\le \hat{z}\mathbb{E}[\hat{H}\hat{Y}]+\hat{z}\hat{y},\quad \forall (z,zH)\in\mathscr{G}^1,$$

respectively. We claim that this $\hat{Y}$ serves as an optimal solution to (3.13). For this end, it is enough to prove the following.

(a) $\hat{Y}\in\mathbb{A}^1_x$;

(b) $\mathbb{E}[\hat{H}\hat{Y}]=x$;

(c) $\hat{Y}$ can be written as the right-hand-side of (3.20).

First, we note that $\hat{y}=-x$. Indeed, observe from (3.41) that

$$(\hat{z}-z)(x+\hat{y})\le \mathbb{E}[(1\wedge\hat{z}\hat{H})G]-\mathbb{E}[(1\wedge K)G]-\mathbb{E}[(\hat{z}\hat{H}-K)\hat{Y}],\quad \forall (z,K)\in\mathbb{L}.$$

If $x+\hat{y}\neq 0$, then letting $z\to\pm\infty$ with K fixed, we could make the left-hand-side $\to\infty$, a contradiction. Therefore $\hat{y}=-x$.

Next, we show that $\hat{Y}$ satisfies $\mathbb{E}[\hat{H}\hat{Y}]=x$. From (3.42) with $\hat{y}=-x$, we have

$$-x(z-\hat{z})+\mathbb{E}[(zH-\hat{z}\hat{H})\hat{Y}]\le 0,\quad \forall (z,zH)\in\mathscr{G}^1.$$

Letting $z=0$, we obtain

$$x\hat{z}-\hat{z}\mathbb{E}[\hat{H}\hat{Y}]\le 0,$$

and hence, by dividing by $\hat{z} > 0$,

$$\mathbb{E}[\hat{H}\hat{Y}] \geq x.$$

Also, from (3.42) with $\hat{y} = -x$, $H = \hat{H}$ and $z = \hat{z} + \varepsilon$ for some $\varepsilon > 0$, we get

$$(\hat{z} + \varepsilon)\mathbb{E}[\hat{H}\hat{Y}] - \hat{z}\mathbb{E}[\hat{H}\hat{Y}] \leq (\hat{z} + \varepsilon)x - \hat{z}x$$

and hence $\mathbb{E}[\hat{H}\hat{Y}] \leq x$. Therefore,

$$\mathbb{E}[\hat{H}\hat{Y}] = x. \tag{3.43}$$

Now, from (3.42) with $z = \hat{z}$, we can see

$$\mathbb{E}[\hat{H}\hat{Y}] \geq \mathbb{E}[H\hat{Y}],$$

for any $H \in \mathscr{D}^1(1)$. This together with (3.43) implies

$$\mathbb{E}[H\hat{Y}] \leq x, \quad \forall H \in \mathscr{D}^1(1),$$

showing $\hat{Y} \in \mathbb{A}^1_x$.

Finally, we show that $\hat{Y}$ can be written as the right-hand-side of (3.20). To show this, we define the random variable A by

$$\hat{Y} = G1_{\{\hat{z}\hat{H}<1\}} + A. \tag{3.44}$$

We need to show that this random variable A satisfies $\mathbb{P}[0 \leq A \leq G] = 1$ and $\mathbb{P}[A \neq 0,\ \hat{z}\hat{H} = 1] = 0$. Observer first that (3.41) with $\hat{y} = -x$ implies

$$-\mathbb{E}[(1 \wedge \hat{z}\hat{H})G] + \mathbb{E}[(1 \wedge K)G] \leq \mathbb{E}[(G1_{\{\hat{z}\hat{H}<1\}} + A)(K - \hat{z}\hat{H})],$$

and hence,

$$\begin{aligned} \mathbb{E}[A(K - \hat{z}\hat{H})] & \\ &\geq \mathbb{E}[(1 \wedge K)G] - \mathbb{E}[(1 \wedge \hat{z}\hat{H})G] - \mathbb{E}[(K - \hat{z}\hat{H})G1_{\{\hat{z}\hat{H}<1\}}] \\ &= \mathbb{E}\left[\left(1_{\{K\geq 1\}} - 1_{\{\hat{z}\hat{H}\geq 1\}}\right) G\right] + \mathbb{E}\left[\left(1_{\{K<1\}} - 1_{\{\hat{z}\hat{H}<1\}}\right) KG\right] \\ &= \mathbb{E}\left[\left(1_{\{K<1\}} - 1_{\{\hat{z}\hat{H}<1\}}\right) (K-1)G\right] \end{aligned} \tag{3.45}$$

for every $K \in \mathbb{L}^1$. In particular, for every $K \in \mathbb{L}^1$ with $\{K < 1\} = \{\hat{z}\hat{H} < 1\}$, the inequality (3.45) reduces to

$$\mathbb{E}[A(K - \hat{z}\hat{H})] \geq 0. \tag{3.46}$$

First, suppose that $\mathbb{P}[\hat{z}\hat{H} < 1,\ A > 0] > 0$, and take

$$K \triangleq -1_{\{\hat{z}\hat{H}<1,\ A>0\}} + \hat{z}\hat{H}1_{\{\hat{z}\hat{H}\geq 1 \text{ or } A\leq 0\}}$$

so that $K \in \mathbb{L}^1$ and $\{K < 1\} = \{\hat{z}\hat{H} < 1\}$, and thus that the inequality (3.46) applies to this K. But under the assumption $\mathbb{P}[\hat{z}\hat{H} < 1,\ A > 0] > 0$, we would also have

$$\mathbb{E}[A(K - \hat{z}\hat{H})] = -\mathbb{E}[A(\hat{z}\hat{H} + 1)1_{\{\hat{z}\hat{H}<1,\,A>0\}}] \leq -\mathbb{E}[A1_{\{\hat{z}\hat{H}<1,\,A>0\}}] < 0,$$

a contradiction. Therefore, it must be that

$$A \leq 0 \quad \text{on} \quad \{\hat{z}\hat{H} < 1\}. \tag{3.47}$$

Next, suppose that $\mathbb{P}[\hat{z}\hat{H} \geq 1,\ A < 0] > 0$, and take

$$K \triangleq \hat{z}\hat{H}1_{\{\hat{z}\hat{H}<1 \text{ or } A\geq 0\}} + 2\hat{z}\hat{H}1_{\{\hat{z}\hat{H}\geq 1,\ A<0\}}.$$

so that $K \in \mathbb{L}^1$ and $\{KG < 1\} = \{\hat{z}\hat{H}G < 1\}$ and thus that the inequality (3.46) applies to this K. But under the assumption $\mathbb{P}[\hat{z}\hat{H} \geq 1,\ A < 0] > 0$ we would also have

$$\mathbb{E}[A(K - \hat{z}\hat{H})] = \mathbb{E}[A\hat{z}\hat{H}1_{\{\hat{z}\hat{H}\geq 1,\,A<0\}}] \leq \mathbb{E}[A1_{\{\hat{z}\hat{H}\geq 1,\,A<0\}}] < 0,$$

a contradiction. Therefore, it must be that

$$A \geq 0 \quad \text{on} \quad \{\hat{z}\hat{H} \geq 1\}. \tag{3.48}$$

Next, suppose $\mathbb{P}[A < 0,\ \hat{z}\hat{H} < 1] > 0$. Then, there exists a number $\delta > 0$ such that

$$\mathbb{E}[A(\hat{z}\hat{H} - 1)1_{\{\hat{z}\hat{H}<1\}}] > \delta. \tag{3.49}$$

For given $\varepsilon > 0$, take

$$K \triangleq (1-\varepsilon)1_{\{\hat{z}\hat{H}<1\}} + 1_{\{\hat{z}\hat{H}\geq 1\}}.$$

Thanks to (3.48) and (3.49), we have then

$$\begin{aligned}
\mathbb{E}[A(K - \hat{z}\hat{H})] &= \mathbb{E}[A((1-\varepsilon)1_{\{\hat{z}\hat{H}<1\}} + 1_{\{\hat{z}\hat{H}\geq 1\}} - \hat{z}\hat{H})] \\
&= -\mathbb{E}[A(\hat{z}\hat{H} - 1)1_{\{\hat{z}\hat{H}<1\}}] - \varepsilon\mathbb{E}[A1_{\{\hat{z}\hat{H}<1\}}] - \mathbb{E}[A(\hat{z}\hat{H} - 1)1_{\{\hat{z}\hat{H}\geq 1\}}] \\
&\leq -\mathbb{E}[A(\hat{z}\hat{H} - 1)1_{\{\hat{z}\hat{H}<1\}}] - \varepsilon\mathbb{E}[A1_{\{\hat{z}\hat{H}<1\}}] \\
&< -\delta - \varepsilon\mathbb{E}[A1_{\{\hat{z}\hat{H}<1\}}] \\
&< 0,
\end{aligned}$$

which contradicts (3.46). Therefore,

$$A = 0 \quad \text{on} \quad \{\hat{z}\hat{H} < 1\}. \tag{3.50}$$

In conjunction with (3.50), the inequality (3.46) becomes

$$\mathbb{E}[A(K - \hat{z}\hat{H})1_{\{\hat{z}\hat{H}\geq 1\}}] \geq 0, \tag{3.51}$$

which is to hold for every $K \in \mathbb{L}^1$ satisfying $\{K < 1\} = \{\hat{z}\hat{H} < 1\}$. Take

$$K \triangleq 1_{\{\hat{z}\hat{H}\geq 1\}}.$$

Then, the inequality (3.51) applies to this K and yields

$$\mathbb{E}[A(1 - \hat{z}\hat{H})1_{\{\hat{z}\hat{H}>1\}}] \geq 0.$$

The integrand is nonpositive because of (3.48). It is strictly negative on the set $\{A > 0, \hat{z}\hat{H} > 1\}$. Since the integral is nonnegative, it follows that

$$A = 0 \quad \text{on} \quad \{\hat{z}\hat{H} > 1\}. \tag{3.52}$$

Then, (3.50) and (3.52) together with (3.45) imply

$$\mathbb{E}[A(K-1)1_{\{\hat{z}\hat{H}=1\}}] \geq \mathbb{E}\left[\left(1_{\{K<1\}} - 1_{\{\hat{z}\hat{H}<1\}}\right)(K-1)G\right], \forall K \in \mathbb{L}^1. \tag{3.53}$$

It remains to show that $A \leq G$ on the set $\{\hat{z}\hat{H} = 1\}$. Suppose that $\mathbb{P}[A > G,\ \hat{z}\hat{H} = 1] > 0$. Then, there exists a number $\delta > 0$ such that

$$\mathbb{E}[A1_{\{A>G,\ \hat{z}\hat{H}=1\}}] > \delta + \mathbb{E}[G1_{\{A>G,\ \hat{z}\hat{H}=1\}}].$$

Let

$$K \triangleq 1_{\{A\leq G,\ \hat{z}\hat{H}=1\}} + (1-\varepsilon)1_{\{\hat{z}\hat{H}\neq 1\}} \tag{3.54}$$

for an arbitrary $\varepsilon > 0$. Then, we can easily see that

$$(K-1) = -1_{\{A>G,\ \hat{z}\hat{H}=1\}} - \varepsilon 1_{\{\hat{z}\hat{H}\neq 1\}}$$

and

$$1_{\{K<1\}} - 1_{\{\hat{z}\hat{H}<1\}} = 1_{\{A>G,\ \hat{z}\hat{H}=1\}} + 1_{\{\hat{z}\hat{H}>1\}}.$$

Using these equalities, we compute both sides of (3.53) as

$$\text{LHS of (3.53)} = -\mathbb{E}[A1_{\{A>G,\ \hat{z}\hat{H}=1\}}] < -\delta - \mathbb{E}[G1_{\{A>G,\ \hat{z}\hat{H}=1\}}], \tag{3.55}$$

and

$$\begin{aligned} &\text{RHS of (3.53)} \\ &\quad = -\mathbb{E}\left[\left(1_{\{A>G,\ \hat{z}\hat{H}=1\}} + 1_{\{\hat{z}\hat{H}>1\}}\right)\left(1_{\{A>G,\ \hat{z}\hat{H}=1\}} + \varepsilon 1_{\{\hat{z}\hat{H}\neq 1\}}\right)G\right] \\ &\quad = -\mathbb{E}\left[\left(1_{\{A>G,\ \hat{z}\hat{H}=1\}} + \varepsilon 1_{\{\hat{z}\hat{H}>1\}}\right)G\right] \end{aligned} \tag{3.56}$$

respectively. Combining the equations (3.55) and (3.56) with (3.53), we obtain

$$0 \leq \text{LHS of (3.53)} - \text{RHS of (3.53)} < -\delta + \varepsilon \mathbb{E}[G 1_{\{\hat{z}\hat{H}>1\}}].$$

This must be true for any $\varepsilon > 0$, which is impossible. Therefore, $A \leq G$ on the set $\{\hat{z}\hat{H} = 1\}$, which completes the proof. $\square$

3.5 Maximizing the Probability of Perfect Hedge

As in the previous section, let G be a cash-settled contingent claim satisfying (3.10). In this section, we consider the problem of maximizing the *probability of perfect hedge* $\mathbb{P}[\ell(X^L_{(x,0,\dots,0)}(T-)) \geq G]$, and the problem of maximizing the *expected success-ratio* $\mathbb{E}[\varphi^L_{(x,0,\dots,0)}]$ with

$$\begin{aligned}(3.57)\quad \varphi^L_{(x,0,\dots,0)} \triangleq\ & 1_{\{\ell(X^L_{(x,0,\dots,0)}(T-))\geq G\}} \\ & + \frac{\ell(X^L_{(x,0,\dots,0)}(T-))}{G} \cdot 1_{\{\ell(X^L_{(x,0,\dots,0)}(T-))<G\}},\end{aligned}$$

over trading strategies $L(\cdot) \in \mathscr{A}((x,0,\dots,0))$ for a given initial endowment $0 < x < G(0)$ in cash (domestic currency). The value functions corresponding to these two optimization problems are

$$(3.58)\quad V_1(x) \triangleq \sup_{L(\cdot)\in\mathscr{A}((x,0,\dots,0))} \mathbb{P}[\ell(X^L_{(x,0,\dots,0)}(T-)) \geq G],$$

and

$$\begin{aligned}(3.59)\quad V_2(x) \triangleq \sup_{L(\cdot)\in\mathscr{A}((x,0,\dots,0))} \mathbb{E}\Bigg[& 1_{\{\ell(X^L_{(x,0,\dots,0)}(T-))\geq G\}} \\ & + \frac{\ell(X^L_{(x,0,\dots,0)}(T-))}{G} \cdot 1_{\{\ell(X^L_{(x,0,\dots,0)}(T-))<G\}}\Bigg],\end{aligned}$$

respectively. These functions can also be written as

$$V_1(x) = \sup_{\xi \in \mathbb{A}_x^1} \mathbb{P}[\xi \geq G] = \sup_{\xi \in \mathbb{A}_x^1} \mathbb{E}[1_{\{\xi \geq G\}}], \tag{3.60}$$

and

$$V_2(x) = \sup_{\xi \in \mathbb{A}_x^1} \mathbb{E}\left[1_{\{\xi \geq G\}} + \frac{\xi}{G} 1_{\{\xi < G\}}\right], \tag{3.61}$$

respectively, where the set $\mathbb{A}_x^1$ was defined in (3.1). Unlike the minimization of expected shortfall, the two problems considered in this section are not based on the first moment, and thus we no longer need the assumption $\mathbb{E}[G] < \infty$.

The difference between the two partial hedging problems of (3.58) and (3.59) is that, while the former considers only the event that the terminal holdings vector successfully hedges the liability, the latter evaluates the performance of the trading strategy even in the case of a not completely successful hedge ("partial credit"). In this sense, the latter is a more sophisticated risk measure than the former. Nevertheless, the probability of perfect hedge is also meaningful as a dynamic version of the *Value-at-Risk* concept, which is commonly used by practitioners.

As in the previous chapter, we adopt the convex duality approach. We first observe that for a given constant $g \geq 0$, the function

$$R_1(u) \triangleq 1_{[g,\infty)}(u) + \frac{u}{g} 1_{[0,g)}(u), \quad u \in [0, \infty)$$

is the smallest concave function that dominates the function

$$R_2(u) \triangleq 1_{[g,\infty)}(u), \quad u \in [0, \infty)$$

(In the definition of $R_1(u)$, we take the last term to be 0, when $g = 0$). Therefore, these two functions $R_1(\cdot)$ and $R_2(\cdot)$ have the common conjugate function $\tilde{R}$: $[0,\infty) \to \mathbb{R}$ given by

$$\begin{aligned} \tilde{R}(v) &\triangleq (1-vg)^+ \\ &= \sup_{u\geq 0}[1_{[g,\infty)}(u) - vu] && (3.62) \\ &= \sup_{u\geq 0}\left[1_{[g,\infty)}(u) + \frac{u}{g}1_{[0,g)}(u) - vu\right], && (3.63) \end{aligned}$$

so that

$$1_{[g,\infty)}(u) \leq 1_{[g,\infty)}(u) + \frac{u}{g}1_{[u,\infty)}(u) \leq (1-vg)^+ + vu. \tag{3.64}$$

The supremum in (3.63) is attained, and thus the second inequality in (3.64) holds as equality, if and only if we have $u \in I_1(v)$ with

$$I_1(v) \triangleq \left\{ \begin{array}{lll} \{0\} & \text{if} & vg > 1 \\ [0,g] & \text{if} & vg = 1 \\ \{g\} & \text{if} & 0 \leq vg < 1 \end{array} \right\}. \tag{3.65}$$

Furthermore, the supremum in (3.62) is attained – and thus all the inequalities in (3.64) hold as equalities – if and only if we have $u \in I_2(v)$ with

$$I_2(v) \triangleq \left\{ \begin{array}{lll} \{0\} & \text{if} & vg > 1 \\ \{0,g\} & \text{if} & vg = 1 \\ \{g\} & \text{if} & 0 \leq vg < 1 \end{array} \right\}. \tag{3.66}$$

From the second inequality of (3.64), for any $\xi \in \mathbb{A}^1_x$, $z > 0$ and $H \in \mathscr{D}^1(1)$, we have

$$1_{\{\xi \geq G\}} + \frac{\xi}{G}1_{\{\xi < G\}} \leq (1 - zHG)^+ + zH\xi,$$

and hence, by taking expectations,

$$\begin{aligned} \mathbb{E}\left[1_{\{\xi\geq G\}} + \frac{\xi}{G}1_{\{\xi<G\}}\right] &\leq \mathbb{E}(1-zHG)^+ + z\mathbb{E}[H\xi] \\ &\leq \mathbb{E}(1-zHG)^+ + zx. \end{aligned} \tag{3.67}$$

Both inequalities in (3.67) hold as equalities, if and only if

$$\xi \in I_1(zH) \quad \text{and} \quad \mathbb{E}[H\xi] = x. \tag{3.68}$$

Note that the first of theses conditions is equivalent to the existence of an $\mathscr{F}(T)$-measurable random variable U with $0 \le U \le G$ such that

$$\xi = G1_{\{zHG<1\}} + U1_{\{zHG=1\}}. \tag{3.69}$$

In a similar fashion, we have

$$\mathbb{P}[\xi \ge G] = \mathbb{E}\left[1_{\{\xi \ge G\}} + \frac{\xi}{G}1_{\{\xi<G\}}\right] = \mathbb{E}(1 - zHG)^+ + zx, \tag{3.70}$$

if and only if

$$\xi \in I_2(zH) \quad \text{and} \quad \mathbb{E}[H\xi] = x. \tag{3.71}$$

The first condition is equivalent to the existence of an $\mathscr{F}(T)$-measurable set E such that

$$\xi = G1_{\{zHG<1\}} + G1_{E\cap\{zHG=1\}}. \tag{3.72}$$

The dual optimization problem, is then given by

$$W(x) \triangleq \inf_{z\ge 0,\, H\in\mathscr{D}^1(1)} \left\{\mathbb{E}(1 - zHG)^+ + zx\right\} = \inf_{z\ge 0} \gamma(z;x), \tag{3.73}$$

where

$$\gamma(z;x) \triangleq \tilde{V}(z) + zx, \quad z \ge 0,\ x \ge 0 \tag{3.74}$$

$$\tilde{V}(z) \triangleq \inf_{H\in\mathscr{D}^1(1)} \mathbb{E}(1 - zHG)^+ = \inf_{K\in\mathscr{D}^1(z)} \mathbb{E}(1 - KG)^+, \quad z \ge 0. \tag{3.75}$$

Notice that this dual problem is common to *both* of the primal problems (3.60) and (3.61).

