Non-addictive habits: optimal consumption-portfolio policies

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Abstract

We formulate a model of preferences with non-addictive habits, where consumption is required to be non-negative at all times, but is allowed to fall below a "standard of living" index that aggregates past consumption. In this context we study the consumption-portfolio choice problem taking account of the non-negativity constraint on consumption, and provide a constructive proof for the existence of an optimal policy on a finite time-horizon $[0, T]$. In particular, we show that the consumption constraint binds up to an endogenously determined stopping time $\tau^* \in [0, T]$, after which it remains slack until T. A decomposition of constrained consumption involving an Asian average-strike capped call-option is demonstrated.

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1 Introduction.

Habit-formation models have traditionally played an important role in economics (e.g. Hicks (1965), Pollak (1970), Ryder and Heal (1973)). The notion that an individual's evaluation of a given consumption bundle depends on past consumption experience is, indeed, appealing on intuitive grounds. Furthermore, studies in economics as well as psychology suggest that past decisions condition current choices. More recently, habit-formation has played an important role for explaining the behavior of consumption and the properties of the equity premium (Sundaresan (1989), Constantinides (1991), Detemple and Zapatero (1991, 1992), Heaton (1992), Chapman (1998), Schroder and Skiadas (2000)).

To a large extent, the existing literature on this subject has relied on a preference model in which habit-formation has an "addictive" flavor. More specifically, it has been typically assumed that the marginal utility from consumption, net of the *standard of living* (i.e., the benchmark based on past consumption history), is infinite at zero. In essence, this assumption implies that (i) optimal consumption can never fall below the standard of living, and that (ii) initial wealth must be sufficiently large to sustain habits and ensure the existence of an optimal policy. The addictive behavior of consumption is very strong and appears counterintuitive. While there is substantial support for the notion that "habits matter", it is doubtful that individuals adopt lifecycle consumption profiles which display systematic excesses over historical averages, even when suitably depreciated. In any case, the fact that consumption may substantially decrease in recessions seems to contradict this "addiction" property. From a theoretical perspective, infinite marginal utility at zero suggests that the consumption space consists of those consumption processes which exceed their associated standard of living. To avoid convexities in preferences and ensure the existence of an optimal policy, one must then penalize consumption plans that fall below the standard of living to the extreme (infinite disutility is assumed) and constrain initial wealth to exceed a strictly positive amount; see Detemple and Zapatero (1991).

In this paper we adopt a specification which allows *finite marginal utility of consumption at zero*. Thus, our consumption space is the usual positive orthant: we only impose the natural constraint that consumption cannot take negative values. Preferences are defined over this space in the usual manner. Thus, consumption plans that fall below the standard of living are admissible, and initial wealth need not be restricted. At the same time, the notion that consumption history has a detrimental effect on the enjoyment of current consumption, i.e. that "habits matter", is retained, and influences the structure and properties of our solution.

Evidently, in a model with non-addictive habits, we expect optimal consumption to fall below the standard of living when state-prices are high (and vice-versa). For instance, if high state-prices follow a period of high consumption expenditures and standard of living buildup, then it is intuitive that an individual would want to reduce consumption so as to *decumulate* habits. In fact, unconstrained optimal consumption may very well become negative, so as to reduce rapidly the standard of living, and thereby the "cost" of habits as well. This cannot happen in our context, since we constrain consumption to be always non-negative. Thus, the non-negativity constraint may bind with positive probability, and this will influence the optimal consumption policy adopted by an agent.

In standard models with von Neumann-Morgenstern preferences, additively separable utility, and finite marginal utility at zero, a non-negativity constraint has a "local effect" on consumption, in the sense that it increases consumption in the constrained state without affecting nearby states. In fact, as pointed out by Cox and Huang (1991), optimal constrained consumption is the same as unconstrained consumption plus a put-option on unconstrained consumption with zero strike, which pays off precisely in those states that are constrained. The optimal consumption profile may then display random times, scattered throughout the time-horizon of the investor, at which the constraint binds.

One would intuitively conjecture that the same behavior materializes in the presence of habits. By analogy with Karatzas, Lehoczky and Shreve (1987) or Cox and Huang (1991), one would expect to see the non-negativity constraint bind at random, endogenously determined times throughout the investment time-horizon. As we shall show, this reasoning turns out to be incorrect. In the presence of habits it is, in fact, optimal for the individual to forgo consumption completely, until an endogenously determined stopping time, after which the constraint on consumption will always cease to bind. In essence, an individual who develops habits will sit on the constraint until the first time at which the "adjusted" (for habit-formation) state price-density in the constrained problem, hits the adjusted state price-density for the unconstrained problem. After that time, unconstrained nonnegative consumption can be sustained throughout the remaining lifetime, and this is the course that the individual optimally follows. Nevertheless, this behavior induces a generalized Cox-Huang decomposition, in which constrained consumption can be synthesized using a path-dependent derivative, namely an Asian capped call-option on the unconstrained policy.

Section 2 presents the model studied in this paper. In section 3 we provide several characterization of the optimal consumption plan involving a class of stopping times. Existence is proved in section 4. In section 5 we examine the economic implications of our model for consumption behavior and provide further intuition for the optimal consumption pattern. Conclusions are formulated in section 6.

2 The model.

We consider the standard financial market model with one riskless asset and d risky securities (stocks). All the uncertainty in this market is generated by a Brownian Motion $W = (W^1, ..., W_d)'$ on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = \{\mathcal{F}(t);\ 0 \le t \le T\}$ denote the information filtration, i.e., the augmentation of the natural filtration $\mathcal{F}^W(t) = \sigma(W(s);\ s \in [0, t])$, for $0 \le t \le T$. In particular, $\mathcal{F}(0) = \{\emptyset, \Omega\}$, mod. P.

The riskless asset pays interest at the rate $r(\cdot)$. Stock prices satisfy the dynamics

$$dS_i(t) = S_i(t)\left[\mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\right], \quad i = 1, ..., d\ ;$$

here $\mu_i(\cdot)$ represents the instantaneous rate of return of the i^{th} stock, and $\{\sigma_{ij}(\cdot)\}_{j=1,\dots,d}$ the set of volatility coefficients of the rate of return processes. The interest rate $r(\cdot)$, the instantaneous rate of return vector $\mu(\cdot) = (\mu_1(\cdot), ..., \mu_d(\cdot))'$, and the volatility matrix $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1\le i,j\le d}$, are assumed to be bounded and $\mathbb{F}-$progressively measurable random processes. In addition, we impose the strong non-degeneracy condition

$$\eta'\sigma(t)\sigma'(t)\eta \ge \delta \|\eta\|^2, \quad \text{for all } \ \eta \in \mathbb{R}^d, \ t \in [0,T]$$

almost surely, for some $\delta > 0$. Under this condition the inverses of $\sigma(\cdot), \sigma'(\cdot)$ exist and are bounded, thus the progressively measurable market price of risk process

$$\vartheta(t) := \sigma(t)^{-1}\left[\mu(t) - r(t)\mathbf{1}\right], \quad t \in [0,T]$$

is bounded as well. In this setting, the state-price density process is

$$H(t) := \exp\left(-\int_0^t r(s)ds - \int_0^t \vartheta'(s)dW(s) - \frac{1}{2}\int_0^t \|\vartheta(s)\|^2 ds\right), \quad t \in [0,T]$$

and Arrow-Debreu prices are given by integrating over $[0,T]\times\Omega$ with respect to $H(t,\omega)\,dt\,dP(\omega)$.

Consider now a "small investor" who receives an endowment (income) $\varepsilon : [0,T]\times\Omega \to (0,\infty)$. This is a bounded, $\mathbb{F}-$progressively measurable random process that satisfies $0 < E\left(\int_0^T H(t)\,\varepsilon(t)\,dt\right) < \infty$. At any time $t \in [0,T]$ the investor has to choose a consumption rate $c(t) \ge 0$, as well as amounts $\pi_i(t)$ to invest in each of the stocks $i = 1, ..., d$. The resulting consumption policy $c : [0,T]\times\Omega \to [0,\infty)$ and portfolio policy $\pi = (\pi_1, ..., \pi_d)' : [0,T]\times\Omega \to \mathbb{R}^d$, are assumed to be $\mathbb{F}-$progressively measurable processes, and to satisfy the integrability condition $\int_0^T \left(c(t) + \|\pi(t)\|^2\right) dt < \infty$ almost surely. A pair of policies (π, c) generates the *wealth process* $X(\cdot) \equiv X^{\pi,c}(\cdot)$, given by the solution of the linear stochastic differential equation

$$dX(t) = r(t)X(t)dt + (\varepsilon(t) - c(t))\,dt + \pi'(t)\sigma(t)\left[\vartheta(t)dt + dW(t)\right] \tag{1}$$

subject to the initial condition $X(0) = 0$. This equation can be written in the equivalent form

$$H(t)X(t) + \int_0^t H(s)\left(c(s) - \varepsilon(s)\right) ds = \int_0^t H(s)\pi'(s)\sigma(s)dW(s), \quad t \in [0,T]. \tag{1$'$}$$

A pair of consumption/portfolio policies (π, c) is called *admissible*, if the investor's total wealth is nonnegative at all times, i.e., if

$$H(t)X(t) + E_t\left(\int_t^T H(s)\varepsilon(s)ds\right) \ge 0, \qquad t \in [0,T] \tag{2}$$

holds almost surely. Here and in the sequel, $E_t[\,\cdot\,]$ represents the conditional expectation $E[\,\cdot \mid \mathcal{F}(t)]$ under the probability measure P, given the σ-algebra $\mathcal{F}(t)$ (information up to time t). Condition (2) is standard: since total wealth equals the portfolio value plus the present value of future income, the condition allows borrowing against future endowment.

