CONNECTIONS BETWEEN BOUNDED-VARIATION CONTROL AND DYNKIN GAMES

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Abstract

A general result is obtained for the existence of saddle-point in a stochastic game of timing, by exploiting its connection with a bounded-variation control problem. Weak compactness arguments prove the existence of an optimal process for the control problem. It is shown that this optimal process generates a pair of stopping times that constitute a saddle-point for the game, using the method of comparing costs at nearby points by switching paths at appropriate random times.

1 Introduction and Summary

In three very important and influential papers, Bensoussan & Friedman [4], [5] and Friedman [11] developed the analytical theory of stochastic differential games with stopping times. Their setting is that of a Markov diffusion process, and of a pair of players, each of whom can chose when to terminate the process. At that time, and depending on who made the decision to stop, one of the players (the "minimizer") pays the other ("maximizer") a certain random amount. The problem then is for one player to minimize, and for the other to maximize, the expected value of this payoff. Bensoussan and Friedman studied the value and saddle-points of such a game using appropriate partial differential equations, variational inequalities, and free-boundary problems.

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Detailed expositions of their approach can be found in their monographs Bensoussan & Lions [6], pp. 462-493, and Friedman [12].

Along a parallel track, the purely probabilistic approach to such games of timing (stopping) was being developed in progressively greater generality, by Dynkin [9], Krylov [19], Neveu [22], Bismut [7], Alario-Nazaret [1], Stettner [24], Alario-Nazaret, Lepeltier & Marchal [2], Lepeltier & Maingeneau [20], Morimoto [21], among others. This theory relied on the martingale approach to the problem of optimal stopping introduced by Snell [23], but now applied in the more challenging setup of two coupled such stopping problems. More recently, this problem has been connected by Cvitanić & Karatzas [8] to the solution of backwards stochastic differential equations with upper- and lower-constraints on the state-process (see also Hamadène & Lepeltier [13], [14]), and has received pathwise (Karatzas [15]) and mixed-strategy (Touzi & Vieille [26]) treatments.

This paper presents another approach to the general, non-Markovian stochastic game of timing ("Dynkin game"), and brings it into contact with a suitable bounded-variation or *singular* control problem. We show that the game has a value, which coincides with the derivative of the value-function of the auxiliary control problem. An optimal process to this latter problem generates in a simple way a pair of stopping times, that constitute a saddle-point for the game of timing; see Theorems 3.1, 3.2. Then a result of Komlós [18] is coupled with weak compactness arguments, to show that such an optimal process indeed exists; cf. Theorem 3.3.

The approach is very direct and "pathwise". It uses the method of *comparing costs at nearby points by switching paths at appropriate random times*, initiated by Karatzas & Shreve ([16], [17]) in the context of optimal-stopping/reflected-follower problems for Brownian motion, and further developed in Baldursson & Karatzas [3]. The connections of singular stochastic control with games of stopping were first noticed by Taksar [25], in a Markovian context and using very different (analytical) methods.

2 The Two Stochastic Optimization Problems

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t \le T}$ which satisfies the usual conditions (of right-continuity and augmentation by $\mathbb{P}$-null sets). We denote by $\mathcal{A}$ the class of increasing, left-continuous and $\mathbb{F}$-adapted processes $\zeta : [0,T] \times \Omega \to [0,\infty)$ with $\zeta(0) = 0$, and by $\mathcal{B}$ the class of processes $\xi : [0,T] \times \Omega \to \mathbb{R}$ that can be written in the form

$$\xi(t) = \xi^+(t) - \xi^-(t), \qquad 0 \le t \le T \tag{2.1}$$

for two processes $\xi^\pm \in \mathcal{A}$. We assume in fact that the decomposition of (2.1) is *minimal*, and thus the total variation of the function $s \mapsto \xi(s)$ on any interval $[0,t]$ is given by

$$\check{\xi}(t) = \xi^+(t) + \xi^-(t), \qquad \forall\ 0 \le t \le T \tag{2.2}$$

almost surely. In the control problem of (2.5), (2.6) below, $\xi^+(t)$ (resp., $\xi^-(t)$) represents the total cumulative push in the positive (resp., negative) direction exerted up to time t, so that

$$X(t) := x + \xi(t), \qquad 0 \le t \le T \tag{2.3}$$

represents the *state* (or position) at time t, when starting at $X(0) = x \in \mathbb{R}$ and employing the "strategy" $\xi \in \mathcal{B}$. We shall denote by $\mathcal{S}$ the set of all $\mathbb{F}$-stopping times $\rho : \Omega \to [0,T]$. In order to formulate the two optimization problems that will occupy us in this paper, let us introduce:

• A random field $H : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$, such that $(t,\omega) \mapsto H(t,\omega,x)$ is $\mathbb{F}$-progressively measurable for every $x \in \mathbb{R}$, and $x \mapsto H(t,\omega,x)$ is convex and of class $\mathcal{C}^1$ for every $(t,\omega) \in [0,T] \times \Omega$.
• Two continuous, $\mathbb{F}$-adapted processes $\gamma : [0,T] \times \Omega \to [0,\infty)$ and $\nu : [0,T] \times \Omega \to [0,\infty)$.
• A random field $G : \Omega \times \mathbb{R} \to \mathbb{R}$, such that $G(\cdot, x)$ is $\mathcal{F}(T)$-measurable for every $x \in \mathbb{R}$, and $G(\omega, \cdot)$ is convex and of class $\mathcal{C}^1$ for every $\omega \in \Omega$.

