THE NUMÉRAIRE PORTFOLIO IN SEMIMARTINGALE FINANCIAL MODELS

IOANNIS KARATZAS AND CONSTANTINOS KARDARAS

Abstract. We study the existence of the *numéraire portfolio* under predictable convex constraints in a general semimartingale model of a financial market. The numéraire portfolio generates a wealth process, with respect to which the relative wealth processes of all other portfolios are supermartingales. Necessary and sufficient conditions for the existence of the numéraire portfolio are obtained in terms of the triplet of predictable characteristics of the asset price process. This characterization is then used to obtain further necessary and sufficient conditions, in terms of an arbitrage-type notion. In particular, the full strength of the "No Free Lunch with Vanishing Risk" (NFLVR) is not needed, only the weaker "No Unbounded Profit with Bounded Risk" (NUPBR) condition that involves the boundedness in probability of the terminal values of wealth processes. We show that this notion is the minimal a-priori assumption required, in order to proceed with utility optimization. The fact that it is expressed entirely in terms of predictable characteristics makes it easy to check, something that the stronger NFLVR condition lacks.

0. Introduction

0.1. **Background and Discussion of Results.** The branch of Probability Theory that goes by the name "Stochastic Finance" is concerned, amongst other things, with finding adequate descriptions of the way financial markets work. There exists a huge literature of such models by now, and we do not attempt to give a history or summary of all the relevant work. There is, however, a broad class of models that have been used extensively: those for which the price processes of certain financial instruments (stocks, bonds, indices, currencies, etc.) are considered to evolve as semimartingales. The concept of semimartingale is a very intuitive one: it connotes a process that can be decomposed into a *finite variation* term that represents the "signal" or "drift", and a *local martingale* term that represents the "noise" or "uncertainty". Discrete-time models can be embedded in this class, as can processes with independent increments and many other Markov processes, such as solutions to stochastic differential equations. Models that are not included and have received attention are, for example, those where price processes are driven by fractional Brownian motion.

There are at least two good reasons for the choice of semimartingales in modeling financial asset price-processes. The first is that semimartingales constitute the largest class of stochastic processes which can be used as integrators, in a theory that resembles as closely as possible the ordinary Lebesgue integration. In

Date: May 30, 2006.

Results in this paper are drawn in part from the second author's PhD thesis [24]. Work was partially supported by the National Science Foundation, under grant NSF-DMS-00-99690.

economic terms, integration with respect to a price process represents the wealth of an investment in the market, the integrand being the strategy that an investor uses. To be more precise, let us denote the price process of a certain tradeable asset by $S = (S_t)_{t\in\mathbb{R}_+}$; for the time being, S could be any random process. An investor wants to invest in this asset. As long as simple "buy-and-hold strategies" are being used, which in mathematical terms are captured by an *elementary integrand* θ, the "stochastic integral" of the strategy θ with respect to S is obviously defined: it is the sum of net gains or losses resulting from the use of the buy-and-hold strategy. Nevertheless, the need arises to consider strategies that are not of that simple and specific structure, but can change continuously in time. If one wishes to extend the definition of the integral to this case, keeping the previous intuitive property for the case of simple strategies and requiring a very mild "dominated convergence" property, the Bichteler-Dellacherie theorem (see for example the book [5]) states that S *has to be* a semimartingale.

A second reason why semimartingale models are ubiquitous, is the pioneering work on no-arbitrage criteria that has been ongoing during the last decades. Culminating with the works [8] and [12] of F. Delbaen and W. Schachermayer, the connection has been established between the economic notion of no arbitrage — which found its ultimate incarnation in the "No Free Lunch with Vanishing Risk" (NFLVR) condition — and the mathematical notion of existence of equivalent probability measures, under which asset prices have some sort of martingale property. In [8] it was shown that if we want to restrict ourselves to the realm of locally bounded stock prices, and agree that we should banish arbitrage by use of simple strategies, the price process again has to be a semimartingale.

In this paper we consider a general semimartingale model and make no further mathematical assumptions. On the economic side, it is part of the assumptions that the asset prices are exogenously determined — in some sense they "fall from the sky", and an investor's behavior has no effect whatsoever on their movement. The usual practice is to assume that we are dealing with small investors and whistle away all the criticism, as we shall do. We also assume a frictionless market, in the sense that transaction costs for trading are non-existent or negligible.

Our main concern will be a problem which can be cast in the mold of dynamic stochastic optimization, though of a highly static and deterministic nature, since the optimization is being done in a path-by-path, pointwise manner. We explore a specific strategy whose wealth appears "better" when compared to the wealth generated by any other strategy, in the sense that the ratio of the two processes is a supermartingale. If such a strategy exists, it is essentially unique and we call it *the numéraire portfolio*.

We derive necessary and sufficient conditions for the numéraire portfolio to exist, in terms of the triplet of *predictable characteristics* of the returns of stock-price processes. These are direct analogues of the drift and volatility coëfficients in continuous-path models. Since we are working here in a more general setting, where jumps are also allowed, it becomes necessary to introduce a third characteristic that measures the intensity of these jumps.

Sufficient conditions for the existence of the numéraire portfolio have already been established in Goll and Kallsen [16], who focused on the (almost equivalent) problem of maximizing expected logarithmic utility. These authors went on to show that their conditions are also necessary, under the following assumptions: the problem of maximizing the expected log-utility from terminal wealth has a finite value, no constraints are enforced on strategies, and NFLVR holds. Becherer [4] also discussed how under these assumptions the numéraire portfolio exists, and coincides with the log-optimal one. In both these papers, deep results of Kramkov and Schachermayer [26] on utility maximization had to be invoked in order to obtain necessary and sufficient conditions.

Here we follow a bare-hands approach which enables us to obtain stronger results. First, the assumption of finite expected log-utility is dropped entirely; there should be no reason for it anyhow, since we are not working on the problem of log-utility optimization. Secondly, general closed convex constraints on portfolio choice can be enforced, as long as these constraints unfold in a predictable manner. Thirdly, and perhaps most controversially, we drop the NFLVR assumption: *no* normative assumption is imposed on the model. It turns out that the numéraire portfolio can exist, even in cases where the classical *No Arbitrage* (NA) condition fails.

In the context of stochastic portfolio theory, we feel there is no need for no-arbitrage assumptions to begin with: if there are arbitrage opportunities in the market, the role of optimization should be to find and utilize them, rather than ban the model. It is actually possible that the optimal strategy of an investor is *not* the arbitrage (an example involves the notorious 3-dimensional Bessel process). The usual practice of assuming that we can invest unconditionally on the arbitrage breaks down because of credit limit constraints: arbitrages are sure to generate, at a fixed future date, more capital than initially invested; but they can do pretty badly in-between, and this imposes an upper bound on the money the investor can bet on them. If the previous reasoning for not banning arbitrage does not satisfy the reader, here is a more severe problem: in very general semimartingale financial markets *there does not seem to exist any computationally feasible way of deciding whether arbitrages exist or not.* This goes hand-in-hand with the fact that the existence of equivalent martingale measures — its remarkable theoretical importance notwithstanding — is a purely normative assumption and *not easy to check*, at least by looking directly at the dynamics of the stock-price process.

Our second main result comes hopefully to shed some light on this situation. Having assumed nothing about the model when initially trying to decide whether the numéraire portfolio exists, we now take a step backwards and in the opposite-than-usual direction: we ask ourselves *what the existence of the numéraire portfolio can tell us about arbitrage-like opportunities in the market.* Here, the necessary and sufficient condition for existence, is the boundedness in probability of the collection of terminal wealths attainable by trading. Readers acquainted with arbitrage notions will recognize this as one of the two conditions that comprise NFLVR; what remains of course is the NA condition. One can go on further, and ask how severe this assumption (of boundedness in probability for the set of terminal wealths) really is. The answer is simple: when this condition fails, one cannot do utility optimization for *any* utility function; conversely if this assumption holds, one can proceed with utility maximization as usual. The main advantage of not assuming the full NFLVR condition is that, *for the weaker condition of boundedness in probability, there is a direct way of checking its validity in terms of the predictable* *characteristics (dynamics) of the price process.* No such characterization exists for the NA condition, as we show by example in subsection 3.3. Furthermore, our result can be used to understand the gap between the concepts of NA and the stronger NFLVR; *the existence of the numéraire portfolio is exactly the bridge needed to take us from NA to NFLVR.* This has already been understood for the continuous-path process case in the paper [9]; here we do it for the general case.

0.2. **Synopsis.** We offer here an overview of what is to come, so the reader does not get lost in technical details and little detours. After this short subsection, the remainder of this section will set up some general notation and reminders of some probabilistic concepts to be used throughout.

Section 1 introduces the financial model, the ways that a financial agent can invest in this market, and the constraints that are faced.

In section 2 we introduce the numéraire portfolio. We discuss how it relates to other notions, and conclude with our main Theorem 2.15; this provides necessary and sufficient conditions for the existence of the numéraire portfolio in terms of the predictable characteristics of the stock-price processes.

Section 3 deals with the connections between the numéraire portfolio and free lunches. The main result there is Theorem 3.12 that can be seen as another version of the Fundamental Theorem of Asset Pricing.

Some of the proofs are not given in sections 2 and 3, as they tend to be quite long; instead, they occupy the next four sections. In section 4 we describe necessary and sufficient conditions for the existence of wealth processes that are increasing and not constant. Section 5 deals with the proof of our main Theorem 2.15. Section 6 contains the proof of a result on the rate of convergence to zero of positive supermartingales; this result is used to study the asymptotic optimality property of the numéraire portfolio. Then, section 7 contains the backbone of the proof of our second main Theorem 3.12.

In an effort to keep the paper as self-contained as possible, we have included Appendices that cover useful results on three topics: (A) measurable random subsets and selections; (B) semimartingales up to infinity and the corresponding "stochastic integration up to infinity"; and (C) σ-localization. These results do not seem to be as widely known as perhaps they deserve to be.

0.3. **General notation.** A vector p of the d-dimensional real Euclidean space $\mathbb{R}^d$ is understood as a $d \times 1$ (column) matrix. The transpose of p is denoted by $\mathrm{p}^\top$, and the usual Euclidean norm is $|\mathrm{p}| := \sqrt{\mathrm{p}^\top \mathrm{p}}$. We use superscripts to denote coordinates: $\mathrm{p} = (\mathrm{p}^1, \dots \mathrm{p}^d)^\top$. By $\mathbb{R}_+$ we denote the positive real half-line $[0, \infty)$.

The symbol "$\wedge$" denotes minimum: $f \wedge g = \min\{f, g\}$; for a real-valued function f its *negative part* is $f^- := -(f \wedge 0)$ and its *positive part* is $f^+ := \max(f, 0)$.

The *indicator function* of a set A is denoted by $\mathbb{I}_A$. To ease notation and the task of reading, subsets of $\mathbb{R}^d$ such as $\{x \in \mathbb{R}^d \mid |x| \leq 1\}$ are "schematically" denoted by $\{|x| \leq 1\}$; for the corresponding indicator function we write $\mathbb{I}_{\{|x| \leq 1\}}$.

A measure ν on $\mathbb{R}^d$ (endowed with its Borel σ-algebra) is called a *Lévy measure*, if $\nu(\{0\}) = 0$ and $\int (1 \wedge |x|^2) \nu(\mathrm{d}x) < +\infty$. A *Lévy triplet* (b, c, ν) consists of a vector $b \in \mathbb{R}^d$, a $(d \times d)$ symmetric, non-negative definite matrix c, and a Lévy measure ν on $\mathbb{R}^d$. Once we have defined the price processes, the elements c and

ν of the Lévy triplet will correspond to the instantaneous covariation rate of the continuous part of the process, and to the instantaneous jump intensity of the process, respectively. The vector b can be thought of as an instantaneous drift rate; one has to be careful with this interpretation, though, since b ignores the drift coming from large jumps of the process. In this respect, see Definition 1.4.

Suppose we have two measurable spaces $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$, a measure μ_1 on $(\Omega_1, \mathcal{F}_1)$, and a *transition measure* $\mu_2 : \Omega_1 \times \mathcal{F}_2 \to \mathbb{R}_+$; this means that for every $\omega_1 \in \Omega_1$, the set function $\mu_2(\omega_1, \cdot)$ is a measure on $(\Omega_2, \mathcal{F}_2)$; and for every $A \in \mathcal{F}_2$ the function $\mu_2(\cdot, A)$ is $\mathcal{F}_1$-measurable. We shall denote by $\mu_1 \otimes \mu_2$ the measure on the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ defined for $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$ as

$$(\mu_1 \otimes \mu_2)(E) := \int \left(\int \mathbb{I}_E(\omega_1, \omega_2) \mu_2(\omega_1, \mathrm{d}\omega_2) \right) \mu_1(\mathrm{d}\omega_1). \tag{0.1}$$

0.4. **Remarks of probabilistic nature.** For results concerning the general theory of stochastic processes described below, we refer the reader to the book [18] of Jacod and Shiryaev (especially the first two chapters).

We are given a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where the filtration $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is assumed to satisfy the usual hypotheses of right-continuity and augmentation by the $\mathbb{P}$-null sets. The probability measure $\mathbb{P}$ will be fixed throughout and will receive no special mention. Every formula, relationship, etc. is supposed to be valid $\mathbb{P}$-a.s. (again, no special mention will be made). The expectation of random variables defined on the measure space $(\Omega, \mathcal{F}, \mathbb{P})$ will be denoted by $\mathbb{E}$.

The set $\Omega \times \mathbb{R}_+$ is the *base space*; a generic element will be denoted by (ω, t). Every process on the stochastic basis can be seen as a function from $\Omega \times \mathbb{R}_+$ with values in $\mathbb{R}^d$ for some $d \in \mathbb{N}$. The *predictable σ-algebra* on $\Omega \times \mathbb{R}_+$ is generated by all the adapted, left-continuous processes; we denote it by $\mathcal{P}$ — if π is a d-dimensional predictable process we write $\pi \in \mathcal{P}(\mathbb{R}^d)$. For any adapted, right-continuous process Y that admits left-hand limits, its *left-continuous version* Y_- is defined by setting $Y_-(0) := Y(0)$ and $Y_-(t) := \lim_{s \uparrow t} Y(s)$ for $t > 0$; this process is obviously predictable, and we define its *jump process* $\Delta Y := Y - Y_-$.

For a d-dimensional semimartingale X and $\pi \in \mathcal{P}(\mathbb{R}^d)$, we shall denote by $\pi \cdot X$ the *stochastic integral* process, whenever this makes sense, in which case we shall be referring to π as being *X-integrable*. Let us note that we are assuming *vector stochastic integration*; a good account of this can be found in [18] as well as in Cherny and Shiryaev [6]. Also, for two semimartingales X and Y, we define their *quadratic covariation* process by $[X, Y] := XY - X_- \cdot Y - Y_- \cdot X$.

Finally, by $\mathcal{E}(Y)$ we shall denote the *stochastic exponential* of the scalar semimartingale Y; $\mathcal{E}(Y)$ is the unique solution Z of the stochastic integral equation $Z = 1 + Z_- \cdot Y$ and is given by the formula

$$\mathcal{E}(Y) = \exp\left\{ Y - \frac{1}{2}[Y^{\mathsf{c}}, Y^{\mathsf{c}}] \right\} \cdot \prod_{s \leq \cdot} \left\{ (1 + \Delta Y_s) \exp(-\Delta Y_s) \right\}, \tag{0.2}$$

where Y^{c} denotes the *continuous martingale part* of the semimartingale Y. The stochastic exponential $Z = \mathcal{E}(Y)$ satisfies $Z > 0$ and $Z_- > 0$ if and only if $\Delta Y > -1$. Given a process Z which satisfies $Z > 0$ and $Z_- > 0$, we can invert the stochastic exponential operator and get the *stochastic logarithm* $\mathcal{L}(Z)$, which is defined as $\mathcal{L}(Z) := (1/Z_-) \cdot Z$ and satisfies $\Delta \mathcal{L}(Z) > -1$. In other words, we have a one-to- one correspondence between the class of semimartingales Y that satisfy $\Delta Y > -1$, and the class of semimartingales Z that satisfy $Z > 0$ and $Z_- > 0$.

1. The Market, Investments, and Constraints

1.1. **The stock-prices model.** On the given stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ we consider d strictly positive semimartingales $\widetilde{S}^1, \ldots, \widetilde{S}^d$ that model the prices of d assets; we shall refer to these as *stocks*. There is also another process $\widetilde{S}^0$ which we regard as representing the *money market* or *bank account*. The only difference between the stocks and the money market is that the latter plays the role of a "benchmark", in the sense that wealth processes will be quoted in units of $\widetilde{S}^0$ and not nominally. We also consider the *discounted price processes* $S^i := \widetilde{S}^i / \widetilde{S}^0$ for $i = 0, \ldots, d$ and denote the d-dimensional vector process $(S^1, \ldots, S^d)$ by S. In this discounted world $S^0 \equiv 1$; in economic language, the interest rate is zero.

Since S^i is strictly positive for all $i = 1, \ldots, d$, there exists another d-dimensional semimartingale $X \equiv (X^1, \ldots, X^d)$ with $X_0 = 0$, $\Delta X^i > -1$ and $S^i = S^i_0 \, \mathcal{E}(X^i)$ for $i = 1, \ldots, d$. We interpret X as the process of **discounted returns** that generate the asset prices S in a multiplicative way.

The infinite time-horizon $\mathbb{R}_+ = [0, \infty[$ is certainly sufficient for our purposes, since any finite time horizon can be easily embedded in it. Nevertheless, in many cases it is much more natural to think of finite time-horizons, and many of our examples will be given in this setting; for this reason we shall be working on $[\![0, T]\!] := \{(\omega, t) \in \Omega \times \mathbb{R}_+ \mid t \leq T(\omega)\}$ where T is a *possibly infinite-valued stopping time*. All processes then will be considered as being constant and equal to their value at T for all times after T, i.e., every process Z is equal to the *stopped process* at time T that is defined via $Z^T_t := Z_{t \wedge T}$ for all $t \in \mathbb{R}_+$. Further, we can assume without loss of generality that $\mathcal{F}_0$ is $\mathbb{P}$-trivial (thus all $\mathcal{F}_0$-measurable random variables are constants) and that $\mathcal{F} = \mathcal{F}_T := \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_{t \wedge T}$.

Remark 1.1. Under our model we have $S^i > 0$ and $S^i_- > 0$; one can argue that this is not the most general semimartingale model, since it does not allow for negative prices (for example, prices of forward contracts can take negative values). This will be most relevant when we consider arbitrage in section 3, to be in par with the earlier work of Delbaen and Schachermayer [8, 12]. The general model should be an *additive* one: $S = S_0 + Y$, where now Y^i represents the cumulative discounted *gains* of S^i after time zero and can be *any* semimartingale (without having to satisfy $\Delta Y^i > -1$ for $i = 1, \ldots, d$).

In our discussion we shall be using the returns process X, not the stock-price process S directly. All the work we shall do carries to the additive model almost vis-a-vis; if there is a slight change we trust that the reader can spot it. We choose to work under the multiplicative model since it is somehow more intuitive and more applicable: almost every model used in practice is written in this way. Nevertheless, if one aims at utmost generality, the additive model is more appropriate. To remove all doubt, there will be a follow-up to this discussion in subsection 3.8.

The predictable characteristics of the returns process X will be important in our discussion. To this end, we fix the *canonical truncation function*[1] $x \mapsto x\mathbb{I}_{\{|x|\le 1\}}$ and write the *canonical decomposition* of the semimartingale X:

$$X = X^{\mathsf{c}} + B + \left[x\mathbb{I}_{\{|x|\le 1\}}\right] * (\mu - \eta) + \left[x\mathbb{I}_{\{|x|>1\}}\right] * \mu\,. \tag{1.1}$$

Some remarks on this representation are in order. First, μ is the *jump measure* of X, i.e., the random counting measure on $\mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$ defined by

$$\mu([0,t]\times A) := \sum_{0\le s\le t} \mathbb{I}_A(\Delta X_s), \quad \text{for } t\in\mathbb{R}_+ \text{ and } A \subseteq \mathbb{R}^d\setminus\{0\}. \tag{1.2}$$

Thus, the last process in (1.1) is just $\left[x\mathbb{I}_{\{|x|>1\}}\right] * \mu \equiv \sum_{0\le s\le \cdot} \Delta X_s \mathbb{I}_{\{|\Delta X_s|>1\}}$, the *sum of the "big" jumps of* X; throughout the paper, the asterisk denotes integration with respect to random measures. Once this term is subtracted from X, what remains is a semimartingale with bounded jumps, thus a *special* semimartingale. This, in turn, can be decomposed uniquely into a *predictable finite variation* part, denoted by B in (1.1), and a *local martingale* part. Finally, this last local martingale part can be decomposed further: into its *continuous* part, denoted by X^{c} in (1.1); and its *purely discontinuous* part, identified as the local martingale $\left[x\mathbb{I}_{\{|x|\le 1\}}\right] * (\mu-\eta)$. Here, η is the *predictable compensator* of the measure μ, so the purely discontinuous part is just a *compensated sum of "small" jumps*.

We introduce the *quadratic covariation* process $C := [X^{\mathsf{c}}, X^{\mathsf{c}}]$ of X^{c} . Then we call (B, C, η) the *triplet of predictable characteristics* of X, and set $G := \sum_{i=1}^d \left(C^{i,i} + \mathrm{Var}(B^i) + [1 \wedge |x^i|^2] * \eta\right)$, This G is a predictable increasing scalar process, and B, C, η are all absolutely continuous with respect to G, thus

$$B = b\cdot G,\ C = c\cdot G,\ \text{and } \eta = G\otimes\nu\,. \tag{1.3}$$

Here b, c and ν are predictable, b is a vector process, c is a positive-definite matrix-valued process and ν is a process with values in the set of Lévy measures; for the product-measure notation $G\otimes\nu$ (see formula (0.1)) we consider the measure induced by G. Let us remark that *any* $\widetilde{G}$ with $\mathrm{d}\widetilde{G}_t \sim \mathrm{d}G_t$ can be used in place of G; the actual choice of increasing process G reflects the notion of *operational clock* (as opposed to the natural time flow, described by t): a rough idea of *how fast the market is moving*. In an abuse of terminology, we shall refer to (b, c, ν) also as the *triplet of predictable characteristics of* X; this depends on G, but the validity of all results not.

Remark 1.2. In *quasi-left-continuous* models (in which the price process does not jump at predictable times), G can be taken to be continuous. Nevertheless, if we want to include discrete-time models in our discussion, we must allow for G to have positive jumps. Since C is a continuous increasing process and (1.1) gives $\mathbb{E}[\Delta X_\tau \mathbb{I}_{\{|\Delta X_\tau|\le 1\}} \mid \mathcal{F}_{\tau-}] = \Delta B_\tau$ for every predictable time τ, we have

$$c = 0 \ \text{ and } \ b = \int x\mathbb{I}_{\{|x|\le 1\}}\nu(\mathrm{d}x), \ \text{ on the predictable set } \{\Delta G > 0\}. \tag{1.4}$$

[1]In principle one could use any bounded Borel function h such that $h(x) = x$ in a neighborhood of $x = 0$; the use of this specific choice facilitates some calculations and notation.

Remark 1.3. We make a small technical observation. The properties of c being a symmetric positive-definite predictable process and ν a predictable process taking values in the set of Lévy processes, in general hold $\mathbb{P} \otimes G$-a.e. We shall assume that they hold *everywhere*, i.e., for all $[\![0, T]\!]$; we can always do this by changing them on a predictable set of $\mathbb{P} \otimes G$-measure zero to be $c \equiv 0$ and $\nu \equiv 0$. This point is also made in Jacod-Shiryaev [18].

The following concept of *drift rate* will be used throughout the paper.

Definition 1.4. Let X be *any* semimartingale with canonical representation (1.1), and consider an operational clock G such that the relationships (1.3) hold. On $\left\{\int |x| \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x) < \infty\right\}$, the *drift rate* (with respect to G) of X is defined as the expression $b + \int x \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x)$.

The range of definition $\left\{\int |x| \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x) < \infty\right\}$ for the drift rate does not depend on the choice of operational clock G, though the drift rate itself does. Whenever the increasing process $\left[|x| \mathbb{I}_{\{|x|>1\}}\right] * \eta = \left(\int |x| \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x)\right) \cdot G$ is *finite*, (this happens if and only if X is a *special* semimartingale), the predictable process $B + \left[x \mathbb{I}_{\{|x|>1\}}\right] * \eta = \left(b + \int x \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x)\right) \cdot G$ is called the *drift* of X. If drifts exist, drift rates exist too; the converse is not true. Semimartingales that are *not* special might have well-defined drift rates; for instance, a σ-martingale is a semimartingale with drift rate identically equal to zero. See Appendix C on σ-localization, for further discussion and intuition.

1.2. **Portfolios and Wealth processes.** A financial agent starts with some strictly positive initial capital, which we normalize to $W_0 = 1$, and can invest in the assets described by the process S by choosing a predictable, d-dimensional and X-integrable process π, which we shall refer to as *portfolio*. The number π_t^i represents the *proportion of current wealth* invested in stock i at time t; the remaining proportion $\pi_t^0 := 1 - \sum_{i=1}^d \pi_t^i$ of wealth is invested in the money market.