Let $\mathcal{A}$ denote the class of admissible pairs (π, c). It follows from (1)$'$ and (2) that the local martingale

$$\begin{aligned} Q(t) &:= H(t)X(t) + E_t\left(\int_t^T H(s)\varepsilon(s)ds\right) + \int_0^t H(s)c(s)\,ds \\ &= \int_0^t H(s)\pi'(s)\sigma(s)dW(s) + E_t\left(\int_0^T H(s)\varepsilon(s)ds\right), \quad t \in [0,T] \end{aligned}$$

is non-negative, thus a supermartingale, for every $(\pi, c) \in \mathcal{A}$. The static budget constraint

$$E\left(\int_0^T H(s)c(s)ds\right) \leq x := E\left(\int_0^T H(s)\varepsilon(s)ds\right) \tag{3}$$

is then satisfied for every $(c, \pi) \in \mathcal{A}$, since

$$E\left(\int_0^T H(s)\varepsilon(s)ds\right) \geq E\left(H(T)X(T) + \int_0^T H(s)c(s)ds\right) \geq E\left(\int_0^T H(s)c(s)ds\right)$$

from the supermartingale property $Q(0) \geq E[Q(T)]$ and the condition (2) for $t = T$.

We shall let $\mathcal{B}$ denote the set of *budget-feasible* consumption policies, i.e., the set of consumption policies $c : [0,T] \times \Omega \to [0,\infty)$ which are progressively measurable and satisfy (3). We have just shown that $c \in \mathcal{B}$ holds for every pair $(\pi, c) \in \mathcal{A}$. The converse also holds, in the following sense (see Karatzas, Lakner, Lehoczky and Shreve (1991), pp. 256-257, or Karatzas and Shreve (1998), pp. 166-169, for a proof).

Lemma 1: *For every progressively measurable process $c : [0,T] \times \Omega \to [0,\infty)$ that satisfies (3), i.e. for every $c \in \mathcal{B}$, there exists a portfolio process $\pi(\cdot)$ such that $(\pi, c) \in \mathcal{A}$. The corresponding wealth process $X(\cdot) \equiv X^{\pi,c}(\cdot)$ is then given by*

$$H(t)X(t) = E(D_0) - E_t(D_t), \quad t \in [0,T]$$

where $D_t := \int_t^T H(s)[\varepsilon(s) - c(s)]ds$ *and* $E(D_0) \geq 0$.

This result ensures the existence of a portfolio process financing any budget-feasible consumption policy.

In the present paper, preferences are represented by the non-separable von Neumann-Morgenstern index

$$U(c) := E\left[\int_0^T u(t, c(t) - z(t; c))dt\right]$$

where $u(t, \cdot)$ denotes the instantaneous utility function. At each time $t \in [0,T]$, utility is a function of the difference between the current instantaneous consumption-rate $c(t)$, and an index $z(t; c)$ which depends on past consumption $c(s)$, $0 \leq s \leq t$. The process $z(\cdot) \equiv z(t; \cdot)$ is commonly referred to as the "standard of living process"; see (4) and (5) below. Joint assumptions on z and $u(t, c-z)$ will be formulated to capture the notion that an individual "develops habits", in the sense that past consumption experience conditions the current felicity from consumption.

The specification of preferences adopted in this paper generalizes the extant literature in two directions, as follows.

First, we assume that the standard of living $z(\cdot) \equiv z(t;\cdot)$ follows the dynamics

$$dz(t) = (\delta c(t) - \alpha z(t))dt + \delta \eta'(t)dW(t), \qquad z(0) = z_0 \tag{4}$$

where $z_0 \geq 0, \alpha \geq 0, \delta \geq 0$ are given real constants and $\eta : [0,T] \times \Omega \to \mathbb{R}^d$ is a bounded, progressively measurable process.[1] Equivalently, we write

$$z(t) \equiv z(t;c) = z_0 e^{-\alpha t} + \delta \int_0^t e^{-\alpha(t-s)} \left[c(s)ds + \eta'(s)dW(s)\right], \tag{5}$$

for the solution of the stochastic differential equation (4). In this formulation, the standard of living at date t is a noisy estimate of a weighted average of past consumption. The presence of noise may reflect imperfect recollection of consumption history, or the influence of other factors in the formation of a benchmark for evaluation of current felicity. Information-gathering (or record-keeping) costs can be invoked, to motivate the absence of a precise recollection of past consumption experience. The influence of business cycles on the evaluation of a given consumption profile (i.e., the notion that a given consumption history has less of an impact on current utility of consumption, if the economy is in a recession) rationalizes the second type of effect.

Secondly, we adopt a more general utility specification. We assume that $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous function with the property that for each $t \in [0,T]$ the function $u(t,\cdot)$ is strictly concave, of class $\mathcal{C}^1$, with $u'(t,\cdot) > 0$, $u'(t,\infty) = 0$ and $u(t,-\infty) = \infty$. Thanks to these assumptions, the inverse $I(t,\cdot) : (0,\infty) \to \mathbb{R}$ of the marginal utility function $u'(t,\cdot) : \mathbb{R} \to (0,\infty)$ is well-defined, continuous and strictly decreasing, with $I(t,\infty) = -\infty$ and $I(t,0) = \infty$.

Thus, we do *not* impose the standard "addiction" assumption, namely, the condition $u'(t,0+) = \infty$. In contrast, our specification insists that consumption be always non-negative, but allows it to fall below the standard of living – thus removing the incentives for a systematic buildup of habits over time and for an optimal consumption perennially in excess of the standard of living. At the same time, the notion that consumption history has a detrimental effect on the enjoyment of current consumption, i.e., that "habits matter", is retained in our model.

In addition to providing a more general and realistic specification of habit-forming preferences and consumption plans, our model also obviates the need to impose strong restrictions on the endowment process to ensure the existence of optimal policies. Recall from Detemple and Zapatero (1991) that, with the standard specification, initial wealth must exceed the cost of the subsistence consumption policy, namely,

[1] Versions of this model with constant coefficients and without noise (i.e. $\eta = 0$) were introduced by Sundaresan (1989) and Constantinides (1990) and also examined by Detemple and Zapatero (1991). The standard model with time-separable preferences is obtained by setting $\delta = z_0 = 0$ and $\eta(\cdot) \equiv 0$.

must exceed the quantity $z_0 \cdot E\left[\int_0^T H(t)\, e^{-(\alpha-\delta)t} dt\right]$; in the absence of this condition, habits cannot be sustained throughout the time-horizon $[0,T]$ in their model (that is, $c(s) - z(s) < 0$ for some $s \in [0,T]$), and their optimization problem is ill-defined. In the present paper the "proxy" for initial wealth, namely, the present value $x = E\left(\int_0^T H(t)\, \varepsilon(t)\, dt\right)$ of future income as in (3), is unrestricted.

Definition 2.a: (Dynamic Optimization). *An admissible pair* $(\pi, c) \in \mathcal{A}$ *is called* optimal *for the dynamic problem, if for any other admissible pair* $(\pi', c') \in \mathcal{A}$ *we have* $U(c') \leq U(c)$.

Definition 2.b: (Static Optimization). *A consumption policy* $c \in \mathcal{B}$ *is called* optimal *for the static problem, if for any other* $c' \in \mathcal{B}$ *we have* $U(c') \leq U(c)$.

Due to Lemma 1, we need only solve the static maximization problem to identify the set of optimal policies.

3 Optimal policies: characterization.

The static optimization problem described above is a typical optimization problem with constraints, namely (3) and $c(\cdot) \geq 0$. As usual with problems of this sort, we shall try to solve it by introducing *Lagrange multipliers:* namely, a real number $y > 0$ to enforce the static budget constraint (3), and a progressively measurable process $\xi : [0,T] \times \Omega \to [0,\infty)$ to enforce the non-negativity constraint $c(\cdot) \geq 0$.

3.1 Duality.

For any Lagrange multipliers $y \in (0,\infty)$ and $\xi : [0,T] \times \Omega \to [0,\infty)$ as above, let us consider the auxiliary functional

$$\begin{aligned} V(c; y, \xi) \quad := \quad & E\left[\int_0^T u(t, c(t) - z(t;c)) dt\right] + y \cdot E\left[\int_0^T H(t)[\varepsilon(t) - c(t)] dt\right] \\ & + E\left[\int_0^T H(t)\xi(t)c(t) dt\right]. \end{aligned} \tag{6}$$

For every consumption-rate process $c(\cdot) \geq 0$ that satisfies (3), we have $V(c; y, \xi) \geq U(c)$, with equality if and only if the conditions

$$\left\{ \begin{array}{c} E\left(\int_0^T H(t)[\varepsilon(t) - c(t)] dt\right) = 0 \\ \xi(t)c(t) = 0, \quad \lambda \otimes P - \text{a.e.} \end{array} \right\} \tag{7}$$

are both satisfied (λ is Lebesgue measure on $[0,T]$). The following duality result then holds.