On these random functions we shall impose throughout the regularity requirements

$$\mathbb{E}\left[\int_0^T |H_x(t,x)|\, dt + \left|G'(x)\right|\right] < \infty, \quad \forall\, x \in \mathbb{R} \qquad \text{and} \qquad \mathbb{E}\left[\sup_{0\le t\le T} \gamma(t) + \sup_{0\le t\le T} \nu(t)\right] < \infty. \tag{2.4}$$

2.1 A Bounded-Variation Control Problem

Suppose that when in state $X(t) = y \in \mathbb{R}$ at time $t \in [0,T)$, we incur a running cost $H(t,y)$ per unit of time, and a running cost $\gamma(t)$ (resp., $\nu(t)$) per unit of fuel spent to push in the positive (resp., negative) direction. Suppose also that we incur a terminal cost $G(x)$ for being in position $X(T) = y$ at the terminal time $t = T$. Then we should try to find a strategy $\xi^* \in \mathcal{B}$ that attains

$$V(x) \stackrel{\triangle}{=} \inf_{\xi\in\mathcal{B}} \mathbb{E}[J(\xi;x)]\,, \tag{2.5}$$

the *minimal expected cost* of our stochastic control problem with

$$J(\xi;x) \stackrel{\triangle}{=} \int_0^T H\big(t, X(t)\big)dt + \int_{[0,T)} \gamma(t) d\xi^+(t) + \int_{[0,T)} \nu(t) d\xi^-(t) + G\big(X(T)\big)\,, \tag{2.6}$$

in the notation of (2.1)-(2.3). If there exists a strategy $\xi^* \in \mathcal{B}$ that attains the infimum in (2.5), it is called *optimal for the initial condition* x.

2.2 A Game of Timing (Dynkin Game)

Consider two players, $\mathcal{N}$ and $\mathcal{G}$, each of whom chooses a stopping time (σ and τ, respectively) in $\mathcal{S}$. The game terminates at $\sigma \wedge \tau$, that is, as soon as one of the players decides to stop. If player $\mathcal{G}$ stops first, he pays $\mathcal{N}$ the amount $\gamma(\tau)$. If player $\mathcal{N}$ stops first, he pays $\mathcal{G}$ the amount $\nu(\sigma)$ (resp. $G'(x)$) when the game terminates before (resp., at) the end of the time-horizon T; and as long as the game is in progress, $\mathcal{N}$ keeps paying $\mathcal{G}$ at the rate $H_x(t,x)$ per unit of time. In other words, the *total payment from* $\mathcal{N}$ *to* $\mathcal{G}$ is given by the expression

$$I(\sigma,\tau;x) \stackrel{\triangle}{=} \int_0^{\sigma\wedge\tau} H_x(t,x)\, dt + \nu(\sigma) 1_{\{\sigma<\tau\}} - \gamma(\tau) 1_{\{\tau<\sigma\}} + G'(x) 1_{\{\sigma=\tau=T\}}, \tag{2.7}$$

a random variable whose expectation $\mathcal{N}$ tries to minimize and $\mathcal{G}$ to maximize. We obtain a *stochastic game of timing* (or "Dynkin game", after Dynkin [9]) with lower- and upper- values

$$\underline{u}(x) \stackrel{\triangle}{=} \sup_{\tau\in\mathcal{S}} \inf_{\sigma\in\mathcal{S}} \mathbb{E}[I(\sigma,\tau;x)] \;\le\; \bar{u}(x) \stackrel{\triangle}{=} \inf_{\sigma\in\mathcal{S}} \sup_{\tau\in\mathcal{S}} \mathbb{E}[I(\sigma,\tau;x)]\,, \tag{2.8}$$

respectively. We say that the game *has value* $u(x)$, if $\underline{u}(x) = \bar{u}(x) = u(x)$. A pair $(\sigma_*, \tau_*) \in \mathcal{S}^2$ is called *saddle-point of the game*, if

$$\mathbb{E}[I(\sigma_*,\tau;x)] \;\le\; \mathbb{E}[I(\sigma_*,\tau_*;x)] \;\le\; \mathbb{E}[I(\sigma,\tau_*;x)] \qquad \text{holds for all } \sigma\in\mathcal{S}\,,\ \tau\in\mathcal{S}\,. \tag{2.9}$$

It is easy to see, that the existence of a saddle-point implies that the game has value

$$u(x) = \mathbb{E}[I(\sigma_*, \tau_*; x)] = \underline{u}(x) = \bar{u}(x). \tag{2.10}$$

3 Results

The stochastic control problem of subsection 2.1 and the stochastic game of subsection 2.2 turn out to be very closely connected. Under the conditions of Section 2, it can be proved that every optimal strategy for the control problem induces a saddle-point for the game (Theorem 3.1), whose value $u(\cdot)$ is then shown to coincide with the derivative $V'(\cdot)$ of the value of the control problem (Theorem 3.2). And under certain additional conditions on the random quantities H, G, γ and ν, it can be shown that such an optimal strategy for the control problem indeed exists (Theorem 3.3). Consequently, *under the conditions of Theorem 3.3, the Dynkin Game of section 2.2 has a saddle-point*, and this is of the type (3.1) below.