The next theorem is the existence result for the optimization problems (3.60) and (3.61). Since this result can be proved in much the same way as Theorem 3.3.1, we present the proof in Appendix A.2.

Theorem 3.5.1. *Let G be a cash-settled contingent claim with $G(0) < \infty$ and x be initial endowment with $0 < x < G(0)$. Then:*

(i) *There exist a number $\hat{z} > 0$ and a random variable $\hat{H} \in \mathscr{D}^1(1)$ that attain the infimum in* (3.73)*;*

(ii) *There exists a random variable $U \in \mathbb{L}^0_+$ with $U \leq G$ a.s. such that the random variable*

$$\hat{\xi} \triangleq G1_{\{\hat{z}\hat{H}G<1\}} + U1_{\{\hat{z}\hat{H}G=1\}},$$

given by the right-hand-side of (3.69) *with $(z, H) \equiv (\hat{z}, \hat{H})$, belongs to $\mathbb{A}^1_x$, satisfies $\mathbb{E}[\hat{H}\hat{\xi}] = x$, and thereby attains the supremum in* (3.61).

(iii) *Furthermore, if there exists a set $E \in \mathscr{F}(T)$ such that $U = G1_E$ a.s., then the random variable $\hat{\xi}$ of* **(ii)** *attains the supremum in* (3.60) *as well.*

The next corollary easily follows once this theorem is proved.

Corollary 3.5.2. *There exists a trading strategy $L(\cdot) \in \mathscr{A}((x, 0, \dots, 0))$ that attains the supremum in* (3.59)*. Furthermore, if the random variable U of Theorem 3.5.1 can be written as $U = G1_E$ almost surely for some set $E \in \mathscr{F}(T)$, then there exists a trading strategy $L(\cdot) \in \mathscr{A}((x, 0, \dots, 0))$ that attains the supremum in* (3.58).

Proof of Corollary 3.5.2. From the part **(ii)** of Theorem 3.5.1, we know that the random variable $\hat{\xi} \in \mathbb{A}^1_x$ attains the infimum in (3.61). Since $\hat{\xi}$ belongs to $\mathbb{A}^1_x$, there exists a trading strategy $\hat{L}(\cdot) \in \mathscr{A}((x, 0, \dots, 0))$ such that

$\ell(X^{L}_{(x,0,\dots,0)}(T-)) \geq \hat{\xi}$, a.s., and hence

$$
\begin{aligned}
V_2(x) &= \mathbb{E}\left[1_{\{\xi \geq G\}} + \frac{\xi}{G} 1_{\{\xi < G\}}\right] \\
&\leq \mathbb{E}\left[1_{\{X^{\hat{L}}_{(x,0,\dots,0)}(T-) \geq G\}} + \frac{X^{\hat{L}}_{(x,0,\dots,0)}(T-)}{G} 1_{\{\xi < G\}}\right] = V_2(x).
\end{aligned}
$$

This shows that the trading strategy $\hat{L}$ attains the supremum in (3.59). Furthermore, when the random variable U of Theorem 3.5.1 can be written as $U = G1_E$ almost surely for some set $E \in \mathscr{F}(T)$, we have

$$
V_1(x) = \mathbb{P}\left[\hat{\xi} \geq G\right] \leq \mathbb{P}\left[X^{\hat{L}}_{(x,0,\dots,0)}(T-) \geq G\right] \leq V_1(x),
$$

and hence the trading strategy $\hat{L}(\cdot)$ attains the supremum in (3.58) as well.

$\square$

Chapter 4 Utility Maximization

4.1 Introduction

In this chapter, we consider utility maximization problems under the general model with transaction costs, which we introduced in Chapter 2. In most of the existing literature, it is assumed that an investor gains utility from the liquidated value of the portfolio holdings vector; a typical formulation of the optimization problem is

$$\text{Maximize } \mathbb{E}[U(\ell(X_x^L(T-)))] \text{ over } L(\cdot) \in \mathscr{A}(x), \tag{4.1}$$

where $U : \mathbb{R} \to \mathbb{R}$ is a pre-specified *uni-variate* utility function and $x \in \mathbb{R}^d$ is a given initial endowment vector. Behind such a formulation is the assumption that there exists a single consumption good, which can be purchased only through the domestic currency. This is why utility depends on the portfolio holdings vector only through the liquidation function ℓ.

The paper [16] recently studied a utility maximization problem in the case where the utility function depends on the terminal holdings in a rather arbitrary way, not just through liquidation; the optimization problem there is

$$\text{Maximize } \mathbb{E}[U(X_x^L(T-))] \text{ over } L(\cdot) \in \mathscr{A}(x), \tag{4.2}$$

where, this time, $U : \mathbb{R}^d \to \mathbb{R}$ is a *multi-variate* utility function. This formulation enables consideration of a situation, where several consumption goods are available and can be purchased with foreign currencies. The results obtained in that paper, which inspired our investigation in this chapter, are quite general but in our view not completely satisfactory: They are formulated in terms of a utility function which depends on the terminal holdings measured in monetary units, i.e., the domestic currency units; this causes the so-called "money illusion", meaning that utility changes if the prices change, even when the physical amount of consumption goods actually consumed stays the same. In fact, the standard microeconomics theory tells that an indirect or derived utility function (utility from wealth), when correctly derived from a direct utility function (utility from consumption), has to depend on the relative price of the consumption goods as well as wealth. Notice that the money illusion can be avoided for the classical case (4.1), simply by taking the consumption good to be the numeraire, which is why we usually consider utility from wealth in mathematical finance.

Furthermore, in the presence of transaction costs, there remain several difficulties in dealing with an indirect utility function even when it is money-illusion free. First, unlike a market without transaction costs, it is essential that the indirect utility function be non-smooth; thus, dealing with an indirect utility function, requires rather heavy tools (much heavier than what we used in the previous chapter) from non-smooth convex analysis; see Sections 6 – 9 of [16]. This is because an indirect utility depends not only on the consumer's preference, but also on the market structure such as the solvency region, which is naturally non-smooth. For example, as a function of the portfolio holdings vector $X_x^L(T-)$, the utility function $U(\ell(\cdot))$ in (4.1) is non-smooth because of the non-smoothness of the function ℓ. Secondly, because of this dependence on the solvency region, it is rather difficult to specify an indirect utility function. In fact, apart from the simplest case of utility from the liquidated wealth, it seems that the only way of giving an appropriate indirect utility function is to derive it via (4.26) below, from some direct utility function. However, as we shall see in Proposition 4.3.5, the utility maximization problem with the indirect utility function given in (4.26) is equivalent to that with the original direct utility function; thus it makes good sense to deal with the direct utility function from the beginning.

The aim of this chapter is thus to propose an alternative approach to the utility maximization problem under transaction costs. As we indicated above, we deal with a direct utility function rather than an indirect utility function. The advantage of dealing with direct utility in the context of markets with transaction costs is that we can specify a utility function independently of the solvency region. In particular, there is no essential reason to specify a utility function as non-smooth. In fact, we shall define a direct utility function as a smooth concave function on $\mathbb{R}^d$; see Definition 4.2.1. It seems that this still covers nearly all interesting cases of utility functions, including the utility from the liquidated terminal wealth studied by [9], [13] and [20]. Also, there are many obvious ways of producing examples of a direct utility function (Example 4.2.6 below is one of them), whereas this is not the case for indirect utility. Furthermore, thanks to the simple structure of direct utility, we can prove the existence of an optimal terminal wealth by using standard tools from convex analysis and the smooth calculus of variations, without recourse to non-smooth analysis.

4.2 The Optimization Problem

Except for Section 4.5, we consider the problem of maximizing expected utility from consumption at the terminal time T. For simplicity, we assume that for each currency, there is one representative consumption good which can be purchased with that currency only. It is clear that without loss of generality, the price of each consumption good is equal to 1 unit of the corresponding currency. For the investor we are concerned with, not all consumption goods have to be actually consumed. In other words, there may be some currencies which are purely investment instruments for the investor. To take into account this possibility, we assume that the first d_1 currencies, where $1 \le d_1 \le d$, are relevant to the consumption goods which are actually to be consumed; thus, the other $d_2 (\triangleq d - d_1)$ currencies are purely investment instruments. We may formulate this idea by making the utility function depend only on the first d_1 components of the consumption vector c.

Definition 4.2.1 (Utility Function). A *(direct) utility function* is a function $U\colon \mathbb{R}_+^{d_1} \times \mathbb{R}_+^{d_2} \to \mathbb{R}$ of the form

$$U(c_1, c_2) = U_1(c_1), \quad (c_1, c_2) \in \mathbb{R}_+^{d_1} \times \mathbb{R}_+^{d_2} \tag{4.3}$$

with $1 \le d_1 \le d$ and $d_1 + d_2 = d$, where the function $U_1\colon \mathbb{R}_+^{d_1} \to \mathbb{R}$ satisfies the following conditions:

(i) U_1 is concave and continuous on $\mathbb{R}_+^{d_1}$ with the properties

$$\sup_{c_1 \in \mathbb{R}_+^{d_1}} U_1(c_1) = \infty \tag{4.4a}$$

$$\inf_{c_1 \in \mathbb{R}_+^{d_1}} U_1(c_1) = U_1(0) = 0; \tag{4.4b}$$

(ii) U_1 is "increasing" in the sense that

$$U_1(c_1) \le U_1(\bar{c}_1) \quad \text{if } c_1^i \le \bar{c}_1^i \text{ for each } i = 1, \dots, d_1; \tag{4.5}$$

(iii) U_1 is of class C^1 and strictly concave on $(0,\infty)^{d_1}$ with the gradient

$$\nabla U_1(\cdot) \triangleq \left(\frac{\partial U_1}{\partial c^1}(\cdot), \dots, \frac{\partial U_1}{\partial c^{d_1}}(\cdot)\right)$$

mapping $(0,\infty)^{d_1}$ bijectively onto itself. We denote by I_1 the inverse function of ∇U_1.

Remark 4.2.2. P. Lakner, in his dissertation [27], considered a market model with several consumption goods but without transaction costs. Apart from some slight differences about the smoothness condition and the behavior at the boundary, the function U_1 is essentially the same as his utility function; see Definition 2.1 of [27].

The main question we are concerned about is the problem of maximizing expected utility $\mathbb{E}[U(C)]$ over terminal consumption vectors C which can be purchased with some admissible trading strategy for a given initial endowment vector $x \in K$. The precise definition for this "financeability" is the following. For "admissibility" of a trading strategy $L(\cdot)$, look back to Definition 2.3.3.

Definition 4.2.3. An $\mathbb{R}^d_+$-valued, $\mathscr{F}(T)$-measurable random vector C is simply called a *terminal consumption vector*. A terminal consumption vector C is called *financeable* for an initial endowment vector $x \in K$, if there exists an admissible trading strategy $L(\cdot) \in \mathscr{A}(x)$ such that

$$X^L_x(T-) \succeq \operatorname{diag}[S(T)]C, \quad \text{a.s.} \tag{4.6}$$

The condition (4.6) states that, starting from the initial endowment x, the investor can purchase at time T the consumption vector C, by investing according to the trading strategy $L(\cdot)$ over time period $[0,T)$ and making a final transfer at time T, if necessary. We denote by $\mathscr{C}(x)$ the set of all financeable terminal consumption vectors.

In terms of the notion of hedgeability, which we introduced in Definition 2.3.4, financeability of a terminal consumption vector C is equivalent to hedgeability of the corresponding contingent claim $\mathrm{diag}[S(T)]C$. Thus, we have

$$\mathscr{C}(x) = \{C \in \mathbb{L}^0(\mathbb{R}^d_+) \mid \mathrm{diag}[S(T)]C \in \mathbb{A}_x\}. \tag{4.7}$$

The utility maximization problem is now expressed by the value function

$$V(x) \triangleq \sup_{C \in \mathscr{C}(x)} \mathbb{E}[U(C)], \quad x \in K. \tag{4.8}$$

For our later analysis via convex duality, we define the convex conjugate $\tilde{U}$ of the utility function U by

$$\tilde{U}(y) \triangleq \sup_{c \in \mathbb{R}^d_+} [U(c) - y \cdot c], \quad y \in \mathbb{R}^d_+. \tag{4.9}$$

The function $\tilde{U}$ has the following properties.

Lemma 4.2.4. **(i)** *The function $\tilde{U}$, defined by* (4.9), *can also be written as*

$$\tilde{U}(y_1, y_2) = \tilde{U}_1(y_1), \quad (y_1, y_2) \in \mathbb{R}^{d_1}_+ \times \mathbb{R}^{d_2}_+, \tag{4.10}$$

where $\tilde{U}_1$ is the convex conjugate of the function U_1, i.e.,

$$\tilde{U}_1(y_1) \triangleq \sup_{c_1 \in \mathbb{R}^{d_1}_+} [U_1(c_1) - y_1 \cdot c_1], \quad y_1 \in \mathbb{R}^{d_1}_+; \tag{4.11}$$

(ii) *When $y_1 \in (0, \infty)^{d_1}$, the supremum in* (4.11) *is uniquely attained at $c_1 = I_1(y_1)$. In particular, we have*

$$\tilde{U}_1(y_1) = U_1(I_1(y_1)) - y_1 \cdot I_1(y_1), \quad \forall y_1 \in (0, \infty)^{d_1}; \tag{4.12}$$

(iii) *$\tilde{U}_1$ is lower semi-continuous and convex on $\mathbb{R}_+^{d_1}$, and is related to the function U_1 via*

$$U_1(c_1) = \inf_{y_1 \in \mathbb{R}_+^{d_1}} [\tilde{U}_1(y_1) + c_1 \cdot y_1], \quad c_1 \in \mathbb{R}_+^{d_1}; \tag{4.13}$$

(iv) *$\tilde{U}_1$ is "decreasing" in the sense that*

$$\tilde{U}_1(y_1) \geq \tilde{U}_1(\bar{y}_1) \quad \textit{if } \; y_1^i \leq \bar{y}_1^i \; \textit{ for each } i = 1, \dots, d_1 \; ; \tag{4.14}$$

(v) *$\tilde{U}_1$ is of class C^1 on $(0,\infty)^{d_1}$ with $\nabla \tilde{U}_1 = -I_1$;*

(vi) *The following equations are valid:*

$$\inf_{y_1 \in \mathbb{R}_+^{d_1}} \tilde{U}_1(y_1) = 0 \tag{4.15a}$$

$$\sup_{y_1 \in \mathbb{R}_+^{d_1}} \tilde{U}_1(y_1) = \tilde{U}_1(0) = \infty. \tag{4.15b}$$

Proof. **(i)** From the equation (4.3), we have

$$\begin{aligned} \tilde{U}(y) &= \sup_{c \in \mathbb{R}_+^{d}} [U(c) - y \cdot c] \\ &= \sup_{c_1 \in \mathbb{R}_+^{d_1}} [U_1(c_1) - y_1 \cdot c_1] - \inf_{c_2 \in \mathbb{R}_+^{d_2}} [y_2 \cdot c_2] \\ &= \sup_{c_1 \in \mathbb{R}_+^{d_1}} [U_1(c_1) - y_1 \cdot c_1] \end{aligned} \tag{4.16}$$

for every $y = (y_1, y_2) \in \mathbb{R}_+^{d_1} \times \mathbb{R}_+^{d_2} = \mathbb{R}_+^{d}$, showing (4.10) with (4.11).

(ii) This follows immediately from the differential property and the strict concavity of the function U_1.

(iii) This follows from Theorem 12.2 of [29].

(iv) Let $\bar{y}_1$ and y_1 be vectors in $\mathbb{R}_+^{d_1}$ such that $y_1^i \leq \bar{y}_1^i$ for each $i = 1, \dots, d_1$. Then, we have

$$U_1(c_1) - y_1 \cdot c_1 \geq U_1(c_1) - \bar{y}_1 \cdot c_1, \quad \forall c_1 \in \mathbb{R}_+^{d_1}.$$

Taking the supremum over $c_1 \in \mathbb{R}_+^{d_1}$, we obtain $\tilde{U}_1(y_1) \geq \tilde{U}_1(\bar{y}_1)$.

(v) From Theorem 23.5 of [29], the supremum in (4.11) is attained at c_1 if and only if $c_1 \in -\partial\tilde{U}_1(y_1)$, where $\partial\tilde{U}_1(y_1)$ is the subdifferential at y_1 of the convex function $\tilde{U}_1$. This, in conjunction with **(ii)** above, gives $-\partial\tilde{U}_1(y_1) = \{I_1(y_1)\}$, $\forall y_1 \in (0,\infty)^{d_1}$, which yields the assertion.

(vi) By taking $c_1 = 0$ in (4.13) and using (4.4b), we obtain (4.15a). Similarly, by taking $y_1 = 0$ in (4.11) and using (4.4a) and (4.14), we obtain (4.15b). □

Corollary 4.2.5. *For* $y = (y_1, y_2) \in (0,\infty)^{d_1} \times (0,\infty)^{d_2} = (0,\infty)^d$*, the supremum on the right-hand-side of* (4.9) *is uniquely attained by* $c = I(y)$*, where*

$$I(y) \triangleq (I_1(y_1), 0). \tag{4.17}$$

In particular, $\tilde{U}$ *is of class* C^1 *on* $(0,\infty)^d$ *with*

$$\nabla\tilde{U}(y) = -I(y), \quad \forall y \in (0,\infty)^d. \tag{4.18}$$

Proof. This immediately follows from (4.16) and **(ii)** of Lemma 4.2.4. □

Example 4.2.6 (Additive Utility). For $i = 1, \ldots, d$, let $\upsilon_i \colon \mathbb{R}_+ \to \mathbb{R}$ be a utility function on $\mathbb{R}_+$, i.e., υ_i is continuous, strictly concave and strictly increasing on $\mathbb{R}_+$ with $\lim_{c^i \uparrow \infty} \upsilon_i(c^i) = \infty$ and $\upsilon_i(0) = 0$, and is of class C^1 on $(0,\infty)$ with $\upsilon_i'(0+) \triangleq \lim_{c^i \downarrow 0} \upsilon_i'(c^i) = \infty$ and $\upsilon_i'(\infty) \triangleq \lim_{c^i \uparrow \infty} \upsilon_i'(c^i) = 0$. Then the function $U \colon \mathbb{R}_+^d \to \mathbb{R}$ given by

$$U(c) \triangleq \sum_{i=1}^{d} \upsilon_i(c^i) \tag{4.19}$$

is a utility function, according to the Definition 4.2.1, with $d_1 = d$. It is readily seen that

$$\tilde{U}(y) = \sum_{i=1}^{d} \tilde{v}_i(y^i) \tag{4.20}$$

with each $\tilde{v}_i(y_i)$ the convex conjugate of v_i, i.e., $\tilde{v}_i(y^i) \triangleq \sup_{c^i \in \mathbb{R}_+}[v_i(c^i) - y^i c^i]$ and that

$$\tilde{U}(y) = \sum_{i=1}^{d} \left\{ v_i(\psi_i(y^i)) - y^i \psi_i(y^i) \right\}, \quad \forall y \in (0,\infty)^d, \tag{4.21}$$

where ψ_i is the inverse function of u_i'. We thus have $I(y) \triangleq (\psi_1(y^1), \ldots, \psi_d(y^d))$, $\forall y \in (0,\infty)^d$; see also Example 2.3 of [27].

In addition to the defining properties in Definition 4.2.1, for our existence theorem (Theorem 4.4.1 below) we shall make the following assumption about the function U_1.