Lemma 3: (Duality) *For a given pair $y > 0$, $\xi(\cdot) \geq 0$ as above, suppose that we can find a progressively measurable process $c^*(\cdot) \geq 0$ that satisfies the conditions of (7) and maximizes the functional (6), namely*

$$V(c^*; y, \xi) \geq V(c; y, \xi), \quad \textit{for all } \ c(\cdot) \in \ \mathcal{B}. \tag{8}$$

Then,

$$U(c^*) = V(c^*; y, \xi) \geq V(c; y, \xi) \geq U(c) \tag{9}$$

for every $c(\cdot) \in \mathcal{B}$. That is, $c^*(\cdot)$ is optimal for the static optimization problem.

Under the stated conditions, the process $c^*(\cdot) \geq 0$ solves the static optimization problem, thus also the dynamic optimization problem. In other words, there exists a portfolio process $\pi^*(\cdot)$, with $(\pi^*, c^*) \in \mathcal{A}$ and associated wealth process $X^*(\cdot) \equiv X^{\pi^*, c^*}(\cdot)$ given by

$$X^*(t) = \frac{1}{H(t)} \cdot E_t \left(\int_t^T H(s)[\varepsilon(s) - c^*(s)]ds \right), \quad t \in [0, T],$$

such that (π^*, c^*) attains the supremum of $U(c)$ over the set of admissible policies $(\pi, c) \in \mathcal{A}$.

3.2 The unconstrained problem.

In order to characterize the solution of the auxiliary optimization problem (8), it is useful to recall briefly the construction of the optimal policy in the *unconstrained case*, i.e., when $c(\cdot)$ is allowed to take negative values. In this instance, a prominent rôle is played by the "adjusted" state-price density process

$$\Gamma(t) := H(t) + \delta \cdot E_t \left(\int_t^T e^{(\delta-\alpha)(s-t)} H(s)ds \right), \quad t \in [0, T] \tag{10}$$

introduced by Detemple and Zapatero (1991). As was shown in that paper, the optimal unconstrained consumption, in excess of the standard of living, can be written as

$$c_u(t) - z_u(t) = e^{(\delta-\alpha)t} \, F(t);$$

we have denoted by $c^u(\cdot)$ the optimal *unconstrained* consumption process, and have set

$$F(t) := e^{(\alpha-\delta)t} I(t, y\Gamma(t)), \quad t \in [0, T] \tag{11}$$

for an appropriate Lagrange multiplier $y = y_u > 0$. In conjunction with the dynamics of (4) for the process $z_u(\cdot)$, it follows that

$$z_u(t) \; = \; e^{(\delta-\alpha)t} \left[z_0 + \delta \int_0^t F(s)ds + \int_0^t \zeta'(s)dW(s) \right]$$

and

$$c_u(t) \; = \; e^{(\delta-\alpha)t} \left[F(t) + z_0 + \delta \int_0^t F(s)ds + \int_0^t \zeta'(s)dW(s) \right]$$

where we have set $\zeta(t) \equiv \delta e^{(\alpha-\delta)t}\eta(t)$. The constant $y = y_u > 0$ of (11) is then determined by the requirement that (3) hold as equality. More precisely, it can be checked that the function

$$y \mapsto \mathcal{X}_u(y) := E\left(\int_0^T H(t)c_u(t)dt\right)$$

is continuous and decreasing on $(0,\infty)$ with $\mathcal{X}_u(0+) = \infty$, $\mathcal{X}_u(\infty) = 0$; thus, we take as y_u the smallest $y > 0$ such that $\mathcal{X}_u(y) = x$.

3.3 A parametric representation of the solution.

Let us return now to the *constrained problem*, which we shall treat under the following conditions:

Standing Assumption 4: *It will be assumed throughout the paper that*

$$E\left(\int_0^T H(t)\left(I(t,y\Gamma(t))\right)^+ dt\right) < \infty, \quad \text{for some } \; y \in (0,\infty)$$

$$E\left(\int_0^T \left|u(t,I(t,y\Gamma(t))\right| dt\right) < \infty, \quad \forall \; y \in (0,\infty)$$

$$E\left(\int_0^T \left|u\left(t,-e^{-\alpha t}\left(z_0+\delta\int_0^t e^{\alpha s}\,\eta'(s)\,dW(s)\right)\right)\right| dt\right) < \infty\,.$$

In particular, by taking $c(\cdot) \equiv 0$ in (9), we see that

$$U(c^*) = V(c^*;y,\xi) \geq E\int_0^T u\left(t,-e^{-\alpha t}\left(z_0+\delta\int_0^t e^{\alpha s}\,\eta'(s)\,dW(s)\right)\right) dt > -\infty\,.$$

In our present context the rôle of the "adjusted" state-price-density, *in the presence of the non-negativity constraint on consumption*, will be played by the process

$$\gamma(t) \equiv \gamma(t;y,\xi) = \frac{1}{y}\left[(y-\xi(t))H(t) + \delta\cdot E_t\left(\int_t^T (y-\xi(s))H(s)e^{(\delta-\alpha)(s-t)}ds\right)\right] \tag{12}$$

for given Lagrange multipliers $y > 0$ and progressively measurable $\xi(\cdot) \geq 0$. Clearly, $\Gamma(\cdot) \geq \gamma(\cdot)$ from (10) and (12). In terms of the process $\gamma(\cdot)$ of (12), the quantity

$$G(t) := e^{(\alpha-\delta)t}\, I(t,y\gamma(t)), \quad t \in [0,T] \tag{13}$$

will now represent the normalized optimal net consumption, in excess of the standard of living, for the constrained problem – in the sense that we shall have

$$c^*(t) - z(t;c^*) \;=\; e^{(\delta-\alpha)t}\, G(t)\,,$$

where $c^*(\cdot)$ is the optimal constrained consumption process. Here is a formalization of these ideas.

Proposition 5: *Suppose that $c^*(\cdot)$ solves the dual problem (8), for some given $y > 0$ and progressively measurable $\xi(\cdot) \geq 0$; namely,*

$$V(c^*; y, \xi) \geq V(c; y, \xi) < \infty, \quad \forall\, c(\cdot) \in \mathcal{B}.$$

Then we have

$$c^*(t) = e^{(\delta-\alpha)t} \left[G(t) + z_0 + \delta \int_0^t G(s)ds + \int_0^t \zeta'(s)dW(s) \right] \tag{14}$$

and

$$\begin{aligned} z^*(t) \equiv z(t; c^*) &= c^*(t) - e^{(\delta-\alpha)t}\, G(t) \\ &= e^{(\delta-\alpha)t} \left[z_0 + \delta \int_0^t G(s)ds + \int_0^t \zeta'(s)dW(s) \right], \end{aligned} \tag{15}$$

in the notation of (13), (12) and with $\zeta(t) \equiv \delta e^{(\alpha-\delta)t}\eta(t)$.

Proof: The optimality of $c^*(\cdot)$ implies that

$$\overline{\lim}_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left[V\Big(c^* + \epsilon(c - c^*); y, \xi\Big) - V\Big(c^*; y, \xi\Big) \right] \leq 0$$

holds for every $c(\cdot) \in \mathcal{B}$, or equivalently

$$\overline{\lim}_{\epsilon \downarrow 0} \frac{1}{\epsilon} E \int_0^T \left[u\Big(t, c^*(t) + \epsilon(c(t) - c^*(t)) - z(t; c^* + \epsilon(c - c^*))\Big) - u\Big(t, c^*(t) - z(t; c^*)\Big) \right] dt$$

$$\leq E \left[\int_0^T H(t)(y - \xi(t))[c(t) - c^*(t)]dt \right].$$

Let us consider from now onwards only those processes $c(\cdot) \in \mathcal{B}$ that satisfy $\sup_{0 \leq t \leq T} |c(t) - c^*(t)| \leq 1$, and denote the resulting class by $\mathcal{B}_1$. From the concavity of the utility function $u(t, \cdot)$, the assumption $V(c^*; y, \xi) < \infty$, and dominated convergence, the above inequality reads

$$\begin{aligned} & E \left[\int_0^T u'(t, c^*(t) - z(t; c^*)) \Big[(c(t) - c^*(t)) - (z(t; c) - z(t; c^*)) \Big] dt \right] \\ \leq\ & E \left[\int_0^T H(t)(y - \xi(t))[c(t) - c^*(t)]dt \right], \end{aligned}$$

or equivalently

$$\begin{aligned} & E \left[\int_0^T \left\{ u'(t, c^*(t) - z(t; c^*)) - H(t)(y - \xi(t)) \right\} [c(t) - c^*(t)]dt \right] \\ \leq\ & E \left[\int_0^T u'(t, c^*(t) - z(t; c^*))\, (z(t; c) - z(t; c^*))\, dt \right], \end{aligned} \tag{16}$$

for every $c(\cdot) \in \mathcal{B}_1$. From (5), the right-hand side of this inequality can be written as

$$\begin{aligned} & E\left[\int_0^T u'(t, c^*(t) - z(t; c^*)) \cdot \delta \left(\int_0^t e^{-\alpha(t-s)}(c(s) - c^*(s))ds \right) dt \right] \\ = \; & E\left[\int_0^T \delta \left(\int_s^T e^{-\alpha(t-s)} u'(t, c^*(t) - z(t; c^*))dt \right) (c(s) - c^*(s))ds \right] \\ = \; & E\left[\int_0^T \delta \cdot E_t \left(\int_t^T e^{-\alpha(s-t)} u'(s, c^*(s) - z(s; c^*))ds \right) (c(t) - c^*(t))dt \right]. \end{aligned}$$