Some restrictions have to be enforced, so that the agent cannot use so-called *doubling strategies*. The assumption prevailing in this context is that the wealth process should be uniformly bounded from below by some constant. This has the very clear financial interpretation of a *credit limit* that the agent has to face. For convenience, we shall set this credit limit at zero.

The above discussion leads to the following definition: a wealth process will be called *admissible*, if it and its left-continuous version stay strictly positive. Let us denote the wealth process generated from such a portfolio by $\widetilde{W}^\pi$; we must have $\widetilde{W}^\pi > 0$ as well as $\widetilde{W}^\pi_- > 0$. From the previous interpretation, we get:

$$\frac{\mathrm{d}\widetilde{W}_t^\pi}{\widetilde{W}_{t-}^\pi} = \sum_{i=0}^d \pi_t^i \frac{\mathrm{d}\widetilde{S}_t^i}{\widetilde{S}_{t-}^i},$$

a linear stochastic differential equation that describes the dynamics od $\widetilde{W}^\pi$. It is much easier (and cleaner) to write this equation in terms of the *discounted* wealth process $W^\pi := \widetilde{W}^\pi / \widetilde{S}^0$; as the reader can check, W^π satisfies the similar-looking linear stochastic differential equation

$$\frac{\mathrm{d}W_t^\pi}{W_{t-}^\pi} \;=\; \sum_{i=0}^{d} \pi_t^i \,\frac{\mathrm{d}S_t^i}{S_{t-}^i} \;=\; \sum_{i=1}^{d} \pi_t^i \,\mathrm{d}X_t^i \;=\; \pi_t^\top \mathrm{d}X_t\,,$$

where everything is now written in terms of discounted quantities. The second equality above holds simply because $S^0 \equiv 1$ and $S^i = S_0^i\,\mathcal{E}(X^i)$, while the last is just a matter of cleaner notation. It follows then that

$$W^\pi = \mathcal{E}(\pi \cdot X). \tag{1.5}$$

From now on we shall only consider discounted wealth processes.

1.3. **Further constraints on portfolios.** We start with an example in order to motivate Definition 1.6 below.

Example 1.5. Suppose that the agent is prevented from selling stock short. In terms of the portfolio used, this means $\pi^i \geq 0$ for all $i = 1, \ldots, d$, or that $\pi(\omega, t) \in (\mathbb{R}_+)^d$ for all $(\omega, t) \in [\![0, T]\!]$. If we further prohibit borrowing from the bank, then also $\pi^0 \geq 0$; setting $\mathfrak{C} := \{\mathrm{p} \in \mathbb{R}^d \mid \mathrm{p}^i \geq 0 \text{ and } \sum_{i=1}^d \mathrm{p}^i \leq 1\}$, the prohibition of short sales and borrowing translates into the requirement $\pi(\omega, t) \in \mathfrak{C}$ for all $(\omega, t) \in [\![0, T]\!]$.

The example leads us to consider all possible constraints that can arise this way; although in this particular case the set $\mathfrak{C}$ was non-random, we shall encounter very soon situations, where the constraints depend on both time and the path.

Definition 1.6. Consider an arbitrary set-valued process $\mathfrak{C} : [\![0, T]\!] \to \mathcal{B}(\mathbb{R}^d)$. The predictable process π will be called *$\mathfrak{C}$-constrained*, if $\pi(\omega, t) \in \mathfrak{C}(\omega, t)$ for all $(\omega, t) \in [\![0, T]\!]$. We denote by $\Pi_{\mathfrak{C}}$ the class of all $\mathfrak{C}$-constrained, predictable and X-integrable processes that satisfy $\pi^\top \Delta X > -1$.

The requirement $\pi^\top \Delta X > -1$ is there to ensure that we can define the admissible wealth process W^π, i.e., that the wealth will remain strictly positive.

Let us use this requirement to give other constraints of this type. Since these actually follow from the definitions, they will not constrain the wealth processes further; the point is that we can always include them in our constraint set.

Example 1.7. Natural Constraints. An admissible strategy generates a wealth process that starts positive and stays positive. Thus, if $W^\pi = \mathcal{E}(\pi \cdot X)$, then we have $\Delta W^\pi \geq -W_-^\pi$, or $\pi^\top \Delta X \geq -1$. Recalling the definition of the random measure ν from (1.3), we see that this requirement is equivalent to

$$\nu[\pi^\top x < -1] \equiv \nu[\{x \in \mathbb{R}^d \mid \pi^\top x < -1\}] = 0, \quad \mathbb{P} \otimes G\text{-almost everywhere}\,;$$

Define now the random set-valued process (the randomness comes through ν)

$$\mathfrak{C}_0 := \{\mathrm{p} \in \mathbb{R}^d \mid \nu\,[\mathrm{p}^\top x < -1] = 0\}, \tag{1.6}$$

which we shall call *the set-valued process of natural constraints*. Since $\pi^\top X > -1$, whenever $\pi \in \Pi_{\mathfrak{C}}$, we always have $\pi \in \Pi_{\mathfrak{C} \cap \mathfrak{C}_0}$ as well.

Note that $\mathfrak{C}_0$ is not deterministic in general. It is now clear that we are not considering random constraints just for the sake of generality, but because they arise naturally in portfolio choice settings. Eventually, in subsection 2.3, we shall impose more structure on the set-valued process $\mathfrak{C}$: namely, convexity, closedness and predictability. The reader can check that the above examples have these properties; the "predictability structure" should be clear for $\mathfrak{C}_0$, which involves the predictable process ν.

2. The Numéraire Portfolio: Definitions, General Discussion, and Predictable Characterization

2.1. **The numéraire portfolio.** The following is a central notion of the paper.

Definition 2.1. A process $\rho \in \Pi_{\mathfrak{C}}$ will be called *numéraire portfolio*, if for every $\pi \in \Pi_{\mathfrak{C}}$ the *relative wealth process* W^{π}/W^{ρ} is a supermartingale.

The term "numéraire portfolio" was first introduced by Long [29]; he defined it as a portfolio ρ that makes W^{π}/W^{ρ} a *martingale* for every portfolio π, then went on to show that this requirement is equivalent, under some additional assumptions, to absence of arbitrage for discrete-time and Itô-process models. Definition 2.1 in this form first appears in Becherer [4], where we send the reader for the history of this concept. A simple observation from that paper is that the wealth process generated by numéraire portfolios is unique: if there are two numéraire portfolios ρ_1 and ρ_2 in $\Pi_{\mathfrak{C}}$, then both W^{ρ_1}/W^{ρ_2} and W^{ρ_2}/W^{ρ_1} are supermartingales and Jensen's inequality shows that they are equal.

Observe that W^{ρ}_T is always well-defined, even on $\{T = \infty\}$, since $1/W^{\rho}$ is a positive supermartingale and the supermartingale convergence theorem implies that W^{ρ}_T exists, thought it might take the value $+\infty$ on $\{T = \infty\}$. A condition of the form $W^{\rho}_T < +\infty$ will be essential when we consider free lunches in section 3.

Remark 2.2. The numéraire portfolio is introduced in Definition 2.1 as the solution to some sort of optimization problem. It has at least four more such optimality properties. If ρ is the numéraire portfolio, then:

- it is *growth-optimal* in the sense that it maximizes the growth rate over all portfolios (see subsection 2.5);
- it maximizes the *asymptotic growth* of the wealth process it generates over all portfolios (see Proposition 2.23);
- it is also the solution of a *log-utility maximization* problem. In fact, if this problem is defined in relative (as opposed to absolute) terms, the two are equivalent. For more infomation, see subsection 2.7; and
- $(W^{\rho})^{-1}$ it minimizes the *reverse relative entropy* among all *supermartingale deflators*, i.e., strictly positive semimartingales D with $D_0 = 1$ such that DW^{π} is a supermartingale for all $\pi \in \Pi_{\mathfrak{C}}$ (see subsection 3.4).

We now state the basic problem that will occupy us in this section; its solution will be the content of Theorem 2.15.

Problem 2.3. *Find necessary and sufficient conditions for the existence of the numéraire portfolio in terms of the triplet of predictable characteristics of the stock-price process S (equivalently, of the returns process X).*

2.2. **Preliminary necessary and sufficient conditions for existence of the numéraire portfolio.** In order to decide whether $\rho \in \Pi_{\mathfrak{C}}$ is the numéraire portfolio, we must check whether W^π/W^ρ is a supermartingale for all $\pi \in \Pi_{\mathfrak{C}}$. Let us then derive a convenient expression for the ratio W^π/W^ρ.

Consider a baseline portfolio $\rho \in \Pi_{\mathfrak{C}}$ that generates a wealth W^ρ, and any other portfolio $\pi \in \Pi_{\mathfrak{C}}$; their relative wealth process is given by the ratio $W^\pi/W^\rho = \mathcal{E}(\pi \cdot X)/\mathcal{E}(\rho \cdot X)$, which can be further expressed as follows.

Lemma 2.4. *Suppose that Y and R are two scalar semimartingales with $\Delta Y > -1$ and $\Delta R > -1$. Then $\mathcal{E}(Y)/\mathcal{E}(R) = \mathcal{E}(Z)$, where*

$$Z = Y - R - [Y^{\mathsf{c}} - R^{\mathsf{c}}, R^{\mathsf{c}}] - \sum_{s \le \cdot} \left\{ \Delta(Y_s - R_s) \frac{\Delta R_s}{1 + \Delta R_s} \right\}. \tag{2.1}$$

Proof. The process $\mathcal{E}(R)^{-1}$ is bounded away from zero, so the stochastic logarithm of $Z = \mathcal{E}(Y)/\mathcal{E}(R)$ exists. Furthermore, the process on the right-hand-side of (2.1) is well-defined and a semimartingale, since $\sum_{s\le\cdot} |\Delta R_s|^2 < \infty$ and $\sum_{s\le\cdot} |\Delta Y_s \Delta R_s| < \infty$. Now, $\mathcal{E}(Y) = \mathcal{E}(R)\mathcal{E}(Z) = \mathcal{E}(R + Z + [R, Z])$, by Yor's formula. The uniqueness of the stochastic exponential implies $Y = R + Z + [R, Z]$.

This is an equation for the process Z; by splitting it into continuous and purely discontinuous parts, one can guess, then easily check, that it is solved by the process on the right-hand side of (2.1). □

It is obvious from this last lemma and (1.5) that we have

$$\frac{W^\pi}{W^\rho} = \mathcal{E}\left((\pi - \rho) \cdot X^{(\rho)}\right), \quad \text{with} \quad X^{(\rho)} := X - (c\rho) \cdot G - \left[\frac{\rho^\top x}{1 + \rho^\top x}\, x\right] * \mu\,;$$

here μ is the jump measure of X in (1.2), and G is the operational clock of (1.3).

We are interested in ensuring that W^π/W^ρ is a supermartingale. This relative wealth process is strictly positive, so the supermartingale property is equivalent to the σ-supermartingale one, which is in turn equivalent to requiring that its drift rate be finite and negative[2]. Since $W^\pi/W^\rho = \mathcal{E}\left((\pi - \rho) \cdot X^{(\rho)}\right)$, the condition of negativity on the drift rate of W^π/W^ρ is equivalent to the requirement that the drift rate of the process $(\pi - \rho) \cdot X^{(\rho)}$ be negative. Straightforward computations show that, when it exists, this drift rate is

$$\mathfrak{rel}(\pi \mid \rho) := (\pi - \rho)^\top b - (\pi - \rho)^\top c\rho + \int \vartheta_{\pi|\rho}(x)\, \nu(\mathrm{d}x)\,. \tag{2.2}$$

The integrand in this expression is defined as

$$\vartheta_{\pi|\rho}(x) := \left[\frac{(\pi - \rho)^\top x}{1 + \rho^\top x} - (\pi - \rho)^\top x \mathbb{I}_{\{|x|\le 1\}}\right] = \frac{1 + \pi^\top x}{1 + \rho^\top x} - 1 - (\pi - \rho)^\top x \mathbb{I}_{\{|x|\le 1\}}\,; \tag{2.3}$$

it is ν-bounded from below by -1 on the set $\{|x| > 1\}$, while on $\{|x| \le 1\}$ (near $x = 0$) it behaves like $(\rho - \pi)^\top x x^\top \rho$, which is comparable to $|x|^2$. It follows that the integral in (2.2) always makes sense, but can take the value $+\infty$; thus the drift rate of W^π/W^ρ either exists (i.e., is finite) or takes the value $+\infty$. In any case,

[2]For drift rates, see Definition 1.4. For the σ-localization technique, see Kallsen [20]; an overview of what is needed here is in Appendix C, in particular, Propositions C.2 and C.3.

the quantity $\mathfrak{rel}(\pi \mid \rho)$ of (2.2) is well-defined. The point of the notation $\mathfrak{rel}(\pi \mid \rho)$ is to serve as a reminder that this quantity is the rate of return of the *relative* wealth process W^π / W^ρ.

The above discussion shows that if π and ρ are two portfolios, then W^π/W^ρ is a supermartingale if and only if $\mathfrak{rel}(\pi \mid \rho) \le 0$, $\mathbb{P} \otimes G$-almost everywhere. Using this last fact we get preliminary necessary and sufficient conditions needed to solve Problem 2.3. In a different, more general form (involving also "consumption") these have already appeared in Goll and Kallsen [16].

Lemma 2.5. *Suppose that the constraints $\mathfrak{C}$ imply the natural constraints of (1.6) (i.e., $\mathfrak{C} \subseteq \mathfrak{C}_0$), and consider a process ρ with $\rho(\omega,t) \in \mathfrak{C}(\omega,t)$ for all $(\omega,t) \in [\![0,T]\!]$. In order for ρ to be the numéraire portfolio in the class $\Pi_{\mathfrak{C}}$, it is necessary and sufficient that the following hold:*

(1) $\mathfrak{rel}(\pi \mid \rho) \le 0$, *$\mathbb{P} \otimes G$-a.e. for every $\pi \in \mathcal{P}(\mathbb{R}^d)$ with $\pi(\omega,t) \in \mathfrak{C}(\omega,t)$;*
(2) *ρ is predictable; and*
(3) *ρ is X-integrable.*

Proof. The three conditions are clearly sufficient for ensuring that W^π/W^ρ is a supermartingale for all $\pi \in \Pi_{\mathfrak{C}}$.

The necessity is trivial, but for the fact that we ask condition (1) to hold not only for all $\pi \in \Pi_{\mathfrak{C}}$, but for *any* predictable process π (which might not even be X-integrable) such that $\pi(\omega,t) \in \mathfrak{C}(\omega,t)$. Suppose condition (1) holds for all $\pi \in \Pi_{\mathfrak{C}}$; as a first step, take any $\xi \in \mathcal{P}$ such that $\xi(\omega,t) \in \mathfrak{C}(\omega,t)$ and $\xi^\top \Delta X > -1$. Then, $\xi_n := \xi \mathbb{I}_{\{|\xi| \le n\}} + \rho \mathbb{I}_{\{|\xi| > n\}}$ belongs to $\Pi_{\mathfrak{C}}$, so that $\mathfrak{rel}(\xi \mid \rho)\mathbb{I}_{\{|\xi| \le n\}} = \mathfrak{rel}(\xi_n \mid \rho)\mathbb{I}_{\{|\xi| \le n\}} \le 0$; sending n to infinity we get $\mathfrak{rel}(\xi \mid \rho) \le 0$. Now pick any $\xi \in \mathcal{P}(\mathbb{R}^d)$ such that $\xi(\omega,t) \in \mathfrak{C}(\omega,t)$; we have $\xi^\top \Delta X \ge -1$ but not necessarily $\xi^\top \Delta X > -1$. Then, for $n \in \mathbb{N}$, $\xi_n := (1 - n^{-1})\xi$ also satisfies $\xi_n \in \mathcal{P}(\mathbb{R}^d)$ and $\xi_n(\omega,t) \in \mathfrak{C}(\omega,t)$ and further $\xi_n^\top \Delta X > -1$; it follows that $\mathfrak{rel}(\xi_n \mid \rho) \le 0$. An application of Fatou's lemma now will give $\mathfrak{rel}(\xi \mid \rho) \le 0$. □

In order to obtain necessary and sufficient conditions for the existence of the numéraire portfolio *in terms of predictable characteristics,* the conditions of Lemma 2.5 will be tackled one by one. For condition (1), it will turn out that one has to solve pointwise (for each fixed $(\omega,t) \in [\![0,T]\!]$) a convex optimization problem over the set $\mathfrak{C}(\omega,t)$. It is obvious that if (1) above is to hold for $\mathfrak{C}$, then it must also hold for the closed convex hull of $\mathfrak{C}$, so we might as well assume that $\mathfrak{C}$ is closed and convex. For condition (2), in order to prove that the solution we get is predictable, the set-valued process $\mathfrak{C}$ must have some predictable structure. We describe in the next subsection how this is done. After that, a simple test will give us condition (3), and we shall be able to provide the solution of Problem 2.3 in Theorem 2.15, all in terms of predictable characteristics.

2.3. **The predictable, closed convex structure of constraints.** Let us start with a remark concerning *degeneracies* that might appear in the market. These have to do with linear dependence that some stocks might exhibit at some points of the base space, causing seemingly different portfolios to produce the exact same wealth processes; such portfolios should then be treated as equivalent.

To formulate this notion, consider two portfolios π_1 and π_2 with $W^{\pi_1} = W^{\pi_2}$. The uniqueness of the stochastic exponential implies that $\pi_1 \cdot X = \pi_2 \cdot X$, so the predictable process $\zeta := \pi_2 - \pi_1$ satisfies $\zeta \cdot X \equiv 0$; this is easily seen to be equivalent to $\zeta \cdot X^{\mathsf{c}} = 0$, $\zeta^\top \Delta X = 0$ and $\zeta \cdot B = 0$, and makes the following definition plausible.

Definition 2.6. For a Lévy triplet (b, c, ν) define the linear subspace of *null investments* $\mathfrak{N}$ to be the set of vectors

$$\mathfrak{N} := \left\{ \zeta \in \mathbb{R}^d \mid \zeta^\top c = 0,\ \nu[\zeta^\top x \neq 0] = 0 \text{ and } \zeta^\top b = 0 \right\} \tag{2.4}$$

for which nothing happens if one invests in them.

Two portfolios π_1 and π_2 satisfy $\pi_2(\omega, t) - \pi_1(\omega, t) \in \mathfrak{N}(\omega, t)$ for $\mathbb{P} \otimes G$-almost every $(\omega, t) \in [\![0, T]\!]$, if and only if $W^{\pi_1} = W^{\pi_2}$; we consider such π_1 and π_2 to be the same. Here are the predictability, closedness and convexity requirements for our set-valued process of constraints.

Definition 2.7. The $\mathbb{R}^d$-set-valued process $\mathfrak{C}$ will be said to impose *predictable closed convex constraints*, if

1. $\mathfrak{N}(\omega, t) \subseteq \mathfrak{C}(\omega, t)$ for *all* $(\omega, t) \in [\![0, T]\!]$,
2. $\mathfrak{C}(\omega, t)$ is a closed convex set, for *all* $(\omega, t) \in [\![0, T]\!]$, and
3. $\mathfrak{C}$ is predictably measurable, in the sense that for any closed $F \subseteq \mathbb{R}^d$, we have $\{\mathfrak{C} \cap F \neq \emptyset\} := \{(\omega, t) \in [\![0, T]\!] \mid \mathfrak{C}(\omega, t) \cap F \neq \emptyset\} \in \mathcal{P}$.

Note the insistence that (1), (2) must hold for *every* $(\omega, t) \in [\![0, T]\!]$, not just in an "almost every" sense. The first requirement in this definition can be construed as saying that we are giving investors *at least* the freedom to do nothing; that is, if an investment is to lead to absolutely no profit or loss, one should be free to do it. In the non-degenerate case this just becomes $0 \in \mathfrak{C}(\omega, t)$ for all $(\omega, t) \in [\![0, T]\!]$.

One can refer to Appendix A for more information about the measurability requirement $\{\mathfrak{C} \cap F \neq \emptyset\} \in \mathcal{P}$ for all closed $F \subseteq \mathbb{R}^d$, where the equivalence with other definitions of measurability is discussed.

The *natural constraints* $\mathfrak{C}_0$ of (1.6) can be easily seen to satisfy the requirements of Definition 2.7; the proof of the predictability requirement is very plausible from the definition. Indeed, one just has to write

$$\mathfrak{C}_0 = \left\{ \mathrm{p} \in \mathbb{R}^d \;\Big|\; \int \kappa(1 + \mathrm{p}^\top x)\nu(\mathrm{d}x) = 0 \right\}, \text{ where } \kappa(x) := \frac{(x^-)^2}{1 + (x^-)^2},$$

then use Lemma A.4 in conjunction with Remark 1.3 which provides a version of the characteristics, such that the integrals in the above representation of $\mathfrak{C}_0$ make sense for *all* $(\omega, t) \in [\![0, T]\!]$. In view of this, we can — and certainly *will* — always assume $\mathfrak{C} \subseteq \mathfrak{C}_0$, since otherwise we can replace $\mathfrak{C}$ by $\mathfrak{C} \cap \mathfrak{C}_0$ (and use the fact that intersections of closed predictable set-valued processes are also predictable — see Lemma A.3 of Appendix A).

2.4. **Unbounded Increasing Profit.** We proceed with an effort to obtain a better understanding of condition (1) in Lemma 2.5. In this subsection we state a sufficient predictable condition for its failure; in the next subsection, when we state our first main theorem about the predictable characterization for the existence of the numéraire portfolio, we shall see that this condition is also necessary. The failure of that condition is intimately related to the existence of wealth processes that start with unit capital, manage to make some wealth with positive probability, and are furthermore increasing. The existence of such a possibility in a financial market amounts to the most egregious form of arbitrage.

Definition 2.8. The predictable set-valued process $\check{\mathfrak{C}} := \bigcap_{a>0} a\mathfrak{C}$ is called the *cone points* (or *recession cone*) of $\mathfrak{C}$. A portfolio $\pi \in \Pi_{\check{\mathfrak{C}}}$ will be said to generate an *unbounded increasing profit* (UIP), if the wealth process W^π is increasing ($\mathbb{P}[W^\pi_s \leq W^\pi_t, \forall\, s < t \leq T] = 1$), and if $\mathbb{P}[W^\pi_T > 1] > 0$. If no such portfolio exists, then we say that the *No Unbounded Increasing Profit* (NUIP) condition holds.

The qualifier "unbounded" stems from the fact that since $\pi \in \Pi_{\check{\mathfrak{C}}}$, an agent has unconstrained leverage on the position π and can invest unconditionally; by doing so, the agent's wealth will be multiplied accordingly. It should be clear that the numéraire portfolio cannot exist, if such strategies exist. To obtain the connection with predictable characteristics, we also give the definition of the immediate arbitrage opportunity vectors in terms of the Lévy triplet.

Definition 2.9. Let (b, c, ν) be any Lévy triplet. The set $\mathfrak{I}$ of *immediate arbitrage opportunities* is defined as the set of vectors $\xi \in \mathbb{R}^d \setminus \mathfrak{N}$ for which the following three conditions hold:

(1) $\xi^\top c = 0$,
(2) $\nu[\xi^\top x < 0] = 0$,
(3) $\xi^\top b - \int \xi^\top x \mathbb{I}_{\{|x|\leq 1\}} \nu(\mathrm{d}x) \geq 0$.

Vectors in the set $\mathfrak{N}$ of (2.4) satisfy these three conditions, but cannot be considered "arbitrage opportunities" since they have zero returns. One can see that $\mathfrak{I}$ is a cone with the whole "face" $\mathfrak{N}$ removed. When we want to make explicit the dependence of the set $\mathfrak{I}$ on the chosen Lévy triplet (b, c, ν), we write $\mathfrak{I}(b, c, \nu)$.

Assume, for simplicity only, that X is a Lévy processes; and that we can find a vector $\xi \in \mathfrak{I}$. The significance of conditions (1) and (2) in Definition 2.9 for the process $\xi \cdot X$ are obvious: the first implies that there is no diffusion part; the second, that there are no negative jumps; and the third condition turns out to imply that $\xi \cdot X$ has finite first variation (though this is not as obvious). Using also the fact that $\xi \notin \mathfrak{N}$, we get that $\xi \cdot X$ is actually non-zero and increasing, and the same will hold for $W^\xi = \mathcal{E}(\xi \cdot X)$; see subsection 4.1 for a thorough discussion.

Proposition 2.10. *The* NUIP *condition of Definition 2.8 is equivalent to the requirement that the predictable set* $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ *be* $\mathbb{P} \otimes G$*-null. Here,*

$$\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\} := \{(\omega, t) \in [\![0, T]\!] \mid \mathfrak{I}\,(b(\omega, t), c(\omega, t), \nu(\omega, t)) \cap \check{\mathfrak{C}}(\omega, t) \neq \emptyset\},$$

and $\check{\mathfrak{C}} := \bigcap_{a\in\mathbb{R}_+} a\mathfrak{C}$ *is the set of cone points of* $\mathfrak{C}$*.*

Section 4 is devoted to the proof of this result. The reader should should go over at least subsection 4.1, which contains one side of the argument: if there exists an unbounded increasing profit, then the set $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ *cannot* be $\mathbb{P} \otimes G$-null. The other direction, though it follows from the same idea, has a "measurable selection" flavor and the reader might wish to skim it.