Assumption 4.2.7. The function U_1 satisfies the following two conditions:

$$AE(U_1) \triangleq \overline{\lim_{b \to \infty}} \sup_{\substack{c_1 \in (0,\infty)^{d_1} \\ m(c_1) \geq b}} \left[\frac{c_1 \cdot \nabla U_1(c_1)}{U_1(c_1)} \right] < 1 \tag{4.22a}$$

$$\underline{\lim_{r \downarrow 0}} \inf_{\substack{y_1 \in (0,\infty)^{d_1} \\ m(y_1) \leq r}} m(I_1(y_1)) = \infty, \tag{4.22b}$$

where we are denoting $m(\eta) \triangleq \min_{1 \leq i \leq d_1} \eta^i$ for any vector $\eta \in \mathbb{R}_+^{d_1}$.

The quantity $AE(U_1)$ is called the *asymptotic elasticity* of the function U_1. When $d_1 = 1$, this notion coincides with that of Definition 2.2 of [22]. The equation (4.22b) states that "$I_1(y_1)$ approaches to infinity as y_1 gets small", where we are measuring the magnitude of the vectors $I_1(y_1)$ and y_1

in terms of their smallest component. When $d_1 = 1$, the equation (4.22b) simply says $I(0+) = \infty$, the assumption which is fairly standard in the literature. If U is an additive utility function as in Example 4.2.6, and if the asymptotic elasticity of each υ_i is less than 1, then U satisfies the conditions in Assumption 4.2.7.

Exactly in the same way as Lemma 6.3 of [22], we can show that under the assumption $AE(U_1) < 1$, there exist constants $b > 0$ and $\beta > 0$ such that

$$\tilde{U}_1(\alpha y_1) < \alpha^{-\beta} \tilde{U}_1(y_1) \tag{4.23}$$

holds for every $0 < \alpha < 1$ and $y \in (0,\infty)^{d_1}$ with $m(I_1(y_1)) > b$; for a similar argument, see the proof of Lemma 4.5.3, especially the derivation of (4.61), below. With such a number $b > 0$ fixed, we may take, under (4.22b), a number $r_b > 0$ such that

$$m(I_1(y_1)) > b$$

for every $y_1 \in (0,\infty)^{d_1}$ with $m(y_1) < r_b$. This implies that we have $m(y_1) \geq r_b$ for every $y_1 \in (0,\infty)^{d_1}$ satisfying $m(I_1(y_1)) \leq b$. In conjunction with (4.14), it follows then that

$$\tilde{U}_1(\alpha y_1) \leq \tilde{U}_1(\alpha r_b(1,\ldots,1)) \tag{4.24}$$

for every $0 < \alpha < 1$ and $y_1 \in (0,\infty)^{d_1}$ with $m(I_1(y_1)) \leq b$. Note that the right-hand-side of (4.24) defines a non-increasing function of $\alpha \in (0,1)$. Combining the two inequalities (4.23) and (4.24), we obtain the following lemma.

Lemma 4.2.8. *Under Assumption 4.2.7, there exist a number $\beta > 0$ and a non-increasing function $\zeta\colon (0,1) \to \mathbb{R}_+$ such that the inequality*

$$\tilde{U}_1(\alpha y_1) \leq \zeta(\alpha) + \alpha^{-\beta} \tilde{U}_1(y_1) \tag{4.25}$$

is valid for every $\alpha \in (0,1)$ and $y_1 \in (0,\infty)^{d_1}$.

4.3 Indirect Utility; Utility from Terminal Wealth

In mathematical finance, it is common to consider utility from *money*. In other words, the utility function takes as the independent variable either wealth or consumption *measured in monetary units*, rather than in the physical amount of consumption goods. Therefore, the utility function is not directly connected to the consumption goods, which are actually consumed, but to the total expenditure allocated to the consumption. Such utility is called the "derived utility" or the "indirect utility", in mathematical economics. In this section, we compare our direct utility approach with the indirect utility approach, and prove that the two are equivalent when the indirect utility function is correctly derived from the direct utility function. However, as we shall illustrate in Example 4.3.4 below, the indirect utility function is typically non-smooth in the presence of transaction costs, which is the reason that we stick to the direct utility function rather than passing through the indirect utility function.

Definition 4.3.1. For a utility function U as in Definition 4.2.1, we define the *indirect utility function* $\bar{U}\colon \mathbb{R}^d \times (0,\infty)^d \to [-\infty,\infty)$ by

(4.26)
$$\begin{aligned}\bar{U}(x,p) &\triangleq \sup\{U(c_1,c_2) \mid (c_1,c_2) \in \mathbb{R}_+^{d_1} \times \mathbb{R}_+^{d_2},\, x \succeq \mathrm{diag}[p](c_1,c_2)\}\\ &= \sup\{U_1(c_1) \mid c_1 \in \mathbb{R}_+^{d_1},\, x \succeq \mathrm{diag}[p](c_1,0)\}, \quad (x,p) \in K \times (0,\infty)^d.\end{aligned}$$

Here, for the physical meaning of c_1 and c_2, recall the first paragraph of Section 4.2.

By definition, $\bar{U}(x,p)$ is the maximal utility that can be derived from some consumption vector c, financeable with the portfolio holdings x under the currency price p. Notice that the indirect utility function depends on the currency price as well as the portfolio holdings vector, which corresponds to the total wealth in the usual setting. If there is only one consumption good, we can avoid this dependence simply by taking the consumption good as the numeraire. However, if there are several consumption goods, we cannot eliminate this dependence and therefore, formulating a utility maximization problem with a utility function depending only on wealth, causes the so-called "money illusion".

Remark 4.3.2. The indirect utility function $\bar{U}$ corresponds to the function $\tilde{U}$ in the equation (2.10) of [27]. Without transaction costs, the indirect utility function depends on the total wealth $\sum_{i=1}^{d} x^i$ allocated to consumption and the price vector p of the consumption goods.

Given $(x,p) \in K \times (0,\infty)^d$, the set of all $c_1 \in \mathbb{R}_+^{d_1}$ satisfying $x - \mathrm{diag}[p](c_1, 0) \in K$, as in (4.26), is compact. Thus, from the continuity of the function U_1 on $\mathbb{R}_+^{d_1}$, we see that the supremum in (4.26) is always attained at some $c_1 \in \mathbb{R}_+^{d_1}$. Furthermore, since U_1 is strictly concave, such a maximizer is unique. For each $(x,p) \in K \times (0,\infty)^d$, we denote by $\phi(x,p)$ the unique maximizer $c_1 \in \mathbb{R}_+^{d_1}$ for (4.26). The function ϕ is called the *Marshallian demand function* in microeconomics.

Lemma 4.3.3. *The "Marshallian demand" function $\phi\colon K \times (0,\infty)^d \to \mathbb{R}_+^{d_1}$ is Borel measurable.*

Proof. Define a multifunction $G\colon K \times (0,\infty)^d \to \mathbb{R}_+^{d_1}$ by

$$G(x,p) \triangleq \{c_1 \in \mathbb{R}_+^{d_1} \mid x \succeq \mathrm{diag}[p](c_1, 0)\}.$$

We claim that G is Borel-measurable, i.e., for every open set $O \subseteq \mathbb{R}_+^{d}$, the set $G^{-1}(O) \triangleq \{(x,p) \in K \times (0,\infty)^d \mid G(x,p) \cap O \neq \emptyset\}$ is a Borel set.

Once this claim is proved, then, noting that the multifunction G is compact-valued and the function U_1 is continuous, we can conclude from the Dubins-Savage selection theorem (see, for example, Theorem 5.3.1 of [34]) that the multifunction $(x, p) \mapsto \{\phi(x, p)\}$ admits a Borel-measurable selection, which simply reduces to the Borel measurability of the function ϕ.

It thus suffices to prove the Borel-measurability of the multifunction G. To this end, consider the Borel-measurable function $h\colon K \times (0, \infty)^d \times \mathbb{R}_+^{d_1} \to \mathbb{R}$ given by

$$h(x, p, c_1) \triangleq \ell(x - \mathrm{diag}[p](c_1, 0)),$$

where ℓ is the liquidation function given by (2.14), so that $h^{-1}([0, \infty))$ is equal to the graph of G, i.e.,

$$h^{-1}([0, \infty)) = \{(x, p, c_1) \in K \times (0, \infty)^d \times \mathbb{R}_+^{d_1} \mid c_1 \in G(x, p)\}.$$

We then have

$$G^{-1}(O) = \mathrm{proj}_{K \times (0,\infty)^d}(E),$$

where $\mathrm{proj}_{K \times (0,\infty)^d}$ is the canonical projection of $K \times (0, \infty)^d \times \mathbb{R}_+^d$ onto $K \times (0, \infty)^d$ and E is the Borel set given by

$$E \triangleq h^{-1}([0, \infty)) \cap [K \times (0, \infty)^d \times O].$$

Now, for each $(x, p) \in K \times (0, \infty)^d$, the section

$$E_{(x,p)} \triangleq \left[(x, p) \times \mathbb{R}_+^{d_1}\right] \cap E = h^{-1}([0, \infty)) \cap [(x, p) \times O]$$

is clearly σ-compact. Therefore, from the Arsenin-Kunugui theorem (see, for example, Theorem 5.12.1 of [34]), we see that $G^{-1}(O)$ is a Borel set, which shows the Borel-measurability of the multifunction G, as claimed. $\square$

Example 4.3.4 (Utility from the Liquidated Terminal Wealth). Consider the case where only the consumption of the domestic consumption good affects the utility, in other words, $d_1 = 1$. From direct computation, we can see

$$\bar{U}(x,p) = U_1\left(\frac{\ell(x)}{p^1}\right),$$

which goes back to utility from the liquidated terminal wealth considered by [9], [13] and [20]. Notice that despite the smoothness of U_1, the indirect utility $\bar{U}$ is not necessarily smooth, which makes the analysis based on $\bar{U}$ rather involved; see [16].

Now, we show the equivalence of our utility maximization problem from terminal consumption and the indirect utility maximization from terminal wealth. Note that, in conjunction with (4.26), the equation (4.27) can be interpreted as the dynamic programming principle at time $T-$.

Proposition 4.3.5. *The value function V defined by* (4.8) *satisfies*

$$V(x) = \sup_{L(\cdot)\in\mathscr{A}(x)} \mathbb{E}[\bar{U}(X_x^L(T-), S(T))], \quad \forall x \in K. \tag{4.27}$$

Proof. It is clear by the definition (4.26) that we have $U(C) \leq \bar{U}(X_x^L(T-), S(T))$ for every $C \in \mathscr{C}(x)$, and thus,

$$V(x) \leq \sup_{L(\cdot)\in\mathscr{A}(x)} \mathbb{E}[\bar{U}(X_x^L(T-), S(T))].$$

To prove the reverse inequality, let $L(\cdot) \in \mathscr{A}(x)$ be given. Define the random variable C by

$$C^i = X_x^{iL}(T-) + \sum_{j=1}^{d}[a^{ji} - (1+\lambda^{ij})a^{ij}], \quad i = 1, \ldots, d$$

with $a = \alpha(\xi)$; here $\xi = X_x^L(T-) - \mathrm{diag}[S(T)]\phi(X_x^L(T-), S(T))$, and the function $\alpha(\cdot)$ is a continuous selection of $A(\cdot)$ given by Proposition 2.2.5.

Since $\phi(x,p) \in G(x,p)$, we have $X_x^L(T-) \succeq \mathrm{diag}[S(T)]\phi(X_x^L(T-), S(T))$ and thus a is well-defined. From the definition of $\alpha(\cdot)$, we have $C = \phi(X_x^L(T-), S(T))$ and thus, in particular, C is $\mathscr{F}(T)$-measurable. Finally, from the definition of $\phi(\cdot)$, we obtain

$$U(C) = U(\phi(X_x^L(T-), S(T))) = \bar{U}(X_x^L(T-), S(T)).$$

This shows

$$V(x) \geq \sup_{L(\cdot)\in\mathscr{A}(x)} \mathbb{E}[\bar{U}(X_x^L(T-), S(T))].$$

□

4.4 Existence via a Dual Optimization Problem

In this section, we state and prove the existence theorem to our optimization problem (4.8). Because of the simple differentiability assumption of the direct utility function, we can prove the existence theorem by the standard tools from convex analysis and the smooth calculus of variations.

As a conjugate of the set $\mathscr{C}(x)$ of financeable terminal consumption vectors (see Definition 4.2.3), we take the set $\mathscr{D}(z)$ defined by

(4.28)
$$\mathscr{D}(z) \triangleq \left\{ H \in \mathbb{L}^0(\mathbb{R}^d_+) \;\middle|\; \begin{array}{l} \mathrm{diag}[S(T)]^{-1}H \in \mathbb{L}^0(K^*), \text{ and} \\ \quad \mathbb{E}[H\cdot C] \leq z\cdot \mathrm{diag}[S(0)]^{-1}x,\ \forall x \in K,\ \forall C \in \mathscr{C}(x) \end{array} \right\},$$

for each $z \in \mathrm{diag}[S(0)]K^*$. From (4.7) and Theorem 2.3.7, we have

$$\mathscr{D}_0(z) \subseteq \mathscr{D}(z), \quad \forall z \in \mathrm{diag}[S(0)]K^* \setminus \{0\}. \tag{4.29}$$

We also set

$$\mathscr{G} \triangleq \{(z, H) \in \mathrm{diag}[S(0)]K^* \times \mathbb{L}^0(\mathbb{R}^d_+) \mid H \in \mathscr{D}(z)\}. \tag{4.30}$$

Notice that if $x \in \mathbb{R}^d_+$, then $X^0_x(t) = \operatorname{diag}[S(t)]\operatorname{diag}[S(0)]^{-1}x \in \mathbb{R}^d_+ \subseteq K$ for all $t \in [0,T)$, which implies that the constant function $L(\cdot) \equiv 0$ belongs to $\mathscr{A}(x)$ and that $C \triangleq \operatorname{diag}[S(0)]^{-1}x \in \mathscr{C}(x)$. It follows from the definition (4.28) of $\mathscr{D}(z)$ that

$$\mathbb{E}[H]\cdot\operatorname{diag}[S(0)]^{-1}x \le z\cdot\operatorname{diag}[S(0)]^{-1}x, \quad \forall H \in \mathscr{D}(z),\ \forall x \in \mathbb{R}^d_+.$$

In particular, by taking x to be the i-th unit vector for each $i = 1,\dots,d$, we obtain

$$0 \le \mathbb{E}[H^i] \le z^i, \quad \forall i = 1,\dots,d, \tag{4.31}$$

which implies that the set $\mathscr{D}(z)$ of (4.28) is bounded in $\mathbb{L}^1(\mathbb{R}^d)$ for each fixed $z \in \operatorname{diag}[S(0)]K^*$. Notice also that the set $\mathscr{G}$ is a convex cone in $\mathbb{R}^d \times \mathbb{L}^0(\mathbb{R}^d)$ and is closed with respect to a.s.-convergence. More precisely, if $\{(z_k,H_k)\}_{k\in\mathbb{N}} \subseteq \mathscr{G}$ with $z_k \to z$ in $\mathbb{R}^d$ and $H_k \to H$ a.s., then we have $(z,H) \in \mathscr{G}$. This follows easily form Fatou's lemma.

Now, from (4.9), (4.28) and (4.30), we have

$$\mathbb{E}[U(C)] \le \mathbb{E}[\tilde{U}(H)] + \mathbb{E}[H\cdot C] \le \mathbb{E}[\tilde{U}(H)] + z\cdot\operatorname{diag}[S(0)]^{-1}x \tag{4.32}$$

for every $x \in K$, $C \in \mathscr{C}(x)$ and $(z,H) \in \mathscr{G}$. Furthermore, from Corollary 4.2.5, the two inequalities in (4.32) hold as equalities if we have

$$C = I(H) \quad \text{and} \quad \mathbb{E}[H\cdot I(H)] = z\cdot\operatorname{diag}[S(0)]^{-1}x, \tag{4.33}$$

as well as $H \in (0,\infty)^d$ a.s. Finally, for each $x \in K$, we define the dual optimization problem by

$$W(x) \triangleq \inf_{(z,H)\in\mathscr{G}} \left(\mathbb{E}[\tilde{U}(H)] + z\cdot\operatorname{diag}[S(0)]^{-1}x\right). \tag{4.34}$$

Here is our basic existence result for the optimization problem (4.8).

Theorem 4.4.1. *Let $x \in \operatorname{int} K$ be an initial endowment vector and assume $W(x) < \infty$. Then there exists a pair $(\hat{z}, \hat{H}) \in \mathscr{G}$ satisfying $\hat{H} \in (0,\infty)^d$ a.s. that attains the infimum for the dual optimization problem* (4.34)*. Furthermore, under Assumption 4.2.7, the random variable $\hat{C} \triangleq I(\hat{H})$ is a solution to the original utility maximization problem* (4.8)*, that is, we have $\hat{C} \in \mathscr{C}(x)$ and*

$$\mathbb{E}[U(\hat{C})] = \sup_{C \in \mathscr{C}(x)} \mathbb{E}[U(C)]. \tag{4.35}$$

The rest of this section consists of the proof of Theorem 4.4.1.

Lemma 4.4.2. *Let $x \in \operatorname{int} K$ and assume $W(x) < \infty$. Then there exists a pair $(\hat{z}, \hat{H}) \in \mathscr{G}$ that attains the infimum in* (4.34)*. Furthermore, we have $\hat{H} \in (0,\infty)^d$ almost surely.*

Proof. Let $\{(z_k, H_k)\}_{k \in \mathbb{N}}$ be a minimizing sequence for (4.34), i.e.,

$$\lim_{k \to \infty} \downarrow \Big(\mathbb{E}[\tilde{U}(H_k)] + z_k \cdot \operatorname{diag}[S(0)]^{-1} x \Big) = W(x). \tag{4.36}$$

Suppose that there exists a subsequence $\{(z_{k(l)}, H_{k(l)})\}_{l \in \mathbb{N}}$ which converges to $(0,0)$ as $l \to \infty$, almost surely. Then, from the lower semi-continuity of $\tilde{U}$ and (4.15b), we have $\lim_{l \to \infty} \tilde{U}(H_{k(l)}) \geq \tilde{U}(0) = \infty$, almost surely. But since the function $\tilde{U}$ is non-negative because of (4.15a) and since $z_{k(l)} \cdot \operatorname{diag}[S(0)]^{-1} x \geq 0$, Fatou's lemma gives

$$\varliminf_{l \to \infty} \Big(\mathbb{E}[\tilde{U}(H_{k(l)})] + z_{k(l)} \cdot \operatorname{diag}[S(0)]^{-1} x \Big) = \infty,$$

and thus $W(x) = \infty$, a contradiction. This shows that the sequence $\{z_k\}_{k \in \mathbb{N}}$ is away from 0 for large k; otherwise, there would exist a subsequence $\{z_{k(l)}\}_{l \in \mathbb{N}}$ such that $z_{k(l)} \to 0$, which would imply from (4.31) that $H_{k(l)} \to 0$ in $\mathbb{L}^1$, and thus, along a further subsequence, we would have $(z_{k(l)}, H_{k(l)}) \to (0,0)$ a.s., a contradiction.