Theorem 3.1. *Suppose that the process $\xi_* \in \mathcal{B}$ is optimal for the control problem of subsection 2.1, i.e., attains the infimum in (2.5). Write this process in its minimal decomposition $\xi_* = \xi_*^+ - \xi_*^-$ as in (2.1), and define the stopping times*

$$\sigma_* \stackrel{\triangle}{=} \inf\left\{t \in [0,T) \;/\; \xi_*^-(t) > 0\right\} \wedge T, \quad \tau_* \stackrel{\triangle}{=} \inf\left\{t \in [0,T) \;/\; \xi_*^+(t) > 0\right\} \wedge T. \tag{3.1}$$

Then the pair $(\sigma_*, \tau_*) \in \mathcal{S}^2$ is a saddle-point for the game of subsection 2.2, whose value is given by

$$\underline{u}(x) = \bar{u}(x) = u(x) \stackrel{\triangle}{=} \mathbb{E}[I(\sigma_*, \tau_*; x)]\,. \tag{3.2}$$

Theorem 3.2. *Under the conditions of Theorem 3.1, the value function $V(\cdot)$ of the control problem of (2.5), (2.6) is differentiable, and we have*

$$V'(x) = u(x), \qquad \forall\, x \in \mathbb{R}. \tag{3.3}$$

Suppose now that, in addition to the conditions in Section 2, the random functions H, G, γ and ν satisfy

$$\mathbb{E}\left[\int_0^T \left(\sup_{x\in\mathbb{R}} H^-(t,x)\right) dt + \left(\sup_{x\in\mathbb{R}} G^-(t,x)\right)\right] < \infty, \tag{3.4}$$

and the a.s. conditions

$$-\gamma(T) \le G'(x) \le \nu(T), \qquad \forall\, x \in \mathbb{R} \tag{3.5}$$

$$\gamma(t) \ge \kappa\,, \quad \nu(t) \ge \kappa\,; \quad \forall\, t \in [0,T] \qquad \text{for some } \kappa > 0. \tag{3.6}$$

Theorem 3.3. *Under the conditions (3.4)-(3.6), in addition to those of Section 2, the control problem of (2.5), (2.6) admits an optimal process $\xi_* \in \mathcal{B}$.*

Corollary 3.1. *With the assumption of Theorem 3.3, the Dynkin game of subsection 2.2 has a saddle-point and a value given by (3.1)-(3.2), and the relation (3.3) holds.*

4 Proofs of Theorems 3.1, 3.2

We begin with a simple observation.

Lemma 4.1. *The function $V(\cdot)$ of (2.5) is proper, convex.*

Proof : Convexity of $V(\cdot)$ is a consequence of the convexity of the functions $H(t,\omega,\cdot)$, $G(\omega,\cdot)$, and follows by taking infima over $\zeta \in \mathcal{B}$, $\eta \in \mathcal{B}$ on the right-hand side of the inequality

$$(4.1)\quad V\big(\lambda x_1+(1-\lambda)x_2\big) \le \mathbb{E}[J\big(\lambda x_1+(1-\lambda)x_2;\lambda\zeta+(1-\lambda)\eta\big)] \le \lambda\,\mathbb{E}[J(x_1;\zeta)]+(1-\lambda)\,\mathbb{E}[J(x_2;\eta)]\,,$$

valid for $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$, $0 \le \lambda \le 1$; and (2.4) implies $V(x) \le \mathbb{E}\left[\int_0^T H(t,x)\,dt + G(x)\right] < \infty$ for every $x \in \mathbb{R}$, which gives properness. □

As a corollary of convexity, the right- and left- derivatives

$$(4.2)\qquad D^{\pm}(x) \triangleq \lim_{h\to 0\pm} \frac{V(x+h)-V(x)}{h} \qquad \text{exist, and we have} \quad D^{-}V(x) \le D^{+}V(x)\,, \quad \forall\, x \in \mathbb{R}.$$

Lemma 4.2. *Under the assumption of Theorem 3.1, we have in the notation of (2.7), (3.1), (4.2):*

$$(4.3)\qquad D^{+}V(x) \;\le\; \mathbb{E}[I(\sigma,\tau_*;x)] \qquad \textit{for every} \quad x \in \mathbb{R}\,,\; \sigma \in \mathcal{S}\,.$$

Proof : We compare costs at nearby points using the technique of "switching paths at appropriate random times" introduced in Karatzas & Shreve [16]. Let us consider the stopping times

$$(4.4)\qquad \tau_\varepsilon \triangleq \inf\big\{t \in [0,T)\;/\;\xi_*^{+}(t) \ge \varepsilon\big\} \wedge T, \qquad 0<\varepsilon<1$$

and notice that $\tau_\varepsilon \downarrow \tau_*$ as $\varepsilon \downarrow 0$, a.s. Introduce also the auxiliary process

$$(4.5)\qquad \xi_\varepsilon(t) \triangleq \left\{ \begin{array}{lcl} -\xi_*^{-}(t) & ; & 0 \le t \le \sigma\wedge\tau_\varepsilon \\ \xi_*(t)-\varepsilon & ; & \sigma\wedge\tau_\varepsilon < t \le T \end{array} \right\} \;\in\; \mathcal{B}$$

for each given $\sigma \in \mathcal{S}$, $0<\varepsilon<1$. The state-process $\tilde{X}_\varepsilon(\cdot) = x+\varepsilon+\xi_\varepsilon(\cdot)$ corresponds to the strategy of "starting at $x+\varepsilon$ and following a modification of the optimal strategy $\xi_*(\cdot)$ for x, whereby we suppress any movement to the right up to time $\sigma\wedge\tau_\varepsilon$; after which we jump onto the optimal path $X_*(\cdot) = x+\xi_*(\cdot)$ for x, and follow it up to time T". The cost associated with this strategy is