Remark 2.11. The reader might wonder what connection the previous result has with our original Problem 2.3. We attempt here a quick answer in a Lévy process setting, so everything is deterministic. We shall show that *if we have* $\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$ (i.e., if the set of cone points of our constraints $\mathfrak{C}$ exposes some immediate arbitrage opportunities), *then one cannot find a process* $\rho \in \mathfrak{C}$ *such that* $\mathfrak{rel}(\pi \mid \rho) \leq 0$ *holds for all* $\pi \in \Pi_{\mathfrak{C}}$.

To this end, let us pick a vector $\xi \in \mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$ and suppose that ρ satisfies $\mathfrak{rel}(\pi \mid \rho) \leq 0$, for all $\pi \in \mathfrak{C}$. Since $\xi \in \check{\mathfrak{C}}$, we have $n\xi \in \mathfrak{C}$ for all $n \in \mathbb{N}$, as well as $(1 - n^{-1})\rho + \xi \in \mathfrak{C}$ from convexity; but $\mathfrak{C}$ is closed, so $\rho + \xi \in \mathfrak{C}$. Now from (2.2) and the definition of $\mathfrak{I} \ni \xi$, we see that

$$\mathfrak{rel}(\rho + \xi \mid \rho) = \ldots = \xi^\top b - \int \xi^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(\mathrm{d}x) + \int \frac{\xi^\top x}{1 + \rho^\top x} \nu(\mathrm{d}x)$$

is strictly positive, which leads to a contradiction: there cannot exist any ρ satisfying $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \mathfrak{C}$.

The converse holds as well; namely, if $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$, then one can find a ρ that satisfies the previous requirement. But the proof of this part is longer and will be discussed in section 5 (a complete argument for the deterministic case of a Lévy triplet can be found in Kardaras [25]).

Example 2.12. Suppose that X is a semimartingale with continuous paths. Then the jump-measure ν is identically equal to zero, and an immediate arbitrage opportunity is a vector $\xi \in \mathbb{R}^d$ with $c\xi = 0$ and $\xi^\top b > 0$. It follows that immediate arbitrage opportunities do not exist, if and only if b lies in the range of c, i.e., if there exists a d-dimensional process ρ with $b = c\rho$; of course, if c is non-singular this always holds and $\rho = c^{-1}b$. It is easy to see that this *"no immediate arbitrage opportunity"* condition is equivalent to $\mathrm{d}B_t \ll \mathrm{d}[X, X]_t$. We refer the reader to Karatzas, Lehoczky and Shreve [22], Appendix B of Karatzas and Shreve [23], and Delbaen and Schachermayer [9] for a more thorough discussion.

Remark 2.13. Let us write $X = A + M$ for the unique decomposition of a special semimartingale X into a predictable finite variation part A and a local martingale M, which we further assume is locally square-integrable. Denoting by $\langle M, M \rangle$ the predictable compensator of $[M, M]$, Example 2.12 shows that the condition for absence of immediate arbitrage opportunities in continuous-path models is the very simple $\mathrm{d}A_t \ll \mathrm{d}\langle M, M \rangle_t$. This should be compared with the more complicated way we have defined this notion for general markets in Definition 2.9.

One wonders whether this simple criterion might work in more general situations. It is easy to see that $\mathrm{d}A_t \ll \mathrm{d}\langle M, M \rangle_t$ is then necessary for absence of immediate arbitrage opportunities; nevertheless, it is not sufficient — it is too weak. Take for example X to be the standard scalar Poisson process. In the absence of constraints on portfolio choice, any positive portfolio is an immediate arbitrage opportunity. Nevertheless, $A_t = t$ and $M_t = X_t - t$ with $\langle M, M \rangle_t = t = A_t$, so that $\mathrm{d}A_t \ll \mathrm{d}\langle M, M \rangle_t$ holds trivially.

2.5. **The growth-optimal portfolio and connection with the numéraire portfolio.** We hinted in Remark 2.11 that if $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ is $\mathbb{P} \otimes G$-null, then one can find a process $\rho \in \Pi_{\mathfrak{C}}$ such that $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \Pi_{\mathfrak{C}}$. It is actually also important to have an algorithmic way of computing this process ρ.

For a portfolio $\pi \in \Pi_{\mathfrak{C}}$, its *growth rate* is defined as the drift rate of the log-wealth process $\log W^\pi$. One can use the stochastic exponential formula (0.2) and formally (since this will not always exist) compute the growth rate of W^π as

$$\mathfrak{g}(\pi) := \pi^\top b - \frac{1}{2}\pi^\top c\pi + \int \left[\log(1+\pi^\top x) - \pi^\top x \mathbb{I}_{\{|x|\le 1\}}\right] \nu(\mathrm{d}x). \tag{2.5}$$

It is well-understood by now that the numéraire portfolio and the portfolio that maximizes in an (ω, t)-pointwise sense the growth rate over all portfolios in $\Pi_{\mathfrak{C}}$ are essentially the same. Let us describe this connection somewhat informally. Consider the case of a deterministic triplet: a vector $\rho \in \mathfrak{C}$ maximizes this concave function $\mathfrak{g}$ if and only if the directional derivative of $\mathfrak{g}$ at the point ρ in the direction of $\pi - \rho$ is negative for any $\pi \in \mathfrak{C}$. This directional derivative can be computed as

$$(\nabla\mathfrak{g})_\rho(\pi-\rho) \;=\; (\pi-\rho)^\top b - (\pi-\rho)^\top c\rho + \int \left[\frac{(\pi-\rho)^\top x}{1+\rho^\top x} - (\pi-\rho)^\top x \mathbb{I}_{\{|x|\le 1\}}\right] \nu(\mathrm{d}x),$$

which is exactly $\mathfrak{rel}(\pi \mid \rho)$.

Of course, we do not know if we can differentiate under the integral appearing in equation 2.5; even worse, we do not know a priori whether the integral is well-defined. Both its positive and negative parts could lead to infinite results. We now describe a class of Lévy measures for which the concave growth rate function $\mathfrak{g}(\cdot)$ of (2.5) *is* well-defined.

Definition 2.14. A Lévy measure ν will be said to *integrate the log*, if

$$\int \log(1+|x|)\, \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x) < \infty.$$

Consider any Lévy measure ν; a sequence $(\nu_n)_{n\in\mathbb{N}}$ of Lévy measures that integrate the log with $\nu_n \sim \nu$, whose densities $f_n := \mathrm{d}\nu_n/\mathrm{d}\nu$ satisfy $0 < f_n \le 1$, $f_n(x) = 1$ for $|x| \le 1$, and $\lim_{n\to\infty} \uparrow f_n = \mathbb{I}$, will be called an *approximating sequence*.

There are many ways to choose the sequence $(\nu_n)_{n\in\mathbb{N}}$, or equivalently the densities $(f_n)_{n\in\mathbb{N}}$; as a concrete example, take $f_n(x) = \mathbb{I}_{\{|x|\le 1\}} + |x|^{-1/n}\mathbb{I}_{\{|x|>1\}}$.

The integral in (2.5) is well defined and finite, when the Lévy measure ν integrates the log. When the growth-optimization problem has infinite value, our strategy will be to solve the optimization problem concerning $\mathfrak{g}(\cdot)$ for a sequence of problems using the approximation described in Definition 2.14, then show that these solutions converge to the solution of the original problem.

2.6. **The first main result.** We are now ready to state the main result of this section, describing the existence of the numéraire portfolio in terms of predictable characteristics. We already discussed condition (1) of Lemma 2.5 and its predictable characterization: *there exists a predictable process ρ with $\rho(\omega, t) \in \mathfrak{C}(\omega, t)$ such that $\mathfrak{rel}(\pi \mid \rho) \le 0$ for all $\pi \in \Pi_{\mathfrak{C}}$, if and only if $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ has zero $\mathbb{P} \otimes G$-measure* (Remark 2.13). If this holds, we construct such a process ρ; the only thing that might keep this ρ from being the numéraire portfolio is failure of X-integrability. To deal with this, define for a given predictable ρ the process

$$\psi^\rho := \nu[\rho^\top x > 1] + \left|\rho^\top b + \int \rho^\top x (\mathbb{I}_{\{|x|>1\}} - \mathbb{I}_{\{|\rho^\top x|>1\}})\nu(\mathrm{d}x)\right|.$$

Here is the statement of the main theorem; its proof is given in section 5.

Theorem 2.15. *Consider a financial model described by a semimartingale returns process X and predictable closed convex constraints $\mathfrak{C}$.*

(1) • *If the predictable set* $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ *has zero* $\mathbb{P} \otimes G$*-measure, then there exists a unique* $\rho \in \mathcal{P}(\mathbb{R}^d)$ *with* $\rho(\omega, t) \in \mathfrak{C} \cap \mathfrak{N}^{\perp}(\omega, t)$ *for all* $(\omega, t) \in [\![0, T]\!]$ *such that* $\mathfrak{rel}(\pi \mid \rho) \leq 0$ *for all* $\pi \in \Pi_{\mathfrak{C}}$*.*

• *On the predictable set* $\left\{\int \log(1 + |x|)\mathbb{I}_{\{|x|>1\}}\nu(\mathrm{d}x) < \infty\right\}$*, this process* ρ *is obtained as the unique solution of the concave optimization problem*

$$\rho = \arg \max_{\pi \in \mathfrak{C} \cap \mathfrak{N}^{\perp}} \mathfrak{g}(\pi)\,.$$

In general, ρ *can be obtained as the limit of solutions to corresponding problems, where one replaces* ν *by* ν_n *an approximating sequence) in the definition of* $\mathfrak{g}$*.*

• *Furthermore, if the process* $\rho \in \mathcal{P}(\mathbb{R}^d)$ *constructed above is such that* $(\psi^{\rho} \cdot G)_t < +\infty$*,* $\mathbb{P}$*-a.s., for all finite* $t \in [0, T]$ *, then* ρ *is* X*-integrable and it is the numéraire portfolio.*

(2) *Conversely, if the numéraire portfolio* ρ *exists in* $\Pi_{\mathfrak{C}}$*, then the predictable set* $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ *has zero* $\mathbb{P} \otimes G$*-measure, and* ρ *satisfies* $(\psi^{\rho} \cdot G)_t < +\infty$*,* $\mathbb{P}$*-a.s., for all finite* $t \in [0, T]$*, as well as* $\mathfrak{rel}(\pi \mid \rho) \leq 0$ *for all* $\pi \in \Pi_{\mathfrak{C}}$*.*

Remark 2.16. Let us pause to comment on the predictable characterization of X-integrability of ρ, which amounts to G-integrability of both processes

$$\psi_1^{\rho} := \nu[\rho^{\top} x > 1] \quad \text{and} \quad \psi_2^{\rho} := \rho^{\top} b + \int \rho^{\top} x (\mathbb{I}_{\{|x|>1\}} - \mathbb{I}_{\{|\rho^{\top} x|>1\}})\nu(\mathrm{d}x). \tag{2.6}$$

The integrability of ψ_1^{ρ} simply states that the process $\rho \cdot X$ cannot make an infinite number of large positive jumps in finite time; but this must obviously be the case if $\rho \cdot X$ is to be well-defined. The second term ψ_2^{ρ} is exactly the drift rate of the part of $\rho \cdot X$ that remains when we subtract all large positive jumps (more than unit in magnitude). This part has to be a special semimartingale, so its drift rate must be G-integrable, which is exactly the requirement $(|\psi_2^{\rho}| \cdot G)_t < \infty$, for all finite $t \in [0, T]$.

Remark 2.17. The conclusion of Theorem 2.15 can be stated succinctly as follows: the numéraire portfolio holds if and only if we have $\Psi(B, C, \eta) < \infty$ for all $(\omega, t) \in [\![0, T]\!]$, for the *deterministic, increasing* functional

$$\Psi(B, C, \eta) \;:=\; \Big(\infty \mathbb{I}_{\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}} + \psi^{\rho} \mathbb{I}_{\{\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset\}}\Big) \cdot G$$

of the triplet of predictable characteristics (B, C, η). This is easily seen not to dependent on the choice of the operational clock G.

Remark 2.18. The requirement $(\psi^{\rho} \cdot G)_t < +\infty$, $\mathbb{P}$-a.s., for all finite $t \in [0, T]$ does *not* imply $(\psi^{\rho} \cdot G)_T < +\infty$ on $\{T = \infty\}$. It is nevertheless easy to see that the stronger requirement $(\psi^{\rho} \cdot G)_T < \infty$ (equivalently, $\Psi_T(B, C, \eta) < \infty$ in the notation of Remark 2.17) means that ρ is X-integrable up to time T, in the terminology of Appendix B. This, in turn, is equivalent to the fact that the numéraire portfolio exists *and* that $W_T^{\rho} < \infty$, a constraint that only makes sense at infinity. We shall return to this when we study arbitrage in the next section.

Example 2.19. Remember the situation with continuous-time semimartingale price process of Example 2.12. Consider the unconstrained case $\mathfrak{C} = \mathbb{R}^d$. The derivative of the growth rate is $(\nabla \mathfrak{g})_{\pi} = b - c\pi = c\rho - c\pi$, which is trivially zero for $\pi = \rho$, and so ρ will be the numéraire portfolio as long as $\big((\rho^\top c \rho) \cdot G\big)_T < \infty$ or, in the case where c^{-1} exists, when $\big((b^\top c^{-1} b) \cdot G\big)_T < \infty$.

2.7. **Relative log-optimality.** In this and the next subsection we give two optimality properties of the numéraire portfolio. Here we show that it exactly the log-optimal portfolio in the relative sense, a more restrictive notion of log-optimality. We proceed with the definition of relative log-optimality where we take into account the fact that we might be in a infinite-time horizon setting.

Definition 2.20. A portfolio $\rho \in \Pi_{\mathfrak{C}}$ will be called *relatively log-optimal*, if

$$\mathbb{E}\left[\limsup_{t \uparrow T} \left(\log \frac{W_t^\pi}{W_t^\rho}\right)\right] \le 0 \quad \text{holds for every } \pi \in \Pi_{\mathfrak{C}}.$$

Here the lim sup is clearly superfluous on $\{T < \infty\}$ but we include it to also cover the infinite time-horizon case. If ρ is relatively log-optimal, the lim sup is actually a finite limit; this is an easy consequence of the followings result.

Proposition 2.21. *A numéraire portfolio exists if and only if a relatively log-optimal problem portfolio exists, in which case the two are the same.*

Proof. In the course of the proof, whenever we write $W_T^{\pi_1}/W_T^{\pi_2}$ for two portfolios π_1 and π_2, we tacitly imply that on $\{T = \infty\}$ the limit of this ratio exists, and we take $W_T^{\pi_1}/W_T^{\pi_2}$ to be exactly that limit.

Suppose ρ is a numéraire portfolio. For any $\pi \in \Pi_{\mathfrak{C}}$ we have $\mathbb{E}[W_T^\pi/W_T^\rho] \le 1$, and Jensen's inequality gives $\mathbb{E}[\log(W_T^\pi/W_T^\rho)] \le 0$, so ρ is also relatively log-optimal.

Let us now assume that *the numéraire portfolio does not exist*; we shall show that a relative log-optimal portfolio does not exist either. By way of contradiction, suppose that $\hat\rho$ was a relatively log-optimal portfolio.

First, we observe that $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ *must* have zero $\mathbb{P} \otimes G$-measure. To see why, suppose the contrary. Then, by Proposition 2.10, we could select a portfolio $\xi \in \Pi_{\mathfrak{C}}$ that leads to unbounded increasing profit. According to Remark 2.11 (actually, the "pathwise" version of the argument we used for the Lévy process setting), we would have $\hat\rho + \xi \in \Pi_{\mathfrak{C}}$ and $\mathfrak{rel}(\hat\rho \mid \hat\rho + \xi) \le 0$, with strict inequality on a predictable set of positive $\mathbb{P} \otimes G$-measure; this would mean that the process $W^{\hat\rho}/W^{\hat\rho+\xi}$ is a non-constant positive supermartingale, and Jensen's inequality again would give $\mathbb{E}[\log(W_T^{\hat\rho}/W_T^{\hat\rho+\xi})] < 0$, contradicting the relative log-optimality of $\hat\rho$.

Continuing, since the numéraire portfolio does not exist and we already showed that $\{\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset\}$ has full $\mathbb{P} \otimes G$-measure, we must have that ρ (the candidate in Theorem 2.15 (1) for being the numéraire portfolio) is not X-integrable. In particular, the predictable set $\{\hat\rho \neq \rho\}$ must have non-zero $\mathbb{P} \otimes G$-measure. But then we can find a predictable set $\Sigma \subseteq \{\hat\rho \neq \rho\}$ such that Σ has non-zero $\mathbb{P} \otimes G$-measure and such that $\rho \mathbb{I}_\Sigma \in \Pi_{\mathfrak{C}}$. This implies $\pi := \hat\rho \mathbb{I}_{[\![0,T]\!] \setminus \Sigma} + \rho \mathbb{I}_\Sigma \in \Pi_{\mathfrak{C}}$, and since $\mathfrak{rel}(\hat\rho \mid \pi) = \mathfrak{rel}(\hat\rho \mid \rho)\mathbb{I}_\Sigma \le 0$, with strict inequality on Σ, the same discussion as in the end of the preceding paragraph shows that $\hat\rho$ cannot be the relatively log-optimal portfolio. $\square$

Example 2.22. Take a one-stock market model with $S_t = \exp(\beta_{T \wedge t})$, where β is a standard, 1-dimensional Brownian motion and T is an a.s. finite stopping time with $\mathbb{E}\left[\beta_T^+\right] = +\infty$. Then, $\mathbb{E}\left[\log S_T\right] = +\infty$ and the classical log-utility optimization problem does not have a unique solution (one can find a multitude of portfolios that achieve infinite expected log-utility). In this case, Example 2.12 shows that $\rho = 1/2$ is both the numéraire and the relative log-optimal portfolio.

2.8. **An asymptotic optimality property.** In this subsection we deal with a purely infinite time-horizon case $T \equiv \infty$ and describe an "asymptotic growth optimality" property of the numéraire portfolio. Note that if ρ is the numéraire portfolio, then for the positive supermartingale W^π / W^ρ the $\lim_{t\to\infty}(W^\pi_t / W^\rho_t)$ exists in $[0, +\infty)$ for every $\pi \in \Pi_{\mathfrak{C}}$. Consequently, for *any* increasing process H with $H_\infty = +\infty$ (H does not even have to be adapted), we have

$$\limsup_{t\to\infty} \left(\frac{1}{H_t} \log \frac{W^\pi_t}{W^\rho_t} \right) \le 0 \,. \tag{2.7}$$

This version of "asymptotic growth optimality" was first observed and proved (for $H_t \equiv t$, but this is not too essential) in Algoet and Cover [1] for the discrete-time case; see also Karatzas and Shreve [23] and Goll and Kallsen [16] for a discussion of (2.7) in the continuous-path and the general semimartingale case, respectively.

Our next result, Proposition 2.23, separates the cases when $\lim_{t\to\infty}(W^\pi_t / W^\rho_t)$ is $(0, \infty)$-valued and when it is zero, and finds a predictable characterization of this dichotomy. Also, in the case of convergence to zero, it quantifies how fast does this convergence takes place. The proof of Proposition 2.23 is given in section 6, actually in a more general and abstract setting.

Proposition 2.23. *Assume that the numéraire portfolio ρ exists for the infinite time-horizon $[0, \infty]$. For any other $\pi \in \Pi_{\mathfrak{C}}$, define the positive, predictable process*

$$h^\pi := -\mathfrak{rel}(\pi \mid \rho) + \frac{1}{2}(\pi - \rho)^\top c(\pi - \rho) + \int q_a \left(\frac{1 + \pi^\top x}{1 + \rho^\top x} \right) \nu(\mathrm{d}x).$$

Here, $q_a(y) := \big[-\log a + (1 - a^{-1})y\big]\, \mathbb{I}_{[0,a)}(y) + [y - 1 - \log y]\, \mathbb{I}_{[a,+\infty)}(y)$ for some $a \in (0, 1)$ is a positive, convex function. Consider the increasing, predictable process $H^\pi := h^\pi \cdot G$. Then we have:

$$\text{on } \{H^\pi_\infty < +\infty\}, \qquad \lim_{t\to\infty} \frac{W^\pi_t}{W^\rho_t} \in (0, +\infty)\,; \ \text{ while}$$

$$\text{on } \{H^\pi_\infty = +\infty\}, \qquad \limsup_{t\to\infty} \left(\frac{1}{H^\pi_t} \log \frac{W^\pi_t}{W^\rho_t} \right) \le -1.$$

Remark 2.24. Some comments are in order. We begin with the "strange looking" function $q_a(\cdot)$, that depends also on the (cut-off point) parameter $a \in (0, 1)$. Ideally we would like to define $q_0(y) = y - 1 - \log y$ for all $y > 0$, since then the predictable increasing process H^π would be exactly the negative of the drift of the semimartingale $\log(W^\pi / W^\rho)$. Unfortunately, a problem arises when the positive predictable process $\int q_0 \left(\frac{1+\pi^\top x}{1+\rho^\top x} \right) \nu(\mathrm{d}x)$ fails to be G-integrable which is equivalent to saying that $\log(W^\pi / W^\rho)$ is *not* a special semimartingale; the problem comes from the fact that $q_0(y)$ explodes to $+\infty$ as $y \downarrow 0$. For this reason, we define $q_a(y)$ to be equal to $q_0(y)$ for all $y \ge a$, and for $y \in [0, a)$ we define it in a linear way so that $q_a(\cdot)$ is continuously differentiable at the "gluing" point a. The functions $q_a(\cdot)$ all are finite-valued at $y = 0$ and satisfy $q_a(\cdot) \uparrow q_0(\cdot)$ as $a \downarrow 0$.

Let us now study h^π and H^π. Observe that h^π is *predictably convex* in π, namely, if π_1 and π_2 are two portfolios and λ is a $[0,1]$-valued predictable process, then $h^{\lambda\pi_1+(1-\lambda)\pi_2} \leq \lambda h^{\pi_1} + (1-\lambda)h^{\pi_2}$. This, together with the fact that $h^\pi = 0$ if and only if $\pi - \rho$ is a null investment, implies that h^π *can be seen as a measure of instantaneous deviation of* π *from* ρ; by the same token, H^π_∞ can be seen as the total (cumulative) deviation of π from ρ. With these remarks in mind, Proposition 2.23 says in effect that, if an investment deviates a lot from the numéraire portfolio ρ (i.e., if $H^\pi_\infty = +\infty$), its performance will lag considerably behind that of ρ. Only if an investment tracks very closely the numéraire portfolio for the whole amount of time (i.e., if $H^\pi_\infty < +\infty$) will the two wealth processes have comparable growth over the whole time-period. Also, in connection with the previous paragraph, letting $a \downarrow 0$ in the definition of H^π we get equivalent measures of distance of a portfolio π from the numéraire portfolio, in the sense that the event $\{H^\pi_\infty = +\infty\}$ does not depend on the choice of a; nevertheless we get ever sharper results, since h^π is increasing for decreasing $a \in (0,1)$.

3. Unbounded Profits with Bounded Risks, Supermartingale Deflators and the Numéraire Portfolio

In this section we proceed to investigate how the existence or non-existence of the numéraire portfolio relates to some concept of "free lunch" in the financial market. We shall eventually prove a version of the Fundamental theorem of Asset Pricing; this is our second main result, Theorem 3.12.

3.1. **Arbitrage-type definitions.** There are two widely-known conditions relating to arbitrage in financial markets: the classical "No Arbitrage" and its stronger version "No Free Lunch with Vanishing Risk". We recall them below, together with yet another notion; this is exactly what one needs to bridge the gap between the previous two, and it will actually be the most important for our discussion.

Definition 3.1. For the following definitions we consider our financial model with constrains $\mathfrak{C}$ on portfolios. When we write W^π_T for some $\pi \in \Pi_\mathfrak{C}$ we tacitly assume that $\lim_{t\to\infty} W^\pi_t$ exists on $\{T = \infty\}$, and set W^π_T equal to that limit.