Next, from (4.36) and $W(x) < \infty$, we have $\mathbb{E}[\tilde{U}(H_k)] + z_k \cdot \mathrm{diag}[S(0)]^{-1}x < W(x) + 1$ for large k, which implies from (4.15a) that

$$z_k \cdot \mathrm{diag}[S(0)]^{-1}x < -\mathbb{E}[\tilde{U}(H_k)] + W(x) + 1 \leq W(x) + 1 =: M < \infty$$

for large k. Note also that since $z_k \in \mathrm{diag}[S(0)]K^*$ and $x \in K$, we have $z_k \cdot \mathrm{diag}[S(0)]^{-1}x \geq 0$. Therefore,

$$0 \leq z_k \cdot \mathrm{diag}[S(0)]^{-1}x < M$$

for large k. Furthermore, since $z_k \in \mathrm{diag}[S(0)]K^* \setminus \{0\} \subseteq (0, \infty)^d$ for large k, we may divide by z_k^1 to obtain

$$0 \leq \frac{\mathrm{diag}[S(0)]^{-1}z_k}{z_k^1} \cdot x < \frac{M}{z_k^1}$$

for large k. This, in conjunction with (2.16), implies

$$0 \leq \ell(x) \leq \frac{\mathrm{diag}[S(0)]^{-1}z_k}{z_k^1} \cdot x < \frac{M}{z_k^1},$$

and thus

$$0 \leq z_k^1 \ell(x) < M$$

for large k. Now, since $x \in \mathrm{int}\, K$, we have $\ell(x) > 0$ from Lemma 3.1 of [16], and therefore,

$$0 < z_k^1 < \frac{M}{\ell(x)} < \infty,$$

which implies that the sequence $\{z_k^1\}_{k \in \mathbb{N}}$ is bounded. Since $z_k \in z_k^1 \Lambda$ and the set Λ is compact in $\mathbb{R}^d$ (see (2.15) and and Lemma 2.2.6), the boundedness of the sequence $\{z_k^1\}_{k \in \mathbb{N}}$ in $\mathbb{R}$ implies the boundedness the sequence $\{z_k\}_{k \in \mathbb{N}}$ in $\mathbb{R}^d$. Then, by passing through a subsequence, we may assume that $z_k \to \hat{z}$ for some $\hat{z} \in \mathbb{R}^d$ as $k \to \infty$. Since K^* is closed and $\{z_k\}_{k \in \mathbb{N}}$ is away from 0 for large k, we obtain $\hat{z} \in \mathrm{diag}[S(0)]K^* \setminus \{0\}$.

Now, from (4.31) and the boundedness of $\{z_k\}_{k\in\mathbb{N}}$, it follows that the sequence $\{H_k\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{L}^1(\mathbb{R}^d)$. Then, by Komlós' theorem (see, for example, [30]), there exists a subsequence $\{H_{k(l)}\}_{l\in\mathbb{N}}$ such that the sequence

$$\Theta_k \triangleq \frac{1}{k}\sum_{j=1}^{k} H_{k(j)}, \quad k \in \mathbb{N}$$

converges almost surely to some random variable $\hat{H} \in \mathbb{L}^0(\mathbb{R}^d)$. Set

$$\zeta_k \triangleq \frac{1}{k}\sum_{j=1}^{k} z_{k(j)}, \quad k \in \mathbb{N}$$

so that $(\zeta_k, \Theta_k) \to (\hat{z}, \hat{H})$ a.s. Since $\mathscr{G}$ is convex, $(\zeta_k, \Theta_k) \in \mathscr{G}$, $\forall k \in \mathbb{N}$. Since $\mathscr{G}$ is closed under a.s.-convergence, it follows that $(\hat{z}, \hat{H}) \in \mathscr{G}$. Fatou's lemma and the convexity of $\tilde{U}$ now give

$$\begin{aligned}
&\mathbb{E}[\tilde{U}(\hat{H})] + \hat{z}\cdot\mathrm{diag}[S(0)]^{-1}x \\
&\quad \le \varliminf_{k\to\infty} \Big(\mathbb{E}[\tilde{U}(\Theta_k)] + \zeta_k\cdot\mathrm{diag}[S(0)]^{-1}x\Big) \\
&\quad \le \varliminf_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} \Big\{\mathbb{E}[\tilde{U}(H_{k(j)})] + z_{k(j)}\cdot\mathrm{diag}[S(0)]^{-1}x\Big\} \\
&\quad = \lim_{k\to\infty} \Big(\mathbb{E}[\tilde{U}(H_k)] + z_k\cdot\mathrm{diag}[S(0)]^{-1}x\Big) = W(x).
\end{aligned}$$

Therefore, $(\hat{z}, \hat{H})$ attains the infimum in (4.34).

Finally, if $\mathbb{P}(\hat{H} = 0) > 0$, then $\mathbb{P}(\tilde{U}(\hat{H}) = \infty) > 0$, which gives $W(x) = \infty$, a contradiction. In conjunction with Corollary 2.2.3 and the fact $\mathrm{diag}[S(T)]^{-1}\hat{H} \in K^*$ a.s., we obtain $\hat{H} \in (0,\infty)^d$ a.s. This completes the proof. $\square$

Proof of Theorem 4.4.1: Let $(\hat{z}, \hat{H}) \in \mathscr{G}$ be a pair given in Lemma 4.4.2 above and set $\hat{C} \triangleq I(\hat{H})$. We first claim that

$$\mathbb{E}[\hat{C}\cdot(H - \hat{H})] \le (z - \hat{z})\cdot\mathrm{diag}[S(0)]^{-1}x, \quad \forall (z, H) \in \mathscr{G}. \tag{4.37}$$

To see this, fix $(z, H) \in \mathscr{G}$ arbitrarily, and set

$$\begin{aligned} (z_\varepsilon, H_\varepsilon) &\triangleq (1-\varepsilon)(\hat{z}, \hat{H}) + \varepsilon(z, H) \\ C_\varepsilon &\triangleq I(H_\varepsilon) \end{aligned} \tag{4.38}$$

for each $0 < \varepsilon < 1$. Then the optimality of $(\hat{z}, \hat{H})$ implies

$$\begin{aligned} 0 &\geq \left(\mathbb{E}[\tilde{U}(\hat{H})] + \hat{z}\cdot\mathrm{diag}[S(0)]^{-1}x\right) - \left(\mathbb{E}[\tilde{U}(H_\varepsilon)] + z_\varepsilon\cdot\mathrm{diag}[S(0)]^{-1}x\right) \\ &= \mathbb{E}[\tilde{U}(\hat{H}) - \tilde{U}(H_\varepsilon)] + (\hat{z} - z_\varepsilon)\cdot\mathrm{diag}[S(0)]^{-1}x. \end{aligned} \tag{4.39}$$

On the other hand, from the convexity of the function $\tilde{U}$, the fact that $H_\varepsilon \in (0, \infty)^d$ a.s., and the equality (4.18), we have

$$\mathbb{E}[\tilde{U}(\hat{H}) - \tilde{U}(H_\varepsilon)] \geq -\mathbb{E}[I(H_\varepsilon)\cdot(\hat{H} - H_\varepsilon)]. \tag{4.40}$$

Combining the inequalities (4.39) and (4.40) and then substituting (4.38), we obtain

$$\begin{aligned} 0 &\geq -\mathbb{E}[I(H_\varepsilon)\cdot(\hat{H} - H_\varepsilon)] + (\hat{z} - z_\varepsilon)\cdot\mathrm{diag}[S(0)]^{-1}x \\ &= \varepsilon\left\{\mathbb{E}[C_\varepsilon\cdot(H - \hat{H})] - (z - \hat{z})\cdot\mathrm{diag}[S(0)]^{-1}x\right\}. \end{aligned}$$

Dividing by ε, we obtain

$$\mathbb{E}[C_\varepsilon\cdot(H - \hat{H})] \leq (z - \hat{z})\cdot\mathrm{diag}[S(0)]^{-1}x. \tag{4.41}$$

Now, let $\varepsilon \downarrow 0$. Then by continuity of I, we have $C_\varepsilon = I(H_\varepsilon) \to I(\hat{H}) = \hat{C}$. We would like to apply Fatou's lemma to (4.41) to obtain

$$\mathbb{E}[\hat{C}\cdot(H - \hat{H})] \leq \varliminf_{\varepsilon\downarrow 0} \mathbb{E}[C_\varepsilon\cdot(H - \hat{H})] \leq (z - \hat{z})\cdot\mathrm{diag}[S(0)]^{-1}x,$$

which will provide (4.37). To justify the applicability of Fatou's lemma, we claim that the random variable $C_\varepsilon\cdot(H - \hat{H})$ is bounded from below by some integrable random variable, uniformly in $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$. Let $0 < \alpha < 1$ be a parameter. Then, for $\varepsilon \in (0, 1-\alpha)$, we have

$$H_\varepsilon + \alpha(H - \hat{H}) = (1 - (\varepsilon + \alpha))\hat{H} + (\varepsilon + \alpha)H \in (0, \infty)^d, \quad \text{a.s.}$$

as well as $H_\varepsilon \in (0, \infty)^d$, a.s. Thus, the convexity of the function $\tilde{U}$ with the equation (4.18) gives

$$\begin{aligned} \tilde{U}((1-\varepsilon-\alpha)\hat{H}) &\geq \tilde{U}(H_\varepsilon + \alpha(H - \hat{H})) \\ &\qquad - I(H_\varepsilon + \alpha(H - \hat{H})) \cdot \Big[(1-\varepsilon-\alpha)\hat{H} - (H_\varepsilon + \alpha(H - \hat{H}))\Big] \\ &= \tilde{U}(H_\varepsilon + \alpha(H - \hat{H})) + (\varepsilon + \alpha) I(H_\varepsilon + \alpha(H - \hat{H})) \cdot H \\ &\geq \tilde{U}(H_\varepsilon + \alpha(H - \hat{H})) \end{aligned} \tag{4.42}$$

as well as

$$\begin{aligned} \tilde{U}(H_\varepsilon + \alpha(H - \hat{H})) &\geq \tilde{U}(H_\varepsilon) - I(H_\varepsilon) \cdot \Big(H_\varepsilon + \alpha(H - \hat{H}) - H_\varepsilon\Big) \\ &= \tilde{U}(H_\varepsilon) - \alpha I(H_\varepsilon) \cdot (H - \hat{H}). \end{aligned} \tag{4.43}$$

In conjunction with (4.15a), the inequality (4.43) implies that

$$\tilde{U}(H_\varepsilon + \alpha(H - \hat{H})) \geq -\alpha I(H_\varepsilon) \cdot (H - \hat{H}). \tag{4.44}$$

Combining (4.42) and (4.44), we obtain

$$I(H_\varepsilon) \cdot (H - \hat{H}) \geq -\frac{1}{\alpha}\tilde{U}((1-\varepsilon-\alpha)\hat{H}).$$

Therefore, if we take $0 < \alpha \leq 1/4$, then for each $\varepsilon \in (0, 1-2\alpha]$, it follows from Lemma 4.2.8 together the equation (4.10) that

$$\begin{aligned} I(H_\varepsilon) \cdot (H - \hat{H}) &\geq -\frac{1}{\alpha}\left[\zeta(1-\varepsilon-\alpha) + (1-\varepsilon-\alpha)^{-\beta}\tilde{U}(\hat{H})\right] \\ &\geq -\frac{1}{\alpha}\left[\zeta(\alpha) + \alpha^{-\beta}\tilde{U}(\hat{H})\right]. \end{aligned} \tag{4.45}$$

Since the right-hand-side is integrable and independent of $\varepsilon \in (0, 1-2\alpha]$, application of Fatou's lemma in (4.41) is now justified, which establishes the inequality (4.37).

Now, by taking $(z, H) = (0, 0)$ in (4.37), we get

$$\mathbb{E}[\hat{H} \cdot \hat{C}] \geq \hat{z} \cdot \mathrm{diag}[S(0)]^{-1} x.$$

Also, by taking $(z, H) = (1+\varepsilon)(\hat{z}, \hat{H})$ in (4.37) for some small $\varepsilon > 0$, we get

$$\mathbb{E}[\hat{H} \cdot \hat{C}] \leq \hat{z} \cdot \mathrm{diag}[S(0)]^{-1} x;$$

recall that the set $\mathscr{G}$ is a cone and thus $(1+\varepsilon)(\hat{z}, \hat{H}) \in \mathscr{G}$. We thus obtain

$$\mathbb{E}[\hat{H} \cdot \hat{C}] = \hat{z} \cdot \mathrm{diag}[S(0)]^{-1} x. \tag{4.46}$$

Finally, (4.37) and (4.46) imply

$$\mathbb{E}[H \cdot \hat{C}] \leq z \cdot \mathrm{diag}[S(0)]^{-1} x, \quad \forall (z, H) \in \mathscr{G}, \tag{4.47}$$

which in particular gives

$$\mathbb{E}[Z \cdot \hat{C}] \leq z \cdot \mathrm{diag}[S(0)]^{-1} x, \quad \forall Z \in \mathscr{D}_0(z) \text{ and } \forall z \in \mathrm{diag}[S(0)] K^* \setminus \{0\}$$

because of (4.29). Theorem 2.3.7 then gives $\hat{C} \in \mathscr{C}(x)$. In conjunction with (4.32) and (4.33), the equation (4.46) now gives (4.35). ☐

4.5 A Generalization; Utility from Inter-Temporal and Terminal Consumption

In this section, we sketch a generalization of the results in the previous section to the case where consumption over the time period $[0, T)$ is allowed as well as at the terminal time T. An *inter-temporal consumption process* is an

$\mathbb{R}^d_+$-valued, progressively measurable process $c(\cdot)$ defined on $[0,T)$ such that $\int_0^T |c(t)|dt < \infty$. For each $t \in [0,T)$ and $i = 1, \dots, d$, the random variable $c^i(t)$ is the rate of consumption of the i-th consumption good at time t. A *terminal consumption vector* is simply an $\mathbb{R}^d_+$-valued, $\mathscr{F}(T)$-measurable random variable as in Definition 4.2.3. A pair (C,c) of terminal/inter-temporal consumption is then an element of the space $\mathscr{L}$ defined by

$$\mathscr{L} \triangleq \mathbb{L}^0((\Omega, \mathscr{F}(T), \mathbb{P}); \mathbb{R}^d_+) \times \mathbb{L}^0(([0,T) \times \Omega, \Sigma_\pi, \mu \otimes \mathbb{P}); \mathbb{R}^d_+), \tag{4.48}$$

where μ is the Lebesgue measure on the interval $[0,T)$ and Σ_π is the σ-field on $[0,T) \times \Omega$ generated by the progressively measurable processes. We identify an element $(Y,y) \in \mathscr{L}$ with the progressively measurable process $\check{y}(t,\omega) \triangleq 1_{[0,T)}(t)y(t,\omega) + 1_{\{T\}}(t)Y(\omega)$ defined on $[0,T] \times \Omega$. Note that we have

$$\left\{ \begin{array}{c} Y_1(\omega) = Y_2(\omega), \quad \mathbb{P}\text{-a.e. } \omega \in \Omega \quad \text{and} \\ y_1(t,\omega) = y_2(t,\omega), \quad (\mu \otimes \mathbb{P})\text{-a.e. } (t,\omega) \in [0,T) \times \Omega \end{array} \right\}$$

if and only if

$$\check{y}_1(t,\omega) = \check{y}_2(t,\omega), \quad (\check{\mu} \otimes \mathbb{P})\text{-a.e. } (t,\omega) \in [0,T] \times \Omega,$$

where the measure $\check{\mu}$ is the sum of the Lebesgue measure on $[0,T]$ and the point mass measure on $\{T\}$. Note that under the identification $(Y,y) = \check{y}$, the space $\mathscr{L}$ is identified with the space $\mathbb{L}^0(([0,T] \times \Omega, \check{\Sigma}_\pi, \check{\mu} \otimes \mathbb{P}); \mathbb{R}^d_+)$, where $\check{\Sigma}_\pi$ is the σ-field on $[0,T] \times \Omega$ generated by the progressively measurable processes. This identification allows us to prove Theorem 4.5.4 below, by line-by-line change of Theorem 4.4.1.

Given an initial endowment vector $x \in \mathbb{R}^d$, a trading strategy $L(\cdot)$ and an inter-temporal consumption process $c(\cdot)$, we define the *portfolio holdings* *process* $X_x^{Lc}(\cdot)$ by the formula

(4.49)
$$\frac{X_x^{iLc}(t)}{S^i(t)} = \frac{x^i}{S^i(0)} + \int_{[0,t]} \frac{1}{S^i(u)} \sum_{j=1}^{d} \left\{dL^{ji}(u) - (1+\lambda^{ij})dL^{ij}(u)\right\} - \int_0^t c^i(u)du,$$
$$t \in [0,T),\ i = 1, \ldots, d.$$

Noting that $X_x^{L0}(\cdot) = X_x^L(\cdot)$, we see easily from (4.49) that

(4.50)
$$X_x^{Lc}(t) = X_x^L(t) - \mathrm{diag}[S(t)] \int_0^t c(u)du, \quad \forall t \in [0,T).$$

Let $x \in K$ be an initial endowment vector and $c(\cdot)$ be an inter-temporal consumption process. With analogy of Definition 4.2.3, we call a trading strategy $L(\cdot)$ *admissible for* (x, c) if the "no-bankruptcy" condition

$$X_x^{Lc}(t) \in K, \quad \forall t \in [0,T),$$

is satisfied almost surely. Also, a pair (C, c) of terminal/inter-temporal consumption is said to be *financeable* for $x \in K$, if there exists an admissible trading strategy $L(\cdot)$ for (x, c) such that

$$X_x^{Lc}(T-) \succeq \mathrm{diag}[S(T)]C, \quad \text{a.s.}$$

We denote by $\mathscr{A}(x, c)$ the set of all admissible trading strategies for (x, c) and by $\check{\mathscr{C}}(x)$ the set of all financeable terminal/inter-temporal consumption pairs for x. The next lemma states that a terminal/inter-temporal consumption pair (C, c) is financeable if and only if the total consumption $C + \int_0^T c(t)dt$ is financeable in the sense of Definition 4.2.3, or equivalently, if and only if the random vector $\mathrm{diag}[S(T)]\left(C + \int_0^T c(t)dt\right)$ defines a contingent claim hedgeable for x.