Substituting back into (16), we deduce that the "utility-gradient"

$$\mathcal{G}(t; c^*) := u'(t, c^*(t) - z(t; c^*)) - \delta \cdot E_t \left(\int_t^T e^{-\alpha(s-t)} u'(s, c^*(s) - z(s; c^*))ds \right)$$

satisfies

$$E\left(\int_0^T \{\mathcal{G}(t; c^*) - H(t)[y - \xi(t)]\} \cdot [c(t) - c^*(t)] \, dt \right) \le 0, \qquad \text{for every } c(\cdot) \in \mathcal{B}_1.$$

This, in turn, shows that $c^*(\cdot)$ should satisfy $\mathcal{G}(\cdot\,; c^*) \equiv H(\cdot)[y - \xi(\cdot)]$, that is

$$u'(t, c^*(t) - z(t; c^*)) - \delta \cdot E_t \left(\int_t^T e^{-\alpha(s-t)} u'(s, c^*(s) - z(s; c^*))ds \right) = H(t)[y - \xi(t)] \tag{17}$$

for all $t \in [0, T]$. One then observes that the "normalized marginal utility" process

$$\gamma(t) := \frac{1}{y} u'(t, c^*(t) - z(t; c^*)), \quad t \in [0, T] \tag{18}$$

solves the *Recursive Linear Stochastic Equation*

$$y\gamma(t) = H(t)[y - \xi(t)] + \delta \cdot E_t \left(\int_t^T e^{-\alpha(s-t)} y\gamma(s)\, ds \right), \quad t \in [0, T]. \tag{19}$$

As shown in the Appendix, the equation (19) can be solved exactly: the solution is provided by the process $\gamma(\cdot) \equiv \gamma(\cdot\,; y, \xi)$ of (12). Inverting (18), we obtain the optimal net consumption

$$c^*(t) - z(t; c^*) = I(t, y\gamma(t)) \qquad \left(= e^{(\delta-\alpha)t}\, G(t), \quad \text{in the notation of (13)} \right),$$

and substituting into (4) we obtain the dynamics

$$dz^*(t) = [I(t, y\gamma(t)) + (\delta - \alpha)z^*(t))]\, dt + \delta\eta'(t)dW(t), \qquad z(0) = z_0 \tag{20}$$

for the standard-of-living process $z^*(\cdot) \equiv z(t; c^*)$.

The solution of the linear stochastic differential equation (20) is given by (15), and the expression (14) for the optimal consumption process $c^*(\cdot)$ follows readily. $\diamondsuit$

The reader should not fail to notice the formal similarity of the expressions in (14), (15) with those provided for $c_u(\cdot)$, $z_u(\cdot)$ in subsection 3.2; in these expressions, the process $G(\cdot)$ of (13) has replaced the process $F(\cdot)$ of (11).

3.4 A characterization of the shadow prices.

The representation of the solution in Proposition 5 expresses the adjusted state-price density in terms of the Lagrange-multipliers $y > 0$ and $\xi(\cdot) \geq 0$. We now complete this characterization by deriving a set of equations that these Lagrange-multipliers will have to satisfy, in order for the process $c^*(\cdot)$ of Proposition 5 to be non-negative and to satisfy the conditions (7).

In order to state this result, we formulate the following bold conjecture:

Conjecture: *There exists a stopping time $\tau \in [0, T]$, such that*

$$\left\{ \begin{array}{lll} \gamma(t) < \Gamma(t), & c^*(t) = 0; & \text{on} \quad [0, \tau) \\ \gamma(t) = \Gamma(t), & \xi(t) = 0; & \text{on} \quad [\tau, T] \end{array} \right\}. \tag{21}$$

In essence, the conjecture postulates that the adjusted state-price-density $\gamma(\cdot)$ is strictly less that the unconstrained one $\Gamma(\cdot)$, thus the optimal consumption is null ($c^*(\cdot) \equiv 0$), until an endogenously determined random time τ. After τ has occurred, the constrained investor follows the unconstrained optimal pattern, in the sense that the constraint will never bind again (i.e., $\xi(\cdot) \equiv 0$ on $[\tau, T]$). The validity of this conjecture constitutes a rather striking result. Indeed, our initial guess was that the consumption constraint $c(\cdot) \geq 0$ would bind repeatedly over time. We shall return to the economic intuition associated with this conjecture in section 4.

Proceeding with the analysis, we obtain the following characterization for the Lagrange multipliers $y > 0$ and $\xi(\cdot) \geq 0$; recall the notation of (10)-(13).

Proposition 6: *Suppose that $c^*(\cdot)$ solves the auxiliary problem (8). Then, it must be that the condition*

$$\Gamma(t) \geq \gamma(t)\,, \tag{22}$$

as well as the conditions

$$G(t) \geq F(t)\,, \tag{23}$$

$$G(t) \geq -z_0 - \delta \int_0^t G(s)ds - \int_0^t \zeta'(s)dW(s) =: Y(t)\,, \tag{24}$$

are satisfied for all $t \in [0, T]$. If, in addition, the conjecture (21) holds, then we also have

$$[G(t) - F(t)][G(t) - Y(t)] = 0 \tag{25}$$

for all $t \in [0, T]$.

With the exception of the last one, the conditions in the proposition are intuitive. In essence, the inequality (22) states that adjusted state-prices in the constrained case can never exceed those in the unconstrained case. This reflects the fact that a binding constraint forces a higher consumption pattern than in the unconstrained case, which means that the corresponding state-prices must be lower. Condition (23) is the counterpart of this restriction, expressed in net-consumption space. Condition (24) mandates that consumption cannot be negative.

The last condition is the most intriguing. Under the conjecture (21), the final requirement (25) corresponds to the *complementary slackness condition*, and is obviously satisfied. Indeed, if the conjecture is valid, events on which the consumption-constraint $c(t) \geq 0$ binds correspond to $G(t) = Y(t)$, while events on which the constraint does not bind entail unconstrained net consumption behavior, i.e., $G(t) = F(t)$. If the conjecture did not hold, then there would be no reason for the individual to behave in an unconstrained manner, in terms of net consumption, in between times at which the constraint binds. In this instance, interim net consumption would account for the possibility of a binding constraint at future dates and would be lower than if unconstrained. In short, condition (25) would become an inequality as opposed to an equality, and complementary slackness would entail a different restriction.

Proof of Proposition 6: Let us write the equation (19) in the form

$$\xi(t) = \frac{y}{H(t)} \left[H(t) - \gamma(t) + \delta \cdot E_t \left(\int_t^T e^{-\alpha(s-t)} \gamma(s) ds \right) \right]. \qquad (19)'$$

The formulae $(19)'$, (14) express the optimal consumption process $c^*(\cdot)$, and the Lagrange multiplier $\xi(\cdot)$ corresponding to it, in terms of the normalized marginal utility process $\gamma(\cdot)$ of (18). Now recall that both $c^*(\cdot)$ and $\xi(\cdot)$ should be non-negative.

From the equation $(19)'$, the requirement $\xi(\cdot) \geq 0$ amounts to

$$\gamma(t) \leq H(t) + \delta \cdot E_t \left(\int_t^T e^{-\alpha(s-t)} \gamma(s) ds \right), \quad \text{for all } t \in [0,T], \qquad (26)$$

which, in turn, is equivalent to the inequality

$$\gamma(t) \leq H(t) + \delta \cdot E_t \left(\int_t^T e^{(\delta-\alpha)(s-t)} H(s) ds \right) \equiv \Gamma(t), \quad \text{for all } t \in [0,T] \qquad (27)$$

(see Appendix). This last process $\Gamma(\cdot)$ of (10) satisfies the Linear Recursive Equation

$$\Gamma(t) = H(t) + \delta \cdot E_t \left(\int_t^T \Gamma(s) e^{-\alpha(s-t)} ds \right), \quad \text{for all } t \in [0,T], \qquad (28)$$

and the requirement $\xi(\cdot) \geq 0$ leads to

$$G(t) \equiv e^{(\alpha-\delta)t} I(t, y\gamma(t)) \geq e^{(\alpha-\delta)t} I(t, y\Gamma(t)) \equiv F(t), \qquad (29)$$

in conjunction with (27).

On the other hand, it follows from (14) that the requirement $c^*(\cdot) \geq 0$ amounts to the inequality (24). Finally, the conjecture (21) leads to $G(\cdot) \equiv Y(\cdot)$ on $[0, \tau)$, and to $\gamma(\cdot) \equiv \Gamma(\cdot)$ (thus also $G(\cdot) \equiv F(\cdot)$) on $[\tau, T]$, which justify the condition (25) of the proposition. $\diamondsuit$

A few additional steps provide the following characterization of the solution.