(1) A portfolio $\pi \in \Pi_\mathfrak{C}$ is said to generate an *arbitrage opportunity*, if it satisfies $\mathbb{P}[W^\pi_T \geq 1] = 1$ and $\mathbb{P}[W^\pi_T > 1] > 0$. If no such wealth process exists we say that the $\mathfrak{C}$-constrained market satisfies the *no arbitrage* condition, which we denote by $\mathrm{NA}_\mathfrak{C}$.
(2) A sequence $(\pi_n)_{n\in\mathbb{N}}$ of portfolios in $\Pi_\mathfrak{C}$ is said to generate an *unbounded profit with bounded risk* (UPBR), if the collection of positive random variables $(W^{\pi_n}_T)_{n\in\mathbb{N}}$ is unbounded in probability, i.e., if
$$\lim_{m\to\infty} \downarrow \sup_{n\in\mathbb{N}} \mathbb{P}[W^{\pi_n}_T > m] > 0.$$
If no such sequence exists, we say that the constrained market satisfies the *no unbounded profit with bounded risk* ($\mathrm{NUPBR}_\mathfrak{C}$) condition.
(3) A sequence $(\pi_n)_{n\in\mathbb{N}}$ of portfolios in $\Pi_\mathfrak{C}$ is said to be a *free lunch with vanishing risk* (FLVR), if there exist an $\epsilon > 0$ and an increasing sequence $(\delta_n)_{n\in\mathbb{N}}$ of real numbers with $0 \leq \delta_n \uparrow 1$, such that $\mathbb{P}[W^{\pi_n}_T > \delta_n] = 1$ as

well as $\mathbb{P}[W_T^{\pi_n} > 1+\epsilon] \geq \epsilon$. If no such sequence exists we say that the market satisfies the *no free lunch with vanishing risk* (NFLVR) condition.

The NFLVR condition was introduced by Delbaen and Schachermayer [8] in a slightly different way. With the above definition of free lunch with vanishing risk and the convexity lemma A 1.1 from that last paper we can further assume that there exists a $[1,+\infty]$-valued random variable f with $\mathbb{P}[f>1]>0$ such that $\mathbb{P}\text{-}\lim_{n\to\infty} W_T^{\pi_n} = f$, and this brings us back to the usual definition.

If UPBR exists, one can find a sequence of wealth processes, each starting with less and less capital (converging to zero) and such that the terminal wealths are unbounded with a fixed probability. Thus, UPBR can be translated as "the *possibility* of making (a considerable) something out of almost nothing"; it should be contrasted with the classical notion of arbitrage, which can be translated as "the *certainty* of making something more out of something".

Observe that $\mathrm{NUPBR}_{\mathfrak{C}}$ can be alternatively stated by using portfolios with *bounded support*, so the requirement of a limit at infinity for the wealth processes on $\{T=\infty\}$ is automatically satisfied. This is relevant because, as we shall see, when $\mathrm{NUPBR}_{\mathfrak{C}}$ holds every wealth process W^π has a limit on $\{T=\infty\}$ and is a semimartingale up to T in the terminology of section B of the Appendix.

None of the two conditions $\mathrm{NA}_{\mathfrak{C}}$ and $\mathrm{NUPBR}_{\mathfrak{C}}$ implies the other, and they are not mutually exclusive. It is easy to see that they are both weaker than $\mathrm{NFLVR}_{\mathfrak{C}}$, and that in fact we have the following result which gives the exact relationship between these notions under the case of cone constraints. Its proof can be found in [8] for the unconstrained case; we include it here for completeness.

Proposition 3.2. *Suppose that $\mathfrak{C}$ enforces predictable closed convex* cone *constraints. Then,* $\mathrm{NFLVR}_{\mathfrak{C}}$ *holds if and only if both* $\mathrm{NA}_{\mathfrak{C}}$ *and* $\mathrm{NUPBR}_{\mathfrak{C}}$ *hold.*

Proof. It is obvious that if either $\mathrm{NA}_{\mathfrak{C}}$ or $\mathrm{NUPBR}_{\mathfrak{C}}$ fail, then $\mathrm{NFLVR}_{\mathfrak{C}}$ fails too. Conversely, suppose that $\mathrm{NFLVR}_{\mathfrak{C}}$ fails. If $\mathrm{NA}_{\mathfrak{C}}$ fails there is nothing more to say, so suppose that $\mathrm{NA}_{\mathfrak{C}}$ holds and let $(\pi^n)_{n\in\mathbb{N}}$ generate a free lunch with vanishing risk. Since we have no arbitrage, the assumption $\mathbb{P}[W_T^{\pi_n} > \delta_n] = 1$ results in the stronger $\mathbb{P}(W_t^{\pi_n} > \delta_n$ for all $t\in[0,T]) = 1$. Construct a new sequence of wealth processes $(W^{\xi_n})_{n\in\mathbb{N}}$ by requiring $W^{\xi_n} = 1 + (1-\delta_n)^{-1}(W^{\pi_n} - 1)$. The reader can readily check that $W^{\xi_n} > 0$ and then that $\xi_n \in \Pi_{\mathfrak{C}}$ (here it is essential that $\mathfrak{C}$ be a cone). Furthermore, $\mathbb{P}[W_T^{\pi_n} \geq 1+\epsilon] \geq \epsilon$ becomes $\mathbb{P}[W_T^{\xi_n} > 1 + (1-\delta_n)^{-1}\epsilon] \geq \epsilon$, meaning that $(\xi_n)_{n\in\mathbb{N}}$ generates an unbounded profit with bounded risk, so $\mathrm{NUPBR}_{\mathfrak{C}}$ fails. □

3.2. **The Fundamental Theorem of Asset Pricing.** The $\mathrm{NFLVR}_{\mathfrak{C}}$ condition has proven very fruitful in understanding cases when we can change the original measure $\mathbb{P}$ to some other equivalent probability measure such that the stock-price processes (or, at least the wealth processes) has some kind of martingale (or maybe only supermartingale) property under $\mathbb{Q}$. The following definition puts us in the proper context for the statement of Theorem 3.4.

Definition 3.3. Consider a financial market model described by a semimartingale discounted stock price process S and predictable closed convex constraints $\mathfrak{C}$ on portfolios. A probability measure $\mathbb{Q}$ will be called a *$\mathfrak{C}$-equivalent supermartingale* *measure* (ESMM$_{\mathfrak{C}}$ for short), if $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_T$, and every W^π for $\pi \in \Pi_{\mathfrak{C}}$ is a $\mathbb{Q}$-supermartingales. The class of ESMM$_{\mathfrak{C}}$ is denoted by $\mathfrak{Q}_{\mathfrak{C}}$.

Similarly, define a $\mathfrak{C}$-*equivalent local martingale measure* (ELMM$_{\mathfrak{C}}$ for short) $\mathbb{Q}$ by requiring $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_T$ and that every W^π for $\pi \in \Pi_{\mathfrak{C}}$ is a $\mathbb{Q}$-local martingale.

In this definition we might as well assume that $\mathfrak{C}$ are cone constraints. The reason is that if ESMM$_{\mathfrak{C}}$ holds, the same holds for the market under constraints $\overline{\mathsf{cone}}(\mathfrak{C})$, the closure of the *cone generated by* $\mathfrak{C}$.

The following theorem is one of the most well-known in mathematical finance; we give the "cone-constrained" version.

Theorem 3.4. (FTAP) *For a financial market model with stock-price process S and predictable closed convex cone constraints $\mathfrak{C}$,* NFLVR$_{\mathfrak{C}}$ *is equivalent to $\mathfrak{Q}_{\mathfrak{C}} \neq \emptyset$.*

In the unconstrained case, one can prove further that there exists a $\mathbb{Q} \sim \mathbb{P}$ such that the stock prices S become σ-martingales under $\mathbb{Q}$ — this version of the theorem is what is called the "Fundamental Theorem of Asset Pricing". The concept of σ-martingale is just a natural equivalent of local martingales for possibly locally unbounded processes; for *positive* processes, σ-martingale and local martingale is the same thing, so the reader can somehow discard this subtle difference. Nevertheless, it should be pointed out that Theorem 3.4 holds for *any* stock-price process, and then the local martingale concept is not sufficient. In any case, we know in particular that $\mathbb{Q}$ will be an equivalent local martingale measure according to Definition 3.3.

As a contrast to the preceding paragraph, let us note that because we are working under constraints, we cannot hope in general for anything better than an equivalent *supermartingale* measure in the statement of Theorem 3.4. One can see this easily in the case where X is a single-jump process which jumps at a stopping time τ with $\Delta X_\tau \in (-1, 0)$ and we are constrained in the cone of positive strategies. Under any measure $\mathbb{Q} \sim \mathbb{P}$, the process $S = \mathcal{E}(X) = W^1$, an admissible wealth process, will be non-increasing and not identically equal zero, which prevents it from being a local martingale.

The implication $\mathfrak{Q}_{\mathfrak{C}} \neq \emptyset \ \Rightarrow$ NFLVR$_{\mathfrak{C}}$ is easy; the reverse is considerably harder for the general semimartingale model. There is a plethora of papers devoted to proving some version of this result. In the generality assumed here, a proof appears in Kabanov [19], although all the crucial work has been done by Delbaen and Schachermayer in [8] and the theorem is certainly due to them. To make sure that Theorem 3.4 can be derived from Kabanov's statement, observe that the class of wealth processes $(W^\pi)_{\pi \in \Pi_{\mathfrak{C}}}$ is convex and closed in the semimartingale (also called "Émery") topology. A careful inspection in Mémin's work [30], of the proof that the set of all stochastic integrals with respect to the d-dimensional semimartingale X is closed under this topology, will convince the reader that one can actually pick the limiting semimartingale from a convergent sequence $(W^{\pi_n})_{n \in \mathbb{N}}$, with $\pi_n \in \Pi_{\mathfrak{C}}$ for all $n \in \mathbb{N}$, to be of the form W^π for some $\pi \in \Pi_{\mathfrak{C}}$.

3.3. **Beyond the Fundamental Theorem of Asset Pricing.** Let us study a little more the assumptions and statement of the FTAP 3.4. We shall be concerned with three questions, which will turn out to have the same answer; this answer will be linked with the NUPBR$_{\mathfrak{C}}$ condition and — as we shall see in Theorem 3.12 — with the existence of the numéraire portfolio.

3.3.1. *Convex but non-conic constraints.* Slightly before the statement of the Fundamental Theorem of Asset Pricing 3.4 we discussed that we *have* to assume that the constraint set is a cone. This is crucial — the theorem is no longer true if we drop the "cone" assumption. Of course, $\mathfrak{Q}_{\mathfrak{C}} \neq \emptyset \Rightarrow$ NFLVR$_{\mathfrak{C}}$ still holds, but the reverse implication fails, as shown in the example below; this is a raw version of a similar example from Kardaras [25].

Example 3.5. Consider a simple discrete-time model with one time-period: there exist only day zero and day one. We have two stocks with discounted stock prices S^1 and S^2 that satisfy $S^1_0 = S^2_0 = 1$, while $S^1_1 = 1 + e$ and $S^2_1 = f$. Here e and f are two independent, exponentially distributed random variables. Using the independence of e and f, the class of portfolios is easily identified with all $(p, q) \in \mathfrak{C}_0 = \mathbb{R}_+ \times [0, 1]$. Since $X^1_1 = S^1_1 - S^1_0 = e > 0$, $\mathbb{P}$-a.s., we have that NA fails for this (non-constrained) market. In other words, for the non-constrained case there can be no ESMM.

Consider now the non-random constraint set $\mathfrak{C} = \{(p, q) \in \mathfrak{C}_0 \mid p^2 \leq q\}$. Observe that $\overline{\mathsf{cone}}(\mathfrak{C}) = \mathbb{R}_+ \times \mathbb{R}$ and thus no ESMM$_{\mathfrak{C}}$ exists; for otherwise an ESMM would exist already for the unconstrained case. We shall nevertheless show in the following paragraph that NFLVR$_{\mathfrak{C}}$ holds for this constrained market.

Start with a sequence of portfolios $\pi_n \equiv (p_n, q_n)_{n \in \mathbb{N}}$ in $\mathfrak{C}$. The wealth at day one will be $W^{\pi_n}_1 = 1 - q_n + q_n f + p_n e$; obviously $\mathbb{P}[W^{\pi_n}_1 \geq 1 - q_n] = 1$, since $1 - q_n$ is the essential infimum of $W^{\pi_n}_1$. It then turns out that in order for $(\pi_n)_{n \in \mathbb{N}}$ to generate a free lunch with vanishing risk we must require $q_n \downarrow 0$ and $\mathbb{P}[W^{\pi_n}_1 > 1 + \epsilon] > \epsilon$ for some $\epsilon > 0$. Observe that we *must* have $q_n > 0$, otherwise $p_n = 0$ as well (because of the constraints) and then $W^{\pi_n}_1 = 1$. Now, because of the constraints again we have $|p_n| \leq \sqrt{q_n}$; since $\mathbb{P}[e > 0] = 1$ the sequence of strategies $\xi_n := (\sqrt{q_n}, q_n)$ will generate a sequence of wealth processes $(W^{\xi_n})_{n \in \mathbb{N}}$ that will dominate $(W^{\pi_n})_{n \in \mathbb{N}}$: $\mathbb{P}[W^{\xi_n}_1 \geq W^{\pi_n}_1] = 1$; this will of course mean that $(W^{\xi_n})_{n \in \mathbb{N}}$ is also a free lunch with vanishing risk. We should then have $\mathbb{P}[1 - q_n + \sqrt{q_n} e + q_n f > 1 + \epsilon] > \epsilon$; using $q_n > 0$ and some algebra we get $\mathbb{P}[e > \sqrt{q_n}(1 - f) + \epsilon / \sqrt{q_n}] > \epsilon$. Since $(q_n)_{n \in \mathbb{N}}$ goes to zero this would imply that $\mathbb{P}[e > M] \geq \epsilon$ for all $M > 0$, which is clearly ridiculous. We conclude that NFLVR$_{\mathfrak{C}}$ holds, although as we have seen $\mathfrak{Q}_{\mathfrak{C}} = \emptyset$.

What can we say then in the case of convex — but non necessarily conic — constraints? It will turn out that for the equivalent of the Fundamental Theorem of Asset Pricing, the assumptions from *both* the economic *and* the mathematical side should be relaxed. The relevant economic notion will be NUPBR$_{\mathfrak{C}}$ and the mathematical one will be the concept of supermartingale deflators — more on this in subsections 3.4 and 3.5.

3.3.2. *Predictability of free lunches.* The reason why "free lunches" are considered economically unsound stems from the following reasoning: if they exist in a market, many agents will try to take advantage of them; then, usual supply-and-demand arguments will imply that some correction on the prices of the assets will occur, and remove these kinds of opportunities. This is a very reasonable line of thought, *provided that one can discover the free lunches that are present.* But is it true that, given a specific model, one is in a position to decide whether free lunches exist or not? In other words, mere knowledge of the *existence* of a free lunch may not be enough to carry the previous economic argument — one should be able to *construct* the free lunch. This goes somewhat hand in hand with the fact that the FTAP is an *existential* result, in the sense that it provides knowledge that *some* equivalent (super)martingale measure exists; in some cases we shall be able to spot it, in others not.

Here is a natural question that almost poses itself at this point: *when free lunches exist, is there a way to construct them from observable quantities in the market?* Of course there are many ways to understand what "observable quantities" means, but here is a partial answer: if $\mathrm{NUPBR}_{\mathfrak{C}}$ fails, then there exists a way to construct the unbounded profit with bounded risk using the triplet of predictable characteristics (B, C, η). The detailed statement will be given later on in subsection 3.6, but let us quickly say here that the *deterministic* positive functional Ψ of Remark 2.17 is such that on the event $\{\Psi_T(B, C, \eta) = \infty\}$ $\mathrm{NUPBR}_{\mathfrak{C}}$ fails (and then we can construct free lunches using the triplet of predictable characteristics and an algorithm), while on $\{\Psi_T(B, C, \eta) < \infty\}$ $\mathrm{NUPBR}_{\mathfrak{C}}$ holds. As a result of this, we get that $\mathrm{NUPBR}_{\mathfrak{C}}$ is somehow a *pathwise* notion.

What we described in the last paragraph for the $\mathrm{NUPBR}_{\mathfrak{C}}$ condition does not apply to the $\mathrm{NA}_{\mathfrak{C}}$ condition as we shall soon show, but not before we recall a famous example where the model admits arbitrage opportunities.

Example 3.6. Consider a one-stock market, where the price process satisfies

$$\mathrm{d}S_t = (1/S_t)\mathrm{d}t + \mathrm{d}\beta_t, \quad S_0 = 1.$$

Here, β is a standard, 1-dimensional Brownian motion, so S is the *3-dimensional Bessel process.* We work on the finite time horizon $[0, 1]$.

For future reference, observe that by using the natural operational clock $G_t = t$ we get $b = S^{-2}$ and $c = S^{-2}$; this follows from $\mathrm{d}S_t/S_t = (1/S_t^2)\mathrm{d}t + (1/S_t)\mathrm{d}\beta_t =: \mathrm{d}X_t$. Thus, the numéraire portfolio for the unconstrained case exists and is $\rho = c^{-1}b = 1$ according to Example 2.19.

This market admits arbitrage. To wit, with the notation

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\mathrm{d}u\,, \quad F(t,x) = \frac{\Phi(x/\sqrt{1-t})}{\Phi(1)}\,, \quad \text{for } x \in \mathbb{R} \text{ and } 0 < t < 1,$$

consider the process $W_t = F(t, S_t)$. Obviously $W_0 = 1$, $W > 0$ and

$$\mathrm{d}W_t = \frac{\partial F}{\partial x}(t, S_t)\mathrm{d}S_t, \text{ and thus } \frac{\mathrm{d}W_t}{W_t} = \left[\frac{1}{F(t,S_t)}\frac{\partial F}{\partial x}(t, S_t)\right]\mathrm{d}S_t$$

by Itô's formula. We conclude that $W = W^{\pi}$ for $\pi_t := (\partial \log F/\partial x)(t, S_t)$, and we clearly have $W_1^{\pi} = 1/\Phi(1) > 1$, i.e., there exists arbitrage in the market.

There is also an indirect way to show that arbitrage exists, proposed by Delbaen and Schachermayer [11]. One has to assume that the filtration $\mathbf{F}$ is the one generated by S (equivalently, by β), and recall that $1/S$ is a *strict local martingale*; i.e., $\mathbb{E}[1/S_t] < 1$ for all $t > 0$. Using the strong martingale representation property of β, it can be seen that $1/S_1$ is the *only* candidate for the density of an equivalent supermartingale measure. Since $1/S_1$ fails to integrate to one, ESMM fails. The

Fundamental Theorem of Asset Pricing 3.4 implies that NFLVR fails. In fact, it is actually NA which fails; this will become clear after our main Theorem 3.12.

We note that *this is one of the rare occasions, when one can compute the arbitrage portfolio concretely.* We were successful in this, because of the very special structure of the 3-dimensional Bessel process; every model has to be attacked in a different way and there is no general theory that will spot the arbitrage. Nevertheless, we refer the reader to Fernholz, Karatzas and Kardaras [14] and Fernholz and Karatzas [13] for many examples of arbitrage relatively to the *market portfolio* (whose wealth process that follows exactly the index $\sum_{i=1}^{d} S^i$ in proportion to the initial investment). This is done under conditions on market structure that are easy to check and descriptive – as opposed to normative, such as ELMM. ◇

We now show that there *cannot* exist a deterministic positive functional Ψ that takes for its arguments triplets of predictable characteristics such that NA holds whenever $\mathbb{P}[\Psi_T(B, C, \eta) < \infty] = 1$. Actually, we shall construct in the next paragraph two stock-price processes on the *same* stochastic basis and with the *same* predictable characteristics and such that NA fails with respect to the one but holds with respect to the other.

Example 3.7. NON-PREDICTABILITY OF ARBITRAGE. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough to accommodate two independent standard 1-dimensional Brownian motions β and γ; the filtration will be the (usual augmentation of the) one generated by the pair (β, γ). We use again the finite time-horizon $[0, 1]$. Let R be the 3-dimensional Bessel process that satisfies the stochastic differential equation $\mathrm{d}R_t = (1/R_t)\mathrm{d}t + \mathrm{d}\beta_t$, with $R_0 = 1$. As R is adapted to the filtration generated by β, it is independent of γ. Start with the market described by the stock-price $S = R$; the triplet of predictable characteristics (B, C, η) consists of $B_t = C_t = \int_0^t (1/R_u)^2 \mathrm{d}u$ and $\eta = 0$. According to Example 3.6, NA fails for this market.

With the same process R, define now a new stock $\widehat{S}$ following the dynamics $\mathrm{d}\widehat{S}_t / \widehat{S}_t = (1/R_t)^2 \mathrm{d}t + (1/R_t)\mathrm{d}\gamma_t$ with $\widehat{S}_0 = 1$. Observe that the new dynamics involve γ, so $\widehat{S}$ is *not* a 3-dimensional Bessel process; nevertheless, it has exactly the same triplet of predictable characteristics as S. But now NA holds for the market that consists only of the stock $\widehat{S}$, and we can actually construct an ELMM. The reason is that the exponential local martingale Z defined by

$$Z_t = \exp\left(- \int_0^t (1/R_u)\mathrm{d}\gamma_u - \frac{1}{2} \int_0^t (1/R_u)^2 \mathrm{d}u \right)$$

is a *true* martingale; this follows from the independence of R and γ, as Lemma 3.8 below will show. We can then define $\mathbb{Q} \sim \mathbb{P}$ via $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = Z_1$, and Girsanov's theorem will imply that $\widehat{S}$ is the stochastic exponential of a Brownian motion under $\mathbb{Q}$ — thus a true martingale.

Lemma 3.8. *On a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ let β be a standard 1-dimensional $\mathbf{F}$-Brownian motion, and α a predictable process which is* independent *of β and satisfies $\int_0^t |\alpha_u|^2 \mathrm{d}u < \infty$, $\mathbb{P}$-a.s. Then, the exponential local martingale $Z = \mathcal{E}(\alpha \cdot \beta)$ satisfies $\mathbb{E}[Z_t] = 1$, i.e., it is a true martingale on $[0, t]$.*

Proof. We begin by enlarging the filtration to $\mathbf{G}$ with $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\alpha_t; t \in \mathbb{R}_+)$, i.e., we throw the *whole* history of α up to the end of time in $\mathbf{F}$. Since α and β are independent, it is easy to see that β is a $\mathbf{G}$-Brownian motion. Of course, α is a $\mathbf{G}$-predictable process and thus the stochastic integral $\alpha \cdot \beta$ is the same seen under $\mathbf{F}$ or $\mathbf{G}$. Then, with $A_n := \{n - 1 \leq \int_0^t |\alpha_u|^2 \mathrm{d}u < n\} \in \mathcal{G}_0$ we have

$$\mathbb{E}[Z_t] = \mathbb{E}[\,\mathbb{E}[Z_t \,|\, \mathcal{G}_0]\,] = \sum_{n=1}^{\infty} \mathbb{E}[\,Z_t \,|\, A_n\,]\,\mathbb{P}[A_n] = 1,$$

the last equality holding in view of $\mathbb{E}[Z_t \,|\, A_n] = 1$, since on A_n the quadratic variation of $\alpha \cdot \beta$ is bounded by n. □

3.3.3. *Connection with utility maximization.* A central problem of mathematical finance is the maximization of expected utility of an economic agent who can invest in the market. To start, let us formalize preference structures. We assume that the agent's preferences can be described by a *utility function*: namely, a *concave* and *strictly increasing* function $U : (0, \infty) \mapsto \mathbb{R}$. We also define $U(0) \equiv U(0+)$ by continuity. Starting with initial capital $w > 0$, the objective of the investor is to find a portfolio $\hat{\rho} \equiv \hat{\rho}(w) \in \Pi_{\mathfrak{C}}$ such that

$$\mathbb{E}[U(wW_T^{\hat{\rho}})] = \sup_{\pi \in \Pi_{\mathfrak{C}}} \mathbb{E}\,[U(wW_T^{\pi})] =: u(w). \tag{3.1}$$

Probably the most important example of a utility function is the logarithmic $U(w) = \log w$. Then, due to this special structure, the optimal portfolio (when it exists) does not depend on the initial capital, or on the given time-horizon T ("myopia"). The relationship between the numéraire and the log-optimal portfolio is well-known and established by now — in fact, we saw in subsection 2.7 that under a suitable reformulation of log-optimality the two notions are equivalent.

The case we have just described is the one of utility maximization from terminal wealth when utilities are only defined in the positive real line (in other words, $U(w) = -\infty$ for $w < 0$). This problem has a long history; let us just mention that it has been solved in a very satisfactory manner for general semimartingale models using previously-developed ideas of martingale duality by Kramkov and Schachermayer [26, 27], where we send the reader for further details.

A common assumption in this context is that the class of equivalent local martingale measures is non-empty, i.e., that NFLVR holds. The 3-dimensional Bessel process Example 3.6 clearly shows that this is not necessary; indeed, since the numéraire portfolio $\rho = 1$ exists and $\mathbb{E}(\log S_1) < \infty$, Proposition 2.21 shows that $\rho = 1$ is the solution to the log-utility optimization problem. Nevertheless, we have seen that NA fails for this market, thus NFLVR fails as well. To recap: an investor with logarithmic utility will choose to hold the stock as the optimal investment and, *even though arbitrage opportunities exist in the market, the investor's optimal choice is clearly* not *an arbitrage.*

In the mathematical theory of economics, the equivalence of no free lunches, equivalent martingale measures, and existence of optimal investments for utility-based preferences, is somewhat of a "folklore theorem". The Fundamental Theorem of Asset Pricing 3.4 deals with the equivalence of the first two of these conditions, but the 3-dimensional Bessel process example shows that this does not necessarily fit well with minimal conditions for utility maximization. In the context of that example, although NA fails, the numéraire and log-optimal portfolios exist. As we shall see in Theorem 3.12, *the existence of the numéraire portfolio is equivalent to the NUPBR condition*. Then, we shall show in subsection 3.7 that NUPBR is actually the *minimal* "no free lunch" notion needed to ensure existence of solution to *any* utility maximization problem. In a loose sense (to become precise there) the problem of maximizing expected utility from terminal wealth, for a rather large class of utility functions, is solvable *if and only if* the special case of the logarithmic utility problem has a solution — which is exactly in the case when NUPBR holds.