Lemma 4.5.1. *A terminal/inter-temporal consumption pair* $(C, c) \in \mathscr{L}$ *is financeable for* x*, if and only if the contingent claim* $C + \int_0^T c(t)dt$ *is hedgeable for* x*. In other words, we have*

$$(C, c) \in \check{\mathscr{C}}(x) \quad \textit{if and only if} \quad \operatorname{diag}[S(T)] \left(C + \int_0^T c(t)dt \right) \in \mathbb{A}_x.$$

Proof (Sufficiency): Suppose that $C + \int_0^T c(t)dt$ is hedgeable for x. Then, there exists a trading strategy $L(\cdot) \in \mathscr{A}(x)$ such that

$$X_x^L(T-) \succeq \operatorname{diag}[S(T)] \left(C + \int_0^T c(t)dt \right)$$

holds almost surely. This implies from (4.50) that

$$X_x^{Lc}(T-) \succeq \operatorname{diag}[S(T)]C,$$

almost surely. We need to show that $L(\cdot) \in \mathscr{A}(x, c)$, or equivalently that $X^{Lc}(t) \succeq 0$, $\forall t \in [0, T)$ holds almost surely. To see this, fix an equivalent martingale measure $\mathbb{Q}$ for the process $S(\cdot)$ and let $z \in \operatorname{diag}[S(0)]K^* \setminus \{0\}$ be an arbitrary constant vector. Let $\rho(\cdot)$ be the Radon-Nikodým density process of $\mathbb{Q}$ with respect to $\mathbb{P}$, given by (2.18), so that the process $Z(\cdot) \triangleq \rho(\cdot) \operatorname{diag}[S(\cdot)] \operatorname{diag}[S(0)]^{-1} z$ belongs to $\mathscr{M}_0$; see the first paragraph of Section 2.3. Then, since $L(\cdot) \in \mathscr{A}(x)$, it follows from Lemma 2.3.2 that the process $Z(\cdot) \operatorname{diag}[S(\cdot)]^{-1} X_x^L(\cdot)$ is a supermartingale. This implies that

$$\begin{aligned}
Z(t) \cdot \operatorname{diag}[S(t)]^{-1} X_x^L(t) &\geq \mathbb{E}[Z(T) \cdot \operatorname{diag}[S(T)]^{-1} X_x^L(T-) \mid \mathscr{F}(t)] \\
&\geq \mathbb{E}\left[Z(T) \cdot \left(C + \int_0^T c(u)du \right) \;\middle|\; \mathscr{F}(t) \right] \\
&= \mathbb{E}\left[Z(T) \cdot \left(C + \int_t^T c(u)du \right) \;\middle|\; \mathscr{F}(t) \right] + Z(t) \cdot \int_0^t c(u)du,
\end{aligned}$$

and hence that

$$Z(t) \cdot \operatorname{diag}[S(t)]^{-1} X_x^{Lc}(t) \geq \mathbb{E}\left[Z(T) \cdot \left(C + \int_t^T c(u)du \right) \;\middle|\; \mathscr{F}(t) \right] \geq 0$$

from (4.50). Since $Z(\cdot) = \rho(\cdot)\,\mathrm{diag}[S(\cdot)]\,\mathrm{diag}[S(0)]^{-1}z$ and $\rho(\cdot)$ is strictly positive, we obtain

$$\mathrm{diag}[S(0)]^{-1}z \cdot X_x^{Lc}(t) \geq 0, \quad \forall t \in [0,T),$$

almost surely, which implies that $X_x^{Lc}(t) \succeq 0$, $\forall t \in [0,T)$ almost surely because $z \in \mathrm{diag}[S(0)]K^* \setminus \{0\}$ is arbitrary. Thus, $L(\cdot) \in \mathscr{A}(x.c)$, and (C,c) is financeable for x. $\square$

Proof (Necessity): Suppose that (C,c) is financeable for x. Then, there exists a trading strategy $L(\cdot) \in \mathscr{A}(x,c)$ such that

$$X_x^{Lc}(T-) \succeq \mathrm{diag}[S(T)]C$$

holds almost surely, which implies from (4.50) that

$$\begin{aligned} X_x^{L}(T-) &= X_x^{Lc}(T-) + \mathrm{diag}[S(T)] \int_0^T c(t)dt \\ &\succeq \mathrm{diag}[S(T)] \left(C + \int_0^T c(t)dt \right), \end{aligned}$$

almost surely. Since $L(\cdot) \in \mathscr{A}(x,c)$, the equation (4.50) also gives

$$X_x^{L}(t) = X_x^{Lc}(t) + \mathrm{diag}[S(t)] \int_0^t c(u)du \succeq 0, \quad \forall t \in [0,T),$$

almost surely, which implies that $L(\cdot) \in \mathscr{A}(x)$. Therefore, the random vector $\mathrm{diag}[S(T)] \left(C + \int_0^T c(t)dt \right)$ is hedgeable. $\square$

A *preference structure* is a pair of functions $U\colon \mathbb{R}_+^d \to \mathbb{R}$ and $u\colon [0,T) \times \mathbb{R}_+^d \to \mathbb{R}$, where $U(\cdot)$ and $u(t,\cdot)$ are utility functions as in Definition 4.2.1 such that the function $u(\cdot,\cdot)$ is $\mathscr{B}([0,T)) \otimes \mathscr{B}(\mathbb{R}_+^d)$-measurable. For simplicity, we assume that the dimension d_1, appearing Definition 4.2.1, does not depend on t, that is, the functions U and u are of the form

$$U(c_1,c_2) = U_1(c_1) \quad \text{and} \quad u(t,c_1,c_2) = u_1(t,c_1), \quad \forall t \in [0,T), \tag{4.51}$$

where $(c_1, c_2) \in \mathbb{R}_+^{d_1} \times \mathbb{R}_+^{d_2}$ and $d_1 + d_2 = d$.

Given a preference structure (U, u), we consider the following utility maximization problem.

$$V(x) \triangleq \sup_{(C,c)\in\check{\mathscr{C}}(x)} \mathbb{E}\left[U(C) + \int_0^T u(t, c(t))dt\right], \quad x \in K. \tag{4.52}$$

As a conjugate of the set $\check{\mathscr{C}}(x)$ of all terminal/inter-temporal consumption pairs (C, c) that are financeable for x, we take for each $z \in \operatorname{diag}[S(0)]K^*$ the set $\check{\mathscr{D}}(z)$ of all pairs $(H, h) \in \mathscr{L}$ such that

(i) $\operatorname{diag}[S(T,\omega)]^{-1}H(\omega) \in K^*$ for $\mathbb{P}$-a.e. $\omega \in \Omega$;

(ii) $\operatorname{diag}[S(t,\omega)]^{-1}h(t,\omega) \in K^*$ for $(\mu \otimes \mathbb{P})$-a.e. $(t,\omega) \in [0,T) \times \Omega$;

(iii) $\mathbb{E}\left[H \cdot C + \int_0^T h(t) \cdot c(t)dt\right] \leq z \cdot \operatorname{diag}[S(0)]^{-1}x$, $\forall x \in K$, $\forall (C,c) \in \check{\mathscr{C}}(x)$.

Under the identification $\check{h} = (H, h)$, which we mentioned right after the equation (4.48), these three conditions can be written simply as

$$\operatorname{diag}[S(t,\omega)]^{-1}\check{h}(t,\omega) \in K^*, \quad (\check{\mu} \otimes \mathbb{P})\text{-a.e. } (t,\omega) \in [0,T] \times \Omega,$$
$$\int_{[0,T]\times\Omega} \check{h}(t,\omega) \cdot \check{c}(t,\omega)(\check{\mu} \otimes \mathbb{P})(dt, d\omega) \leq z \cdot \operatorname{diag}[S(0)]^{-1}x, \quad \forall x \in K,\ \forall \check{c} \in \check{\mathscr{C}}(x).$$

With analogy of (4.30), we set

$$\check{\mathscr{G}} \triangleq \{(z, (H,h)) \in \operatorname{diag}[S(0)]K^* \times \mathscr{L} \mid (H,h) \in \check{\mathscr{D}}(z)\}. \tag{4.53}$$

It is easy to see that $\check{\mathscr{G}}$ is a convex cone in $\mathbb{R}^d \times \mathscr{L}$ and is closed under a.e.-convergence with respect to $(\check{\mu} \otimes \mathbb{P})$, that is, if $\{(z_k, (H_k, h_k)\}_{k\in\mathbb{N}} \in \check{\mathscr{G}}$ is a sequence such that $z_k \to z$ in $\mathbb{R}^d$, $H_k \to H$ $\mathbb{P}$-a.s. and that $h_k \to h$ $(\mu \otimes \mathbb{P})$-a.e., then we have $(z, (H,h)) \in \check{\mathscr{G}}$. Let $x \in \mathbb{R}_+^d \subseteq K$ and consider the pair

$$(C, c) \triangleq \frac{1}{2} \operatorname{diag}[S(0)]^{-1}\left(x, \frac{x}{T}\right).$$

Then for this pair (C, c) and the constant function $L(\cdot) \equiv 0$, we see from (4.49) that

$$X_x^{iLc}(t) = \frac{S^i(t)}{S^i(0)}\left(1 - \frac{t}{2T}\right)x^i \geq 0, \quad \forall t \in [0, T), \forall i = 1, \dots, d,$$

which implies that $L(\cdot) \equiv 0 \in \mathscr{A}(x, c)$. Since we also have $\operatorname{diag}[S(T)]^{-1}X_x^{Lc}(T-) = 1/2\operatorname{diag}[S(0)]^{-1}x = C$, we obtain $(C, c) \in \check{\mathscr{C}}(x)$. Therefore, for every pair $(H, h) \in \check{\mathscr{D}}(z)$, we must have

$$\begin{aligned}
z \cdot \operatorname{diag}[S(0)]^{-1}x &\geq \mathbb{E}\left[H \cdot C + \int_0^T h(t) \cdot c(t)dt\right] \\
&= \frac{1}{2}\mathbb{E}\left[H \cdot \operatorname{diag}[S(0)]^{-1}x + \frac{1}{T}\left(\int_0^T h(t)dt\right) \cdot \operatorname{diag}[S(0)]^{-1}x\right] \\
&= \frac{1}{2}\mathbb{E}\left[H + \frac{1}{T}\int_0^T h(t)dt\right] \cdot \operatorname{diag}[S(0)]^{-1}x \\
&\geq \frac{1}{2}\left(1 \wedge \frac{1}{T}\right)\mathbb{E}\left[H + \int_0^T h(t)dt\right] \cdot \operatorname{diag}[S(0)]^{-1}x,
\end{aligned}$$

and thus

$$\mathbb{E}\left[H + \int_0^T h(t)dt\right] \cdot \operatorname{diag}[S(0)]^{-1}x \leq 2(1 \vee T)z \cdot \operatorname{diag}[S(0)]^{-1}x.$$

By taking x to be the i-th unit vector, we obtain

$$0 \leq \mathbb{E}\left[H^i + \int_0^T h^i(t)dt\right] \leq 2(1 \vee T)z^i, \quad \forall i = 1, \dots, d. \tag{4.54}$$

Thus, the set $\check{\mathscr{D}}(z)$ is bounded in $\mathscr{L}^1 \triangleq \mathbb{L}^1(([0, T] \times \Omega, \check{\Sigma}_\pi, \check{\mu} \otimes \mathbb{P}); \mathbb{R}^d)$.

Now, denote by $\tilde{U}(\cdot)$ and $\tilde{u}(t, \cdot)$ the convex conjugates of the utility functions $U(\cdot)$ and $u(t, \cdot)$, respectively, and define the function $I(\cdot)$ and $\imath(t, \cdot)$ accordingly as in (4.17), so that we have

$$\begin{aligned}
&\mathbb{E}\left[U(C) + \int_0^T u(t, c(t))dt\right] \\
&\qquad\qquad \leq \mathbb{E}\left[\tilde{U}(H) + \int_0^T \tilde{u}(t, h(t))dt\right] + z \cdot \operatorname{diag}[S(0)]^{-1}x,
\end{aligned} \tag{4.55}$$

with equality if we have

$$
\begin{aligned}
C &= I(H) \\
c(\cdot) &= \imath(\cdot, h(\cdot))
\end{aligned}
\tag{4.56}
$$

and

$$\mathbb{E}\left[H\cdot C + \int_0^T h(t)\cdot c(t)dt\right] = z\cdot \mathrm{diag}[S(0)]^{-1}x,$$

as well as $H \in (0,\infty)^d$ a.s. and $h \in (0,\infty)^d$ $(\mu\otimes\mathbb{P})$-a.e. Given $x \in K$, we set

(4.57)

$$W(x) \triangleq \inf_{(z,(H,y))\in\mathscr{G}} \left\{\mathbb{E}\left[\tilde{U}(H) + \int_0^T \tilde{u}(t,h(t))dt\right] + z\cdot\mathrm{diag}[S(0)]^{-1}x\right\}.$$

As in the case of the utility maximization from terminal consumption, for the function U_1, we make the same assumption as Assumption 4.2.7. For the inter-temporal part u_1, we assume that the function $u_1(t,\cdot)$ satisfies the same conditions as U_1, "uniformly in $t \in [0,T)$". More precisely, we make the following assumption.

Assumption 4.5.2. The function u_1 satisfies the following conditions:

$$AE(u_1) \triangleq \overline{\lim_{b\to\infty}} \sup_{\substack{c_1 : [0,T)\to(0,\infty)^{d_1} \\ \inf_{t\in[0,T)} m(c_1(t))\geq b}} \left\{\sup_{t\in[0,T)}\left[\frac{c_1(t)\cdot\nabla u_1(t,c_1(t))}{u_1(t,c_1(t))}\right]\right\} < 1, \tag{4.58a}$$

$$\underline{\lim}_{r\downarrow 0} \inf_{\substack{y_1 : [0,T)\to(0,\infty)^{d_1} \\ \inf_{t\in[0,T)} m(y_1(t))\leq r}} \left\{\inf_{t\in[0,T)} m(\imath_1(t,y_1(t)))\right\} = \infty, \tag{4.58b}$$

$$\int_0^T \tilde{u}_1(t,y_1)dt < \infty, \quad \forall y_1 \in (0,\infty)^{d_1}. \tag{4.58c}$$

Here, as in Assumption 4.2.7, we are denoting $m(\eta) \triangleq \min_{1\leq i\leq d_1} \eta^i$ for any vector $\eta \in \mathbb{R}_+^{d_1}$.

As an example, let u_1 be "separable", i.e., $u_1(t,c_1) = \beta(t)\,\Upsilon_1(c_1)$, where the function $\beta\colon [0.T) \to (0,\infty)$ is bounded, continuous and uniformly away from 0, and the function Υ_1 satisfies the conditions for U_1 in Definition 4.2.1 as well as Assumption 4.2.7. Then, it is easy to check that u_1 satisfies the conditions (4.58a) – (4.58c).

We have an analogy of Lemma 4.2.8.

Lemma 4.5.3. *Under Assumption 4.5.2, there exist a number $\beta > 0$ and a function $\zeta\colon [0,T) \times (0,1) \to \mathbb{R}_+$ such that for every fixed $t \in [0,T)$, the function $\zeta(t,\cdot)$ is non-increasing on $(0,1)$, for every fixed $\alpha \in (0,1)$ the function $\zeta(\cdot,\alpha)$ is integrable on $[0,T)$, and that the inequality*

$$\tilde{u}_1(t,\alpha y_1(t)) \le \zeta(t,\alpha) + \alpha^{-\beta}\tilde{u}_1(t,y_1(t)) \tag{4.59}$$

is valid for every $t \in [0,T)$, $\alpha \in (0,1)$ and $y_1\colon [0,T) \to (0,\infty)^{d_1}$.

Proof. From (4.58a), there exists numbers $\gamma \in (0,1)$ and $b > 0$ such that

$$c_1(t)\cdot\nabla u_1(t,c_1(t)) < \gamma u_1(t,c_1(t)), \quad \forall t \in [0,T), \tag{4.60}$$

whenever $\inf_{t\in[0,T)} m(c_1(t)) > b$. Then, for every function $y_1\colon [0,T) \to (0,\infty)^{d_1}$ satisfying $\inf_{t\in[0,T)} m(\imath_1(t,y_1(t)) > b$, we have

$$\begin{aligned}
\imath_1(t,y_1(t))\cdot y_1(t) &= \imath_1(t,y_1(t))\cdot\nabla u_1(t,\imath_1(t,y_1(t))) \\
&< \gamma u_1(t,\imath_1(t,y_1(t))) \\
&= \gamma[\tilde{u}_1(t,y_1(t)) - y_1(t)\cdot\imath_1(t,y_1(t))],
\end{aligned}$$

and thus

$$\tilde{u}_1(t,y_1(t)) < \frac{1-\gamma}{\gamma} y_1(t)\cdot\imath_1(t,y_1(t)),$$

for every $t \in [0,T)$. We claim that for every such function $y_1\colon [0,T) \to (0,\infty)^{d_1}$ with the property $\inf_{t\in[0,T)} m(\imath_1(t,y_1(t)) > b$, the inequality

$$\tilde{u}_1(t,\alpha y_1(t)) < \alpha^{-\gamma/(1-\gamma)}\tilde{u}_1(t,y_1(t)) \tag{4.61}$$

holds for every number $\alpha \in (0,1)$. To see this, fix arbitrary $t \in [0,T)$, and define functions f and g on $(0,\infty)$ by

$$\begin{aligned} f(\alpha) &\triangleq \tilde{u}_1(t, \alpha y_1(t)) \\ g(\alpha) &\triangleq \alpha^{-\gamma/(1-\gamma)} \tilde{u}_1(t, y_1(t)). \end{aligned}$$

Then f and g are of class C^1 with $f(1) = g(1)$ and

$$\begin{aligned} f'(\alpha) &= -y_1(t) \cdot \imath_1(t, \alpha y_1(t)) < 0 \\ g'(\alpha) &= -\frac{\gamma}{1-\gamma} \alpha^{-\gamma/(1-\gamma)-1} \tilde{u}_1(t, y_1(t)) < 0, \end{aligned}$$

and thus, in particular, $f'(1) > g'(1)$. This implies that there exists a number $\delta > 0$ such that $f(\alpha) < g(\alpha)$, $\forall \alpha \in (1-\delta, 1)$. Let $\hat{\alpha} \triangleq \sup\{\alpha \in (0,1) \mid f(\alpha) = g(\alpha)\}$ and assume that $\hat{\alpha} > 0$. Then, by continuity, we would have $f(\hat{\alpha}) = g(\hat{\alpha})$. We would also have $f'(\hat{\alpha}) \le g'(\hat{\alpha})$ because $f(\alpha) < g(\alpha)$ for $\alpha \in (\hat{\alpha}, 1)$. But this is impossible because

$$\begin{aligned} g'(\hat{\alpha}) &= -\frac{\gamma}{1-\gamma} \hat{\alpha}^{-\gamma/(1-\gamma)-1} \tilde{u}_1(t, y_1(t)) \\ &< -\frac{\gamma}{1-\gamma} \hat{\alpha}^{-\gamma/(1-\gamma)-1} \frac{1-\gamma}{\gamma} y_1(t) \cdot \imath_1(t, y_1(t)) \\ &= -\hat{\alpha}^{-\gamma/(1-\gamma)-1} y_1(t) \cdot \imath_1(t, y_1(t)) \\ &= -\hat{\alpha}^{-\gamma/(1-\gamma)-1} f'(\hat{\alpha}) \\ &< f'(\hat{\alpha}). \end{aligned}$$

Thus, $f(\alpha) < g(\alpha)$, $\forall \alpha \in (0,1)$, which provides (4.61).

Next, notice that from (4.58b), there exists a number $r > 0$ such that the inequality $\inf_{t\in[0,T)} m(\imath_1(t, y_1(t))) > b$ holds for every function $y_1 \colon [0,T) \to (0,\infty)^{d_1}$ with $\inf_{t\in[0,T)} m(y_1(t)) < r$. This implies that if $\inf_{t\in[0,T)} m(\imath_1(t, y_1(t))) \le b$, then we must have $\inf_{t\in[0,T)} m(y_1(t)) \ge r$, and thus, $m(y_1(t)) \ge r$, $\forall t \in$

$[0,T)$. In conjunction with (4.14), we obtain

$$\tilde{u}_1(t, \alpha y_1(t)) \leq \tilde{u}_1(t, \alpha r(1, \cdots, 1)) =: \zeta(t, \alpha), \quad \forall t \in [0,T), \tag{4.62}$$

for every such function $y_1 \colon [0,T) \to (0,\infty)^{d_1}$ with $\inf_{t\in[0,T)} m(\imath_1(t, y_1(t)) \leq b$. Note that $\zeta(t,\cdot)$ is non-increasing. Furthermore, from (4.58c), we have $\int_0^T \zeta(t,\alpha)dt < \infty$, $\forall \alpha \in (0,\infty)$. Combining the two inequalities (4.61) and (4.62), we obtain the assertion. □

The existence theorem under this setting is the following.

Theorem 4.5.4. *Let $x \in \operatorname{int} K$ be a given initial endowment vector and assume $W(x) < \infty$. Then there exists a triplet $(\hat{z}, (\hat{H}, \hat{h})) \in \mathscr{G}$ with $\hat{H} \in (0,\infty)^d$ $\mathbb{P}$-a.e. and $\hat{h} \in (0,\infty)^d$ $(\mu \otimes \mathbb{P})$-a.e. that attains the infimum in* (4.57)*. Furthermore, under Assumptions 4.2.7 and 4.5.2, the pair $(\hat{C}, \hat{c}(\cdot)) \triangleq (I(\hat{H}), \imath(\cdot, \hat{h}(\cdot))$ is a solution to the original utility maximization problem* (4.52)*, that is, we have $(\hat{C}, \hat{c}) \in \check{\mathscr{C}}(x)$ and*

$$\mathbb{E}\left[U(\hat{C}) + \int_0^T u(t, \hat{c}(t))dt\right] = \sup_{(C,c)\in\check{\mathscr{C}}(x)} \mathbb{E}\left[U(C) + \int_0^T u(t, c(t))dt\right]. \tag{4.63}$$

We can prove Theorem 4.5.4 in the same manner as Theorem 4.4.1 and shall present the proof in Appendix B.1.