$$\begin{aligned}
(4.6)\qquad J(x+\varepsilon;\xi_\varepsilon) &= \int_0^{\sigma\wedge\tau_*} H\big(t,x+\varepsilon-\xi_*^{-}(t)\big)\,dt + \int_{\sigma\wedge\tau_*}^{\sigma\wedge\tau_\varepsilon} H\big(t,x+\varepsilon-\xi_*^{-}(t)\big)\,dt \\
&+ \int_{\sigma\wedge\tau_\varepsilon}^{T} H\big(t,x+\xi_*^{+}(t)-\xi_*^{-}(t)\big)\,dt \;+\; G\big(x+\xi_*^{+}(T)-\xi_*^{-}(T)\big)\cdot 1_{\{\sigma\wedge\tau_\varepsilon<T\}} \\
&+ G\big(x+\varepsilon-\xi_*^{-}(T)\big)\cdot 1_{\{\sigma=\tau_*=T\}} \;+\; G\big(x+\varepsilon-\xi_*^{-}(T)\big)\cdot 1_{\{\tau^*<\sigma=\tau_\varepsilon=T\}} \\
&+ 1_{\{\tau_\varepsilon\le\sigma\}}\left(\gamma(\tau_\varepsilon)\cdot\big(\xi_*^{+}(\tau_\varepsilon+)-\varepsilon\big) + \int_{(\tau_\varepsilon,T)} \gamma(t)\,d\xi_*^{+}(t) + \int_{[0,T)} \nu(t)\,d\xi_*^{-}(t)\right) \\
&+ 1_{\{\tau_*\le\sigma<\tau_\varepsilon\}}\left(\int_{(\sigma,T)} \gamma(t)\,d\xi_*^{+}(t) \;+\; \nu(\sigma)\cdot\big(\varepsilon-\xi_*^{+}(\sigma+)\big) + \int_{[0,T)} \nu(t)\,d\xi_*^{-}(t)\right) \\
&+ 1_{\{\sigma<\tau_*\}}\left(\int_{[\tau_*,T)} \gamma(t)\,d\xi_*^{+}(t) + \int_{[0,T)} \nu(t)\,d\xi_*^{-}(t) + \varepsilon\nu(\sigma)\right).
\end{aligned}$$

In a similar vein, we have the decomposition

$$
\begin{aligned}
(4.7)\qquad J(x;\xi_*) \;=\; & \int_0^{\sigma\wedge\tau_*} H\big(t, x-\xi_*^-(t)\big)\,dt + \int_{\sigma\wedge\tau_*}^{\sigma\wedge\tau_\varepsilon} H\big(t, x+\xi_*^+(t)-\xi_*^-(t)\big)\,dt \\
+ & \int_{\sigma\wedge\tau_\varepsilon}^{T} H\big(t, x+\xi_*^+(t)-\xi_*^-(t)\big)\,dt + G\big(x+\xi_*^+(T)-\xi_*^-(T)\big)\cdot 1_{\{\sigma\wedge\tau_\varepsilon<T\}} \\
+ & \; G\big(x-\xi_*^-(T)\big)\cdot 1_{\{\sigma=\tau_*=T\}} + G\big(x+\xi_*^+(T)-\xi_*^-(T)\big)\cdot 1_{\{\tau_*<\sigma=\tau_\varepsilon=T\}} \\
+ & \; 1_{\{\tau_\varepsilon\le\sigma\}}\Bigg(\int_{[\tau_*,\tau_\varepsilon)}\gamma(t)\,d\xi_*^+(t) + \gamma(\tau_\varepsilon)\cdot\big(\xi_*^+(\tau_\varepsilon+)-\xi_*^+(\tau_\varepsilon)\big) \\
& \qquad\qquad + \int_{(\tau_\varepsilon,T)}\gamma(t)\,d\xi_*^+(t) + \int_{[0,T)}\nu(t)\,d\xi_*^-(t)\Bigg) \\
+ & \; 1_{\{\tau_*\le\sigma<\tau_\varepsilon\}}\Bigg(\int_{[\tau_*,\sigma)}\gamma(t)\,d\xi_*^+(t) + \int_{(\sigma,T)}\gamma(t)\,d\xi_*^+(t) + \int_{[0,T)}\nu(t)\,d\xi_*^-(t)\Bigg) \\
+ & \; \gamma(\sigma)\,[\,\xi_*^+(\sigma+)-\xi_*^+(\sigma)\,]\,1_{\{\tau_*\le\sigma<\tau_\varepsilon\}} \;+\; 1_{\{\sigma<\tau_*\}}\Bigg(\int_{[\tau_*,T)}\gamma(t)\,d\xi_*^+(t) + \int_{[0,T)}\nu(t)\,d\xi_*^-(t)\Bigg)
\end{aligned}
$$

for the cost corresponding to the optimal process $\xi_*\in\mathcal{B}$ at position $x\in\mathbb{R}$. Comparing (4.7) with (4.6), we see that $V(x+\varepsilon)-V(x)$ is dominated by $\mathbb{E}[J(x+\varepsilon;\xi_\varepsilon)]-\mathbb{E}[J(x;\xi_*)]$, namely