Accordingly, the existence of an equivalent (local) martingale measure will have to be substituted by the weaker requirement, the existence of a *supermartingale deflator*. This notion is the subject of the next subsection.

3.4. **Supermartingale deflators.** We hope to have made it clear up to now that the NUPBR condition of Definition 3.1 is interesting. In the spirit of the Fundamental Theorem of Asset Pricing, we would like also to find an equivalent mathematical condition. The next concept, closely related to that of equivalent supermartingale measures but weaker, will be exactly what we shall need.

Definition 3.9. The class of *equivalent supermartingale deflators* is defined as

$$\mathfrak{D}_{\mathfrak{C}} := \{D \geq 0 \mid D_0 = 1,\ D_T > 0,\ \text{and } DW^\pi \text{ is supermartingale } \forall \pi \in \Pi_{\mathfrak{C}}\}\,.$$

If there exists an element $D^* \in \mathfrak{D}_{\mathfrak{C}}$ of the form $D^* \equiv 1/W^\rho$ for some $\rho \in \Pi_{\mathfrak{C}}$, we call D^* a *tradeable supermartingale deflator*.

If a tradeable supermartingale deflator $D^* \equiv 1/W^\rho$ exists, then the wealth process W^ρ is such that the relative wealth process W^π/W^ρ is a supermartingale for all $\pi \in \Pi_{\mathfrak{C}}$, i.e., ρ is the numéraire portfolio. We conclude that a tradeable supermartingale deflator exists, if and only if a numéraire portfolio ρ exists and $W^\rho_T < \infty$, $\mathbb{P}$-a.s., and that if it exists it is unique.

An equivalent supermartingale measure $\mathbb{Q}$ generates an equivalent supermartingale deflator through the positive martingale $D_t = (\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})\,|_{\mathcal{F}_t}$; we have then have $\mathfrak{Q}_{\mathfrak{C}} \subseteq \mathfrak{D}_{\mathfrak{C}}$, and thus $\mathfrak{Q}_{\mathfrak{C}} \neq \emptyset \Rightarrow \mathfrak{D}_{\mathfrak{C}} \neq \emptyset$. In general, the elements of $\mathfrak{D}_{\mathfrak{C}}$ are just supermartingales, not martingales, and the inclusion $\mathfrak{Q}_{\mathfrak{C}} \subseteq \mathfrak{D}_{\mathfrak{C}}$ is strict; more importantly, the implication $\mathfrak{D}_{\mathfrak{C}} \neq \emptyset \Rightarrow \mathfrak{Q}_{\mathfrak{C}} \neq \emptyset$ does *not* hold, as we now show.

Example 3.10. Consider the the 3-dimensional Bessel process Example 3.6 on the finite time-horizon $[0, 1]$. Since $\rho = 1$ is the numéraire portfolio, $D^* = 1/S$ is a tradeable supermartingale deflator, so $\mathfrak{D}_{\mathfrak{C}} \neq \emptyset$. As we have already seen, NA fails, thus we must have $\mathfrak{Q}_{\mathfrak{C}} = \emptyset$.

The set $\mathfrak{D}_{\mathfrak{C}}$ of equivalent supermartingale deflators appears as the range of optimization in the "dual" of the utility maximization problem (3.1) in Kramkov and Schachermayer [26]. As we shall see soon, it is the condition $\mathfrak{D}_{\mathfrak{C}} \neq \emptyset$, rather than $\mathfrak{Q}_{\mathfrak{C}} \neq \emptyset$, that is needed in order to solve (3.1).

The existence of an equivalent supermartingale deflator has some consequences for the class of admissible wealth processes

Proposition 3.11. *If $\mathfrak{D}_{\mathfrak{C}} \neq \emptyset$, then for all $\pi \in \Pi_{\mathfrak{C}}$ the wealth process W^π is a semimartingale up to time T. In particular, $\lim_{t\to\infty} W^\pi_t$ exists on $\{T = \infty\}$.*

Proof. Pick $D \in \mathfrak{D}_{\mathfrak{C}}$ and $\pi \in \Pi_{\mathfrak{C}}$. Since DW^{π} is a positive supermartingale, Lemma B.2 from the Appendix gives that DW^{π} is a semimartingale up to T. Now, D is itself a positive supermartingale with $D_T > 0$ and again Lemma B.2 gives that $1/D$ is a semimartingale up to T. We conclude that $W^{\pi} = (1/D)DW^{\pi}$ is a semimartingale up to T. □

In order to keep the discussion complete, let us mention that if a tradeable supermartingale deflator D^* exists, the supermartingale property of $DW^{\rho} \equiv D/D^*$ for all $D \in \mathfrak{D}_{\mathfrak{C}}$ easily gives that the tradeable supermartingale deflator satisfies

$$\mathbb{E}[-\log D_T^*] = \inf_{D \in \mathfrak{D}_{\mathfrak{C}}} \mathbb{E}[-\log D_T].$$

This result can be seen as an optimality property of the tradeable supermartingale deflator, dual to the log-optimality of the numéraire portfolio in subsection 2.7. Also, we can consider it as a *minimal reverse relative entropy* property of D^* in the class $\mathfrak{D}_{\mathfrak{C}}$. Let us explain: in case an element $D \in \mathfrak{D}_{\mathfrak{C}}$ is actually a probability measure $\mathbb{Q}$, i.e., $\mathrm{d}\mathbb{Q} = D_T \mathrm{d}\mathbb{P}$, then the quantity $H(\mathbb{P} \mid \mathbb{Q}) := \mathbb{E}^{\mathbb{Q}}[D_T^{-1} \log \left(D_T^{-1}\right)] = \mathbb{E}[-\log D_T]$ is the relative entropy of $\mathbb{P}$ with respect to $\mathbb{Q}$. Thus in general, even when D_T is not a probability density, we could regard $\mathbb{E}[-\log D_T]$ as the relative entropy of $\mathbb{P}$ with respect to D. The qualifier "reverse" comes from the fact that one usually considers minimizing the entropy of *another* equivalent probability measure $\mathbb{Q}$ with respect to the *original* $\mathbb{P}$ called *minimal entropy measure*). We refer the reader to Example 7.1 of Karatzas and Kou [21] for further discussion.

3.5. **The second main result.** Here is our second main result, which places the numéraire portfolio in the context of arbitrage.

Theorem 3.12. *For a financial model described by the stock-price process S and the predictable closed convex constraints $\mathfrak{C}$, the following are equivalent:*

(1) *The numéraire portfolio exists and $W_T^{\rho} < \infty$.*
(2) *The set $\mathfrak{D}_{\mathfrak{C}}$ of equivalent supermartingale deflators is non-empty.*
(3) *The* NUPBR *condition holds.*

The proofs of (1) ⇒ (2) and (2) ⇒ (3) are very easy; we do them right now.

The implication (1) ⇒ (2) is trivial: $(W^{\rho})^{-1}$ is an element of $\mathfrak{D}_{\mathfrak{C}}$ (observe that we need $W_T^{\rho} < \infty$ to get $(W_T^{\rho})^{-1} > 0$ as required in the definition of $\mathfrak{D}_{\mathfrak{C}}$).

For the implication (2) ⇒ (3), start by assuming that $\mathfrak{D}_{\mathfrak{C}} \neq \emptyset$ and pick $D \in \mathfrak{D}_{\mathfrak{C}}$. We wish to show that the collection of terminal values of positive wealth processes that start with unit capital is bounded in probability. Since $D_T > 0$, this is equivalent to showing that the collection $\{D_T W_T^{\pi} \mid \pi \in \Pi_{\mathfrak{C}}\}$ is bounded in probability. But this is obvious, since every process DW^{π} for $\pi \in \Pi_{\mathfrak{C}}$ is a positive supermartingale and so, for all $a > 0$,

$$\mathbb{P}[D_T W_T^{\pi} > a] \leq \frac{\mathbb{E}[D_T W_T^{\pi}]}{a} \leq \frac{\mathbb{E}[D_0 W_0^{\pi}]}{a} = \frac{1}{a}\ ;$$

this last estimate does not depend on $\pi \in \Pi_{\mathfrak{C}}$, and we are done.

It remains to prove the implication (3) ⇒ (1). This is considerably harder; one will have to analyze what happens when the numéraire portfolio fails to exist. We do this in the next subsection, but the major result that we use will be proved much later, in section 7.

Theorem 3.12 gives a satisfactory answer to the first question we posed in subsection 3.3 regarding the equivalent of the Fundamental Theorem of Asset Pricing when we only have convex, but not necessarily conic, constraints. Further, since the existence of a numéraire portfolio ρ with $W_T^\rho < \infty$ is equivalent to $\Psi_T(B, C, \eta) < \infty$ according to Remark 2.17, we have a partial answer to our second question from subsection 3.3 regarding predictability of free lunches; the full answer will be given in the next subsection 3.6. Finally, the third question on utility maximization will be tackled in subsection 3.7.

Remark 3.13. Conditions (2) and (3) of Theorem 3.12 remain invariant by an equivalent change of probability measure. Thus, the existence of the numéraire portfolio remains unaffected also, although the numéraire portfolio itself of course will change. Although this would have been a pretty reasonable conjecture to have been made from the outset, it does not follow directly from the definition of the numéraire portfolio by any trivial considerations.

Note that the discussion of the previous paragraph does not remain valid if we only consider *absolutely continuous* changes of measure (unless the price process is continuous). Even though one would rush to say that NUPBR would hold, let us remark that non-equivalent changes of measure might change the structure of admissible wealth processes, since now it will be easier for wealth processes to satisfy the positivity condition (in effect, the natural constraints set $\mathfrak{C}_0$ can be larger). Consider, for example, a finite time-horizon case where, under $\mathbb{P}$, X is a driftless compound Poisson process with $\nu(\{-1/2\}) = \nu(\{1/2\}) > 0$. It is obvious that $\mathfrak{C}_0 = [-2, 2]$ and X itself is a martingale. Now, consider the simple absolutely continuous change of measure that transforms the jump measure to $\nu_1(\mathrm{d}x) := \mathbb{I}_{\{x>0\}}\nu(\mathrm{d}x)$; then, $\mathfrak{C}_0 = (-2, \infty]$ and of course NUIP fails.

Remark 3.14. Theorem 3.12 together with Proposition 3.11 imply that under $\mathrm{NUPBR}_{\mathfrak{C}}$ all wealth processes W^π for $\pi \in \Pi_{\mathfrak{C}}$ are semimartingales up to infinity. Thus, under $\mathrm{NUPBR}_{\mathfrak{C}}$ the assumption about existence of $\lim_{t\to\infty} W_t^\pi$ on $\{T = \infty\}$ needed for the NA and the NFLVR conditions in Definition 3.1 is superfluous.

3.6. **Consequences of non-existence of the numéraire portfolio.** In order to finish the proof of Theorem 3.12, we need to describe *what goes wrong if the numéraire portfolio fails to exist*. This can happen in two ways. First, the set $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ may not have zero $\mathbb{P} \otimes G$-measure; in this case, Proposition 2.10 shows that one can construct an unbounded increasing profit, the most egregious form of arbitrage. Secondly, in case the $\mathbb{P} \otimes G$-measure of $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ is zero, the constructed predictable process ρ can fail to be X-integrable (up to time T). The next definition prepares the ground for the statement of Proposition 3.16, which describes what happens in this latter case.

Definition 3.15. Consider a sequence $(f_n)_{n\in\mathbb{N}}$ of random variables. Its *superior limit in the probability sense*, $\mathbb{P}\text{-}\limsup_{n\to\infty} f_n$, is defined as the essential infimum of the collection $\{g \in \mathcal{F} \mid \lim_{n\to\infty} \mathbb{P}[f_n \leq g] = 1\}$.

It is obvious that the sequence $(f_n)_{n\in\mathbb{N}}$ of random variables is unbounded in probability if and only if $\mathbb{P}\text{-}\limsup_{n\to\infty} |f_n| = +\infty$ with positive probability.

Of course, the $\mathbb{P}$-lim inf can be defined analogously, and the reader can check that $(f_n)_{n\in\mathbb{N}}$ converges in probability if and only if its $\mathbb{P}$-lim inf and $\mathbb{P}$-lim sup coincide, but these last facts will not be used below.

Proposition 3.16. *Assume that the predictable set $\{\mathfrak{I}\cap\check{\mathfrak{C}}\neq\emptyset\}$ has zero $\mathbb{P}\otimes G$-measure, and let ρ be the predictable process constructed as in Theorem 2.15. Pick any sequence $(\theta_n)_{n\in\mathbb{N}}$ of $[0,1]$-valued predictable processes with $\lim_{n\to\infty}\theta_n=\mathbb{I}$ holding $\mathbb{P}\otimes G$-almost everywhere, and such that $\rho_n:=\theta_n\rho$ has bounded support and is X-integrable for all $n\in\mathbb{N}$. Then, $\overline{W}^{\rho}_T:=\mathbb{P}\text{-}\limsup_{n\to\infty}W^{\rho_n}_T$ is a $(0,+\infty]$-valued random variable, and does not depend on the choice of the sequence $(\theta_n)_{n\in\mathbb{N}}$. On the event $\{(\psi^\rho\cdot G)_T<+\infty\}$ the random variable $\overline{W}^{\rho}_T$ is an actual limit in probability, and we have*

$$\{\overline{W}^{\rho}_T=+\infty\}=\{(\psi^\rho\cdot G)_T=+\infty\}\,;$$

in particular, $\mathbb{P}[\,\overline{W}^{\rho}_T=+\infty\,]>0$ if and only if ρ fails to be X-integrable up to T.

Observe that there are many ways to choose the sequence $(\theta_n)_{n\in\mathbb{N}}$. A particular example is $\theta_n:=\mathbb{I}_{\Sigma_n}$ with $\Sigma_n:=\{(\omega,t)\in[\![0,T\wedge n]\!]\mid|\rho(\omega,t)|\le n\}$.

The proof of Proposition 3.16 is the content of section 7. The above result says, in effect, that *closely following the numéraire portfolio, when it is not X-integrable up to time T, one can make arbitrarily large gains with fixed, positive probability.*

Remark 3.17. In the context of the statement of Proposition 3.16 we suppose that $\{\mathfrak{I}\cap\check{\mathfrak{C}}\neq\emptyset\}$ has zero $\mathbb{P}\otimes G$-measure. The failure of ρ to be X-integrable up to time T can happen in two distinct ways. Define the stopping time $\tau:=\inf\{t\in[0,T]\mid(\psi^\rho\cdot G)_t=+\infty\}$ and in a similar fashion define a whole sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times by $\tau_n:=\inf\{t\in[0,T]\mid(\psi^\rho\cdot G)_t\ge n\}$. We consider two cases.

First, let us suppose that $\tau>0$ and $(\psi^\rho\cdot G)_\tau=+\infty$; then $\tau_n<\tau$ for all $n\in\mathbb{N}$ and $\tau_n\uparrow\tau$. By using the sequence $\rho_n:=\rho\mathbb{I}_{[\![0,\tau_n]\!]}$ it is easy to see that $\lim_{n\to\infty}W^{\rho_n}_\tau=+\infty$ almost surely — this is a consequence of the supermartingale property of $\{(W^\rho_t)^{-1},\,0\le t<\tau\}$. An example of a situation when this happens in finite time (say, in $[0,1]$) is when the returns process X satisfies the dynamics $\mathrm{d}X_t=(1-t)^{-1/2}\mathrm{d}t+\mathrm{d}\beta_t$, where β is a standard 1-dimensional Brownian motion. Then $\rho_t=(1-t)^{-1/2}$ and thus $(\psi^\rho\cdot G)_t=\int_0^t(1-u)^{-1}\mathrm{d}u$, which gives $\tau\equiv1$.

Nevertheless, this is not the end of the story. With the notation set-up above we will give an example with $(\psi^\rho\cdot G)_\tau<+\infty$. Actually, we shall only time-reverse the example we gave before and show that in this case $\tau\equiv0$. To wit, take the stock-returns process now to be $\mathrm{d}X_t=t^{-1/2}\mathrm{d}t+\mathrm{d}\beta_t$; then $\rho_t=t^{-1/2}$ and $(\psi^\rho\cdot G)_t=\int_0^t u^{-1}\mathrm{d}u=+\infty$ for all $t>0$ so that $\tau=0$. In this case we cannot invest in ρ as before in a "forward" manner, because it has a "singularity" at $t=0$ and we cannot take full advantage of it. This is basically what makes the proof of Proposition 3.16 non-trivial.

Let us remark further that in the case of a continuous-path semimartingale X without portfolio constraints (as the one described in this example), Delbaen & Schachermayer [9] and Levental & Skorohod [28] show that one can actually create "instant arbitrage": this is a non-constant wealth process that never falls below its initial capital (almost the definition of an increasing unbounded profit, but weaker, since the wealth process is not assumed to be increasing). For the case of jumps it is an open question whether one can still construct this instant arbitrage — we could not. $\diamond$

Proof of Theorem 3.12: Assuming Proposition 3.16, we are now in a position to show the implication (3) $\Rightarrow$ (1) of Theorem 3.12, completing its proof. Suppose then that the numéraire portfolio fails to exist. Then, we either we have opportunities for unbounded increasing profit, in which case NUPBR certainly fails; or ρ exists but is not X-integrable up to time T, in which case Proposition 3.16 gives that NUPBR fails again. $\square$

Remark 3.18. Proposition 3.16 gives the final answer to the question regarding the predictability of free lunches raised in subsection 3.3: When $\mathrm{NUPBR}_{\mathfrak{C}}$ fails (equivalently, when the numéraire portfolio fails to exist, or exists but $\mathbb{P}[W^\rho_T = \infty] > 0$), then there is an *algorithmic* way to construct the unbounded profit with bounded risk (UPBR) using knowledge of the triplet of predictable characteristics.

3.7. **Application to Utility Optimization.** Here we tackle the third question that we raised in subsection 3.3. We show that *NUPBR* (we drop the subscript "$\mathfrak{C}$" in this subsection) *is the minimal condition that allows one solve the utility maximization problem (3.1).*

Remark 3.19. The optimization problem (3.1) makes sense only if its value function u is finite. Due to the concavity of U, if $u(w) < +\infty$ for *some* $w > 0$, then $u(w) < +\infty$ for *all* $w > 0$ and u is continuous, concave and increasing. When $u(w) = \infty$ holds for some (equivalently, all) $w > 0$, there are two cases. Either the supremum in (3.1) is not attained, so there is no solution; or, in case there exists a portfolio with infinite expected utility, the concavity of U will imply that there will be infinitely many of them.

We first state and prove the *negative* result: when NUPBR fails, there is no hope in solving (3.1).

Proposition 3.20. *Assume that* NUPBR *fails. Then, for any utility function U, the corresponding utility maximization problem either does not have a solution, or has an infinity of solutions.*
More precisely: If $U(\infty) = +\infty$, then $u(w) = +\infty$ for all $w > 0$, so we either have no solution (when the supremum is not attained) or infinitely many of them (when the supremum is attained); whereas, if $U(\infty) < +\infty$, there is no solution.

Proof. Since NUPBR fails, pick an $\epsilon > 0$ and a sequence $(\pi_n)_{n\in\mathbb{N}}$ of elements of $\Pi_{\mathfrak{C}}$ such that, with $A_n := \{W^{\pi_n}_T \geq n\}$, we have $\mathbb{P}[A_n] \geq \epsilon$, for all $n \in \mathbb{N}$.

If $U(\infty) = +\infty$, then it is obvious that, for all $w > 0$ and $n \in \mathbb{N}$, we have $u(w) \geq \mathbb{E}[U(wW^{\pi_n}_T)] \geq \epsilon U(wn)$; thus $u(w) = +\infty$ and we have the result stated in the proposition in view of Remark 3.19.

Now suppose $U(\infty) < \infty$; then of course $U(w) \leq u(w) \leq U(\infty) < \infty$ for all $w > 0$. Furthermore, u is also concave, thus continuous. Pick any $w > 0$, suppose that $\pi \in \Pi_{\mathfrak{C}}$ is optimal for U with initial capital w, and observe

$$u(w + n^{-1}) \geq \mathbb{E}[U(wW^\pi_T + n^{-1}W^{\pi_n}_T)] \geq \mathbb{E}[U(wW^\pi_T + \mathbb{I}_{A_n})].$$

Pick $M > 0$ large enough so that $\mathbb{P}[wW^\pi_T > M] \leq \epsilon/2$; since U is concave we know that for any $y \in (0, M]$ we have $U(y + 1) - U(y) \geq U(M + 1) - U(M)$.

Set $a := (U(M+1) - U(M))\epsilon/2$ - this is a strictly positive because U is strictly increasing. Then, $\mathbb{E}[U(wW_T^\pi + \mathbb{I}_{A_n})] \geq \mathbb{E}[U(wW_T^\pi) + a] = u(w) + a$; this implies $u(w + n^{-1}) \geq u(w) + a$ for all $n \in \mathbb{N}$, and contradicts the continuity of u. $\square$

Having resolved the situation when NUPBR fails, let us now assume that it holds. We shall have to assume a little more structure on the utility functions we consider, so let us suppose that they are continuously differentiable and satisfy the *Inada conditions* $U'(0) = +\infty$ and $U'(+\infty) = 0$.

The NUPBR condition is equivalent to the existence of a numéraire portfolio ρ. Since all wealth processes become supermartingales when divided by W^ρ, we conclude that the change of numéraire that utilizes W^ρ as a benchmark produces a market for which the *original* $\mathbb{P}$ is a supermartingale measure (see Delbaen and Schachermayer [10] for this "change of numéraire" technique). In particular, NFLVR holds and the "optional decomposition under convex constraints" results of the Föllmer and Kramkov [15] allow us to write down the *superhedging duality*, valid for any positive, $\mathcal{F}_T$-measurable random variable H:

$$\inf\{w > 0 \mid \exists\, \pi \in \Pi_{\mathfrak{C}} \text{ with } wW_T^\pi \geq H\} = \sup_{D \in \mathfrak{D}_{\mathfrak{C}}} \mathbb{E}[D_T H].$$

This "bipolar" relationship allows one to show that the utility optimization problems admit a solution (when their value is finite). We send the reader to the papers [26, 27] of Kramkov and Schachermayer for more information and detail.

3.8. **A word on the additive model.** All the results stated in this and the previous section 2 hold also in the case where the stock-price processes S^i are *not necessarily positive* semimartingales. Indeed, suppose that we start with initial prices S_0, introduce $Y := S - S_0$, and define the admissible (discounted) wealth processes class to be generated by strategies $\theta \in \mathcal{P}(\mathbb{R}^d)$ via $W = 1 + \theta \cdot S = 1 + \theta \cdot Y$, where we assume that $W > 0$ and $W_- > 0$. Here, θ is the *number* of shares of stocks we keep in our portfolio. But then, with $\pi := (1/W_-)\theta$ it follows that we can write $W = \mathcal{E}(\pi \cdot Y)$. Of course, we do not necessarily have $\Delta Y > -1$ anymore in general, but this fact was never used anywhere; the important thing is that admissibility implies $\pi^\top Y > -1$. Observe that now π does not have a nice interpretation as it had in the case of the multiplicative model.

A final note on constraints. One way to enforce them here is to require $\theta \in W_- \mathfrak{C}$, which is completely equivalent to $\pi \in \mathfrak{C}$, and we can continue as before. Nevertheless, a more natural way would be to enforce them on the proportion of investment, i.e., to require $(\theta^i S_-^i / W_-)_{1 \leq i \leq d} \in \mathfrak{C}$, in which case we get under the additive model that $\pi \in \widehat{\mathfrak{C}}$, where $\widehat{\mathfrak{C}} := \{x \in \mathbb{R}^d \mid (x^i S_-^i)_{1 \leq i \leq d} \in \mathfrak{C}\}$ is still predictable if $\mathfrak{C}$ is and we can again proceed as before.

4. Proof of Proposition 2.10 on the NUIP Condition

4.1. **If $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ is $\mathbb{P} \otimes G$-null, then NUIP holds.** Let us suppose that π is a portfolio with unbounded increasing profit; we shall show that $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ is not $\mathbb{P} \otimes G$-null. By definition then $\{\pi \in \check{\mathfrak{C}}\}$ has full $\mathbb{P} \otimes G$-measure, so we wish to prove that $\{\pi \in \mathfrak{I}\}$ has strictly positive $\mathbb{P} \otimes G$-measure.

Now W^{π} has to be a non-decreasing process, which means that the same holds for $\pi \cdot X$. We also have $\pi \cdot X \neq 0$ with positive probability. This means that the predictable set $\{\pi \notin \mathfrak{N}\}$ has strictly positive $\mathbb{P} \otimes G$-measure, and it will suffice to show that properties (1)–(3) of Definition 2.9 hold $\mathbb{P} \otimes G$-a.e.