Corollary 7: *Suppose that the conjecture (21) holds. Then the process $Y(\cdot)$ of (24) solves the stochastic integral equation*

$$Y(t) = -z_0 - \delta \int_0^t (Y(s) \vee F(s))ds - \int_0^t \zeta'(s)dW(s), \quad t \in [0,T]. \tag{30}$$

Furthermore, the optimal consumption, the associated standard of living, the adjusted state-price density, and the Lagrange-multiplier process that enforces the non-negativity constraint on consumption, are given by

$$c^*(t) = e^{(\delta-\alpha)t}\left(G(t) - Y(t)\right) = e^{(\delta-\alpha)t}\left(F(t) - Y(t)\right)^+ \tag{31}$$

$$z^*(t) = -Y(t)\, e^{(\delta-\alpha)t}, \tag{32}$$

$$\gamma(t) \;=\; \frac{1}{y}\, u'\left(t, e^{(\delta-\alpha)t}(Y(t) \vee F(t))\right), \tag{33}$$

and

$$\begin{aligned}\xi(t) \;&=\; y - \frac{u'\left(t, e^{(\delta-\alpha)t}(Y(t) \vee F(t))\right)}{H(t)} \\ &+ \frac{\delta}{H(t)} \cdot E_t\left(\int_t^T e^{-\alpha(s-t)} u'\Big(t, e^{(\delta-\alpha)t}(Y(s) \vee F(s))\Big)ds\right),\end{aligned} \tag{34}$$

respectively, for $t \in [0,T]$.

Proof: Conditions (23)-(25) mandate

$$G(\cdot) = Y(\cdot) \vee F(\cdot), \tag{35}$$

which leads to the equation (30). Since the process $F(\cdot)$ is completely described by (10) and (11), the equation (30) can be construed as a *Stochastic Integral Equation with random drift coefficient* for the process $Y(\cdot)$. This equation can be shown to admit a pathwise unique, strong solution, for every given $y > 0$ (see proof of Lemma 8 in the next section). Once the solution $Y(\cdot)$ of this equation has been determined, we can obtain the processes $c^*(\cdot), z^*(\cdot) \equiv z(\cdot; c^*), \gamma(\cdot) = \gamma(\cdot; c^*)$, and $\xi(\cdot)$ by substituting in the formulae of Propositions 5 and 6. $\diamondsuit$

Proposition 6 and Corollary 7 provide a characterization of the optimal policy, based on the conjecture (21). In order to validate these characterizations and to demonstrate the existence of an optimal policy, we now must

1. verify the Conjecture (21) – thus also the property $c^*(\cdot)\xi(\cdot) = 0$ – and

2. select the Lagrange multiplier $y > 0$ so that $c^*(\cdot)$ satisfies the budget-constraint (3) as an *equality.*

This program will be carried out in the next section.

4 Optimal policies: existence and construction.

We now provide a constructive proof for the existence of an optimal consumption-portfolio policy in the dynamic problem. Recall the process $\Gamma(\cdot)$ of (10), as well as the process

$$F^y(t) := e^{(\alpha-\delta)t} I(t, y\Gamma(t)), \quad t \in [0,T] \tag{36}$$

as in (29), parametrized by $y \in (0,\infty)$. Our first result establishes the existence, uniqueness and properties of the solution to the Stochastic Integral Equation (30).

Lemma 8: *For each $y \in (0,\infty)$, the stochastic equation*

$$Y^y(t) = -z_0 - \delta \int_0^t (Y^y(s) \vee F^y(s))ds - \int_0^t \zeta'(s)dW(s), \quad t \in [0,T] \tag{37}$$

has a pathwise unique strong solution $Y^y(\cdot)$, which is a continuous semimartingale. Furthermore, the mapping $(0,\infty) \ni y \mapsto Y^y(\cdot) \in \mathcal{C}([0,T])$ is continuous and increasing:

$$0 < y_1 \le y_2 < \infty \quad \Longrightarrow \quad Y^{y_1}(t) \le Y^{y_2}(t), \quad \forall\, t \in [0,T]. \tag{38}$$

Proof: The proof follows from classical results about existence, uniqueness and comparison, for solutions of one-dimensional stochastic differential equations, once we write (37) in the form

$$dY^y(t) = b(t, Y^y(t); y)dt - \zeta'(t)dW(t), \quad Y^y(0) = -z_0$$

with $b(t,x;y) \equiv -\delta \max(x, F^y(t))$; see, for instance, Ikeda and Watanabe (1989), Chapter VI. This drift function satisfies the usual Lipschitz and linear growth conditions, is progressively measurable in (t,ω), continuous in the triple (t,x,y), and increasing in the argument y, since the mapping $y \mapsto F^y(\cdot)$ of (36) is clearly decreasing. $\diamondsuit$

Having constructed the process $Y^y(\cdot)$, we can now define for each $y \in (0,\infty)$ the processes

$$G^y(t) := Y^y(t) \vee F^y(t) \tag{39}$$

$$c^y(t) := e^{(\delta-\alpha)t}\,(G^y(t) - Y^y(t)) = e^{(\delta-\alpha)t}\,(F^y(t) - Y^y(t))^+, \quad z^y(t) := -e^{(\delta-\alpha)t}\,Y^y(t), \tag{40}$$

so that $c^y(t) - z^y(t) = e^{(\delta-\alpha)t}\,G^y(t)$, as well as

$$0 < \gamma^y(t) := \frac{1}{y}\,u'(t, e^{(\delta-\alpha)t}\,G^y(t)) \;=\; \frac{1}{y}\,u'(t,\, c^y(t) - z^y(t)) \tag{41}$$

$$\xi^y(t) := \frac{y}{H(t)}\left[H(t) - \gamma^y(t) + \delta \cdot E_t\left(\int_t^T e^{-\alpha(s-t)}\gamma^y(s)ds\right)\right] \tag{42}$$

for $t \in [0,T]$, by analogy with (35) and (32)-(34). Comparing (41) with (36), written in the equivalent form $y\Gamma(t) = u'(t, e^{(\delta-\alpha)t} F^y(t))$, we see that

$$0 < \gamma^y(t) \le \Gamma(t), \quad t \in [0,T] \tag{43}$$

from (39), as well as

$$\left\{ \begin{array}{lllll} \gamma^y(t) < \Gamma(t) & \Leftrightarrow & F^y(t) < Y^y(t) & \Longrightarrow & c^y(t) = 0 \\ \gamma^y(t) = \Gamma(t) & \Leftrightarrow & F^y(t) \ge Y^y(t) & \Longleftarrow & c^y(t) > 0 \end{array} \right\} \tag{44}$$

from (40). Let us also recall the Recursive Equation (28), for the process $\Gamma(\cdot)$ of (27); with its help we can rewrite (42) as

$$\xi^y(t) = \frac{y}{H(t)} \left[\Gamma(t) - \gamma^y(t) - \delta \cdot E_t \left(\int_t^T e^{-\alpha(s-t)} (\Gamma(s) - \gamma^y(s)) ds \right) \right] \tag{45}$$

or, equivalently, in the form

$$y\left[\Gamma(t) - \gamma^y(t)\right] = H(t)\xi^y(t) + \delta \cdot E_t \left(\int_t^T H(s)\xi^y(s) e^{(\delta-\alpha)(s-t)} ds \right), \quad t \in [0,T] \quad (45)'$$

of a Linear Recursive Equation for the process $\xi^y(\cdot)$.

Lemma 9: *The continuous, $\mathbb{F}-$adapted process $\xi^y(\cdot)$ of (42) satisfies*

$$0 \le \xi^y(t) \le \frac{y}{H(t)} \left[\Gamma(t) - \gamma^y(t)\right], \quad t \in [0,T].$$

Proof: From (43) and (45)$'$ we obtain the Recursive Stochastic Inequality

$$h(t) + \delta \cdot E_t \left(\int_t^T h(s) ds \right) \ge 0, \quad t \in [0,T]$$

for the continuous, $\mathbb{F}-$adapted process $h(t) \equiv H(t)\xi^y(t)e^{(\delta-\alpha)t}$, $0 \le t \le T$. The "stochastic version of Gronwall's inequality" (Appendix B in Duffie and Epstein (1992)) then implies $h(\cdot) \ge 0$, which proves $\xi^y(\cdot) \ge 0$; the second inequality follows directly from (45), (43). This completes the proof of the Lemma. $\diamondsuit$

Proposition 10: *For each $y \in (0,\infty)$, consider the $\mathbb{F}-$stopping time*

$$\tau^y := \inf\{t \in [0,T) : F^y(t) \ge Y^y(t)\} \wedge T = \inf\{t \in [0,T) : \Gamma(t) = \gamma^y(t)\} \wedge T. \tag{46}$$

Then the mapping $y \mapsto \tau^y$ is increasing, and we have

$$c^y(t) = 0\,; \quad t \in [0, \tau^y) \tag{47}$$

$$\gamma^y(t) = \Gamma(t), \;\; \xi^y(t) = 0; \quad t \in [\tau^y, T]\,. \tag{48}$$

Proof: From the continuity of the random functions $\Gamma(\cdot)$ and $\gamma^y(\cdot)$, we have $\gamma^y(\tau^y) = \Gamma(\tau^y)$. Therefore, Lemma 9 gives $\xi^y(\tau^y) = 0$, and back in $(45)'$ it implies in turn

$$\xi^y(\cdot) = 0, \quad \text{thus also} \quad \Gamma(\cdot) = \gamma^y(\cdot), \quad \text{on } [\tau^y, T].$$

This establishes (48), whereas (47) is a consequence of the definition (46) and the implications in (44). The increase of the map $y \mapsto \tau^y$ is an immediate consequence of the decrease of $y \mapsto F^y(\cdot) - Y^y(\cdot)$. $\diamondsuit$