$$
\begin{aligned}
V(x+\varepsilon) \;-\; V(x) \;\le\; & \mathbb{E}\Bigg[\int_0^{\sigma\wedge\tau_*}\big[H\big(t,x+\varepsilon-\xi_*^-(t)\big)-H\big(t,x-\xi_*^-(t)\big)\big]\,dt\Bigg] \\
& + \int_{\sigma\wedge\tau_*}^{\sigma\wedge\tau_\varepsilon}\big[H\big(t,x+\varepsilon-\xi_*^-(t)\big)-H\big(t,x+\xi_*^+(t)-\xi_*^-(t)\big)\big]\,dt \\
(4.8)\qquad & + \big[G\big(x+\varepsilon-\xi_*^-(T)\big)-G\big(x-\xi_*^-(T)\big)\big]\cdot 1_{\{\sigma=\tau_*=T\}} \\
& + \big[G\big(x+\varepsilon-\xi_*^-(T)\big)-G\big(x+\xi_*^+(T)-\xi_*^-(T)\big)\big]\cdot 1_{\{\tau_*<\sigma=\tau_\varepsilon=T\}} \\
& - 1_{\{\tau_\varepsilon\le\sigma\}}\Bigg(\gamma(\tau_\varepsilon)\,\big(\varepsilon-\xi_*^+(\tau_\varepsilon)\big)+\int_{[\tau_*,\tau_\varepsilon)}\gamma(t)\,d\xi_*^+(t)\Bigg) + \varepsilon\nu(\sigma)\cdot 1_{\{\sigma<\tau_*\}} \\
& -\Bigg[1_{\{\tau_*\le\sigma<\tau_\varepsilon\}}\Bigg([\gamma(\sigma)+\nu(\sigma)]\,\big(\xi_*^+(\sigma+)-\varepsilon\big)+\gamma(\sigma)\,(\varepsilon-\xi_*^+(\sigma))+\int_{[\tau_*,\sigma)}\gamma(t)\,d\xi_*^+(t)\Bigg)\Bigg]\,.
\end{aligned}
$$

Let us look at all the terms under the expectation on the right-hand side of (4.8), one by one. From the convexity of the functions $H(t,\omega,\cdot)$ and $G(\omega,\cdot)$, we deduce that the first and second terms are dominated by

$$
\varepsilon\int_0^{\sigma\wedge\tau_*} H_x(t,x+\varepsilon)\,dt \qquad\text{and}\qquad \int_{\sigma\wedge\tau_*}^{\sigma\wedge\tau_\varepsilon}\big(\varepsilon-\xi_*^+(t)\big)\,H_x(t,x+\varepsilon)\,dt\,,
$$

while the third and fourth are dominated by

$$
\varepsilon G'(x+\varepsilon)\cdot 1_{\{\sigma=\tau_*=T\}} \qquad\text{and}\qquad \big(\varepsilon-\xi_*^+(T)\big)\,G'(x+\varepsilon)\cdot 1_{\{\tau_*<\sigma=\tau_\varepsilon=T\}}\,,
$$

respectively. We also observe that the fifth term can be written in the form

$$-\varepsilon\gamma(\tau_*)1_{\{\tau_*<\sigma\}}+\varepsilon\left[\gamma(\tau_*)1_{\{\tau_*<\sigma\}}-\gamma(\tau_\varepsilon)1_{\{\tau_\varepsilon\le\sigma\}}\right]+1_{\{\tau_\varepsilon\le\sigma\}}\left[\gamma(\tau_\varepsilon)\xi_*^+(\tau_\varepsilon)-\int_{[\tau_*,\tau_\varepsilon)}\gamma(t)\,d\xi_*^+(t)\right].$$

We can now put these observations together, and deduce from (4.8) the inequality

$$\frac{V(x+\varepsilon)-V(x)}{\varepsilon}\;\le\;\mathbb{E}[I(\sigma,\tau_*;x)]+\sum_{j=1}^{7}L_j(\varepsilon),\qquad\text{where}\tag{4.9}$$

$$\begin{aligned}
L_1(\varepsilon)&\triangleq\mathbb{E}\int_0^{\sigma\wedge\tau_*}\left[H_x(t,x+\varepsilon)dt-H_x(t,x)\right]dt\,,\qquad L_2(\varepsilon)\triangleq\mathbb{E}\int_{\sigma\wedge\tau_*}^{\sigma\wedge\tau_\varepsilon}|H_x(t,x+\varepsilon)|\,dt\\
L_3(\varepsilon)&\triangleq\mathbb{E}\left|\,\gamma(\tau_*)\,1_{\{\tau_*<\sigma\}}\;-\;\gamma(\tau_\varepsilon)\,1_{\{\tau_\varepsilon\le\sigma\}}\,\right|\\
L_4(\varepsilon)&\triangleq\mathbb{E}\left[\frac{1}{\varepsilon}\cdot\left|\gamma(\tau_\varepsilon)\xi_*^+(\tau_\varepsilon)\;-\;\int_{[\tau_*,\tau_\varepsilon)}\gamma(t)d\xi_*^+(t)\right|\,1_{\{\tau_\varepsilon\le\sigma\}}\right]\\
L_5(\varepsilon)&\triangleq\mathbb{E}\left[\frac{1}{\varepsilon}\cdot\left|\nu(\sigma)\big(\varepsilon-\xi_*^+(\sigma+)\big)-\int_{[\tau_*,\sigma)}\gamma(t)d\xi_*^+(t)\right|\,1_{\{\tau_*\le\sigma<\tau_\varepsilon\}}\right]\\
L_6(\varepsilon)&\triangleq\mathbb{E}\left|G'(x+\varepsilon)\cdot1_{\{\tau_*<\tau^\varepsilon\}}\right|\,,\qquad L_7(\varepsilon)\triangleq\mathbb{E}\left|G'(x+\varepsilon)-G'(x)\right|\,.
\end{aligned}$$