Because $\pi \cdot X$ is increasing, we get $\mathbb{I}_{\{\pi^\top x < 0\}} * \mu = 0$, so that $\nu[\pi^\top x < 0] = 0$, $\mathbb{P} \otimes G$-a.e. In particular, $\pi \cdot X$ is of finite variation, so we must have $\pi \cdot X^{\mathsf{c}} = 0$, and this translates into $\pi^\top c = 0$, $\mathbb{P} \otimes G$-a.e. For the same reason, one can decompose

$$\pi \cdot X \;=\; \Big(\pi \cdot B - [\pi^\top x \mathbb{I}_{\{|x| \leq 1\}}] * \eta\Big) + [\pi^\top x] * \mu \,. \tag{4.1}$$

The last term $[\pi^\top x] * \mu$ in this decomposition is a pure-jump increasing process, while for the sum of the terms in parentheses we have from (1.4):

$$\Delta\Big(\pi \cdot B - [\pi^\top x \mathbb{I}_{\{|x| \leq 1\}}] * \eta\Big) = \Big(\pi^\top b - \int \pi^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(\mathrm{d}x)\Big) \Delta G = 0 \,.$$

It follows that the term in parentheses on the right-hand side of equation (4.1) is the continuous part of $\pi \cdot X$ (when seen as a finite variation process) and thus has to be increasing. This translates into the requirement $\pi^\top b - \int \pi^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(\mathrm{d}x) \geq 0$, $\mathbb{P} \otimes G$-a.e., and ends the proof.

4.2. **The set-valued process $\mathfrak{I}$ is predictable.** In order to prove the other half of Proposition 2.10, we need to select a predictable process from the set $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$. For this, we shall have to prove that $\mathfrak{I}$ is a predictable set-valued process, and then to make this selection. However, $\mathfrak{I}$ is not closed, and it is usually helpful to work with closed sets when trying to apply measurable selection results. For this reason we have to go through some technicalities first.

Given a triplet (b, c, ν) of predictable characteristics and $a > 0$, define $\mathfrak{I}^a$ to be the set of vectors of $\mathbb{R}^d$ such that (1)–(3) of Definition 2.9 hold, and where we also require that

$$\xi^\top b + \int \frac{\xi^\top x}{1 + \xi^\top x} \mathbb{I}_{\{|x| > 1\}} \nu(\mathrm{d}x) \;\geq\; \frac{1}{a}. \tag{4.2}$$

The following lemma sets forth properties of these sets that we shall find useful.

Lemma 4.1. *With the previous definition we have:*

1. *The sets $\mathfrak{I}^a$ are increasing in $a > 0$; we have $\mathfrak{I}^a \subseteq \mathfrak{I}$ and $\mathfrak{I} = \bigcup_{a>0} \mathfrak{I}^a$. In particular, $\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$ if and only if $\mathfrak{I}^a \cap \check{\mathfrak{C}} \neq \emptyset$ for all large enough $a > 0$.*
2. *For all $a > 0$, the set $\mathfrak{I}^a$ is closed and convex.*

Proof. Because of conditions (1)–(3) of Definition 2.9, the left-hand-side of (4.2) is well-defined (the integrand is positive since $\nu[\xi^\top x < 0] = 0$) and has to be positive. In fact, for $\xi \in \mathfrak{I}$, it has to be *strictly* positive, otherwise we would have $\xi \in \mathfrak{N}$. The fact that $\mathfrak{I}^a$ is increasing for $a > 0$ is trivial, and part (1) of this lemma follows immediately.

For the second part, we show first that $\mathfrak{I}^a$ is closed. It is obvious that the subset of $\mathbb{R}^d$ consisting of vectors ξ such that $\xi^\top c = 0$ and $\nu[\xi^\top x < 0] = 0$ is closed. On this last set, the functions $\xi^\top x$ are ν-positive. For a sequence $(\xi_n)_{n \in \mathbb{N}}$ in $\mathfrak{I}^a$ with $\lim_{n \to \infty} \xi_n = \xi$, Fatou's lemma gives

$$\int \xi^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(\mathrm{d}x) \leq \liminf_{n \to \infty} \int \xi_n^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(\mathrm{d}x) \leq \liminf_{n \to \infty} \Big(\xi_n^\top b\Big) \;=\; \xi^\top b \,,$$

so that ξ satisfies (3) of Definition 2.9 also. The measure $\mathbb{I}_{\{|x|>1\}}\nu(\mathrm{d}x)$ (the "large jumps" part of the Lévy measure ν) is finite, and bounded convergence gives

$$\xi^\top b + \int \frac{\xi^\top x}{1+\xi^\top x}\mathbb{I}_{\{|x|\geq 1\}}\nu(\mathrm{d}x) = \lim_{n\to\infty}\left\{\xi_n^\top b + \int \frac{\xi_n^\top x}{1+\xi_n^\top x}\mathbb{I}_{\{|x|\geq 1\}}\nu(\mathrm{d}x)\right\} \geq a^{-1}.$$

This establishes that $\mathfrak{I}^a$ is closed. Convexity follows from the fact that the function $x \mapsto x/(1+x)$ is concave on $(0,\infty)$. $\square$

For the remainder of this section, we shall use $\mathfrak{I}$ to denote the $\mathbb{R}^d$-set-valued process $\mathfrak{I}\left(b(\omega,t), c(\omega,t), \nu(\omega,t)\right)$; same for $\mathfrak{I}^a$. In view of $\mathfrak{I} = \bigcup_{n\in\mathbb{N}} \mathfrak{I}^n$ and Lemma A.3, in order to prove the predictability of $\mathfrak{I}$ we only have to prove that of $\mathfrak{I}^a$.

To this end, we define the following real-valued functions, with arguments in $(\Omega\times\mathbb{R}_+)\times\mathbb{R}^d$ (suppressing their dependence on $(\omega,t)\in[\![0,T]\!]$):

$$\begin{aligned} z_1(\mathrm{p}) &= \mathrm{p}^\top c, \qquad z_2(\mathrm{p}) = \int \frac{((\mathrm{p}^\top x)^-)^2}{1+((\mathrm{p}^\top x)^-)^2}\nu(\mathrm{d}x),\\ z_3^n(\mathrm{p}) &= \mathrm{p}^\top b - \int \mathrm{p}^\top x \mathbb{I}_{\{n^{-1}<|x|\leq 1\}}\nu(\mathrm{d}x), \text{ for all } n\in\mathbb{N}, \text{ and}\\ z_4(\mathrm{p}) &= \mathrm{p}^\top b + \int \frac{\mathrm{p}^\top x}{1+\mathrm{p}^\top x}\mathbb{I}_{\{|x|>1\}}\nu(\mathrm{d}x). \end{aligned}$$

Observe that all these functions are predictably measurable in $(\omega,t)\in\Omega\times\mathbb{R}_+$ and continuous in p (follows from applications of the dominated convergence theorem).

In a limiting sense, consider formally $z_3(\mathrm{p}) \equiv z_3^\infty(\mathrm{p}) = \mathrm{p}^\top b - \int \mathrm{p}^\top x \mathbb{I}_{\{|x|\leq 1\}}\nu(\mathrm{d}x)$; observe though that this function might not be well-defined, since both the positive and negative parts of the integrand might have infinite ν-integral. Consider also the sequence of set-valued processes

$$\mathfrak{A}_n^a := \left\{\mathrm{p}\in\mathbb{R}^d \mid z_1(\mathrm{p})=0,\ z_2(\mathrm{p})=0,\ z_3^n(\mathrm{p})\geq 0,\ z_4(\mathrm{p})\geq a^{-1}\right\}$$

for $n\in\mathbb{N}$, of which the "infinite" version coincides with $\mathfrak{I}^a$:

$$\mathfrak{I}^a \equiv \mathfrak{A}_\infty^a := \left\{\mathrm{p}\in\mathbb{R}^d \mid z_1(\mathrm{p})=0,\ z_2(\mathrm{p})=0,\ z_3(\mathrm{p})\geq 0,\ z_4(\mathrm{p})\geq a^{-1}\right\}.$$

Because of the requirement $z_2(\mathrm{p})=0$, the function z_3 can be considered well-defined (though not necessarily finite, since it can take the value $-\infty$). In any case, it is seen that for any p with $z_2(\mathrm{p})=0$, we have $\lim_{n\to\infty}\downarrow z_3^n(\mathrm{p}) = z_3(\mathrm{p})$; so the sequence $(\mathfrak{A}_n^a)_{n\in\mathbb{N}}$ is decreasing, and $\lim_{n\to\infty}\downarrow \mathfrak{A}_n^a = \mathfrak{I}^a$. But each $\mathfrak{A}_n^a$ is closed and predictable (refer to Lemmata A.3 and A.4), and thus so is $\mathfrak{I}^a$.

Remark 4.2. Since $\{\mathfrak{I}\cap\check{\mathfrak{C}}\neq\emptyset\} = \bigcup_{n\in\mathbb{N}}\{\mathfrak{I}^n\cap\check{\mathfrak{C}}\neq\emptyset\}$ and the random set-valued processes $\mathfrak{I}^n$ and $\check{\mathfrak{C}}$ are closed and predictable, Appendix A shows that the set $\{\mathfrak{I}\cap\check{\mathfrak{C}}\neq\emptyset\}$ is predictable.

4.3. **NUIP implies that $\{\mathfrak{I}\cap\check{\mathfrak{C}}\neq\emptyset\}$ is $\mathbb{P}\otimes G$-null.** We are now ready to finish the proof of Proposition 2.10. Let us suppose that $\{\mathfrak{I}\cap\check{\mathfrak{C}}\neq\emptyset\}$ is not $\mathbb{P}\otimes G$-null; we shall construct an unbounded increasing profit.

Since $\mathfrak{I} = \bigcup_{n\in\mathbb{N}}(\{\mathrm{p}\in\mathbb{R}^d \mid |\mathrm{p}|\leq n\}\cap\mathfrak{I}^n)$, where $\mathfrak{I}^n$ is the set-valued process of Lemma 3.1, there exists some $n\in\mathbb{N}$ such that the *convex, closed* and *predictable* set-valued process $\mathcal{B}^n := \{\mathrm{p}\in\mathbb{R}^d \mid |\mathrm{p}|\leq n\}\cap\mathfrak{I}^n\cap\check{\mathfrak{C}}$ has $(\mathbb{P}\otimes G)(\{\mathcal{B}^n\neq\emptyset\})>0$. According to Theorem A.5, there exists a predictable process π with $\pi(\omega,t)\in\mathcal{B}^n(\omega,t)$ when $\mathcal{B}^n(\omega,t)\neq\emptyset$, and $\pi(\omega,t)=0$ if $\mathcal{B}^n(\omega,t)=\emptyset$. This π is bounded, so $\pi \in \Pi_{\mathfrak{C}}$. Using the reasoning of subsection 4.1 "in reverse", we get that $\pi \cdot X$ is a non-decreasing process, and the same is then true of W^π. Now, we *must* have $\mathbb{P}[W^\pi_\infty > 1] > 0$, otherwise $\pi \cdot X \equiv 0$, which is impossible since $(\mathbb{P} \otimes G)(\{\pi \notin \mathfrak{N}\}) > 0$ by construction.

5. Proof of the Main Theorem 2.15

5.1. **The Exponential Lévy market case.** We saw in Lemma 2.5 that if the numéraire portfolio ρ exists, it has to satisfy $\mathfrak{rel}(\pi \mid \rho) \le 0$ pointwise, $\mathbb{P} \otimes G$-a.e. In order to find necessary and sufficient conditions for the existence of a (predictable) process ρ that satisfies this inequality, it makes sense first to consider the corresponding static, deterministic problem. Since Lévy processes correspond to constant, deterministic triplets of characteristics with respect to the natural time flow $G(t) = t$, we shall regard in this subsection X as a Lévy process with characteristic triplet (b, c, ν); this means $B_t = bt$, $C_t = ct$ and $\eta(\mathrm{d}t, \mathrm{d}x) = \nu(\mathrm{d}x)\mathrm{d}t$ in the notation of subsection 1.1. We also take $\mathfrak{C}$ to be a closed convex subset of $\mathbb{R}^d$; recall that $\mathfrak{C}$ can be enriched, so as to accommodate the natural constraints $\mathfrak{C}_0 = \{\pi \in \mathbb{R}^d \mid \nu[\pi^\top x < -1] = 0\}$.

The following result in [25] is the deterministic analogue of Theorem 2.15.

Lemma 5.1. *Let (b, c, ν) be a Lévy triplet and $\mathfrak{C}$ a closed convex subset of $\mathbb{R}^d$. Then the following are equivalent:*

(1) $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$.
(2) *There exists a unique vector $\rho \in \mathfrak{C} \cap \mathfrak{N}^\perp$ with $\nu[\rho^\top x \le -1] = 0$ such that $\mathfrak{rel}(\pi \mid \rho) \le 0$ for all $\pi \in \mathfrak{C}$.*

If the Lévy measure ν integrates the logarithm, the vector ρ is characterized as $\rho = \arg\max_{\pi \in \mathfrak{C} \cap \mathfrak{N}^\perp} \mathfrak{g}(\pi)$. In general, ρ is the limit of a series of problems, in which ν is replaced by a sequence of approximating measures.

We have already shown that if (1) fails, then (2) fails as well (Remark 2.11). The implication (1) $\Rightarrow$ (2) is subtler. For the convenience of the reader we sketch the steps of the proof. We first show the sufficiency of the condition $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$ in solving $\mathfrak{rel}(\pi \mid \rho) \le 0$ for the case of a Lévy measure that integrates the log, then extend to the general case.

Thus, let us start by assuming $\int \log(1 + |x|)\mathbb{I}_{\{|x|>1\}}\nu(\mathrm{d}x) < \infty$. We call $\mathfrak{g}_*$ the supremum of the concave growth rate function

$$\mathfrak{g}(\pi) := \pi^\top b - \frac{1}{2}\pi^\top c\pi + \int \left[\log(1 + \pi^\top x) - \pi^\top x \mathbb{I}_{\{|x|\le 1\}}\right] \nu(\mathrm{d}x)$$

over all $\pi \in \mathfrak{C}$ (the assumption that ν integrates the log gives $\mathfrak{g}(\pi) < \infty$ for all $\pi \in \mathfrak{C}$, but we *might* have $\mathfrak{g}(\pi) = -\infty$ on the boundary of $\mathfrak{C}$). Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of vectors in $\mathfrak{C}$ with $\lim_{n\to\infty} \mathfrak{g}(\rho_n) = \mathfrak{g}_*$. For any $\pi \in \mathfrak{C}$ and any $\zeta \in \mathfrak{N}$ we have $\mathfrak{g}(\pi + \zeta) = \mathfrak{g}(\pi)$, so we can choose the sequence ρ_n to be $\mathfrak{N}^\perp$-valued. It can be shown that this sequence is bounded in $\mathbb{R}^d$; otherwise one can find a $\xi \in \mathfrak{I} \cap \check{\mathfrak{C}}$, contradicting our assumption $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$. Without loss of generality, suppose that $(\rho_n)_{n\in\mathbb{N}}$ converges to a point $\rho \in \mathfrak{C}$; otherwise, choose a convergent subsequence. The concavity of $\mathfrak{g}$ implies that $\mathfrak{g}_*$ is a finite number, and it is obvious from continuity that $\mathfrak{g}(\rho) = \mathfrak{g}_*$. Of course ρ satisfies $\nu[\rho^\top x \le -1] = 0$, otherwise $\mathfrak{g}(\rho) = -\infty$. Finally, since ρ is a point of maximum, the directional derivative of $\mathfrak{g}$ at ρ in the direction of $\pi - \rho$ cannot be positive for any $\pi \in \mathfrak{C}$; this directional derivative is exactly $\mathfrak{rel}(\pi \mid \rho)$.

Now we drop the assumption that ν integrates the log, and pick a sequence of approximating triplets (b, c, ν_n) with $\nu_n(\mathrm{d}x) := f_n(x)\nu(\mathrm{d}x)$ for $n \in \mathbb{N}$; all these measures integrate the log. It is easy to see that $\mathfrak{I}(b, c, \nu) \cap \check{\mathfrak{C}} = \emptyset$ is equivalent to $\mathfrak{I}(b, c, \nu_n) \cap \check{\mathfrak{C}} = \emptyset$ for all $n \in \mathbb{N}$; the discussion of the previous paragraph gives then unique vectors $\rho_n \in \mathfrak{C} \cap \mathfrak{N}^\perp$ such that $\mathfrak{rel}_n(\pi \mid \rho_n) \le 0$ for all $\pi \in \mathfrak{C}$, where $\mathfrak{rel}_n$ is associated with the triplet (b, c, ν_n). As before, the constructed sequence $(\rho_n)_{n\in\mathbb{N}}$ can be shown to be bounded, otherwise $\mathfrak{I} \cap \check{\mathfrak{C}} \ne \emptyset$. Then one can assume that $(\rho_n)_{n\in\mathbb{N}}$ converges to a point $\rho \in \mathfrak{C} \cap \mathfrak{N}^\perp$, picking a subsequence if needed; and Fatou's lemma gives $\mathfrak{rel}(\pi \mid \rho) \le 0$ from $\mathfrak{rel}_n(\pi \mid \rho_n) \le 0$, $\forall\ n \in \mathbb{N}$. We have thus shown $\mathfrak{rel}(\pi \mid \rho) \le 0$, $\forall\ \pi \in \mathfrak{C}$ for the limit ρ of a subsequence of $(\rho_n)_{n\in\mathbb{N}}$. Nevertheless, carrying the previous steps we see that *every* subsequence of $(\rho_n)_{n\in\mathbb{N}}$ has a further convergent subsequence, whose limit $\hat{\rho} \in \mathfrak{C} \cap \mathfrak{N}^\perp$ satisfies $\mathfrak{rel}(\pi \mid \hat{\rho}) \le 0$ for all $\pi \in \mathfrak{C}$. It is easy to see that the vector $\rho \in \mathfrak{C} \cap \mathfrak{N}^\perp$ that satisfies $\mathfrak{rel}(\pi \mid \rho) \le 0$ for all $\pi \in \mathfrak{C}$ must be unique; thus $\hat{\rho} = \rho$. We finally conclude that the whole sequence $(\rho_n)_{n\in\mathbb{N}}$ converges to ρ.

5.2. **Integrability of the numéraire portfolio.** We are close to the proof of our main result. We start with a predictable characterization of X-integrability that the predictable process ρ, our candidate for numéraire portfolio, must satisfy. The following general result is proved in Cherny and Shiryaev [7].

Theorem 5.2. *Let X be a d-dimensional semimartingale with characteristic triplet (b, c, ν) with respect to the canonical truncation function and the operational clock G. A process $\rho \in \mathcal{P}(\mathbb{R}^d)$ is X-integrable, if and only if the condition $(|\widehat{\psi}_i^\rho| \cdot G)_t < \infty$, for all $t \in [\![0, T]\!]$ holds for the predictable processes $\widehat{\psi}_1^\rho := \rho^\top c \rho$,*

$$\widehat{\psi}_2^\rho := \int (1 \wedge |\rho^\top x|^2)\nu(\mathrm{d}x), \quad \textit{and} \quad \widehat{\psi}_3^\rho := \rho^\top b + \int \rho^\top x (\mathbb{I}_{\{|x|>1\}} - \mathbb{I}_{\{|\rho^\top x|>1\}})\nu(\mathrm{d}x)\,.$$

The process $\widehat{\psi}_1^\rho$ controls the quadratic variation of the continuous martingale part of $\rho \cdot X$; the process $\widehat{\psi}_2^\rho$ controls the quadratic variation of the "small-jump" purely discontinuous martingale part of $\rho \cdot X$ and the intensity of the "large jumps"; whereas $\widehat{\psi}_3^\rho$ controls the drift term of $\rho \cdot X$ when the large jumps are subtracted (it is actually the drift rate of the bounded-jump part). This theorem is very general. We shall use it to prove Lemma 5.3 below, which provides a necessary and sufficient condition for X-integrability of the candidate for numéraire portfolio.

Lemma 5.3. *Suppose that ρ is a predictable process with $\nu[\rho^\top x \le -1] = 0$ and $\mathfrak{rel}(0 \mid \rho) \le 0$. Then ρ is X-integrable, if and only if the condition $(\psi^\rho \cdot G)_t(\omega) < \infty$, for all $(\omega, t) \in [\![0, T]\!]$, holds for the increasing, predictable process*

$$\psi^\rho := \nu[\rho^\top x > 1] + \left| \rho^\top b + \int \rho^\top x (\mathbb{I}_{\{|x|>1\}} - \mathbb{I}_{\{|\rho^\top x|>1\}})\nu(\mathrm{d}x) \right| .$$

Proof. According to Theorem 5.2, only the sufficiency has to be proved, since the necessity holds trivially (recall $\nu[\rho^\top x \le -1] = 0$). Furthermore, from the same theorem, the sufficiency will be established if we can prove that the predictable processes $\widehat{\psi}^\rho_1$ and $\widehat{\psi}^\rho_2$ are G-integrable (note that $\widehat{\psi}^\rho_3$ is already covered by ψ^ρ_2).

Dropping the "ρ" superscripts, we embark on proving the G-integrability of $\widehat{\psi}_1$ and $\widehat{\psi}_2$, assuming the G-integrability of ψ_1 and ψ_2. The process $\widehat{\psi}_2$ will certainly be G-integrable, if one can show that the positive process

$$\widetilde{\psi}_2 := \int \frac{(\rho^\top x)^2}{1+\rho^\top x}\,\mathbb{I}_{\{|\rho^\top x|\leq 1\}}\nu(\mathrm{d}x) \;+\; \int \frac{\rho^\top x}{1+\rho^\top x}\,\mathbb{I}_{\{\rho^\top x>1\}}\nu(\mathrm{d}x)$$

is G-integrable. Since both $-\mathfrak{rel}(0 \mid \rho)$ and $\widehat{\psi}_1$ are positive processes, we get that $\widehat{\psi}_1$ and $\widehat{\psi}_2$ will certainly be G-integrable, if we can show that $\widehat{\psi}_1 + \widetilde{\psi}_2 - \mathfrak{rel}(0 \mid \rho)$ is G-integrable. But one can compute this last sum to be equal to

$$\rho^\top b + \int \rho^\top x(\mathbb{I}_{\{|x|>1\}} - \mathbb{I}_{\{|\rho^\top x|>\}})\nu(\mathrm{d}x) \;+\; 2\int \frac{\rho^\top x}{1+\rho^\top x}\,\mathbb{I}_{\{\rho^\top x>1\}}\nu(\mathrm{d}x)\,;$$

the sum of the first two terms equals ψ_2, which is G-integrable, and the last (third) term is G-integrable because $\psi_1 = \nu[\rho^\top x > 1]$ is. □

Remark 5.4. In the context of Lemma 5.3, if we wish ρ to be X-integrable up to T and not simply X-integrable we have to impose the condition $\psi^\rho_T < \infty$. Of course, this follows because from the equivalent characterization of X-integrability up to T in Theorem 5.2 as proved in Cherny and Shiryaev [7].

Remark 5.5. Theorem 5.2 should be contrasted with Lemma 5.3, where one does not have to worry about the large negative jumps of $\rho \cdot X$, about the quadratic variation of its continuous martingale part, or about the quadratic variation of its small-jump purely discontinuous parts. This might look surprising, but follows because in Lemma 5.3 we assume $\nu[\rho^\top x \leq -1] = 0$ and $\mathfrak{rel}(0 \mid \rho) \leq 0$: there are not many negative jumps (none above unit magnitude), and the drift dominates the quadratic variation.

5.3. **Proof of Theorem 2.15.** The fact that $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ is predictable has been shown in Remark 4.2. The claim (2) follows directly from Lemmata 2.5 and 5.3.

For the claims (1.i)–(1.iii), suppose that $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$ has zero $\mathbb{P} \otimes G$-measure. We first assume that ν integrates the log, $\mathbb{P} \otimes G$-a.e. We set $\rho = 0$ on $\{\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset\}$. On the set $\{\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset\}$, according to Theorem 5.1, there exists a (uniquely defined) process ρ with $\rho^\top \Delta X > -1$ that satisfies $\mathfrak{rel}(\pi \mid \rho) \leq 0$, and $\mathfrak{g}(\rho) = \max_{\pi \in \mathfrak{C} \cap \mathfrak{N}^\perp} \mathfrak{g}(\pi)$; by Theorem A.5, this process ρ is predictable and we are done.

Next, we drop the assumption that ν integrates the log. By considering an approximating sequence $(\nu_n)_{n\in\mathbb{N}}$ and keeping every ν_n predictable (this is easy to do, if all densities f_n are deterministic), we get a sequence of processes $(\rho_n)_{n\in\mathbb{N}}$ that take values in $\mathfrak{C} \cap \mathfrak{N}^\perp$ and solve the corresponding approximating problems. As was discussed in subsection 5.1, the sequence $(\rho_n)_{n\in\mathbb{N}}$ will converge pointwise to a process ρ; this will be predictable and satisfy $\mathfrak{rel}(\pi \mid \rho) \leq 0\,,\ \forall\, \pi \in \Pi_{\mathfrak{C}}$.