Proposition 10 vindicates the conjecture (21), so all that remains is to show that we can select the scalar Lagrange multiplier $y > 0$ in such a way as to satisfy the requirement $\mathcal{X}(y) = x$, with $x \in (0, \infty)$ defined in (3), where we have set

$$\mathcal{X}(y) := E\left[\int_0^T H(t)c^y(t)dt\right] = E\left[\int_0^T H(t)e^{(\delta-\alpha)t}(F^y(t) - Y^y(t))^+ dt\right] \tag{49}$$

for $0 < y < \infty$. The mappings $y \mapsto F^y(\cdot)$ and $y \mapsto (F^y(\cdot) - Y^y(\cdot))^+$ are decreasing, thus so is $\mathcal{X}(\cdot)$. As $y \to \infty$ we have $F^y(t) \downarrow -\infty$, $Y^y(t) \uparrow Y^\infty(t)$ and thus $c^y(t) \downarrow 0$, where

$$Y^\infty(t) := -z_0 e^{-\delta t} - \int_0^t e^{-\delta(t-s)}\zeta'(s)dW(s), \quad t \in [0, T] \tag{50}$$

solves the linear equation $dY^\infty(t) = -\delta Y^\infty(t)dt - \zeta'(t)dW(t)$ with $Y^\infty(0) = -z_0$. Consequently, $\mathcal{X}(\infty) = 0$ by Monotone Convergence. On the other hand,

$$F^y(t) \uparrow \infty, \quad Y^y(t) \downarrow -\infty \quad \text{and thus} \quad c^y(t) \uparrow \infty\,, \quad \text{as } y \downarrow 0$$

for all $t \in [0, T]$, so that $\mathcal{X}(0+) = \infty$ by Monotone Convergence. Therefore, $\mathcal{X}(\cdot)$ is a continuous and decreasing function on $(\underline{y}, \infty)$ with $\mathcal{X}(\underline{y}) = \infty$; we have set $\underline{y} := \inf\{y > 0 : \mathcal{X}(y) < \infty\}$, a number in $[0, \infty)$ thanks to the Standing Assumption 4. Clearly then

$$y^* := \inf\{y \in (\underline{y}, \infty) : \mathcal{X}(y) = x\}$$

satisfies the requirement $\mathcal{X}(y^*) = x$.

Corresponding to this $y^* > 0$, let us consider the consumption-rate process $c^*(\cdot) \equiv c^{y^*}(\cdot)$ as in (40) and note that

$$E\left[\int_0^T H(t)c^*(t)dt\right] = E\left[\int_0^T H(t)\varepsilon(t)dt\right]$$

by the choice of y^*. We know then that there exists a portfolio process $\pi^*(\cdot)$ such that $(\pi^*, c^*) \in \mathcal{A}$, and with corresponding wealth process

$$X^*(t) = \frac{1}{H(t)} \cdot E_t\left[\int_t^T H(s)(\varepsilon(s) - c^*(s))ds\right].$$

In conclusion, we obtain the following result.

Theorem 11: *The pair (π^*, c^*) is optimal for the dynamic consumption-portfolio problem; equivalently, we have $U(c') \le U(c^*) < \infty$ for any $c'(\cdot) \in \mathcal{B}$.*

Proof: Define $z^*(\cdot) := z^{y^*}(\cdot)$, $\xi^*(\cdot) := \xi^{y^*}(\cdot)$, $\gamma^*(\cdot) := \gamma^{y^*}(\cdot)$ and $\tau^* := \tau^{y^*}$ as in (40)-(42) and (46). From the Standing Assumption 4, we have clearly $U(c^*) < \infty$. For an arbitrary pair $(\pi', c') \in \mathcal{A}$ and with $z'(\cdot) = z(\cdot; c')$ as in (5), the concavity of $u(t, \cdot)$ gives

$$u(t, c^*(t) - z^*(t)) - u(t, c'(t) - z'(t)) \ge u'(t, c^*(t) - z^*(t)) \left[c^*(t) - c'(t) - (z^*(t) - z'(t)) \right],$$

and thus, with the help of (19), we obtain

$$\begin{aligned}
U(c^*) - U(c') &\ge E \int_0^T u'(t, c^*(t) - z^*(t)) \left\{ c^*(t) - c'(t) - \delta \int_0^t (c^*(s) - c'(s)) e^{-\alpha(t-s)} ds \right\} ds \\
&= E \int_0^T u'(t, c^*(t) - z^*(t))(c^*(t) - c'(t)) dt \\
&\quad - \delta \cdot E \int_0^T \left(\int_s^T u'(t, c^*(t) - z^*(t)) e^{-\alpha(t-s)} dt \right) (c^*(s) - c'(s)) ds \\
&= E \int_0^T u'(t, c^*(t) - z^*(t))(c^*(t) - c'(t)) dt \\
&\quad - \delta \cdot E \int_0^T E_t \left(\int_t^T u'(s, c^*(s) - z^*(s)) e^{-\alpha(s-t)} ds \right) (c^*(t) - c'(t)) dt \\
&= E \int_0^T \left(y\gamma^*(t) - \delta \cdot E_t \left(\int_t^T y\gamma^*(s) e^{-\alpha(s-t)} ds \right) \right) (c^*(t) - c'(t)) dt \\
&= E \int_0^T (y - \xi^*(t)) H(t) (c^*(t) - c'(t)) dt \\
&= y \cdot E \int_0^T H(t)(c^*(t) - c'(t)) dt - E \int_0^T H(t) \xi^*(t) (c^*(t) - c'(t)) dt \\
&= y \left(x - E \int_0^T H(t) c'(t) dt \right) + E \int_0^T H(t) \xi^*(t) c'(t) dt \ge 0. \qquad \diamondsuit
\end{aligned}$$

Remark 12: From (37), (38) and (50) we can re-cast the stopping time of (46) in the form

$$\begin{aligned}
\tau^y &= \inf \{ t \in [0, T) \, : \, F^y(t) \ge Y^\infty(t) \} \wedge T \\
&= \inf \left\{ t \in [0, T) \, : \, u' \left(t, -e^{-\alpha t} \left(z_0 + \delta \int_0^t e^{\alpha s} \eta'(s) \, dW(s) \right) \right) \ge y \Gamma(t) \right\} \wedge T.
\end{aligned} \tag{51}$$

In particular,

$$\tau^y > 0 \quad \Leftrightarrow \quad y > \overline{y} := \frac{u'(0, -z_0)}{1 + \delta \cdot E \left(\int_0^T H(t) e^{(\delta - \alpha)t} dt \right)}. \tag{52}$$

5 Economic properties of optimal consumption.

5.1 A decomposition of optimal consumption.

In the standard model without habit-formation and without the so-called "Inada condition" $u'(t, 0+) = \infty$, the non-negativity condition on consumption changes the optimal policy in an interesting but straightforward manner. Indeed, the constrained policy equals the positive part of the unconstrained policy (equivalently, can be expressed as a call-option with zero-strike, written on the unconstrained policy; see Cox and Huang (1989)). As we show below, the relationship between unconstrained and constrained policies in the presence of habit-formation involves an additional correction with path-dependent payoff.

In order to state this result, let us fix the multiplier y^* corresponding to the solution of the constrained model. Recall from subsection 3.2 that the optimal unconstrained consumption process (which, by definition, does not have to obey the non-negativity constraint), associated with the Lagrange multiplier y^* as in Section 4, is

$$c_u^{y^*}(t) := e^{(\delta-\alpha)t}\left(F^{y^*}(t) - Y_u^{y^*}(t)\right),$$

where

$$Y_u^{y^*}(t) := -z_u^{y^*}(t)\, e^{(\alpha-\delta)t} = -z_0 - \delta\int_0^t F^{y^*}(s)ds - \int_0^t \zeta'(s)dW(s)$$

and $z_u^{y^*}(\cdot)$ is the associated standard of living process. As we shall clarify below, $c_u^{y^*}(\cdot)$ represents the unconstrained-optimal policy corresponding to an "adjusted portion of the constrained investor's future endowment". With this notation, we have the following decomposition.

Proposition 13: (Consumption decomposition). *The optimal constrained consumption can be written as the sum of two call-options:*

$$c^*(t) = (c_u^{y^*}(t))^+ + (c_u^{y^*}(t) \wedge 0 - A(t))^+, \quad t \in [0, T]. \tag{53}$$

Here $c_u^{y^*}(\cdot)$ is the unconstrained consumption policy corresponding to the Lagrange multiplier y^* of Section 4,

1. $(c_u^{y^*}(\cdot))^+$ *is a call-option, written on the unconstrained policy with zero strike-price, and*

2. $(c_u^{y^*}(\cdot) \wedge 0 - A(\cdot))^+$ *is an Asian average-strike, capped call-option on the unconstrained policy, with path-dependent strike-price*

$$A(t) := -\delta e^{(\delta-\alpha)t}\int_0^t (Y^{y^*}(s) - F^{y^*}(s))^+ ds. \tag{54}$$

The decomposition (53) of the sources of consumption highlights the differences between the standard model and the one with habit-formation. Foremost, it should be noted that existence of habits implies a change in the structure of the optimal policy when the consumption constraint is taken into account. This structural effect is captured by the Asian average-strike, capped call-option. Because of history-dependence, a binding consumption constraint in the past implies a different standard of living at date t, and therefore a deviation from the unconstrained policy. This deviation is captured by the quantity $-A(t)$, the negative of the strike-price in (53), (54). Since past constrained consumption cannot be less than past unconstrained consumption, the quantity $-A(t)$ is nonnegative. Nonnegative consumption at date t could be ensured by adding a put-option written on the sum of the unconstrained policy and this deviation term, leading to the decomposition

$$c^*(t) = c_u^{y^*}(t) - A(t) + (c_u^{y^*}(t) - A(t))^-.$$

Alternatively, it can be enforced by adding a put-option on the unconstrained policy, along with the Asian capped call-option, yielding

$$c^*(t) = c_u^{y^*}(t) + (c_u^{y^*}(t))^- + (c_u^{y^*}(t) \wedge 0 - A(t))^+$$

and thus (53). The Asian option pays off when the unconstrained policy exceeds the strike. There are two possible scenarios in this event. If the cap is inactive (hence $c_u^{y^*}(t) \leq 0$), the payoff is $c_u^{y^*}(t) - A(t)$ and therefore $c^*(t) = c_u^{y^*}(t) - A(t)$. In this instance the Asian option pays off the unconstrained policy, which is intuitive since the other two components in the decomposition sum to zero. If the cap is active (hence $c_u^{y^*}(t) > 0$) the payoff is $-A(t)$ and again $c^*(t) = c_u^{y^*}(t) - A(t)$. The cap thus limits the upside payoff of the Asian option when unconstrained consumption meets the constraint. In this case the Asian option pays off the deviation incurred due to past binding constraints, and this is consistent as the other two components sum to the unconstrained policy.