It is not hard to see that $\lim_{\varepsilon\downarrow0}L_j(\varepsilon)=0$, for all $j=1,\cdots,7$, thanks to the dominated convergence theorem and the conditions of (2.4). For instance, $L_3(\varepsilon)\;\le\;\mathbb{E}\left[|\gamma(\tau_\varepsilon)-\gamma(\tau_*)|+|\gamma(\tau_*)|1_{\{\tau_*<\sigma<\tau_\varepsilon\}}\right]$,

$$\begin{aligned}
L_4(\varepsilon)&\le\mathbb{E}\left[\frac{\xi_*^+(\tau_\varepsilon)}{\varepsilon}\cdot\sup_{\tau_*\le t<\tau_\varepsilon}|\gamma(\tau_\varepsilon)-\gamma(t)|\right]\;\le\;\mathbb{E}\left[\sup_{\tau_*\le t<\tau_\varepsilon}|\gamma(\tau_\varepsilon)-\gamma(t)|\right],\\
L_5(\varepsilon)&\le\mathbb{E}\left[1_{\{\tau_*\le\sigma<\tau_\varepsilon\}}\cdot\left(\nu(\sigma)\left(1-\frac{\xi_*^+(\sigma+)}{\varepsilon}\right)+\frac{\xi_*^+(\sigma)}{\varepsilon}\cdot\max_{\tau_*\le t\le\sigma}\gamma(t)\right)\right]\\
&\le\mathbb{E}\left[\max_{0\le t\le T}\big(\gamma(t)+\nu(t)\big)\cdot1_{\{\tau_*\le\sigma<\tau_\varepsilon\}}\right],
\end{aligned}$$

all tend to zero, as $\varepsilon\downarrow0$; and (4.3) follows. □

Lemma 4.3. *With the same assumptions and notation as in Lemma 4.2, we have*

$$D^-V(x)\;\ge\;\mathbb{E}[I(\sigma_*,\tau;x)]\,,\quad\textit{for every}\;\;x\in\mathbb{R}\,,\;\tau\in\mathcal{S}\,.\tag{4.10}$$

Sketch of Proof: The situation is completely symmetric to that of Lemma 4.2, so we just sketch the broad outline of the argument. For each given $\tau\in\mathcal{S}$, $0<\varepsilon<1$, we introduce the analogue $\sigma_\varepsilon\triangleq\inf\left\{t\in[0,T)\;/\;\xi_*^-(t)\ge\varepsilon\right\}\wedge T$, $0<\varepsilon<1$ of the stopping time in (4.4), as well as the analogue

$$\vartheta_\varepsilon(t)\triangleq\left\{\begin{array}{ccc}\xi_*^+(t)&;&0\le t\le\tau\wedge\sigma_\varepsilon\\ \xi_*(t)+\varepsilon&;&\tau\wedge\sigma_\varepsilon<t\le T\end{array}\right\}$$

of the auxiliary process in (4.5), and note that $\mathbb{P}[\lim_{\varepsilon\downarrow0}\downarrow\sigma_\varepsilon=\sigma_*]=1$. The state-process $X'_\varepsilon(\cdot)=x-\varepsilon+\vartheta_\varepsilon(\cdot)$ corresponds then to the strategy of "starting at $x-\varepsilon$ and following a modification of $\xi_*(\cdot)$, whereby we suppress any movement to the left, up to the time $\tau \wedge \sigma_\varepsilon$; after which we follow the optimal state-process $X_*(\cdot) = x + \xi_*(\cdot)$ for x, up to time T". Comparing the cost $J(x-\varepsilon;\vartheta_\varepsilon)$ of this strategy to the cost $J(x;\xi_*)$ of the optimal strategy at x, taking expectations, dividing by $\varepsilon > 0$, and then letting $\varepsilon \downarrow 0$, we arrive at (4.10) – much as we derived (4.3) through the comparisons (4.6)–(4.9).

Proof of Theorems 3.1, 3.2 : From (4.2), (4.3) and (4.10), we obtain

$$D^- V(x) \;\leq\; D^+ V(x) \;\leq\; \mathbb{E}[I(\sigma_*, \tau_*; x)] \;\leq\; D^- V(x), \qquad \forall\, x \in \mathbb{R}\,.$$

This shows that $V(\cdot)$ is differentiable with $V'(x) = \mathbb{E}[I(\sigma_*, \tau_*; x)]$, proving (3.3). Again from (4.2), (4.3) we conclude now $\mathbb{E}[I(\sigma_*, \tau; x)] \leq V'(x) = \mathbb{E}[I(\sigma_*, \tau_*; x)] \leq \mathbb{E}[I(\sigma, \tau_*; x)]$ for every $\sigma \in \mathcal{S}$, $\tau \in \mathcal{S}$, proving (3.2). □

5 Proof of Theorem 3.3

Let $\{\eta_n(\cdot)\}_{n\in\mathbb{N}} \subseteq \mathcal{B}$ be a minimizing sequence of processes for the control problem of (2.5)-(2.6), that is, $\lim_{n\to\infty} \mathbb{E}[J(\eta_n;x)] = V(x) < \infty$. From the conditions (3.4) and (3.6), we have