Now that we have our candidate ρ for numéraire portfolio, we only need to check its X-integrability; according to Lemma 5.3 this is covered by the predictable criterion $(\phi^\rho \cdot G)_t < +\infty$ for all $t \in [\![0, T]\!]$. In light of Lemma 2.5, we are done. □

6. On Rates of Convergence to Zero for Positive Supermartingales

Every positive supermartingale converges as time tends to infinity. The following result gives a predictable characterization of whether this limit is zero or not, and estimates the rate of convergence to zero, if this is the case.

Proposition 6.1. *Let Z be a local supermartingale with $\Delta Z > -1$ and Doob-Meyer decomposition $Z = M - A$, where A is an increasing, predictable process. With $\hat{C} := [Z^{\mathsf{c}}, Z^{\mathsf{c}}]$ the quadratic covariation of the continuous local martingale part of Z and $\hat{\eta}$ the predictable compensator of the jump measure $\hat{\mu}$, define the increasing predictable process $H := A + \hat{C}/2 + q(1+x) * \hat{\eta}$, where $q : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is the convex function $q(y) := \big[-\log a + (1 - a^{-1})y\big]\, \mathbb{I}_{[0,a)}(y) + [y - 1 - \log y]\, \mathbb{I}_{[a,+\infty)}(y)$ for some $a \in (0,1)$.*

Consider also the positive supermartingale $Y = \mathcal{E}(Z)$. Then we have,

$$\text{on } \{H_\infty < +\infty\}, \qquad \lim_{t\to\infty} Y_t \in (0, +\infty)\,;$$
$$\text{on } \{H_\infty = +\infty\}, \qquad \limsup_{t\to\infty} \big(H_t^{-1} \log Y_t\big) \le -1.$$

Remark 6.2. This result can be seen as an abstract version of Proposition 2.23; to obtain that Proposition from it, one has to notice that W^π / W^ρ is a positive supermartingale, and to identify the elements $A, \hat{C}$ and $q(1+x) * \hat{\eta}$ of Proposition 6.1 below with $\mathfrak{rel}(\pi \mid \rho) \cdot G$, $(\pi - \rho)^\top c(\pi - \rho) \cdot G$ and $\left(\int q_a \left(\frac{1+\pi^\top x}{1+\rho^\top x}\right) \nu(\mathrm{d}x)\right) \cdot G$.

Remark 6.3. If we further assume $\Delta Z \ge -1 + \delta$ for some $\delta > 0$, then by considering $q(x) = x - \log(1+x)$ in the definition of H we obtain $\lim_{t\to\infty}(H_t^{-1} \log Y_t) = -1$ on the set $\{H_\infty = +\infty\}$; i.e., we get the exact rate of convergence of $\log Y$ to $-\infty$.

Remark 6.4. In the course of the proof, we shall make heavy use of the following fact: for a locally square integrable martingale N with angle-bracket process $\langle N, N \rangle$, then, on the event $\{\langle N, N \rangle_\infty < +\infty\}$ the limit N_∞ exists and is finite, while on the event $\{\langle N, N \rangle_\infty = +\infty\}$ we have $\lim_{t\to\infty} N_t / \langle N, N \rangle_t = 0$. Note also that if $N = v(x) * (\hat{\mu} - \hat{\eta})$, then $\langle N, N \rangle \le v(x)^2 * \hat{\eta}$ (equality holds if and only if N is quasi-left-continuous). Combining this with the previous remarks we get that on $\{(v(x)^2 * \hat{\eta})_\infty < +\infty\}$ the limit N_∞ exists and is finite, while on $\{(v(x)^2 * \hat{\eta})_\infty = +\infty\}$ we have $\lim_{t\to\infty} N_t / (v(x)^2 * \hat{\eta})_t = 0$.

Proof. For the supermartingale $Y = \mathcal{E}(Z)$, the stochastic exponential formula (0.2) gives $\log Y = Z - [Z^{\mathsf{c}}, Z^{\mathsf{c}}]/2 - \sum_{s \le \cdot} [\Delta Z_s - \log(1 + \Delta Z_s)]$, or equivalently

$$\log Y = -A + (M^{\mathsf{c}} - \hat{C}/2) + \big(x * (\hat{\mu} - \hat{\eta}) - [x - \log(1+x)] * \hat{\mu}\big). \tag{6.1}$$

Let us start with the continuous local martingale part. We use Remark 6.4 twice: first, on $\{\hat{C}_\infty < +\infty\}$, M^{c}_∞ exists and is real-valued; secondly, we get that on $\{\hat{C}_\infty = +\infty\}$, $\lim_{t\to\infty}(M^{\mathsf{c}}_t - \hat{C}_t/2)/(\hat{C}_t/2) = -1$.

We now deal with the purely discontinuous local martingale part. Let us first define the two indicator functions $l := \mathbb{I}_{[-1,-1+a)}$ and $r := \mathbb{I}_{[-1+a,+\infty)}$, where l and r stand as mnemonics for $l$eft and $r$ight. Define the two semimartingales

$$\begin{aligned} E &= [l(x) \log(1+x)] * \hat{\mu} - [l(x)x] * \hat{\eta}, \\ F &= [r(x) \log(1+x)] * (\hat{\mu} - \hat{\eta}) + [r(x) q(1+x)] * \hat{\eta}. \end{aligned}$$

and observe that $x * (\hat{\mu} - \hat{\eta}) - [x - \log(1+x)] * \hat{\mu} = E + F$.

We claim that on $\{(q(1+x) * \hat{\eta})_\infty < +\infty\}$, both E_∞ and F_∞ exist and are real-valued. For E, this happens because $([l(x)q(1+x)] * \hat{\eta})_\infty < +\infty$ implies that there will only be a finite number of times when $\Delta Z \in (-1, -1+a]$ so that both terms in the definition of E will have a limit at infinity. Turning to F, the second term in its definition is obviously finite-valued at infinity while for the local martingale term $[r(x)\log(1+x)] * (\hat{\mu} - \hat{\eta})$ we need only observe that it has finite predictable quadratic variation (because of the set inclusion $\{([r(x)q(1+x)] * \hat{\eta})_\infty < +\infty\} \subseteq \{\left([r(x)\log^2(1+x)] * \hat{\eta}\right)_\infty < +\infty\}$) and use Remark 6.4.

Now we turn attention to the event $\{(q(1+x) * \hat{\eta})_\infty = +\infty\}$; there, at least one of $([l(x)q(1+x)] * \hat{\eta})_\infty$ and $([r(x)q(1+x)] * \hat{\eta})_\infty$ must be infinite.

On the event $\{([r(x)q(1+x)] * \hat{\eta})_\infty = \infty\}$, use of the definition of F and Remark 6.4 easily gives that $\lim_{t\to\infty} F_t / ([r(x)q(1+x)] * \hat{\eta})_t = -1$.

Now let us work on the event $\{([l(x)q(1+x)] * \hat{\eta})_\infty = \infty\}$. We know that the inequality $\log y \leq y - 1 - q(y)$ holds for $y > 0$; using this last inequality in the first term in the definition of E we get $E \leq [l(x)(x - q(1+x))] * \hat{\mu} - [l(x)x] * \hat{\eta}$, or further that $E \leq [l(x)(x - q(1+x))] * (\hat{\mu} - \hat{\eta}) - [l(x)q(1+x)] * \hat{\eta}$. From this last inequality and Remark 6.4 we get $\limsup_{t\to\infty} E_t / ([l(x)q(1+x)] * \hat{\eta})_t \leq -1$.

Let us summarize the last paragraphs on the purely discontinuous part. On the event $\{(q(1+x) * \hat{\eta})_\infty < +\infty\}$, the limit $(x * (\hat{\mu} - \hat{\eta}) - [x - \log(1+x)] * \hat{\mu})_\infty$ exists and is finite; on the other hand, on the event $\{(q(1+x) * \hat{\eta})_\infty = +\infty\}$, we have $\limsup_{t\to\infty} \left(x * (\hat{\mu} - \hat{\eta}) - [x - \log(1+x)] * \hat{\mu}\right)_t / \left(q(1+x) * \hat{\eta}\right)_t \leq -1$.

¿From the previous discussion on the continuous and the purely discontinuous local martingale parts of $\log Y$ and the definition of H, the result follows. $\square$

7. Proof of Proposition 3.16

7.1. **The proof.** Start by defining the two events $\Omega_0 := \{(\psi^\rho \cdot G)_T < \infty\}$ and $\Omega_A := \{(\psi^\rho \cdot G)_T = \infty\} = \Omega \setminus \Omega_0$.

First, we show the result for Ω_0. Assume that $\mathbb{P}[\Omega_0] > 0$, and call $\mathbb{P}_0$ the probability measure one gets by conditioning $\mathbb{P}$ on the set Ω_0. The process ρ of course remains predictable when viewed under the new measure; and because we are restricting ourselves on Ω_0, ρ is X-integrable up to T under $\mathbb{P}_0$.

By a simple use of the dominated convergence theorems for Lebesgue and for stochastic integrals, all three sequences of processes $\rho_n \cdot X$, $[\rho_n \cdot X^{\mathsf{c}}, \rho_n \cdot X^{\mathsf{c}}]$ and $\sum_{s \leq \cdot} \left[\rho_n^\top \Delta X_s - \log(1 + \rho_n^\top \Delta X_s)\right]$ converge uniformly (in $t \in [0,T]$) in $\mathbb{P}_0$-measure to three processes, that do not depend on the sequence $(\rho_n)_{n\in\mathbb{N}}$. Then, the stochastic exponential formula (0.2) gives that $W_T^{\rho_n}$ converges in $\mathbb{P}_0$-measure to a random variable, that does not depend on the sequence $(\rho_n)_{n\in\mathbb{N}}$. Since the limit of the sequence $(\mathbb{I}_{\Omega_0} W_T^{\rho_n})_{n\in\mathbb{N}}$ is the same under both the $\mathbb{P}$-measure and the $\mathbb{P}_0$-measure, we conclude that, on Ω_0, the sequence $(W_T^{\rho_n})_{n\in\mathbb{N}}$ converges in $\mathbb{P}$-measure to a real-valued random variable, independently of the choice of the sequence $(\rho_n)_{n\in\mathbb{N}}$.

Now we have to tackle the set Ω_A, which is trickier. We shall have to further use a "helping sequence of portfolios". Suppose $\mathbb{P}[\Omega_A] > 0$, otherwise there is nothing to prove. Under this assumption, there exist a sequence of $[0,1]$-valued predictable processes $(h_n)_{n\in\mathbb{N}}$ such that each $\pi_n := h_n\rho$ is X-integrable up to

T and such that the sequence of terminal values $((\pi_n \cdot X)_T)_{n\in\mathbb{N}}$ is unbounded in probability (readers unfamiliar with this fact should consult, for instance, the book [5]: the result that we mention can be seen as a rather direct consequence of Corollary 3.6.10, page 128 of that book. It is reasonable to believe (but wrong in general, and a little tedious to show in our case) that unboundedness in probability of the terminal values $((\pi_n \cdot X)_T)_{n\in\mathbb{N}}$ implies that the sequence of the terminal values for the *stochastic* exponentials $(W_T^{\pi_n})_{n\in\mathbb{N}}$ is unbounded in probability as well. We shall show this in Lemma 7.1 of the next subsection; for the time being, we accept this as fact. Then $\mathbb{P}[\limsup_{n\to\infty} W_T^{\pi_n} = +\infty] > 0$, where the lim sup is taken in probability and not almost surely (see Definition 3.15).

Let us return to our original sequence of portfolios $(\rho_n)_{n\in\mathbb{N}}$ with $\rho_n = \theta_n \rho$ and show that $\{\limsup_{n\to\infty} W_T^{\pi_n} = +\infty\} \subseteq \{\limsup_{n\to\infty} W_T^{\rho_n} = +\infty\}$. Both of these upper limits, and in fact all the lim sup that will appear until the end of the proof, are supposed to be in $\mathbb{P}$-measure. Since each θ_n is $[0,1]$-valued and $\lim_{n\to\infty} \theta_n = \mathbb{I}$, one can choose an increasing sequence $(k(n))_{n\in\mathbb{N}}$ of natural numbers such that the sequence $\big(W_T^{\theta_{k(n)}\pi_n}\big)_{n\in\mathbb{N}}$ is unbounded in $\mathbb{P}$-measure on the set $\{\limsup_{n\to\infty} W_T^{\pi_n} = +\infty\}$. Now, each process $W^{\theta_{k(n)}\pi_n}/W^{\rho_{k(n)}}$ is a positive supermartingale, since $\mathfrak{rel}(\theta_{k(n)}\pi_n \mid \rho_{k(n)}) = \mathfrak{rel}(\theta_{k(n)} h_n \rho \mid h_n \rho) \le 0$, the last inequality due to the fact that $[0,1] \ni u \mapsto \mathfrak{g}(u\rho)$ is increasing, and so the sequence of random variables $\big(W_T^{\theta_{k(n)}\pi_n}/W_T^{\rho_{k(n)}}\big)_{n\in\mathbb{N}}$ is bounded in probability. From the last two facts follows that the sequence of random variables $(W_T^{\rho_{k(n)}})_{n\in\mathbb{N}}$ is also unbounded in $\mathbb{P}$-measure on $\{\limsup_{n\to\infty} W_T^{\pi_n} = +\infty\}$.

Up to now we have shown that $\mathbb{P}[\limsup_{n\to\infty} W_T^{\rho_n} = +\infty] > 0$ and we also know that $\{\limsup_{n\to\infty} W_T^{\rho_n} = +\infty\} \subseteq \Omega_A$; it remains to show that the last set inclusion is actually an equality (mod $\mathbb{P}$). Set $\Omega_B := \Omega_A \setminus \{\limsup_{n\to\infty} W_T^{\rho_n} = +\infty\}$ and assume that $\mathbb{P}[\Omega_B] > 0$. Working under the conditional measure on Ω_B (denote by $\mathbb{P}_B$), and following the exact same steps we carried out two paragraphs ago, we find predictable processes $(h_n)_{n\in\mathbb{N}}$ such that each $\pi_n := h_n\rho$ is X-integrable up to T under $\mathbb{P}_B$ and such that the sequence of terminal values $((\pi_n \cdot X)_\infty)_{n\in\mathbb{N}}$ is unbounded in $\mathbb{P}_B$-probability; then $\mathbb{P}_B[\limsup_{n\to\infty} W_T^{\rho_n} = +\infty] > 0$, which contradicts the definition of Ω_B and we are done. □

7.2. **Unboundedness for Stochastic Exponentials.** We still owe one thing in the previous proof: at some point we were presented with a sequence of random variables $((\pi_n \cdot X)_T)_{n\in\mathbb{N}}$ that was unbounded in probability, and wanted to show that the sequence $(\mathcal{E}(\pi_n \cdot X)_T)_{n\in\mathbb{N}}$ is unbounded in probability as well. One has to be careful with statements like that because, as shown later in Remark 7.3, the stochastic — unlike the usual — exponential is *not* a monotone operation.

We have to work harder to prove the following lemma and finish the proof of Proposition 3.16. To make the connection, observe that with $R_n := \pi_n \cdot X$ we have that the collection $(R_n)_{n\in\mathbb{N}}$ is such that $\Delta R_n > -1$ and $\mathcal{E}(R_n)^{-1}$ is a positive supermartingale for all $n \in \mathbb{N}$.

Lemma 7.1. *Let $\mathcal{R}$ be a collection of 1-dimensional semimartingales such that $R_0 = 0$, $\Delta R > -1$ and $\mathcal{E}(R)^{-1}$ is a (positive) supermartingale for all $R \in \mathcal{R}$ (in particular, $\mathcal{E}(R)_T$ exists and takes values in $(0,\infty]$). Then, the collection of* *processes $\mathcal{R}$ is unbounded in probability (see remark below) if and only if the collection of positive random variables $\{\mathcal{E}(R)_T \mid R \in \mathcal{R}\}$ is unbounded in probability.*

Remark 7.2. A class $\mathcal{R}$ of semimartingales will be called "unbounded in probability", if the collection of random variables $\{\sup_{t\in[0,T]} |R_t| \mid R \in \mathcal{R}\}$ is unbounded in probability. Similar definitions will apply for (un)boundedness from above and below, taking one-sided suprema. Without further comment, we shall only consider boundedness notions "in probability" through the course of the proof.

Proof. Since $R \geq \log \mathcal{E}(R)$ for all $R \in \mathcal{R}$, one side of the equivalence is trivial, and we only have to prove that if $\mathcal{R}$ is unbounded then $\{\mathcal{E}(R)_T \mid R \in \mathcal{R}\}$ is unbounded. We split the proof of this into four steps.

As a first step, observe that since $\left\{\mathcal{E}(R)^{-1} \mid R \in \mathcal{R}\right\}$ is a collection of positive supermartingales, it follows that it is bounded from above, and we get that $\{\log \mathcal{E}(R) \mid R \in \mathcal{R}\}$ is bounded from below. Since $R \geq \log \mathcal{E}(R)$ for all $R \in \mathcal{R}$ and $\mathcal{R}$ is unbounded, it follows that it *must* be unbounded from above.

Let us now show that the collection of *random variables* $\{\mathcal{E}(R)_T \mid R \in \mathcal{R}\}$ is unbounded if and only if the collection of *semimartingales* $\{\mathcal{E}(R) \mid R \in \mathcal{R}\}$ is unbounded (from above, of course, since they are positive). One direction is trivial: if the semimartingale class is unbounded, the random variable class is unbounded too; we only need show the other direction. Unboundedness of $\{\mathcal{E}(R) \mid R \in \mathcal{R}\}$ means that we can pick an $\epsilon > 0$ so that, for any $n \in \mathbb{N}$, there exists a semimartingale $R^n \in \mathcal{R}$ such that for the stopping times $\tau_n := \inf\{t \in [0,T] \mid \mathcal{E}(R^n)_t \geq n\}$ (as usual, we set $\tau_n = \infty$ where the last set is empty) we have $\mathbb{P}[\tau_n < \infty] \geq \epsilon$. Each $\mathcal{E}(R^n)^{-1}$ is a supermartingale, so

$$\mathbb{P}[\mathcal{E}(R^n)_T^{-1} \leq n^{-1/2}] \geq \mathbb{P}[\mathcal{E}(R^n)_T^{-1} \leq n^{-1/2} \mid \tau_n < \infty]\ \mathbb{P}[\tau_n < \infty] \geq \epsilon(1 - n^{-1/2}),$$

which shows that the sequence $(\mathcal{E}(R^n)_T)_{n\in\mathbb{N}}$ is unbounded and the claim of this paragraph is proved.

We want to prove now that, if $\mathcal{R}$ is unbounded, then $\{\mathcal{E}(R) \mid R \in \mathcal{R}\}$ is unbounded too. Define the class $\mathcal{Z} := \left\{\mathcal{L}\left(\mathcal{E}(R)^{-1}\right) \mid R \in \mathcal{R}\right\}$; we have $Z_0 = 0$, $\Delta Z > -1$ and that Z is a local supermartingale for all $Z \in \mathcal{Z}$.

For our third step, we show that if the collection $\mathcal{Z}$ is bounded from below, then it is also bounded from above. To this end, pick any $\epsilon > 0$. We can find an $M \in \mathbb{R}_+$ such that the stopping times $\tau_Z := \inf\{t \in [0,T] \mid Z_t \leq -M + 1\}$ (again, we set $\tau_Z = \infty$ where the last set is empty) satisfy $\mathbb{P}[\tau_Z < \infty] \leq \epsilon/2$ for all $Z \in \mathcal{Z}$. Since $\Delta Z > -1$, we have $Z_{\tau_Z} \geq -M$ and so each stopped process Z^{τ_Z} is a supermartingale (it is a local supermartingale bounded uniformly from below). Then, with $y_\epsilon := 2M/\epsilon$ we have

$$\mathbb{P}\Big[\sup_{t\in[0,T]} Z_t > y_\epsilon\Big] \leq (\epsilon/2) + \mathbb{P}\Big[\sup_{t\in[0,T]} Z_t^{\tau_Z} > y_\epsilon\Big] \leq (\epsilon/2) + (1 + y_\epsilon/M)^{-1} \leq \epsilon\,,$$

and thus $\mathcal{Z}$ is bounded from above too.

Now we have all the ingredients for the proof. Suppose that $\mathcal{R}$ is unbounded; we discussed that it has to be unbounded from above. Using Lemma 2.4 with $Y \equiv 0$, we get that every $Z \in \mathcal{Z}$ is of the form

$$Z = -R + [R^{\mathsf{c}}, R^{\mathsf{c}}] + \sum_{s\leq\cdot} \frac{|\Delta R_s|^2}{1 + \Delta R_s}. \tag{7.1}$$

When $\mathcal{Z}$ is unbounded from below, things are pretty simple, because $\log \mathcal{E}(Z) \leq Z$ for all $Z \in \mathcal{Z}$ so that $\{\log \mathcal{E}(\mathcal{Z}) \mid Z \in \mathcal{Z}\}$ is unbounded from below and thus $\{\mathcal{E}(R) \mid R \in \mathcal{R}\} = \{\exp(-\log \mathcal{E}(Z)) \mid Z \in \mathcal{Z}\}$ is unbounded from above.

It remains to see what happens if $\mathcal{Z}$ is bounded from below. The third step (two paragraphs ago) of this proof implies that $\mathcal{Z}$ must be bounded from above as well. Then, because of equation (2.1) and the unboundedness from above of $\mathcal{R}$, this would mean that the collection $\{\ [R^{\mathsf{c}}, R^{\mathsf{c}}] + \sum_{s\leq\cdot} \left[|\Delta R_s|^2 / \left(1+\Delta R_s\right)\right] \ \mid R \in \mathcal{R}\ \}$ of increasing processes is also unbounded. Now, for $Z \in \mathcal{Z}$ we have

$$\log \mathcal{E}(Z) = -\log \mathcal{E}(R) = -R + \frac{1}{2}[R^{\mathsf{c}}, R^{\mathsf{c}}] + \sum_{s\leq\cdot} \left[\Delta R_s - \log(1+\Delta R_s)\right]$$

from (7.1) and the stochastic exponential formula, so that

$$Z - \log \mathcal{E}(Z) = \frac{1}{2}[R^{\mathsf{c}}, R^{\mathsf{c}}] + \sum_{s\leq\cdot} \left[\log(1+\Delta R_s) - \frac{\Delta R_s}{1+\Delta R_s}\right].$$

The collection of increasing processes on the right-hand-side of this last equation is unbounded because $\{\ [R^{\mathsf{c}}, R^{\mathsf{c}}] + \sum_{s\leq\cdot} \left[(\Delta R_s)^2 / \left(1+\Delta R_s\right)\right] \ \mid R \in \mathcal{R}\ \}$ is unbounded too, as we discussed. Since we have $\mathcal{Z}$ being bounded, this means that $\{\log \mathcal{E}(\mathcal{Z}) \mid Z \in \mathcal{Z}\}$ is unbounded from below, and we conclude again as before. $\square$

Remark 7.3. Without the assumption that $\left\{\mathcal{E}(R)^{-1} \mid R \in \mathcal{R}\right\}$ consists of supermartingales, this lemma is not longer true. In fact, take $T \equiv +\infty$ and $\mathcal{R}$ to have only one element R with $R_t = at + \beta_t$, where $a \in (0, 1/2)$ and β is a standard 1-dimensional Brownian motion. Then, R is bounded from below and unbounded from above, nevertheless $\log \mathcal{E}(R)_t = (a - 1/2)t + \beta_t$ is bounded from above, and unbounded from below.

Appendix A. Measurable Random Subsets

Throughout this section we shall be working on a measurable space $(\tilde{\Omega}, \mathcal{P})$; although the results are general, for us $\tilde{\Omega}$ will be the base space $\Omega \times \mathbb{R}_+$ and $\mathcal{P}$ the *predictable* σ-algebra. The metric of the Euclidean space $\mathbb{R}^d$, its denoted by "dist" and its generic point by z.

The proofs of the results below will not be given, but can be found (in greater generality) in Chapter 17 of the excellent book by Border and Aliprantis [2]; for shorter proofs of the specific results, see Kardaras [25]. The subject of measurable random subsets and measurable selection is slightly gory in its technicalities, but the statements should be intuitively clear.

A *random subset of* $\mathbb{R}^d$ is just a random variable taking values in $2^{\mathbb{R}^d}$, the powerset (class of all subsets) of $\mathbb{R}^d$. Thus, a random subset of $\mathbb{R}^d$ is a function $\mathfrak{A} : \tilde{\Omega} \mapsto 2^{\mathbb{R}^d}$. A random subset $\mathfrak{A}$ of $\mathbb{R}^d$ will be called *closed* (resp., *convex*) if the set $\mathfrak{A}(\tilde{\omega})$ is closed (resp., convex) for *every* $\tilde{\omega} \in \tilde{\Omega}$.

We have to impose some measurability requirement on processes of this type, so we must place some measurable structure on the space $2^{\mathbb{R}^d}$. We endow it with the smallest σ-algebra that makes the mappings

$$2^{\mathbb{R}^d} \ni A \mapsto \mathrm{dist}(z, A) \in \mathbb{R}_+ \cup \{+\infty\}$$

measurable for all $z \in \mathbb{R}^d$ (by definition, $\mathrm{dist}(z, \emptyset) = +\infty$). The following equivalent formulations are sometimes useful.