A Appendix for Chapter 3

A.1 On the Bipolar Theorem on the Space $\mathbb{L}^0_+$

In Section 3.2, we defined the dual space $\mathscr{D}^1(1)$ of the set A^1_1 by taking the bipolar of the set $\mathscr{D}^1_0(1)$ of (3.3). The reason for considering the bipolar set was that we needed closedness to ensure the existence of a solution to the dual optimization problem. There, we used the following bipolar theorem of [5].

Theorem A.1.1 (Bipolar Theorem). *For any subset $A \subseteq \mathbb{L}^0_+$, its bipolar set $A^{\circ\circ} \equiv (A^\circ)^\circ$ in the notation of* (3.6)*, coincides with the smallest convex, solid, closed (under the topology of convergence in probability) subset $\tilde{A}$ of $\mathbb{L}^0_+$, that contains A.*

While the bipolar theorem quoted above gives a duality characterization of the set $\tilde{A}$, the next proposition gives an "inner" description of it. We used this result in the proof of Proposition 3.2.4. For functions f, $g \in \mathbb{L}^0_+$, we write $f \le g$ if $f(\omega) \le g(\omega)$ for almost every $\omega \in \Omega$. Also, for a subset A of $\mathbb{L}^0_+$, we define the *solid hull* $S(A)$ of A by

$$S(A) \triangleq \{g \in \mathbb{L}^0_+ \mid g \le f \text{ for some } f \in A\}. \tag{A.1}$$

We denote by $\mathrm{cl}[A]$ and $\mathrm{co}[A]$ the closure (under the topology of convergence in probability) and the convex hull, respectively, of the set A.

Proposition A.1.2. *Let $A \subseteq \mathbb{L}^0_+$. Then the smallest convex, solid, closed (under the topology of convergence in probability) subset $\tilde{A}$ of $\mathbb{L}^0_+$, that contains A, is given by*

$$\tilde{A} = \mathrm{cl}[\mathrm{co}[S(A)]]. \tag{A.2}$$

Proof. Since the set $\tilde{A}$ is convex and contains $S(A)$, we have $\mathrm{co}[S(A)] \subseteq \tilde{A}$. But then, since $\tilde{A}$ is also closed, it follows that $\mathrm{cl}[\mathrm{co}[S(A)]] \subseteq \tilde{A}$. On the other hand, it is clear that the set $\mathrm{cl}[\mathrm{co}[S(A)]]$ is convex, closed and contains A. Thus, in order to prove $\mathrm{cl}[\mathrm{co}[S(A)]] \supseteq \tilde{A}$, it suffices to show that $\mathrm{cl}[\mathrm{co}[S(A)]]$ is solid. But since $S(A)$ is solid, this follows from Lemmas A.1.3 – A.1.5 below. □

The first two lemmas are essentially the same as Theorems 1.2 – 1.3 of [1].

Lemma A.1.3. *If $g \leq f_1 + \cdots + f_n$ holds for g, $f_1, \ldots, f_n \in \mathbb{L}^0_+$, then there exist $g_1, \ldots, g_n \in \mathbb{L}^0_+$ such that $g_i \leq f_i$ for $i = 1, \ldots, n$ and $g = g_1 + \cdots + g_n$.*

Proof. For $n = 1$, the assertion is trivial. For $n \geq 2$, we prove it by induction. First, assume $g \leq f_1 + f_2$ and put $g_1 \triangleq g \wedge f_1$ and $g_2 \triangleq g - g_1$. Then by definition, we have $0 \leq g_1 \leq f_1$ and $g_1 + g_2 = g$. Also, we have

$$g_2 = g - g_1 \geq g - g = 0$$

and

$$g_2 = g - g_1 = g - (g \wedge f_1) = (g - f_1)1_{\{g \geq f_1\}} \leq f_2.$$

Therefore, g_1 and g_2 satisfy the required conditions. This prove the assertion for $n = 2$. Next, let $n \geq 3$ and assume that the assertion is true for $n-1$. Let $g \leq f_1 + \cdots + f_n = f_1 + (f_2 + \cdots + f_n)$. From the previous case for $n = 2$, there exist g_1, $h \in \mathbb{L}^0_+$ such that $g_1 \leq f_1$, $h \leq f_2 + \cdots + f_n$ and $g_1 + h = g$. Now, by the induction assumption, there exist $g_2, \dots, g_n \in \mathbb{L}^0_+$ such that $g_i \le f_i$ for $i = 2, \dots, n$, and $g_2 + \cdots + g_n = h$. But then $g_1 + \cdots + g_n = g_1 + h = g$, and we are done. $\square$

Lemma A.1.4. *If S is a solid subset of $\mathbb{L}^0_+$, then so is its convex hull* $\mathrm{co}[S]$.

Proof. Let $f \in \mathrm{co}[S]$, $g \in \mathbb{L}^0_+$ and $g \le f$. There exist $f_1, \dots, f_n \in S$, $\lambda_1, \dots, \lambda_n \in (0,1]$ with $\sum_{i=1}^n \lambda_i = 1$ such that

$$f = \sum_{i=1}^n \lambda_i f_i.$$

From the previous lemma, there exist $g_1, \dots, g_n \in \mathbb{L}^0_+$ such that $g_i \le \lambda_i f_i$ for $i = 1, \dots, n$ and $g_1 + \cdots + g_n = g$. Put $h_i \triangleq g_i / \lambda_i$ for $i = 1, \dots, n$. Then since

$$h_i \le \frac{1}{\lambda_i} \lambda_i f_i = f_i \in S,$$

we have $h_i \in S$ because of the solidity of the set S. Now, we have

$$g = \sum_{i=1}^n g_i = \sum_{i=1}^n \lambda_i h_i \in \mathrm{co}[S].$$

Therefore, $\mathrm{co}[S]$ is solid. $\square$

Lemma A.1.5. *If S is a solid subset of $\mathbb{L}^0_+$, then so is its closure* $\mathrm{cl}[S]$.

Proof. Let $f \in \mathrm{cl}[S]$, $g \in \mathbb{L}^0_+$ and $g \le f$. There exists a sequence $\{f_n\}_{n=1}^\infty \subseteq S$ such that $f_n \to f$ in probability. Let

$$g_n \triangleq g \wedge f_n, \quad n \in \mathbb{N}.$$

Then $g_n \le f_n$ and hence $g_n \in S$, $\forall n \in \mathbb{N}$. Also, for every $\varepsilon > 0$, we have

$$\begin{aligned} \mathbb{P}[|g - g_n| \ge \varepsilon] &= \mathbb{P}\left[(g - f_n) 1_{\{g \ge f_n\}} \ge \varepsilon\right] \\ &\le \mathbb{P}\left[f - f_n \ge \varepsilon\right] \\ &\le \mathbb{P}\,]|f - f_n| \ge \varepsilon] \\ &\to 0, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore, $g \in \mathrm{cl}[S]$, which proves the assertion. $\square$

A.2 Proof of Theorem 3.5.1

We first prove, in Lemma A.2.1, the existence of an optimal $\hat{H}$ for the dual problem (3.75).

Lemma A.2.1. *For each $z > 0$, there exists a random variable $\hat{H} \equiv \hat{H}(z) \in \mathscr{D}^1(1)$ that attains the infimum in* (3.75).

Proof. Take a sequence $\{H_n\}_{n=1}^{\infty} \subseteq \mathscr{D}^1(1)$ such that $\lim_{n\to\infty} \downarrow \mathbb{E}(1 - zH_n G)^+ = \tilde{V}(z)$. As in the proof of Lemma 3.4.1, from Komlós' theorem, there exists a subsequence $\{H_{n_k}\}_{k=1}^{\infty}$ of $\{H_n\}_{n=1}^{\infty}$ such that the sequence $\{\Theta_k\}_{k=1}^{\infty}$ defined by

$$\Theta_k \triangleq \frac{1}{k} \sum_{i=1}^{k} H_{n_i}, \quad k \in \mathbb{N}$$

converges, almost surely, to some random variable $\hat{H} \in \mathscr{D}^1(1)$. The convexity of the functional

$$\mathscr{D}^1(1) \ni H \mapsto \mathbb{E}(1 - zHG)^+ \in [0, \infty) \tag{A.3}$$

and the fact $\Theta_k \in \mathscr{D}^1(1)$ imply that

$$\tilde{V}(z) \le \mathbb{E}(1 - z\Theta_k G)^+ \le \frac{1}{k} \sum_{i=1}^{k} \mathbb{E}(1 - zH_{n_i} G)^+, \quad \forall k \in \mathbb{N}. \tag{A.4}$$

Let $k \to \infty$. Then, being the Cesàro average of the set of numbers $\{\mathbb{E}(1 - zH_{n_i} G)^+\}_{i=1}^{k}$, the right-most-side of (A.4) converges to $\tilde{V}(z)$, which gives

$$\tilde{V}(z) = \lim_{k\to\infty} \mathbb{E}(1 - z\Theta_k G)^+.$$

On the other hand, since $0 \leq (1 - z\Theta_k G)^+ \leq 1$, $\forall k \in \mathbb{N}$, the bounded convergence theorem gives

$$\mathbb{E}(1 - z\hat{H}G)^+ = \lim_{k\to\infty} \mathbb{E}(1 - z\Theta_k G)^+.$$

We thus obtain

$$\tilde{V}(z) = \mathbb{E}(1 - z\hat{H}G)^+.$$

This shows that $\hat{H}$ attains the infimum in (3.75). □

Next, we investigate the behavior of the function $\gamma(\cdot\,;x)$ of (3.74).

Lemma A.2.2. *The function $\gamma(\cdot\,;x)$ is continuous and convex on $[0,\infty)$, and satisfies*

$$\lim_{z\downarrow 0} \frac{\gamma(z;x)-1}{z} = -G(0) + x < 0 \tag{A.5}$$

as well as

$$\lim_{z\to\infty} \frac{\gamma(z;x)-1}{z} = x. \tag{A.6}$$

Proof of the convexity of $\gamma(\cdot\,;x)$: It suffices to prove the convexity of $\tilde{V}(\cdot)$. Let z_1, $z_2 > 0$, $K_1 \in z_1\mathscr{D}^1(1)$, $K_2 \in z_2\mathscr{D}^1(1)$ and $0 < \alpha < 1$. Then, from (3.30), we have $\alpha K_1 + (1-\alpha)K_2 \in \mathscr{D}^1(\alpha z_1 + (1-\alpha)z_2)$, which, together with the convexity of the map $K \mapsto (1 - KG)^+$, gives

$$\begin{aligned}\tilde{V}(\alpha z_1 + (1-\alpha)z_2) &\leq \mathbb{E}(1 - (\alpha K_1 + (1-\alpha)K_2)G)^+ \\ &\leq \alpha\mathbb{E}(1 - K_1G)^+ + (1-\alpha)\mathbb{E}(1 - K_2G)^+.\end{aligned}$$

Taking the infimum of the right-hand-side over $K_1 \in z_1\mathscr{D}^1(1)$ and $K_2 \in z_2\mathscr{D}^1(1)$, we obtain

$$\tilde{V}(\alpha z_1 + (1-\alpha)z_2) \leq \alpha\tilde{V}(z_1) + (1-\alpha)\tilde{V}(z_2).$$

□

Proof of the continuity of $\gamma(\cdot\,; x)$*:* It suffices to prove that $\lim_{z\downarrow 0}\tilde{V}(z) = 1$. For given $\varepsilon > 0$, take $H_\varepsilon \in \mathscr{D}^1(1)$ such that

$$1 \geq \tilde{V}(z) = \inf_{H\in\mathscr{D}^1(1)} \mathbb{E}(1 - zHG)^+ > \mathbb{E}(1 - zH_\varepsilon G)^+ - \varepsilon.$$

From the bounded convergence theorem, we have $\lim_{z\downarrow 0}\mathbb{E}(1 - zH_\varepsilon G)^+ = 1$, which implies

$$1 \geq \varlimsup_{z\downarrow 0}\tilde{V}(z) \geq \varliminf_{z\downarrow 0}\tilde{V}(z) \geq 1 - \varepsilon.$$

By letting $\varepsilon \downarrow 0$, we obtain $\lim_{z\downarrow 0}\tilde{V}(z) = 1$. $\square$

Proof of (A.5)*:* By definition, for every $z > 0$ we have

$$\begin{aligned}
\frac{\gamma(z;x) - 1}{z} &= \frac{1}{z}\inf_{H\in\mathscr{D}^1(1)} \mathbb{E}[(1 - zHG)^+ - 1] + x \\
&= -\frac{1}{z}\sup_{H\in\mathscr{D}^1(1)} \mathbb{E}(1 \wedge zHG) + x \\
&\geq -\sup_{H\in\mathscr{D}^1(1)} \mathbb{E}[HG] + x \\
&= -G(0) + x,
\end{aligned}$$

and hence

$$\varliminf_{z\downarrow 0}\frac{\gamma(z;x) - 1}{z} \geq -G(0) + x. \tag{A.7}$$

To show that the reverse inequality for the limit-superior, we take, for each $\varepsilon > 0$, a random variable $H_\varepsilon \in \mathscr{D}^1(1)$ such that

$$\mathbb{E}[H_\varepsilon G] > \sup_{H\in\mathscr{D}^1(1)} \mathbb{E}[HG] - \varepsilon = G(0) - \varepsilon.$$

Then, for each $z > 0$, we have

$$\begin{aligned}
\frac{\gamma(z;x) - 1}{z} &= -\frac{1}{z}\sup_{H\in\mathscr{D}^1(1)} \mathbb{E}(1 \wedge zHG) + x \\
&\leq -\sup_{H\in\mathscr{D}^1(1)} \mathbb{E}[HG1_{\{zHG\leq 1\}}] + x \\
&\leq -\mathbb{E}[H_\varepsilon G1_{\{zH_\varepsilon G\leq 1\}}] + x.
\end{aligned}$$

Letting $z \downarrow 0$, we obtain from Fatou's lemma

$$\overline{\lim_{z\downarrow 0}} \frac{\gamma(z;x)-1}{z} \leq -\underline{\lim}_{z\downarrow 0} \mathbb{E}[H_\varepsilon G 1_{\{zH_\varepsilon G\leq 1\}}]+x \leq -\mathbb{E}[H_\varepsilon G]+x < -G(0)+\varepsilon+x.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\overline{\lim_{z\downarrow 0}} \frac{\gamma(z;x)-1}{z} \leq -G(0)+x. \tag{A.8}$$

In conjunction with (A.7), the inequality (A.8) now gives (A.5). □

Proof of (A.6)*:* By definition, we have

$$\frac{\gamma(z;x)-1}{z} = \inf_{H\in\mathscr{D}^1(1)} \mathbb{E}\left(\frac{1}{z} - HG\right)^+ - \frac{1}{z} + x.$$

But for an arbitrarily fixed $H' \in \mathscr{D}^1(1)$, we have by the dominated convergence theorem,

$$0 \leq \inf_{H\in\mathscr{D}^1(1)} \mathbb{E}\left(\frac{1}{z} - HG\right)^+ \leq \mathbb{E}\left(\frac{1}{z} - H'G\right)^+ \to 0, \quad \text{as} \quad z \to \infty.$$

Therefore,

$$\lim_{z\to\infty} \frac{\gamma(z;x)-1}{z} = x.$$

□

Based on the previous lemma, we show that $\gamma(\cdot\,;x)$ assumes the infimum at some $\hat{z} > 0$.

Lemma A.2.3. *The function $\gamma(\cdot\,;x)$ attains its infimum at some $0 < \hat{z} < \infty$.*

Proof. Because of (A.6), we cannot have $\inf_{0<\zeta<\infty} \gamma(\zeta;x) = \lim_{\zeta\uparrow\infty} \gamma(\zeta;x) < \gamma(z;x)$, $\forall z \in (0,\infty)$. Therefore, the convex function $\gamma(\cdot\,;x)$ either attains its infimum at some $0 < \hat{z} < \infty$, or else $\gamma(z;x) \geq \gamma(0+;x) = 1$, $\forall z > 0$. Suppose the latter is true. Then $\gamma(z;x)/z \geq 1$, $\forall z > 0$; but this is again impossible, because of (A.5). Therefore, the function $\gamma(\cdot\,;x)$ must attain its infimum at some $0 < \hat{z} < \infty$. □

With the help of Lemmas A.2.1 – A.2.3, we now proceed to prove Theorem 3.5.1.

Proof of Theorem 3.5.1: Let $0 < \hat{z} < \infty$ be the number given in Lemma A.2.3, and denote by $\hat{H} \equiv \hat{H}(\hat{z}) \in \mathscr{D}^1(1)$ the random variable given in Lemma A.2.1. Then it is clear that the pair $(\hat{z}, \hat{H})$ attains the first infimum in (3.73). This proves **(i)**. Once **(ii)** is proved, **(iii)** immediately follows from (3.72). It thus remains to prove **(ii)**.

In order to prove **(ii)**, we introduce the space $\mathbb{L}$ given by (3.33), and consider its subset $\mathscr{G}^1$ given by (3.35). Recall that the set $\mathscr{G}^1 \subseteq \mathbb{L}$ is convex and closed in $\mathbb{L}$ under the norm topology. We also consider the functional $\tilde{U} : \mathbb{L} \to \mathbb{R}$ given by

$$\tilde{U}(z, K) \triangleq \mathbb{E}(1 - KG)^+ + zx, \quad (z, K) \in \mathbb{L}. \tag{A.9}$$

It is easy to check that this functional is convex, proper and lower semi-continuous in the norm topology $\mathbb{L}$. Moreover, from Lemma A.2.1 and Lemma A.2.3, we know that $\tilde{U}$ attains the infimum over $\mathscr{G}^1$ at the point $(\hat{z}, \hat{z}\hat{H}) \in \mathscr{G}^1 \backslash \{(0,0)\}$. Notice also that the dual space of $\mathbb{L}$ is given by $\mathbb{L}^* = \mathbb{R} \times \mathbb{L}^\infty$, and that the subdifferential of $\tilde{U}$ and the normal cone of $\mathscr{G}^1$ at the point $(\hat{z}, \hat{z}\hat{H})$ are given by

(A.10)
$$\partial\tilde{U}(\hat{z}, \hat{z}\hat{H}) \triangleq \left\{ (y, Y) \in \mathbb{L}^* \;\middle|\; \begin{array}{r} \left[\mathbb{E}(1 - KG)^+ + zx\right] - \left[\mathbb{E}(1 - \hat{z}\hat{H}G)^+ + \hat{z}x\right] \\ \geq (z - \hat{z})y + \mathbb{E}[(K - \hat{z}\hat{H})Y], \ \forall (z, K) \in \mathbb{L} \end{array} \right\}$$

and

(A.11)
$$N(\hat{z}, \hat{z}\hat{H}) \triangleq \left\{ (y, Y) \in \mathbb{L}^* \;\middle|\; (z - \hat{z})y + \mathbb{E}[(zH - \hat{z}\hat{H})Y] \leq 0, \ \forall (z, zH) \in \mathscr{G}^1 \right\},$$

respectively; see Propositions 4.4.4 and 4.3.3 of [2]. The optimality of $(\hat{z}, \hat{z}\hat{H})$ implies the existence of a pair $(\hat{y}, \hat{Y}) \in \mathbb{L}^*$ such that

$$-(\hat{y}, \hat{Y}) \in \partial\tilde{U}(\hat{z}, \hat{z}\hat{H}), \tag{A.12}$$

and

$$(\hat{y}, \hat{Y}) \in N(\hat{z}, \hat{z}\hat{H}); \tag{A.13}$$

see Corollary 4.6.3 of [2]. From (A.10) and (A.11), these are equivalent to

$$\begin{aligned} \mathbb{E}(1 - KG)^+ - \mathbb{E}(1 - \hat{z}\hat{H}G)^+ + (z - \hat{z})x \qquad& \\ \geq -(z - \hat{z})\hat{y} - \mathbb{E}[(K - \hat{z}\hat{H})\hat{Y}], \quad& \forall (z, K) \in \mathbb{L}, \end{aligned} \tag{A.14}$$

and

$$(z - \hat{z})\hat{y} + \mathbb{E}[(zH - \hat{z}\hat{H})\hat{Y}] \leq 0, \quad \forall (z, zH) \in \mathscr{G}^1, \tag{A.15}$$

respectively.