Finally, note that the Asian option has null payoff when preferences are additively separable over time. Indeed, $\delta = 0$ implies $A(t) = 0$ for all $t \in [0, T]$ and hence $(c_u^{y^*}(t) \wedge 0 - A(t))^+ = 0$. In this instance we retrieve the standard representation for the constrained-optimal policy in the context of time-separable utilities, namely $c^*(t) = (c_u^{y^*}(t))^+$.

Remark: *A counterpart of the consumption formula (53) is a decomposition of optimal wealth into several parts, serving to finance the respective consumption components. Specifically, we can write*

$$X^*(t) = X_1(t) - X_2(t) - X_3(t)$$

with

$$X_1(t) = \frac{1}{H(t)} \cdot E_t\left(\int_t^T H(v)\varepsilon(v)dv\right)$$

$$X_2(t) = \frac{1}{H(t)} \cdot E_t\left(\int_t^T H(v)(c_u^{y^*}(v))^+ dv\right)$$

$$X_3(t) = \frac{1}{H(t)} \cdot E_t \left(\int_t^T H(v) \Big(c_u^{y^*}(v) \wedge 0 - A(v) Big)^+ dv \right) .$$

Here $X_1(\cdot)$ *is the value of the future endowment process,* $X_2(\cdot)$ *is the cost of the call-option, and* $X_3(\cdot)$ *the cost of the Asian, capped call-option. The choice of Lagrange multiplier* y^* *ensures that, at time* $t = 0$*, the total cost of these last two components (namely,* $X_2(0) + X_3(0)$*) equals the value* $X_1(0) = x$ *of the endowment process* $\varepsilon(\cdot)$.

We can also use the results of Proposition 10 about the stopping time τ^*, in order to sharpen the decompositions above. Since the processes $\gamma^*(\cdot) \equiv \gamma^{y^*}(\cdot)$ and $\Gamma(t)$ coincide for $t \geq \tau^*$, it must be that

$$A(t) = e^{-(\alpha-\delta)(t-\tau^*)} A(\tau^*) ,$$

and hence

$$c^*(t) \;=\; (c_u^{y^*}(t))^- + \left(c_u^{y^*}(t) \wedge 0 - e^{-(\alpha-\delta)(t-\tau^*)} A(\tau^*) \right)^+ .$$

for $t \geq \tau^*$. Thus, after time τ^*, the strike in the decomposition of consumption becomes “deterministic”: it is equal to the $\mathcal{F}(\tau^*)-$measurable random variable $A(\tau^*)$, discounted at the constant rate $\alpha - \delta$. However, it is worth emphasizing that the quantity $A(\tau^*)$ depends on the paths of $Y^{y^*}(\cdot)$ and $F^{y^*}(\cdot)$ up to the stopping time τ^*.

5.2 Consumption behavior.

The consumption pattern that emerges from the previous analysis entails constrained (null) consumption up to the random time τ^*, after which the optimal net consumption $c^*(t) - z^*(t) = I(t, y^*\gamma^*(t))$ mimics, in some sense to be made precise below, the net consumption $c_u(t) - z_u(t) = I(t, y_u\Gamma(t))$ which is optimal in an unconstrained model. On the surface such behavior may appear surprising, since one might have expected random alternance between periods of constrained and unconstrained consumption, as in the standard model without habits. However, this pattern is perfectly understandable, if we recall that $\tau^* := \tau^{y^*}$ is the first time at which the processes $\gamma^*(\cdot)$ and $\Gamma(\cdot)$ coincide. From this definition it follows that the normalized utility-gradients must also coincide at and after $t = \tau^*$, namely,

$$\begin{aligned} & \frac{1}{y^*} \left[u'(t, c^*(t) - z^*(t)) - \delta \cdot E_t \int_t^T e^{-\alpha(s-t)} u'(s, c^*(s) - z^*(s)) ds \right] \qquad (55) \\ = \; & \frac{1}{y^u} \left[u'(t, c_u(t) - z_u(t)) - \delta \cdot E_t \int_t^T e^{-\alpha(s-t)} u'(s, c_u(s) - z_u(s)) ds \right] , \quad \tau^* \leq t \leq T. \end{aligned}$$

But recall that these normalized gradients incorporate the impact of current consumption choice on future utilities (“habit-effect”). The coincidence of these expressions implies that future normalized marginal utilities cannot differ, namely

$$\frac{1}{y^*} u'(t, c^*(t) - z^*(t)) \;=\; \frac{1}{y^u} u'(t, c_u(t) - z_u(t)) , \quad \text{for} \quad \tau^* \leq t \leq T. \qquad (56)$$

Thus τ^* can also be viewed as *the first time at which the investor is able to sustain the unconstrained net consumption patterns at all future times.*

It is actually possible to go beyond (55), (56), and try to compare actual and/or net consumption in the constrained and unconstrained problems. In order to make some headway in this direction, let us recall the functions $\mathcal{X}(\cdot)$ and $\mathcal{X}_u(\cdot)$ from (49) and subsection 3.2, respectively; observe that the difference

$$\begin{aligned}\mathcal{X}(y) - \mathcal{X}_u(y) &= E \int_0^T H(t) e^{(\delta-\alpha)t} \left[G^y(t) - F^y(t) + \delta \int_0^t (G^y(s) - F^y(s))\, ds \right] dt \\ &= E \int_0^T H(t) e^{(\delta-\alpha)t} \left[(Y^y(t) - F^y(t))^+ + \delta \int_0^t (Y^y(s) - F^y(s))^+\, ds \right] dt\end{aligned}$$

is non-negative for all $0 \le y < \infty$ and strictly positive if and only if $\overline{y} < y < \infty$, where $\overline{y} \in (0, \infty)$ is the constant of (52), Remark 12. It follows immediately that $y^* \ge y_u$. Let us also introduce the notation $\tau_u := \tau^{y_u}$, and recall $0 \le \tau_u \le \tau^* \le T$ from Proposition 10 and $y^* \ge y_u$. Two cases suggest themselves naturally, according to whether the value $x = E \int_0^T H(t) \varepsilon(t) dt$ of the endowment at time $t = 0$ is sufficiently large, or not.

Proposition 14: *Suppose that* $x \ge \mathcal{X}(\overline{y})$. *Then we have* $c^*(\cdot) \equiv c_u(\cdot)$ *on* $[0, T]$.

Proposition 15: *Suppose that* $x < \mathcal{X}(\overline{y})$. *Then the optimal consumption has the following properties:*

1. *for* $t \ge \tau^*$ *we have* $c^*(t) - z^*(t) < c_u(t) - z_u(t)$,
2. *for* $t \le \tau_u$ *we have* $0 = c^*(t) > c_u(t)$, $z^*(t) > z_u(t)$, *and* $c^*(t) - z^*(t) \ge c_u(t) - z_u(t)$*; this last inequality is strict for* $t \in [0, \tau_u)$, *and holds as equality for* $t = \tau_u$.

In other words, for a sufficiently well-endowed investor, Proposition 14 shows that the optimal unconstrained consumption process $c_u(\cdot)$ is non-negative, and is thus optimal in the constrained problem as well; such an investor is effectively not bound by the constraint. For an investor who is not sufficiently well-endowed, Proposition 15 provides information about the manner in which the difference in net consumptions $[c^*(t) - z^*(t)] - [c_u(t) - z_u(t)]$ changes its sign during the time-interval $[0, T]$ so that the inequality

$$E \int_0^T u(t, c_u(t) - z_u(t))\, dt \ge E \int_0^T u(t, c^*(t) - z^*(t))\, dt$$

is satisfied (i.e., the investor os better off when unconstrained). Proposition 15 also provides further intuition about the persistence of a binding constraint until τ^*. Indeed, prior to time τ^* the investor wishes to choose a net consumption policy as close to the unconstrained net consumption as possible. The best that can be done is to build down the standard of living as much as possible, which means forgoing consumption completely. This intuition prevails until τ^*: this is the first time at which net consumption is, effectively, unconstrained.