Proposition A.1. *The constructed σ-algebra on $2^{\mathbb{R}^d}$ is also the smallest σ-algebra that makes any of the following three classes of functions measurable.*

(1) $2^{\mathbb{R}^d} \in A \mapsto \mathbb{I}_{\{A \cap K \neq \emptyset\}}$, *for every compact* $K \subseteq \mathbb{R}^d$.
(2) $2^{\mathbb{R}^d} \in A \mapsto \mathbb{I}_{\{A \cap F \neq \emptyset\}}$, *for every closed* $F \subseteq \mathbb{R}^d$.
(3) $2^{\mathbb{R}^d} \in A \mapsto \mathbb{I}_{\{A \cap G \neq \emptyset\}}$, *for every open* $G \subseteq \mathbb{R}^d$.

A mapping from $\tilde{\Omega} \times \mathbb{R}^d$ into some other topological space (with its Borel σ-algebra), measurable with respect to the first argument (keeping the second fixed) and continuous with respect to the second (keeping the first fixed), will be called a *Carathéodory function*. From the definition of the σ-algebra on $2^{\mathbb{R}^d}$, the random subset $\mathfrak{A}$ of $\mathbb{R}^d$ is measurable, if and only if the function

$$\tilde{\Omega} \times \mathbb{R}^d \ni (\tilde{\omega}, z) \mapsto \mathrm{dist}(z, \mathfrak{A}(\tilde{\omega})) \in \mathbb{R}_+ \cup \{+\infty\}$$

is Carathéodory (continuity in z is evident from the triangle inequality).

¿From Proposition A.1, a random subset $\mathfrak{A}$ of $\mathbb{R}^d$ is measurable if for any compact $K \subseteq \mathbb{R}^d$, the set $\{\mathfrak{A} \cap K \neq \emptyset\} := \{\tilde{\omega} \in \tilde{\Omega} \mid \mathfrak{A}(\tilde{\omega}) \cap K \neq \emptyset\}$ is $\mathcal{P}$-measurable.

Remark A.2. Suppose that random subset $\mathfrak{A}$ is a *singleton* $\mathfrak{A}(\tilde{\omega}) = \{a(\tilde{\omega})\}$ for some $a : \tilde{\Omega} \mapsto \mathbb{R}^d$. Then, $\mathfrak{A}$ is measurable if and only if $\{a \in K\} \in \mathcal{P}$ for all closed $K \subseteq \mathbb{R}^d$, i.e., if and only if a is $\mathcal{P}$-measurable.

We now deal with unions and intersections of random subsets of $\mathbb{R}^d$.

Lemma A.3. *Suppose that $(\mathfrak{A}_n)_{n \in \mathbb{N}}$ is a sequence of measurable random subsets of $\mathbb{R}^d$. Then, the union $\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$ is also measurable. If, furthermore, each random subset $\mathfrak{A}_n$ is closed, then the intersection $\bigcap_{n \in \mathbb{N}} \mathfrak{A}_n$ is measurable.*

The following lemma gives a way to construct measurable, closed random subsets of $\mathbb{R}^d$. To state it, we shall need a slight generalization of the notion of Carathéodory function. For a measurable closed random subset $\mathfrak{A}$ of $\mathbb{R}^d$, a mapping f of $\tilde{\Omega} \times \mathbb{R}^d$ into another topological space will be called *Carathéodory on* $\mathfrak{A}$, if it is measurable (with respect to the product σ-algebra on $\tilde{\Omega} \times \mathbb{R}^d$), and if $z \mapsto f(\tilde{\omega}, z)$ is continuous, for each $\tilde{\omega} \in \tilde{\Omega}$. Of course, if $\mathfrak{A} \equiv \mathbb{R}^d$, we recover the usual notion of a Carathéodory function.

Lemma A.4. *Let E be any topological space, $F \subseteq E$ a closed subset, and $\mathfrak{A}$ a closed and convex random subset of $\mathbb{R}^d$. If $f : \tilde{\Omega} \times \mathbb{R}^d \to E$ is a Carathéodory function on $\mathfrak{A}$, then $\mathfrak{C} := \{z \in \mathfrak{A} \mid f(\cdot, z) \in F\}$ is closed and measurable.*

The last result focuses on the measurability of the "argument" process in random optimization problems.

Theorem A.5. *Suppose that $\mathfrak{C}$ is a closed and convex, measurable, non-empty random subset of $\mathbb{R}^d$, and $f : \tilde{\Omega} \times \mathbb{R}^d \mapsto \mathbb{R} \cup \{-\infty\}$ is a Carathéodory function on $\mathfrak{C}$. For the optimization problem $f_*(\tilde{\omega}) = \max_{z \in \mathfrak{C}} f(\tilde{\omega}, z)$, we have:*

(1) *The value function f_* is $\mathcal{P}$-measurable.*
(2) *Suppose that $f_*(\tilde{\omega})$ is finite for all $\tilde{\omega}$, and that there exists a unique $z_*(\tilde{\omega}) \in \mathfrak{C}(\tilde{\omega})$ satisfying $f(\tilde{\omega}, z_*(\tilde{\omega})) = f_*(\tilde{\omega})$. Then $\tilde{\omega} \mapsto z_*(\tilde{\omega})$ is $\mathcal{P}$-measurable.*

In particular, if $\mathfrak{C}$ is a closed and convex, measurable, non-empty random subset of $\mathbb{R}^d$, we can find a $\mathcal{P}$-measurable $h : \tilde{\Omega} \to \mathbb{R}^d$ with $h(\tilde{\omega}) \in \mathfrak{C}(\tilde{\omega})$ for all $\tilde{\omega} \in \tilde{\Omega}$.

For the "particular" case of the last theorem one can use for example the function $f(x) = -|x|$ and the result first part of the theorem. In case the maximizer is not unique, one can still measurably select from the set of maximizers. This result is more difficult; in any case we shall not be using it.

Appendix B. Semimartingales and Stochastic Integration up to $+\infty$

We recall here a few important concepts from [7] and prove a few useful results.

Definition B.1. Let $X = (X_t)_{t \in \mathbb{R}_+}$ be a semimartingale such that $X_\infty := \lim_{t \to \infty} X_t$ exists. Then X will be called a *semimartingale up to infinity* if the process $\tilde{X}$ defined on the time interval $[0,1]$ by $\tilde{X}_t = X_{\frac{t}{1-t}}$ (of course, $\tilde{X}_1 = X_\infty$) is a semimartingale relative to the filtration $\tilde{\mathbf{F}} = (\tilde{\mathcal{F}}_t)_{t \in [0,1]}$ defined by

$$\tilde{\mathcal{F}}_t := \begin{cases} \mathcal{F}_{\frac{t}{1-t}}, & \text{for } 0 \le t < 1; \\ \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_t, & \text{for } t = 1. \end{cases}$$

We define similarly local martingales up to infinity, processes of finite variation up to infinity, etc., if the corresponding process $\tilde{X}$ has the property.

Until the end of this subsection, a "tilde" over a process, means that we are considering the process of the previous definition, with the new filtration $\tilde{\mathbf{F}}$.

To appreciate the difference between the concepts of (plain) semimartingale and semimartingale up to infinity, consider the simple example where X is the deterministic, continuous process $X_t := t^{-1} \sin t$; it is obvious that X is a semimartingale and that $X_\infty = 0$, but $\mathrm{Var}(X)_\infty = +\infty$ and thus X cannot be a semimartingale up to infinity (recall that a deterministic semimartingale must be of finite variation).

Every semimartingale up to infinity X can be written as the sum $X = A + M$, where A is a process of finite variation up to infinity (which simply means that $\mathrm{Var}(A)_\infty < \infty$) and M is a local martingale up to infinity (which means that there exists an increasing sequence of stopping times $(T_n)_{n \in \mathbb{N}}$ with $\{T_n = +\infty\} \uparrow \Omega$ such that each of the stopped processes M^{T_n} is a uniformly integrable martingale).

Here are examples of semimartingales up to infinity.

Lemma B.2. *If Z is a positive supermartingale, then it is a special semimartingale up to infinity. If furthermore $Z_\infty > 0$, then $\mathcal{L}(Z)$ is also a special semimartingale up to infinity, and both processes Z^{-1} and $\mathcal{L}(Z^{-1})$ are semimartingales up to infinity.*

Proof. We start with the Doob-Meyer decomposition $Z = M - A$, where M is a local martingale with $M_0 = Z_0$ and A is an increasing, predictable process. Since M is a positive local martingale, it is a supermartingale too, and we can infer that both limits Z_∞ and M_∞ exist and are integrable. This means that A_∞ exists and actually $\mathbb{E}[A_\infty] = \mathbb{E}[M_\infty] - \mathbb{E}[Z_\infty] < \infty$, so A is a predictable process of integrable variation up to infinity. It remains to show that M is a local martingale up to infinity. Set $T_n := \inf\{t \ge 0 \mid M_t \ge n\}$; this obviously satisfies $\{T_n = +\infty\} \uparrow \Omega$ (the supremum of a positive supermartingale is finite).

Since $\sup_{0\le t\le T_n} M_t \le n + M_{T_n}\mathbb{I}_{\{T_n<\infty\}}$ and by the optional sampling theorem $\mathbb{E}[M_{T_n}\mathbb{I}_{\{T_n<\infty\}}] \le \mathbb{E}[M_0] < \infty$, we get $\mathbb{E}[\sup_{0\le t\le T_n} M_t] < \infty$. Thus, the local martingale M^{T_n} is actually a uniformly integrable martingale and thus Z is a special semimartingale up to infinity.

Now assume that $Z_\infty > 0$. Since Z is a supermartingale, this will mean that both $\tilde{Z}$ and $\tilde{Z}_-$ are bounded away from zero. Since $\tilde{Z}_-^{-1}$ is locally bounded and $\tilde{Z}$ is a special semimartingale, $\mathcal{L}(\tilde{Z}) = \tilde{Z}_-^{-1}\cdot\tilde{Z}$ will be a special semimartingale as well, meaning that $\mathcal{L}(Z)$ is a special semimartingale up to infinity. Furthermore, Itô's formula applied to the inverse function $(0,\infty)\ni x\mapsto x^{-1}$ implies that $\tilde{Z}^{-1}$ is a semimartingale up to infinity and since $\tilde{Z}_-$ is locally bounded, $\mathcal{L}(\tilde{Z}^{-1}) = \tilde{Z}_-\cdot\tilde{Z}^{-1}$ is a semimartingale, which finishes the proof. □

Consider a d-dimensional semimartingale X. A predictable process H will be called *X-integrable up to infinity* if it is X-integrable and the semimartingale $H\cdot X$ is a semimartingale up to infinity.

Remark B.3. In the course of the paper we consider "semimartingales up to time T" and "stochastic integration up to time T" where T is a possibly infinite stopping time rather than "semimartingales up to infinity" and "stochastic integration up to infinity". Of course, one can use all the results of this section applying them to the processes stopped at time T — differences between the usual notion of integrability appears only when $\mathbb{P}[T=\infty]>0$.

Appendix C. σ-Localization

A good account of the concept of σ-localization is given in Kallsen [20]. Here we recall briefly what is needed for our purposes. For a semimartingale Z and a predictable set Σ, define $Z^\Sigma := Z_0\mathbb{I}_\Sigma(0) + \mathbb{I}_\Sigma\cdot Z$.

Definition C.1. Let $\mathcal{Z}$ be a class of semimartingales. Then, the corresponding σ-localized class $\mathcal{Z}_\sigma$ is defined as the set of all semimartingales Z for which there exists an increasing sequence $(\Sigma_n)_{n\in\mathbb{N}}$ of predictable sets, such that $\Sigma_n\uparrow\Omega\times\mathbb{R}_+$ (up to evanescence) and $Z^{\Sigma_n}\in\mathcal{Z}$ for all $n\in\mathbb{N}$.

When the corresponding class $\mathcal{Z}$ has a name (like "supermartingales") we baptize the class $\mathcal{Z}_\sigma$ with the same name preceded by "σ-" (like "σ-supermartingales").

The concept of σ-localization is a natural extension of the well-known concept of localization along a sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times, as one can easily see by considering the predictable sets $\Sigma_n \equiv [\![0,\tau_n]\!] := \{(\omega,t)\mid 0\le t\le\tau_n(\omega)<\infty\}$.

Let us define the set $\mathcal{U}$ of semimartingales Z, such that the collection of random variables $\{Z_\tau \mid \tau \text{ is a stopping time}\}$ is uniformly integrable — also known in the literature as semimartingales of class (D). The elements of $\mathcal{U}$ admit the *Doob-Meyer decomposition* $Z = A + M$ into a predictable finite variation part A with $A_0 = 0$ and $\mathbb{E}[\mathrm{Var}(A)_\infty]<\infty$ and a uniformly integrable martingale M. It is then obvious that the localized class $\mathcal{U}_{\mathsf{loc}}$ corresponds to all special semimartingales; they are exactly the ones which admit a Doob-Meyer decomposition as before, but where now A is only a predictable, finite variation process with $A_0=0$ and M a local martingale. Let us remark that the local supermartingales (resp., local submartingales) correspond to these elements of $\mathcal{U}_{\mathsf{loc}}$ with A decreasing (resp., increasing). This last result can be found for example in Jacod's book [17].

One can have very intuitive interpretation of some σ-localized classes in terms of the predictable characteristics of Z.

Proposition C.2. *Consider a scalar semimartingale Z, and let (b, c, ν) be the triplet of predictable characteristics of Z relative to the canonical truncation function and the operational clock G. Then*

(1) *Z belongs to $\mathcal{U}_{\mathsf{loc}}$ if and only if the predictable process $\int |x| \, \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x)$ is G-integrable;*
(2) *Z belongs to $\mathcal{U}_\sigma$ if and only if $\int |x| \, \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x) < \infty$; and*
(3) *Z is a σ-supermartingale, if and only if $\int |x| \, \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x) < +\infty$ and $b + \int x \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x) \leq 0$.*

Proof. The first statement follows from the fact that a 1-dimensional semimartingale Z is a special semimartingale (i.e., a member of $\mathcal{U}_{\mathsf{loc}}$) if and only if $\left[|x| \, \mathbb{I}_{\{|x|>1\}}\right] * \hat{\eta}$ is a finite, increasing predictable process (one can consult Jacod [17] for this fact). The second statement follows easily from the first and σ-localization. Finally, the third follows for the fact that for a process in $\mathcal{U}_{\mathsf{loc}}$ the predictable finite variation part is given by the process $\left(b + \int \left[x \mathbb{I}_{\{|x|>1\}}\right] \nu(\mathrm{d}x)\right) \cdot G$, using the last remark before the proposition, the first part of the proposition, and σ-localization. $\square$

Results like the last proposition are very intuitive, because $b + \int x \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x)$ represents the infinitesimal drift rate of the semimartingale Z; we expect this rate to be negative (resp., positive) in the case of σ-supermartingales (resp., σ-submartingales). The importance of σ-localization is that it allows us to talk directly about drift *rates* of processes, rather than about drifts. Sometimes drift rates exist, but cannot be integrated to give a drift process; this is when the usual localization technique fails, and the concept of σ-localization becomes useful.

The following result gives sufficient conditions for a σ-supermartingale to be a local supermartingale (or even plain supermartingale).

Proposition C.3. *Suppose that Z is a scalar semimartingale with triplet of predictable characteristics (b, c, ν).*

(1) *Suppose that Z is a σ-supermartingale. Then, the following are equivalent:*
    (a) *Z is a local supermartingale.*
    (b) *The positive, predictable process $\int (-x) \mathbb{I}_{\{x<-1\}} \nu(\mathrm{d}x)$ is G-integrable.*
(2) *If Z is a σ-supermartingale (resp., σ-martingale) and bounded from below by a constant, then it is a local supermartingale (resp., local martingale). If furthermore $\mathbb{E}[Z_0^+] < \infty$, it is a supermartingale.*
(3) *If Z is bounded from below by a constant, then it is a supermartingale if and only if $\mathbb{E}[Z_0^+] < \infty$ and $b + \int x \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x) \leq 0$.*

Proof. For the proof of (1), the implication (a) $\Rightarrow$ (b) follows from part (1) of Proposition C.2. For (b) $\Rightarrow$ (a), assume that $\int (-x) \mathbb{I}_{\{x<-1\}} \nu(\mathrm{d}x)$ is G-integrable. Since Z is a σ-supermartingale, it follows from part (3) of Proposition C.2 that $\int x \mathbb{I}_{\{x>1\}} \nu(\mathrm{d}x) \leq -b + \int (-x) \mathbb{I}_{\{x<-1\}} \nu(\mathrm{d}x)$. Now, this last inequality implies that $\int |x| \, \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x) \leq -b + 2 \int (-x) \mathbb{I}_{\{x<-1\}} \nu(\mathrm{d}x)$; the last dominating process is

G-integrable, thus $Z \in \mathcal{U}_{\mathsf{loc}}$ (again, part (1) of Proposition C.2). The special semimartingale Z has predictable finite variation part equal to $\big(b + \int x \mathbb{I}_{\{x>1\}} \nu(\mathrm{d}x)\big) \cdot G$, which is decreasing, so that Z is a local supermartingale.

For part (2), we can of course assume that Z is positive. We discuss the case of a σ-supermartingale; the σ-martingale case follows in the same way. According to part (1) of this proposition, we only need to show that $\int (-x) \mathbb{I}_{\{x<-1\}} \nu(\mathrm{d}x)$ is G-integrable. But since the negative jumps of Z are bounded in magnitude by Z_-, we have that $\int (-x) \mathbb{I}_{\{x<-1\}} \nu(\mathrm{d}x) \leq (Z_-) \nu\,[x < -1]$, which is G-integrable, because $\nu\,[x < -1]$ is G-integrable and Z_- is locally bounded. Now, if we further assume that $\mathbb{E}[Z_0] < \infty$, Fatou's lemma for conditional expectations gives us that the positive local supermartingale Z is a supermartingale.

Let us move on to part (3) and assume that Z is positive. First assume that Z is a supermartingale. Then, of course we have $\mathbb{E}[Z_0] < \infty$ and that Z is an element of $\mathcal{U}_\sigma$ (and even of $\mathcal{U}_{\mathsf{loc}}$) and part (3) of Proposition C.2 ensures that $b + \int x \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x) \leq 0$. Now, assume that Z is a positive semimartingale with $\mathbb{E}[Z_0] < \infty$ and that $b + \int x \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x) \leq 0$. Then, of course we have that $\int x \mathbb{I}_{\{x>1\}} \nu(\mathrm{d}x) < \infty$. Also, since Z is positive we always have that $\nu\,[x < -Z_-] = 0$ so that $\int (-x) \mathbb{I}_{\{x<-1\}} \nu(\mathrm{d}x) < \infty$ too. Part (2) of Proposition C.2 will give us that $Z \in \mathcal{U}_\sigma$, and part (3) of the same proposition that Z is a σ-supermartingale. Finally, part (2) of this proposition gives us that Z is a supermartingale. □

Proposition C.3 has been known for some time and made its first appearance in Ansel and Stricker [3]. The authors did not deal directly with σ-martingales, but with semimartingales Z which are of the form $Z = Z_0 + H \cdot M$, where M is a martingale and H is M-integrable (a *martingale transform*). Of course, martingale transforms are σ-martingales and vice-versa. The corollary of Proposition C.3 when the σ-martingale Z is bounded from below by a constant, is sometimes called "The Ansel-Stricker theorem". The case when Z is a σ-supermartingale bounded from below with $\mathbb{E}[Z_0^+] < \infty$ is proved in [20].

References

[1] P. Algoet, Tom M. Cover (1988). "Asymptotic optimality and asymptotic equipartition property of log-optimal investment", Annals of Probability **16**, pp. 876–898.

[2] Charalambos D. Aliprantis, Kim C. Border (1999). "Infinite Dimensional Analysis: A Hitchhicker's Guide", Second Edition, Springer Verlag.

[3] Jean-Pascal Ansel, Christophe Stricker (1994). "Couverture des actifs contigents et prix maximum", Annales de l' Institute Henri Poincaré **30**, p. 303–315.

[4] Dirk Becherer (2001). "The numéraire portfolio for unbounded semimartingales", Finance and Stochastics 5, p. 327–341.

[5] Klaus Bichteler (2002) . "Stochastic Integration with Jumps", Cambridge University Press.

[6] Alexander S. Cherny, Albert N. Shiryaev (2002). "Vector Stochastic Integrals and the Fundamental Theorems of Asset Pricing", Proceedings of the Steklov Mathematical Institute, 237, p. 12–56.

[7] Alexander S. Cherny, Albert N. Shiryaev (2004). "On Stochastic Integrals up to infinity and predictable criteria for integrability", Lecture Notes in Mathematics **1857**, p. 165–185.

[8] Freddy Delbaen, Walter Schachermayer (1994). "A General Version of the Fundamental Theorem of Asset Pricing", Mathematische Annalen **300**, p. 463–520.

[9] Freddy Delbaen, Walter Schachermayer (1995). "The Existence of Absolutely Continuous Local Martingale Measures", Annals of Applied Probability **5**, n° 4, p. 926–945.

[10] Freddy Delbaen, Walter Schachermayer (1995). "The No-Arbitrage Property under a Change of Numéraire", Stochastics and Stochastics Reports 53, n^{os} 3–4, pp. 213–226.
[11] Freddy Delbaen, Walter Schachermayer (1995). "Arbitrage Possibilities in Bessel Processes and their Relations to Local Martingales", Probability Theory and Related Fields 102, n^o 3, 357–366.
[12] Freddy Delbaen, Walter Schachermayer (1998). "The Fundamental Theorem of Asset Pricing for Unbounded Stochastic Processes", Mathematische Annalen **312**, n^o 2, p. 215–260.
[13] Robert Fernholz, Ioannis Karatzas (2005). "Relative arbitrage in volatility-stabilized markets", Annals of Finance n^o 1, pp. 149–177.
[14] Robert Fernholz, Ioannis Karatzas, Constantinos Kardaras (2005). "Diversity and Relative Arbitrage in Equity Markets", Finance and Stochastics, n^o 9, pp. 1–27.
[15] Hans Föllmer, Dmitry Kramkov (1997). "Optional decompositions under constraints", Probability Theory and Related Fields, Vol 109, pp. 1–25.
[16] Thomas Goll, Jan Kallsen (2003)."A Complete Explicit Solution to the Log-Optimal Portfolio Problem", The Annals of Applied Probability **13**, p. 774–799.
[17] Jean Jacod (1979). "Calcul Stochastique et Problèmes de Martingales". Lecture Notes in Mathematics **714**, Springer, Berlin.
[18] Jean Jacod, Albert N. Shiryaev (2003). "Limit Theorems for Stochastic Processes", Second Edition. Springer-Verlag.
[19] Yuri M. Kabanov (1997). "On the FTAP of Kreps-Delbaen-Schachermayer", Statisics and Control of Random Processes. The Liptser Festschrift. Proceedings of Steklov Mathematical Institute Seminar. World Scientific, p. 191–203.
[20] Jan Kallsen (2004). "σ-Localization and σ-Martingales", Theory of Probability and Its Applications **48**, n^o 1, pp. 152–163.
[21] Ioannis Karatzas, Steve G. Kou (1996). "On the pricing of contingent claims under constraints". Annals of Applied Probability, Vol. 6, n^o 2, 321–369.
[22] Ioannis Karatzas, John P. Lehoczky, Steven E. Shreve (1991). "Equilibrium Models with Singular Asset Prices". Mathematical Finance **1**, n^o 3, 11–29.
[23] Ioannis Karatzas, Steven E. Shreve (1998). "Methods of Mathematical Finance", Springer-Verlag.
[24] Constantinos Kardaras (2006). "The numéraire portfolio and arbitrage in semimartingale models of financial markets". Ph.D. Dissertation, Columbia University
[25] Constantinos Kardaras (2006). "No Free Lunch equivalences for exponential Lévy models of financial markets", preprint, Boston University.
[26] Dmitry Kramkov, Walter Schachermayer (1999). "The Asymptotic Elasticity of Utility functions and Optimal Investment in Incomplete Markets", The Annals of Applied Probability **9**, n^o 9, p. 904–950.
[27] Dmitry Kramkov, Walter Schachermayer (2003). "Necessary and sufficient conditions in the problem of optimal investment in incomplete markets", The Annals of Applied Probability, Vol 13, n^o 4, p. 1504–1516.
[28] Shlomo Levental, Anatolii V. Skorohod (1995). "A Necessary and Sufficient Condition for Absence of Arbitrage with Tame Portfolios", Annals of Applied Probability 5, n^o 4, p. 906–925.
[29] John B. Long, Jr. (1990). "The numéraire portfolio", Journal of Financial Economics **26**, n^o 1, p. 29–69.
[30] Jean Mémin (1980). "Espaces de semimartingales et changement de probabilité", Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 52, n^o 1, p. 9–39.

Mathematics and Statistics Departments, Columbia University, NY 10027
E-mail address: `ik@math.columbia.edu`

Mathematics and Statistics Department, Boston University, MA 02215
E-mail address: `kardaras@bu.edu`