First, we note that $\hat{y} = -x$. Indeed, observe from (A.14) that

$$(z - \hat{z})(x + \hat{y}) \geq \mathbb{E}(1 - \hat{z}\hat{H}G)^+ - \mathbb{E}(1 - KG)^+ - \mathbb{E}[(K - \hat{z}\hat{H})\hat{Y}], \quad \forall (z, K) \in \mathbb{L}.$$

If $x + \hat{y} \neq 0$ then, letting $z \to \pm\infty$ with K fixed, we could make the left-hand-side go to $-\infty$, a contradiction. Therefore, we have $\hat{y} = -x$, which implies from (A.15) that

$$-x(z - \hat{z}) + \mathbb{E}[(zH - \hat{z}\hat{H})\hat{Y}] \leq 0, \quad \forall (z, zH) \in \mathscr{G}^1.$$

Taking $z = 0$ and dividing by $\hat{z} > 0$, we obtain

$$\mathbb{E}[\hat{H}\hat{Y}] \geq x.$$

Also, from (A.15) with $\hat{y} = -x$, $H = \hat{H}$ and $z = \hat{z} + \varepsilon$ for some $\varepsilon > 0$, we get

$$-\varepsilon x + \varepsilon \mathbb{E}[\hat{H}\hat{Y}] \leq 0,$$

and hence $\mathbb{E}[\hat{H}\hat{Y}] \leq x$. Therefore,

$$\mathbb{E}[\hat{H}\hat{Y}] = x. \tag{A.16}$$

Also, (A.15) with $z = \hat{z}$ implies $\mathbb{E}[H\hat{Y}] \leq \mathbb{E}[\hat{H}\hat{Y}]$, $\forall H \in \mathscr{D}^1(1)$. This, together with (A.16) gives $\mathbb{E}[H\hat{Y}] \leq \mathbb{E}[\hat{H}\hat{Y}] = x$, $\forall H \in \mathscr{D}^1(1)$, and proves $\hat{Y} \in A^1_x$.

Now, we show that $\hat{Y}$ can be written as in the right-hand-side of (3.69), so that it can serve as an optimal solution $\hat{\xi}$ to the original problem (3.61). First, we define the random variable A by

$$\hat{Y} = G 1_{\{\hat{z}\hat{H}G<1\}} + A. \tag{A.17}$$

Then, (A.14) with $\hat{y} = -x$ implies

$$\mathbb{E}(1 - KG)^+ - \mathbb{E}(1 - \hat{z}\hat{H}G)^+ \geq -\mathbb{E}[(G1_{\{\hat{z}\hat{H}G<1\}} + A)(K - \hat{z}\hat{H})],$$

and hence,

(A.18)

$$\begin{aligned}\mathbb{E}[A(K - \hat{z}\hat{H})] &\geq \mathbb{E}(1 - \hat{z}\hat{H}G)^+ - \mathbb{E}(1 - KG)^+ - \mathbb{E}[(K - \hat{z}\hat{H})G1_{\{\hat{z}\hat{H}G<1,\, G>0\}}] \\ &= \mathbb{E}\left[\left(1_{\{KG<1\}} - 1_{\{\hat{z}\hat{H}G<1\}}\right)(KG - 1)\right]\end{aligned}$$

for every $K \in \mathbb{L}^1$. In particular, for any $K \in \mathbb{L}^1$ with $\{KG < 1\} = \{\hat{z}\hat{H}G < 1\}$, the right-hand-side of (A.18) vanishes, and the inequality (A.18) reduces to

$$\mathbb{E}[A(K - \hat{z}\hat{H})] \geq 0. \tag{A.19}$$

In the rest of the proof, we shall show that the random variable A must vanish outside the event $\{\hat{z}\hat{H}G = 1\}$ and satisfy $0 \leq A \leq G$ on $\{\hat{z}\hat{H}G = 1\}$.

First, let us suppose $\mathbb{P}[\hat{z}\hat{H}G < 1,\ A > 0] > 0$. Take

$$K \triangleq -1_{\{\hat{z}\hat{H}G<1,\ A>0\}} + \hat{z}\hat{H}1_{\{\hat{z}\hat{H}G\geq 1 \text{ or } A\leq 0\}}.$$

Then, $K \in \mathbb{L}^1$ and $\{KG < 1\} = \{\hat{z}\hat{H}G < 1\}$. But we also have

$$\mathbb{E}[A(K - \hat{z}\hat{H})] = -\mathbb{E}[A(1 + \hat{z}\hat{H})1_{\{A>0,\ \hat{z}\hat{H}G<1\}}] \leq -\mathbb{E}[A1_{\{A>0,\ \hat{z}\hat{H}G<1\}}] < 0,$$

which contradicts (A.19). Therefore,

$$A \leq 0 \quad \text{on} \quad \{\hat{z}\hat{H}G < 1\}. \tag{A.20}$$

Next, suppose that $\mathbb{P}[\hat{z}\hat{H}G \geq 1,\ A < 0] > 0$. Then

$$\mathbb{E}\left[\frac{A}{G}1_{\{\hat{z}\hat{H}G\geq 1,\ A<0\}}\right] < 0.$$

Take

$$K \triangleq \hat{z}\hat{H}1_{\{\hat{z}\hat{H}G<1 \text{ or } A\geq 0\}} + 2\hat{z}\hat{H}1_{\{\hat{z}\hat{H}G\geq 1,\ A<0\}}.$$

Then, $K \in \mathbb{L}^1$ and $\{KG < 1\} = \{\hat{z}\hat{H}G < 1\}$. But we also have

$$\mathbb{E}[A(K - \hat{z}\hat{H})] = \mathbb{E}[A\hat{z}\hat{H}1_{\{\hat{z}\hat{H}G\geq 1,\ A<0\}}] \leq \mathbb{E}\left[\frac{A}{G}1_{\{\hat{z}\hat{H}G\geq 1,\ A<0\}}\right] < 0,$$

which contradicts (A.19). Therefore,

$$A \geq 0 \quad \text{on} \quad \{\hat{z}\hat{H}G \geq 1\}. \tag{A.21}$$

Next, suppose $P[A > 0,\ G = 0] > 0$. Then $\mathbb{E}[A1_{\{G=0,\ A>0\}}] > 0$. Take

$$K \triangleq \hat{z}\hat{H} - 1_{\{G=0,\ A>0\}}.$$

Then, $K \in \mathbb{L}^1$ and $\{KG < 1\} = \{\hat{z}\hat{H}G < 1\}$. From (A.19), we obtain

$$0 \leq \mathbb{E}[A(K - \hat{z}\hat{H})] = -\mathbb{E}[A1_{\{G=0,\ A>0\}}] < 0,$$

a contradiction. Therefore, $\mathbb{P}[A > 0,\ G = 0] = 0$, which together with (A.20) implies

$$A = 0 \quad \text{on} \quad \{G = 0\}. \tag{A.22}$$

Now, suppose $\mathbb{P}[A < 0,\ \hat{z}\hat{H}G < 1] > 0$. Then, there exist a real number $\delta > 0$ and an integer $n_0 \in \mathbb{N}$ such that

$$\mathbb{E}\left[\frac{A}{G}\left(\hat{z}\hat{H}G - 1\right)1_{\{\hat{z}\hat{H}G<1,\ G\geq\frac{1}{n}\}}\right] > \delta, \quad \forall n \geq n_0. \tag{A.23}$$

Otherwise, for any $\delta > 0$ there would exist infinitely many integers n such that

$$\mathbb{E}\left[\frac{A}{G}\left(\hat{z}\hat{H}G - 1\right)1_{\{\hat{z}\hat{H}G<1,\ G\geq\frac{1}{n}\}}\right] \leq \delta.$$

By taking the limit-inferior as $n \to \infty$ and then letting $\delta \downarrow 0$, we would obtain

$$\varliminf_{n\to\infty} \mathbb{E}\left[\frac{A}{G}\left(\hat{z}\hat{H}G - 1\right)1_{\{\hat{z}\hat{H}G<1,\ G\geq\frac{1}{n}\}}\right] \leq 0.$$

But since we know from (A.20) that $A \leq 0$ on the set $\{\hat{z}\hat{H}G < 1\}$, the integrand is nonnegative, and Fatou's lemma would yield

$$0 \leq \mathbb{E}\left[\frac{A}{G}\left(\hat{z}\hat{H}G - 1\right)1_{\{\hat{z}\hat{H}G<1,\ G>0\}}\right] \leq \varliminf_{n\to\infty} \mathbb{E}\left[\frac{A}{G}\left(\hat{z}\hat{H}G - 1\right)1_{\{\hat{z}\hat{H}G<1,\ G\geq\frac{1}{n}\}}\right] \leq 0,$$

which would imply that $A = 0$ on $\{\hat{z}\hat{H}G < 1,\ G > 0\}$, a contradiction. This gives the implication $\mathbb{P}[A < 0,\ \hat{z}\hat{H}G < 1] > 0 \Rightarrow$ (A.23). Now, for given $\varepsilon > 0$ and $n \in \mathbb{N}$, take

$$K \triangleq \frac{1}{G}\left[(1-\varepsilon)1_{\{\hat{z}\hat{H}G<1,\ G\geq\frac{1}{n}\}} + 1_{\{\hat{z}\hat{H}G\geq 1\}}\right].$$

This $K \in \mathbb{L}^1$ satisfies $\{KG < 1\} = \{\hat{z}\hat{H}G < 1\}$, and the equations (A.19), (A.21) and (A.23) yield

$$
\begin{aligned}
\text{(A.24)}\quad 0 &\ge -\mathbb{E}[A(K - \hat{z}\hat{H})] \\
&= -\mathbb{E}\left[A\left(\frac{1-\varepsilon}{G}1_{\{\hat{z}\hat{H}G<1,\, G\ge\frac{1}{n}\}} + \frac{1}{G}1_{\{\hat{z}\hat{H}G\ge 1\}} - \hat{z}\hat{H}\right)\right] \\
&= \mathbb{E}\left[\frac{A}{G}(\hat{z}\hat{H}G-1)1_{\{\hat{z}\hat{H}G<1,\, G\ge\frac{1}{n}\}}\right] + \varepsilon\mathbb{E}\left[\frac{A}{G}1_{\{\hat{z}\hat{H}G<1,\, G\ge\frac{1}{n}\}}\right] \\
&\quad + \mathbb{E}\left[\frac{A}{G}\left(\hat{z}\hat{H}G-1\right)1_{\{\hat{z}\hat{H}G\ge 1\}}\right] + \mathbb{E}\left[\hat{z}\hat{H}A1_{\{\hat{z}\hat{H}G<1,\, G<\frac{1}{n}\}}\right] \\
&> \delta + \varepsilon\mathbb{E}\left[\frac{A}{G}1_{\{\hat{z}\hat{H}G<1,\, G\ge\frac{1}{n}\}}\right] + \mathbb{E}\left[\hat{z}\hat{H}A1_{\{\hat{z}\hat{H}G<1,\, G<\frac{1}{n}\}}\right], \\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall n \ge n_0.
\end{aligned}
$$

As for the last expectation in (A.24), we have

$$
\begin{aligned}
|\hat{z}\hat{H}A1_{\{\hat{z}\hat{H}G<1,\, G<\frac{1}{n}\}}| &\le |\hat{z}\hat{H}A1_{\{\hat{z}\hat{H}G<1,\, G>0\}}| \\
&= |\hat{z}\hat{H}\hat{Y}1_{\{\hat{z}\hat{H}G<1\}} - \hat{z}\hat{H}G1_{\{\hat{z}\hat{H}G<1,\, G>0\}}| \\
&\le |\hat{z}\hat{H}\hat{Y}1_{\{\hat{z}\hat{H}G<1\}}| + |\hat{z}\hat{H}G1_{\{\hat{z}\hat{H}G<1\}}| \\
&\le |\hat{z}\hat{H}|\|\hat{Y}\|_\infty + 1 \,\in\, \mathbb{L}^1
\end{aligned}
$$

and, from the dominated convergence theorem, (A.21) and (A.22), we obtain

$$
0 \ge \mathbb{E}[\hat{z}\hat{H}A1_{\{\hat{z}\hat{H}G<1,\, G<\frac{1}{n}\}}] \to \mathbb{E}[\hat{z}\hat{H}A1_{\{G=0\}}] = 0, \quad \text{as} \quad n \to \infty.
$$

This implies that there exists an integer $n_1 \ge 1$ with

$$
\text{(A.25)}\qquad 0 \ge \mathbb{E}[\hat{z}\hat{H}A1_{\{\hat{z}\hat{H}G<1,\, G<\frac{1}{n}\}}] > -\frac{\delta}{2}, \quad \forall n \ge n_1.
$$

It then follows from (A.24) and (A.25) that

$$
\begin{aligned}
0 > \delta + \varepsilon\mathbb{E}\left[\frac{A}{G}1_{\{\hat{z}\hat{H}G<1,\, G\ge\frac{1}{n}\}}\right] - \frac{\delta}{2} = \frac{\delta}{2} + \varepsilon\mathbb{E}\left[\frac{A}{G}1_{\{\hat{z}\hat{H}G<1,\, G\ge\frac{1}{n}\}}\right], \\
\forall n \ge \max(n_0, n_1),
\end{aligned}
$$

a contradiction, since $\varepsilon > 0$ is arbitrary. Therefore, $\mathbb{P}[A < 0,\ \hat{z}\hat{H}G < 1] = 0$, and hence from (A.20), we obtain

$$A = 0 \quad \text{on} \quad \{\hat{z}\hat{H}G < 1\}. \tag{A.26}$$

In conjunction with (A.26), the inequality (A.19) becomes

$$\mathbb{E}[A(K - \hat{z}\hat{H})1_{\{\hat{z}\hat{H}G \geq 1\}}] \geq 0, \tag{A.27}$$

which is to hold for every $K \in \mathbb{L}^1$ satisfying $\{KG < 1\} = \{\hat{z}\hat{H}G < 1\}$. Take

$$K \triangleq \frac{1_{\{\hat{z}\hat{H}G \geq 1\}}}{G}.$$

Then, since $\mathbb{E}|K| = \mathbb{E}\left[1_{\{\hat{z}\hat{H}G \geq 1\}}/G\right] \leq \mathbb{E}\left[\hat{z}\hat{H}1_{\{\hat{z}\hat{H}G \geq 1\}}\right] < \infty$ and since $\{KG < 1\} = \{\hat{z}\hat{H}G < 1\}$, we may apply (A.27) to obtain

$$\mathbb{E}\left[\frac{A}{G}\left(1 - \hat{z}\hat{H}G\right)1_{\{\hat{z}\hat{H}G > 1\}}\right] \geq 0.$$

Now, the integrand is nonpositive because of (A.21). It is strictly negative on the set $\{A < 0,\ \hat{z}\hat{H}G > 1\}$. Since the integral is nonnegative, it follows that

$$A = 0 \quad \text{on} \quad \{\hat{z}\hat{H}G > 1\}. \tag{A.28}$$

The inequalities (A.26) and (A.28) together with (A.18) then imply

$$\mathbb{E}\left[\frac{A}{G}(KG - 1)1_{\{\hat{z}\hat{H}G = 1\}}\right] \geq \mathbb{E}\left[\left(1_{\{KG < 1\}} - 1_{\{\hat{z}\hat{H}G < 1\}}\right)(KG - 1)\right] \tag{A.29}$$

It remains to show that $A \leq G$ on the set $\{\hat{z}\hat{H}G = 1\}$. Suppose $\mathbb{P}[A > G,\ \hat{z}\hat{H}G = 1] > 0$. Then, there exists a number $\delta > 0$ such that

$$\mathbb{E}\left[\frac{A}{G}1_{\{\hat{z}\hat{H}G = 1,\ A > G\}}\right] > \delta + \mathbb{E}[1_{\{\hat{z}\hat{H}G = 1,\ A > G\}}]. \tag{A.30}$$

For any $\varepsilon > 0$ and $n \in \mathbb{N}$, take

$$K \triangleq \frac{1}{G}\left[1_{\{\hat{z}\hat{H}G=1,\, A\leq G\}} + (1-\varepsilon)1_{\{\hat{z}\hat{H}G\neq 1,\, G\geq \frac{1}{n}\}}\right].$$

Then we can easily see that

$$KG - 1 = -1_{\{\hat{z}\hat{H}G=1,\, A>G\}} - 1_{\{\hat{z}\hat{H}G\neq 1,\, G<\frac{1}{n}\}} - \varepsilon 1_{\{\hat{z}\hat{H}G\neq 1,\, G\geq \frac{1}{n}\}}$$

and

$$1_{\{KG<1\}} - 1_{\{\hat{z}\hat{H}G<1\}} = 1_{\{\hat{z}\hat{H}G=1,\, A>G\}} + 1_{\{\hat{z}\hat{H}G>1\}}.$$

Using these inequalities, we now compute the both sides of (A.29) as

$$\text{LHS of (A.29)} = -\mathbb{E}\left[\frac{A}{G}1_{\{\hat{z}\hat{H}G=1,\, A>G\}}\right] < -\delta - \mathbb{E}[1_{\{\hat{z}\hat{H}G=1,\, A>G\}}] \tag{A.31}$$

and

$$\begin{aligned} &\text{RHS of (A.29)} \\ &\quad = -\mathbb{E}\Big[\Big(1_{\{\hat{z}\hat{H}G=1,\, A>G\}} + 1_{\{\hat{z}\hat{H}G\neq 1,\, G<\frac{1}{n}\}} + \varepsilon 1_{\{\hat{z}\hat{H}G\neq 1,\, G\geq\frac{1}{n}\}}\Big) \\ &\qquad\qquad \times \Big(1_{\{\hat{z}\hat{H}G=1,\, A>G\}} + 1_{\{\hat{z}\hat{H}G>1\}}\Big)\Big] \\ &\quad = -\mathbb{E}\Big[1_{\{\hat{z}\hat{H}G=1,\, A>G\}} + 1_{\{\hat{z}\hat{H}G>1,\, G<\frac{1}{n}\}} + \varepsilon 1_{\{\hat{z}\hat{H}G>1,\, G\geq\frac{1}{n}\}}\Big] \end{aligned} \tag{A.32}$$

It follows from (A.29), (A.31) and (A.32) that

$$0 \leq \text{LHS of (A.29)} - \text{RHS of (A.29)} < -\delta + \varepsilon\mathbb{E}[1_{\{\hat{z}\hat{H}G>1,\, G\geq\frac{1}{n}\}}] + \mathbb{E}[1_{\{\hat{z}\hat{H}G>1,\, G<\frac{1}{n}\}}].$$

By letting $n \to \infty$, we obtain

$$0 \leq -\delta + \varepsilon\mathbb{E}[1_{\{\hat{z}\hat{H}G>1\}}] + \mathbb{P}[\hat{z}\hat{H}G > 1,\ G = 0] = -\delta + \varepsilon\mathbb{E}[1_{\{\hat{z}\hat{H}G>1\}}].$$

Finally, by letting $\varepsilon \downarrow 0$, we obtain $\delta \leq 0$, a contradiction. Therefore, $\mathbb{P}[A > G,\ \hat{z}\hat{H}G = 1] = 0$, and hence, in conjunction with (A.21), we obtain

$$0 \leq A \leq G \quad \text{on} \quad \{\hat{z}\hat{H}G = 1\}. \tag{A.33}$$

This completes the proof. $\square$

B Appendix for Chapter 4

B.1 Proof of Theorem 4.5.4

We begin with the existence result of the dual optimization problem (4.57).