(5.1)

$$\sup_{n\in\mathbb{N}} \mathbb{E}[\check{\eta}_n(T)] \;\leq\; \frac{1}{\kappa}\left[\sup_{n\in\mathbb{N}} \mathbb{E}[J(\eta_n;x)] + \mathbb{E}\int_0^T \left(\sup_{x\in\mathbb{R}} H^-(t,x)\right) dt + \mathbb{E}\left(\sup_{x\in\mathbb{R}} G^-(x)\right)\right] =: M < \infty.$$

It follows that $\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^T |\eta_n(T)|\, dt \leq \sup_{n\in\mathbb{N}} \mathbb{E}\int_0^T \check{\eta}_n(T)\, dt \leq MT < \infty$, so that *the sequence of processes* $\{\eta_n(\cdot)\}_{n\in\mathbb{N}}$ *is bounded in* $\mathbb{L}^1(\lambda \otimes \mathbb{P})$, where λ denotes Lebesgue measure on $[0,T]$. Thus, thanks to a theorem of Komlós [18], there exists a subsequence (relabelled, and still denoted by $\{\eta_n(\cdot)\}_{n\in\mathbb{N}}$) and a pair of $\mathcal{B}([0,T]) \otimes \mathcal{F}(T)-$measurable processes $\vartheta^\pm : [0,T]\times\Omega \to [0,\infty)$, such that the Cesàro sequences of processes

$$\left\{ \xi_n^\pm(\cdot) \;\triangleq\; \frac{1}{n}\sum_{j=1}^n \eta_j^\pm(\cdot) \right\}_{n\in\mathbb{N}} \subseteq \mathcal{A} \quad \text{converge} \;\; (\lambda\otimes\mathbb{P})-\text{a.e. to} \;\; \vartheta^\pm(\cdot). \tag{5.2}$$

Define $\xi_n(\cdot) \triangleq \xi_n^+(\cdot) - \xi_n^-(\cdot) \in \mathcal{B}$, for every $n \in \mathbb{N}$. ¿From the convexity of $J(\cdot, x)$ in (2.6), it is clear that $\{\xi_n(\cdot)\}_{n\in\mathbb{N}} \subseteq \mathcal{B}$ *is also a mininizing sequence*, namely

$$\lim_{n\to\infty} \mathbb{E}[J(\xi_n;x)] = V(x) < \infty\,, \tag{5.3}$$

and from (5.2) that the sequence of processes

$$\left\{ \xi_n(\cdot) \triangleq \frac{1}{n}\sum_{j=1}^n \eta_j(\cdot) \right\}_{n\in\mathbb{N}} \quad \text{converges} \;\; (\lambda\otimes\mathbb{P})-\text{a.e. to} \;\; \vartheta(\cdot) \triangleq \vartheta^+(\cdot) - \vartheta^-(\cdot)\,. \tag{5.4}$$

Using Komlós's theorem one last time, we deduce from (5.1) the existence of two $\mathcal{F}(T)-$measurable random variables $\zeta^\pm : \Omega \to [0,\infty)$, such that the sequences of random variables

$$\left\{ \xi_n^\pm(T) \;=\; \frac{1}{n}\sum_{j=1}^n \eta_j^\pm(T) \right\}_{n\in\mathbb{N}} \quad \text{converge} \quad \mathbb{P}-\text{a.s. to} \;\; \zeta^\pm \tag{5.5}$$

and thus $\lim_{n\to\infty} \xi_n(T) = \zeta \triangleq \zeta^+ - \zeta^-$, $\mathbb{P}-$a.s. (possibly after passing to a further, relabelled, subsequence, for which each of (5.2), (5.4) and (5.5) holds).

Proposition 5.1. *The processes $\vartheta^\pm(\cdot)$ of (5.2) have modifications $\xi_*^\pm(\cdot) \in \mathcal{A}$ (i.e., $\mathbb{F}$-adapted, left-continuous, increasing, with $\xi_*^\pm(0) = 0$) that satisfy*

$$\lim_{n\to\infty} \xi_n^\pm(t) = \xi_*^\pm(t), \qquad \mathbb{P} - a.s. \tag{5.6}$$

for $\lambda-a.e.$ $t \in [0,T]$. In particular, we have the analogue

$$\lim_{n\to\infty} \xi_n(t) \; = \; \xi_*(t) \; = \; \xi_*^+(t) - \xi_*^-(t)\,, \qquad \mathbb{P}-a.s. \tag{5.7}$$

of (5.6), for $\lambda-a.e.$ $t \in [0,T]$, where the process $\xi_*(\cdot) \triangleq \xi_*^+(\cdot) - \xi_*^-(\cdot)$ belongs to $\mathcal{B}$, as well as

$$\xi_*^\pm(T) \; \le \; \zeta^\pm\,, \qquad \mathbb{P}-a.s. \tag{5.8}$$

This result can be proved in a manner completely analogous to Lemmata 4.5–4.7, pp. 867–869 in Karatzas & Shreve [16].