Proof of Proposition 14: Under the assumption of the proposition, we have $y^* = y_u \le \overline{y}$, thus $\tau_u = \tau^* = 0$; and with $y = y_u = y^*$, we have

$$\begin{aligned} z^*(t) - z_u(t) &= \delta\, e^{(\delta-\alpha)t} \int_0^t (G^y(s) - F^y(s))\, ds \\ &= \delta\, e^{(\delta-\alpha)t} \int_0^t (Y^y(s) - F^y(s))^+\, ds \;=\; 0 \end{aligned}$$

and

$$\begin{aligned} [c^*(t) - z^*(t)] - [c_u(t) - z_u(t)] &= e^{(\delta-\alpha)t}\,(G^y(t) - F^y(t)) \\ &= e^{(\delta-\alpha)t}\,(Y^y(t) - F^y(t))^+ \;=\; 0 \end{aligned}$$

for all $0 \le t \le T$.

Proof of Proposition 15: Under the assumption of the proposition, we have $0 < \overline{y} < y_u < y^* < \infty$ and $0 < \tau_u < \tau^* \le T$. The first claim follows directly from (56). For the other claims, observe that we have

$$Y^{y^*}(t) = Y^{y_u}(t) = Y^{\infty}(t) > F^{y_u}(t) > F^{y^*}(t), \qquad \text{for} \quad 0 \le t < \tau_u$$

(with the first inequality valid as equality for $t = \tau_u$), thus

$$\begin{aligned} z^*(t) - z_u(t) &= \delta\, e^{(\delta-\alpha)t} \int_0^t \left(G^{y^*}(s) - F^{y_u}(s)\right) ds \\ &= \delta\, e^{(\delta-\alpha)t} \int_0^t (Y^{\infty}(s) - F^{y_u}(s))^+\, ds \;>\; 0\,, \quad 0 \le t \le \tau_u \end{aligned}$$

and

$$\begin{aligned} [c^*(t) - z^*(t)] - [c_u(t) - z_u(t)] &= e^{(\delta-\alpha)t} \left(G^{y^*}(t) - F^{y_u}(t)\right) \\ &= e^{(\delta-\alpha)t}\,(Y^{\infty}(t) - F^{y_u}(t))^+ \;>\; 0\,, \quad 0 \le t < \tau_u \end{aligned}$$

(with equality for $t = \tau_u$). All claims now follow.

6 Conclusion.

In this paper we resolved the consumption-portfolio problem when the investor's preferences exhibit non-addictive habits. Existence of an optimal consumption-portfolio policy was demonstrated. The consumption function was found to have unusual properties. Of particular interest is the fact that the optimal consumption is null up to an endogenously determined stopping time, after which the non-negativity constraint ceases to be binding. The source of this striking behavior is the non-separability of the utility function over time. Another unusual feature is the decomposition of the optimal consumption policy, which involves an Asian average-strike capped call-option. The rationale for the presence of a path-dependent option in this decomposition, is again a consequence of the influence of habits on the investor's behavior.

Our formulation of preferences has also important implications for the study of equilibrium prices and allocations. This becomes clear, once we note that the standard model with an Inada condition $u'(t,0+) = \infty$ is unable to accommodate an aggregate dividend process which is lognormally distributed. Thus, typical assumptions about aggregate consumption are simply unsustainable in the standard pure exchange model with habit formation. In contrast, our setting can readily accommodate any non-negative aggregate dividend process. Our model can then be easily calibrated and used in order to quantify the effects of habits on the structure and properties of prices and the interest rate. The equilibrium implications of this work will be taken up in a subsequent paper.

7 Appendix.

Proof of the equivalence of (26) and (27): Suppose that (26) holds, and set $\varphi(t) \equiv e^{-\alpha t}(\gamma(t) - H(t))$ as well as $\widetilde{H}(t) \equiv \delta e^{-\alpha t} H(t), t \in [0,T]$. Then (26) reads

$$\varphi(t) \le E\left(\int_t^T \left(\delta\varphi(s) + \widetilde{H}(s)\right) ds\right), \quad t \in [0,T].$$

From Appendix B in Duffie and Epstein (1992), this last inequality implies

$$\varphi(t) \le E\left(\int_t^T e^{\delta(s-t)} \widetilde{H}(s) ds\right), \quad t \in [0,T]$$

which is (27). On the other hand, suppose that (27) holds, and set

$$\widehat{\varphi}(t) \equiv e^{(\delta-\alpha)t}(\gamma(t) - H(t)) \quad \text{and} \quad \widehat{\gamma}(t) \equiv \delta e^{(\delta-\alpha)t}\gamma(t);$$

then (27) becomes

$$\widehat{\varphi}(t) \le E_t\left(\int_t^T (\widehat{\gamma}(s) - \delta\widehat{\varphi}(s)) ds\right),$$

and Appendix B from Duffie and Epstein (1992) implies $\widehat{\varphi}(t) \le E\left(\int_t^T e^{-\delta(s-t)}\widehat{\gamma}(s) ds\right)$, $t \in [0,T]$, which is (26). $\diamondsuit$

The solution of Equation (19): With $h(t) \equiv y\gamma(t)e^{-\alpha t}$, $g(t) \equiv (y - \xi(t))H(t)e^{-\alpha t}$, we can rewrite the Backwards Stochastic Equation (19) in the form

$$h(t) = g(t) + \delta \cdot E_t\left(\int_t^T h(s) ds\right), \quad t \in [0,T].$$

Clearly $h(T) = g(T)$, and

$$h(t) + \delta \int_0^t h(s)ds = g(t) + \delta\, M(t)\,, \quad t \in [0,T]$$

where $M(t) := E_t \left(\int_0^T h(s)ds \right)$, $t \in [0,T]$ is a martingale. Now this integral equation is solved via integration-by-parts, as follows:

$$\begin{aligned} h(T)\, e^{\delta T} \; - \; h(t)\, e^{\delta t} \; &= \; \int_t^T e^{\delta s}[dh(s) + \delta\, h(s)ds] \\ &= \; \int_t^T e^{\delta s}[dg(s) + \delta\, dM(s)] \\ &= \; g(T)\, e^{\delta T} \; - \; g(t)\, e^{\delta t} - \delta \int_t^T e^{\delta s} g(s)ds + \delta \int_t^T e^{\delta s} dM(s)\,, \end{aligned}$$

or equivalently

$$h(t) = g(t) + \delta \int_t^T e^{\delta(s-t)} g(s)ds - \delta \int_t^T e^{\delta(s-t)} dM(s).$$

Taking conditional expectations, we obtain $h(t) = g(t) + \delta \cdot E_t \left(\int_t^T e^{\delta(s-t)} g(s)ds \right)$, $t \in [0,T]$, which is (12). $\diamondsuit$

8 References.

1. Chapman, D. A. "Habit Formation and Aggregate Consumption," *Econometrica*, 66, 1998: 1223-1230.

2. Constantinides, G. M. "Habit Formation: A Resolution of the Equity Premium Puzzle," *Journal of Political Economy*, 98, 1990: 519-543.

3. Cox, J. C., and Huang, C.-F. "Optimal consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process," *Journal of Economic Theory*, 49, 1989: 33-83

4. Detemple, J. B. and Zapatero, F. "Asset Prices in an Exchange Economy with Habit Formation," *Econometrica*, 59, 1991: 1633-1657.

5. Detemple, J. B. and Zapatero, F. "Optimal Consumption-Portfolio Policies with Habit Formation," *Mathematical Finance*, 2, 1992: 35-58.

6. Duffie, D. and Epstein, L., "Stochastic Differential Utility," *Econometrica*, 60, 1992: 353-394.

7. Heaton, J. "The Interaction between Time-Nonseparable Preferences and Time Aggregation," *Econometrica*, 61, 1993: 353-385.

8. Hicks, J. *Capital and Growth*, New York, Oxford University Press, 1965.

9. Ikeda, N. and Watanabe, S., *Stochastic Differential Equations and Diffusion Processes*, Second Edition, North-Holland, Amsterdam, and Kodansha Ltd., Tokyo, 1989.

10. Karatzas, I., Lakner, P., Lehoczky, J. P. and Shreve, S. E. "Dynamic Equilibrium in a Multi-Agent Economy: Construction and Uniqueness," in *Stochastic Analysis: Liber Amicorum for Moshe Zakai* (Meyer-Wolf, Schwartz and Zeitouni, eds.), Academic Press, New York, 1991: 245-272.

11. Karatzas, I., Lehoczky, J. P. and Shreve, S. E. "Optimal Portfolio and Consumption Decisions for a 'Small Investor' on a Finite Horizon," *SIAM Journal of Control and Optimization*, 25, 1987: 1557-1586.

12. Karatzas, I. and Shreve, S. E., *Methods of Mathematical Finance*, Springer-Verlag, New York, 1998.

13. Merton, R. C. "Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case," *Review of Economics and Statistics*, 51, 1969: 247-257.

14. Merton, R. C. "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory*, 3, 1971: 373-413.

15. Pollak R. A. "Habit Formation and Dynamic Demand Functions," *Journal of Political Economy*, 78, 1970: 745-763.

16. Ryder, H. E. and Heal, G. M. "Optimal Growth with Intertemporally Dependent Preferences," *Review of Economic Studies*, 40, 1973: 1-33.

17. Schroder, M. and Skiadas, C. "An Isomorphism Between Asset-Pricing with and without Habit-Formation," *Preprint,* Northwestern University, Evanston, 2000.

18. Sundaresan, S. M. "Intertemporally Dependent Preferences and the Volatility of consumption and Wealth," *Review of Financial Studies* , 2, 1989: 73-89.