Lemma B.1.1. *Let $x \in \operatorname{int} K$ and assume $W(x) < \infty$. Then there exists a triplet $(\hat{z}, (\hat{H}, \hat{h})) \in \mathscr{G}$ that attains the infimum in (4.57). Furthermore, we have $\hat{H} \in (0,\infty)^d$, $\mathbb{P}$-a.e. and $\hat{h}(t,\omega) \in (0,\infty)^d$, $(\mu \otimes \mathbb{P})$-a.e.*

Proof. Let $\{(z_k, (H_k, h_k))\}_{k\in\mathbb{N}}$ be a minimizing sequence for (4.57), i.e.,

$$\lim_{k\to\infty} \downarrow \left(\mathbb{E}\left[\tilde{U}(H_k) + \int_0^T \tilde{u}(t, h_k(t))dt \right] + z_k \cdot \operatorname{diag}[S(0)]^{-1} x \right) = W(x). \tag{B.1}$$

Suppose that there exists a subsequence $\{(z_{k(l)}, (H_{k(l)}, h_{k(l)}))\}_{l\in\mathbb{N}}$ which converges to $(0,(0,0))$, $(\breve{\mu}\otimes\mathbb{P})$-a.e., that is, $z_{k(l)} \to 0$ in $\mathbb{R}^d$, $H_{k(l)} \to 0$ $\mathbb{P}$-a.e. and $h_{k(l)} \to 0$ $(\mu\otimes\mathbb{P})$-a.e. Then, from the lower semi-continuity of the functions $\tilde{U}(\cdot)$ and $\tilde{u}(t,\cdot)$ and from the equation (4.15b), we have $\lim_{l\to\infty} \tilde{U}(H_{k(l)}) \geq \tilde{U}(0) = \infty$ $\mathbb{P}$-a.e. and $\lim_{l\to\infty} \tilde{u}(t, h_{k(l)}(t)) \geq \tilde{u}(t,0) = \infty$, $(\mu\otimes\mathbb{P})$-a.e. But since $\tilde{U}$ and $\tilde{u}$ are non-negative because of (4.15a) and since $z_{k(l)} \cdot \operatorname{diag}[S(0)]^{-1}x \geq 0$, Fatou's lemma gives

$$\varliminf_{l\to\infty} \left(\mathbb{E}\left[\tilde{U}(H_{k(l)}) + \int_0^T \tilde{u}(t, h_{k(l)})dt \right] + z_{k(l)} \cdot \operatorname{diag}[S(0)]^{-1} x \right) = \infty,$$

and thus $W(x) = \infty$, a contradiction. This shows that the sequence $\{z_k\}_{k\in\mathbb{N}}$ is away from 0 for large k; otherwise, there would exist a subsequence

$\{z_{k(l)}\}_{l\in\mathbb{N}}$ such that $z_{k(l)} \to 0$, which would imply from (4.54) that $(H_{k(l)}, h_{k(l)}) \to (0,0)$ in $\mathscr{L}^1$, and thus, along a further subsequence, we would have $(z_{k(l)}, (H_{k(l)}, h_{k(l)})) \to (0,(0,0))$ $(\check{\mu}\otimes\mathbb{P})$-a.e., a contradiction.

Next, from (B.1) and $W(x) < \infty$, we have $\mathbb{E}\left[\tilde{U}(H_k) + \int_0^T \tilde{u}(t, h_k(t))dt\right] + z_k\cdot\mathrm{diag}[S(0)]^{-1}x < W(x) + 1$ for large k, which implies from (4.15a) that

$$\begin{aligned} z_k\cdot\mathrm{diag}[S(0)]^{-1}x &< -\mathbb{E}\left[\tilde{U}(H_k) + \int_0^T \tilde{u}(t, h_k(t))dt\right] + W(x) + 1 \\ &\le W(x) + 1 =: M < \infty. \end{aligned}$$

Note also that since $z_k \in \mathrm{diag}[S(0)]K^*$ and $x \in K$, we have $z_k\cdot\mathrm{diag}[S(0)]^{-1}x \ge 0$. Therefore,

$$0 \le z_k\cdot\mathrm{diag}[S(0)]^{-1}x < M$$

for large k. Furthermore, since $z_k \in \mathrm{diag}[S(0)]K^* \setminus \{0\} \subseteq (0,\infty)^d$ for large k, we may divide by z_k^1 to obtain

$$0 \le \frac{\mathrm{diag}[S(0)]^{-1}z_k}{z_k^1}\cdot x < \frac{M}{z_k^1}$$

for large k. This, in conjunction with (2.16), implies

$$0 \le \ell(x) \le \frac{\mathrm{diag}[S(0)]^{-1}z_k}{z_k^1}\cdot x < \frac{M}{z_k^1},$$

and thus

$$0 \le z_k^1\ell(x) < M$$

for large k. Now, since $x \in \mathrm{int}\, K$, we have $\ell(x) > 0$ from Lemma 3.1 of [16], and therefore,

$$0 < z_k^1 < \frac{M}{\ell(x)} < \infty,$$

which implies that the sequence $\{z_k^1\}_{k\in\mathbb{N}}$ is bounded. Since $z_k \in z_k^1\Lambda$ and the set Λ is compact in $\mathbb{R}^d$, the boundedness of the sequence $\{z_k^1\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ implies the boundedness of the sequence $\{z_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^d$. Then, by passing through a subsequence, we may assume that $z_k \to \hat{z}$ for some $\hat{z} \in \mathbb{R}^d$ as $k \to \infty$. Since K^* is closed and $\{z_k\}_{k\in\mathbb{N}}$ is away from 0 for large k, we obtain $\hat{z} \in K^* \setminus \{0\}$.

Now, from (4.54) and the boundedness of $\{z_k\}_{k\in\mathbb{N}}$, it follows that the sequence $\{(H_k, h_k)\}_{k\in\mathbb{N}}$ is bounded in $\mathscr{L}^1$. Then, by Komlós' theorem (see, for example, [30]), there exists a subsequence $\{(H_{k(l)}, h_{k(l)})\}_{l\in\mathbb{N}}$ such that the sequence

$$(\Theta_k, \theta_k) \triangleq \frac{1}{k}\sum_{j=1}^{k}(H_{k(j)}, h_{k(j)}), \quad k \in \mathbb{N}$$

converges to some $(\hat{H}, \hat{h}) \in \mathscr{L}$ for $(\breve{\mu} \otimes \mathbb{P})$-a.e., that is, $\Theta_k \to \hat{H}$, $\mathbb{P}$-a.e. and $\theta_k \to \hat{h}$, $(\mu \otimes \mathbb{P})$-a.e. Set

$$\zeta_k \triangleq \frac{1}{k}\sum_{j=1}^{k} z_{k(j)}, \quad k \in \mathbb{N}$$

so that $(\zeta_k, (\Theta_k, \theta_k)) \to (\hat{z}, (\hat{H}, \hat{h}))$ $(\breve{\mu} \otimes \mathbb{P})$-a.e. Since $\breve{\mathscr{G}}$ is convex, we have $(\zeta, (\Theta_k, \theta_k)) \in \breve{\mathscr{G}}$, $\forall k \in \mathbb{N}$. Since $\breve{\mathscr{G}}$ is closed under a.e.-convergence with respect to $(\breve{\mu} \otimes \mathbb{P})$, it follows that $(\hat{z}, (\hat{H}, \hat{h})) \in \breve{\mathscr{G}}$. Fatou's lemma and the convexity of the functions $\tilde{U}$ and $\tilde{u}$ now give

$$\begin{aligned}
&\mathbb{E}\left[\tilde{U}(\hat{H}) + \int_0^T \tilde{u}(t, \hat{h})dt\right] + \hat{z}\cdot\mathrm{diag}[S(0)]^{-1}x \\
&\quad\leq \varliminf_{k\to\infty}\left(\mathbb{E}\left[\tilde{U}(\Theta_k) + \int_0^T \tilde{u}(t, \theta_k)dt\right] + \zeta_k\cdot\mathrm{diag}[S(0)]^{-1}x\right) \\
&\quad\leq \lim_{k\to\infty}\left(\frac{1}{k}\sum_{j=1}^{k}\mathbb{E}\left[\tilde{U}(H_{k(j)}) + \int_0^T \tilde{u}(t, h_{k(j)})dt\right] + z_{k(j)}\cdot\mathrm{diag}[S(0)]^{-1}x\right) \\
&\quad= \lim_{k\to\infty}\left(\mathbb{E}\left[\tilde{U}(H_k) + \int_0^T \tilde{u}(t, h_k)dt\right] + z_k\cdot\mathrm{diag}[S(0)]^{-1}x\right) = W(x).
\end{aligned}$$

Therefore, $(\hat{z}, (\hat{H}, \hat{h}))$ attains the infimum in (4.57).

Finally, if $\mathbb{P}(\hat{H} = 0) > 0$ or $(\mu \otimes \mathbb{P})(\hat{h} = 0) > 0$, then $\mathbb{P}[\tilde{U}(\hat{H}) + \int_0^T \tilde{u}(t, y(t))dt = \infty] > 0$ which gives $W(x) = \infty$, a contradiction. In conjunction with Corollary 2.2.3 and the fact $\mathrm{diag}[S(T)]^{-1}\hat{H} \in K^*$ and $\mathrm{diag}[S(t)]^{-1}\hat{h} \in K^*$ we obtain $\hat{H} \in (0,\infty)^d$ $\mathbb{P}$-a.e. and $\hat{h} \in (0,\infty)^d$ $(\mu\otimes\mathbb{P})$-a.e. This completes the proof. $\square$

Proof of Theorem 4.5.4: Let $(\hat{z}, (\hat{H}, \hat{h})) \in \check{\mathscr{G}}$ be a triplet given in Lemma B.1.1 above and set $\hat{C} \triangleq I(\hat{H})$ and $\hat{c}(t) \triangleq \imath(t, \hat{h}(t))$, $t \in [0,T)$. We first claim that

$$\text{(B.2)} \quad \mathbb{E}\left[\hat{C}\cdot(H - \hat{H}) + \int_0^T \hat{c}(t)\cdot(h(t) - \hat{h}(t))dt\right] \le (z - \hat{z})\cdot\mathrm{diag}[S(0)]^{-1}x, \quad \forall (z, (H,h)) \in \check{\mathscr{G}}.$$

To see this, fix $(z, (H,h)) \in \check{\mathscr{G}}$ arbitrarily, and set

$$\text{(B.3)} \quad \begin{aligned} (z_\varepsilon, (H_\varepsilon, h_\varepsilon)) &\triangleq (1-\varepsilon)(\hat{z}, (\hat{H}, \hat{h})) + \varepsilon(z, (H,h)) \\ C_\varepsilon &\triangleq I(H_\varepsilon) \\ c_\varepsilon(t) &\triangleq \imath(t, h_\varepsilon(t)), \quad t \in [0,T). \end{aligned}$$

for each $0 < \varepsilon < 1$. Then the optimality of $(\hat{z}, (\hat{H}, \hat{h}))$ implies

$$\text{(B.4)} \quad \begin{aligned} 0 \ge &\left\{\mathbb{E}\left[\tilde{U}(\hat{H}) + \int_0^T \tilde{u}(t, \hat{h}(t))dt\right] + \hat{z}\cdot\mathrm{diag}[S(0)]^{-1}x\right\} \\ &- \left\{\mathbb{E}\left[\tilde{U}(H_\varepsilon) + \int_0^T \tilde{u}(t, h_\varepsilon(t))dt\right] + z_\varepsilon\cdot\mathrm{diag}[S(0)]^{-1}x\right\} \\ = &\ \mathbb{E}\left[\tilde{U}(\hat{H}) - \tilde{U}(H_\varepsilon) + \int_0^T \left\{\tilde{u}(t, \hat{h}(t)) - \tilde{u}(t, h_\varepsilon(t))\right\} dt\right] \\ &+ (\hat{z} - z_\varepsilon)\cdot\mathrm{diag}[S(0)]^{-1}x. \end{aligned}$$

Note that as in Corollary 4.2.5, the functions $I(\cdot)$ and $\imath(t,\cdot)$ satisfy

$$I(y) = -\nabla\tilde{U}(y) \quad \text{and} \quad \imath(t,y) = -\nabla\tilde{u}(t,y)$$

for every $t \in [0,\infty)$ and $y \in (0,\infty)^d$, where $\nabla\tilde{u}(t,y)$ denotes the partial derivative with respect to y. It then follows from the convexity of the functions $\tilde{U}(\cdot)$ and $\tilde{u}(t,\cdot)$ that

$$\text{(B.5)} \qquad \mathbb{E}\left[\tilde{U}(\hat{H}) - \tilde{U}(H_\varepsilon)\right] \geq -\mathbb{E}\left[I(H_\varepsilon)\cdot(\hat{H} - H_\varepsilon)\right]$$
$$\mathbb{E}\left[\int_0^T \left\{\tilde{u}(t,\hat{h}(t)) - \tilde{u}(t,h_\varepsilon(t))\right\} dt\right] \geq -\mathbb{E}\left[\int_0^T \imath(t,h_\varepsilon(t))\cdot(\hat{h}(t) - h_\varepsilon(t))dt\right].$$

Combining the inequalities (B.4) and (B.5) and then substituting (B.3), we obtain

$$\begin{aligned}
0 \geq &-\mathbb{E}\left[I(H_\varepsilon)\cdot(\hat{H} - H_\varepsilon) + \int_0^T \imath(t,h_\varepsilon(t))\cdot(\hat{h}(t) - h_\varepsilon(t))dt\right] \\
&+ (\hat{z} - z_\varepsilon)\cdot\mathrm{diag}[S(0)]^{-1}x \\
= &\ \varepsilon\left\{\mathbb{E}\left[C_\varepsilon\cdot(H - \hat{H}) + \int_0^T c_\varepsilon(t)\cdot(h(t) - \hat{h}(t))dt\right] - (z - \hat{z})\cdot\mathrm{diag}[S(0)]^{-1}x\right\}.
\end{aligned}$$

Dividing by ε, we obtain

$$\text{(B.6)} \quad \mathbb{E}\left[C_\varepsilon\cdot(H - \hat{H}) + \int_0^T c_\varepsilon(t)\cdot(h(t) - \hat{h}(t))dt\right] \leq (z - \hat{z})\cdot\mathrm{diag}[S(0)]^{-1}x.$$

Now, let $\varepsilon \downarrow 0$. Then by continuity of I and $\imath(t,\cdot)$, we have $C_\varepsilon = I_1(H_\varepsilon) \to I_1(\hat{H}) = \hat{C}$ and $c_\varepsilon(t) = I_2(t,h_\varepsilon(t)) \to I_2(t,\hat{h}(t)) = \hat{c}(t)$. We would like to apply Fatou's lemma to (B.6) to obtain

$$\begin{aligned}
&\mathbb{E}\left[\hat{C}\cdot(H - \hat{H}) + \int_0^T \hat{c}(t)\cdot(h(t) - \hat{h}(t))dt\right] \\
&\qquad \leq \varliminf_{\varepsilon\downarrow 0} \mathbb{E}\left[C_\varepsilon\cdot(H - \hat{H}) + \int_0^T c_\varepsilon(t)\cdot(h(t) - \hat{h}(t))dt\right] \leq x\cdot(z - \hat{z}),
\end{aligned}$$

which will provide (B.2). To justify the applicability of Fatou's lemma, we may proceed as in the argument around (4.42) – (4.45) and use Lemmas 4.2.8

and 4.5.3 to obtain the inequalities

$$\text{(B.7)}\qquad \begin{aligned} I(H_\varepsilon)\cdot(H-\hat{H}) &\geq -\frac{1}{\alpha}\left[\zeta_1(\alpha)+\alpha^{-\beta_1}\tilde{U}(\hat{H})\right] \\ \imath(t,h_\varepsilon(t))\cdot(h(t)-\hat{h}(t)) &\geq -\frac{1}{\alpha}\left[\zeta_2(t,\alpha)+\alpha^{-\beta_2}\tilde{u}(t,\hat{h}(t))\right], \end{aligned}$$

which are valid for every $\varepsilon \in (0, 1-2\alpha]$ and $t \in [0,T)$, where α is any fixed constant in $(0,1/4]$, and the function ζ_1 and the number β_1 (respectively, ζ_2 and β_2) are given in Lemma 4.2.8 (respectively, Lemma 4.5.3). Notice that the right-hand-side of (B.7) is independent of $\varepsilon \in (0,1-2\alpha]$. It is $(\check{\mu}\otimes\mathbb{P})$-integrable as well because

$$\begin{aligned} &\mathbb{E}\left[\zeta_1(\alpha)+\alpha^{-\beta_1}\tilde{U}(\hat{H})\right]+\mathbb{E}\left[\int_0^T\left\{\zeta_2(t,\alpha)+\alpha^{-\beta_2}\tilde{u}(t,\hat{h}(t))\right\}dt\right] \\ &\qquad \leq \mathbb{E}\left[\zeta_1(\alpha)+\int_0^T\zeta_2(t,\alpha)dt\right]+\mathbb{E}\left[\tilde{U}(\hat{H})+\int_0^T\tilde{u}(t,\hat{h}(t))dt\right]<\infty. \end{aligned}$$

Application of Fatou's lemma in (B.6) is now justified, which establishes the inequality (B.2).

Now, by taking $(z,(H,h)) = (0,(0,0))$ in (B.2), we get

$$\mathbb{E}\left[\hat{H}\cdot\hat{C}+\int_0^T\hat{h}(t)\cdot\hat{c}(t)dt\right]\geq \hat{z}\cdot\mathrm{diag}[S(0)]^{-1}x.$$

Also, by taking $(z,(H,h)) = (1+\varepsilon)(\hat{z},(\hat{H},\hat{h}))$ in (B.2) for some small $\varepsilon>0$, we get

$$\mathbb{E}\left[\hat{H}\cdot\hat{C}+\int_0^T\hat{h}(t)\cdot\hat{c}(t)dt\right]\leq \hat{z}\cdot\mathrm{diag}[S(0)]^{-1}x;$$

recall that the set $\check{\mathscr{G}}$ is a cone and thus $(1+\varepsilon)(\hat{z},(\hat{H},\hat{h}))\in\check{\mathscr{G}}$. We thus obtain

$$\text{(B.8)}\qquad \mathbb{E}\left[\hat{H}\cdot\hat{C}+\int_0^T\hat{h}(t)\cdot\hat{c}(t)dt\right]= \hat{z}\cdot\mathrm{diag}[S(0)]^{-1}x.$$

Now, it follows from (B.2) and (B.8) that

$$\text{(B.9)} \quad \mathbb{E}\left[H \cdot \hat{C} + \int_0^T h(t) \cdot \hat{c}(t)dt\right] \leq z \cdot \mathrm{diag}[S(0)]^{-1}x, \quad \forall (z, (H, h)) \in \check{\mathscr{G}},$$

which in particular gives

$$\mathbb{E}\left[Z(T) \cdot \hat{C} + \int_0^T Z(t) \cdot \hat{c}(t)dt\right] \leq Z(0) \cdot \mathrm{diag}[S(0)]^{-1}x$$

for every $Z(\cdot) \in \mathscr{M}_0$. From the martingale property of $Z(\cdot)$, we can rewrite the second integral in the left-hand-side as

$$\begin{aligned}
\mathbb{E}\left[\int_0^T Z(t) \cdot \hat{c}(t)dt\right] &= \mathbb{E}\left[\int_0^T \mathbb{E}[Z(T)|\mathscr{F}(t)] \cdot \hat{c}(t)dt\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_0^T Z(T) \cdot \hat{c}(t)dt \,\middle|\, \mathscr{F}(t)\right]\right] \\
&= \mathbb{E}\left[Z(T) \cdot \int_0^T \hat{c}(t)dt\right],
\end{aligned}$$

which yields

$$\mathbb{E}\left[Z(T) \cdot \left(\hat{C} + \int_0^T \hat{c}(t)dt\right)\right] \leq Z(0) \cdot \mathrm{diag}[S(0)]^{-1}x$$

for every $Z(\cdot) \in \mathscr{M}_0$. It then follows from Theorem 2.3.7 that the random vector $\mathrm{diag}[S(T)] \left(\hat{C} + \int_0^T \hat{c}(t)dt\right)$ is a contingent claim which is super-hedgeable for x. We thus obtain $(\hat{C}, \hat{c}) \in \check{\mathscr{C}}(x)$ from Lemma 4.5.1. In conjunction with (4.55) and (4.56), the equation (B.8) now gives (4.63). $\square$

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