Proposition 5.2. *For the process $\xi_*^\pm(\cdot) \in \mathcal{A}$, $\xi_*(\cdot) \in \mathcal{B}$ of Proposition 5.1, we have*

$$V(x) \; \ge \; \mathbb{E}\left[\int_0^T H\big(t, x+\xi_*(t)\big)\,dt + \int_{[0,T)} \gamma(t) d\xi_*^+(t) + \int_{[0,t)} \nu(t) d\xi_*^-(t) + G\big(x+\xi_*(T)\big)\right]. \tag{5.9}$$

Corollary 5.3: *The process $\xi_*(\cdot)$ of (5.7) is optimal for the control problem of (2.5), (2.6).*

Proof: This is clear if the decomposition $\xi_*(\cdot) = \xi_*^+(\cdot) - \xi_*^-(\cdot)$ is minimal, because then (5.9) reads $V(x) \ge \mathbb{E}[J(\xi_*; x)]$. But even if this decomposition is not minimal, the right-hand side of (5.9) dominates $\mathbb{E}[J(\xi_*; x)]$, and the conclusion follows as before. $\square$

Proof of Proposition 5.2 : From (5.5), (5.7), the assumption (3.4), and Fatou's Lemma, we obtain

$$\mathbb{E}\int_0^T H\big(t, x+\xi_*(t)\big)\,dt \;\; \le \;\; \liminf_{n\to\infty}\; \mathbb{E}\int_0^T H\big(t, x+\xi_n(t)\big)\,dt\,, \tag{5.10}$$

$$\mathbb{E}[G(x+\zeta)] \;\; \le \;\; \liminf_{n\to\infty}\; \mathbb{E}[G\big(x+\xi_n(T)\big)]. \tag{5.11}$$

It can also be shown that we have

$$\mathbb{E}\left[\textstyle\int_{[0,T)} \gamma(t) d\xi_*^+(t) + \gamma(T)\left(\zeta^+ - \xi_*^+(T)\right)\right] \;\; \le \;\; \liminf_{n\to\infty}\; \mathbb{E}\textstyle\int_{[0,T)} \gamma(t) d\xi_n^+(t) \tag{5.12}$$

$$\mathbb{E}\left[\textstyle\int_{[0,T)} \nu(t) d\xi_*^-(t) + \nu(T)\left(\zeta^- - \xi_*^-(T)\right)\right] \;\; \le \;\; \liminf_{n\to\infty}\; \mathbb{E}\textstyle\int_{[0,T)} \nu(t) d\xi_n^-(t)\,. \tag{5.13}$$

Indeed, denote by $\mathcal{T}$ the set of full Lebesgue measure in $\mathcal{B}\big([0,T]\big)$ on which (5.6) and (5.7) hold. For any partition $\Pi = \{t_i\}_{i=1}^m \subseteq \mathcal{T}$ of $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, we have

$$\sum_{i=1}^m \left(\inf_{t_{i-1}\le t\le t_i} \gamma(t)\right) \cdot \left[\xi_n^+(t_i) - \xi_n^+(t_{i-1})\right] + \left(\inf_{t_m \le t \le T} \gamma(t)\right) \left[\xi_n^+(T) - \xi_n^+(t_m)\right] \le \int_{[0,T)} \gamma(t) d\xi_n^+(t)$$

$\mathbb{P}-$a.s. and, by (5.5) and Fatou's lemma, $\liminf_{n\to\infty} \mathbb{E}\int_{[0,T)} \gamma(t)d\xi_n^+(t)$ dominates the expression

$$\mathbb{E}\left[\sum_{i=1}^{m}\left(\inf_{t_{i-1}\le t\le t_i}\gamma(t)\right)\cdot\left[\xi_*^+(t_i)-\xi_*^+(t_{i-1})\right]+\left(\inf_{t_m\le t\le T}\gamma(t)\right)\cdot\left[\zeta^+-\xi_n^+(t_m)\right]\right].$$

As we let $\|\Pi\| \stackrel{\triangle}{=} \max_{0\le i\le m}|t_{i+1}-t_i| \to 0$, this expression approaches the left-hand side of (5.12), which is then established; (5.13) is proved similarly. Now let us put (5.10)-(5.13) together; in conjunction with (5.3), and using repeatedly the inequality $\liminf_n x_n + \limsup_n y_n \le \limsup_n(x_n + y_n)$, we obtain

$$\mathbb{E}\left[\int_{[0,T)} H\big(t, x+\xi_*(t)\big) + \int_{[0,T)}\gamma(t)d\xi_*^+(t) + \int_{[0,T)}\nu(t)d\xi_*^-(t) + G\big(x+\xi_*(T)\big)\right] + \mathbb{E}[\Theta]$$

$$\le \limsup_{n\to\infty} \mathbb{E}[J(\xi_n;x)] = V(x), \quad \text{where} \tag{5.14}$$

$$\Theta \stackrel{\triangle}{=} G(x+\zeta) - G\big(x+\xi_*(T)\big) + \gamma(T)\left[\zeta^+ - \xi_*^+(T)\right] + \nu(T)\left[\zeta^- - \xi_*^-(T)\right]. \tag{5.15}$$

In order to deduce (5.9) from (5.14), it suffices to show that the random variable of (5.15) satisfies $\mathbb{P}[\Theta \ge 0] = 1$; indeed, (3.5), (5.5), (5.8) and the convexity of $G(\omega,\cdot)$ imply

$$\begin{aligned} G(x+\zeta) - G\big(x+\xi_*(T)\big) &= G(x+\zeta^+-\zeta^-) - G\big(x+\xi_*(T)^+ - \xi_*(T)^-\big) \\ &\ge G'\big(x-\zeta^-\big)\left[\zeta^+ - \xi_*(T)^+\right] - G'\big(x+\zeta^+\big)\left[\zeta^- - \xi_*(T)^-\right] \\ &\ge -\gamma(T)\left[\zeta^+ - \xi_*^+(T)\right] - \nu(T)\left[\zeta^- - \xi_*^-(T)\right], \qquad \text{a.s.} \end{aligned}$$

whence $\Theta \ge 0$, a.s. □

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