A NOTE ON BAYESIAN DETECTION OF CHANGE-POINTS WITH AN EXPECTED MISS CRITERION *

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Synopsis

A process X is observed continuously in time; it behaves like Brownian motion with drift, which changes from zero to a known constant $\vartheta > 0$ at some time τ that is not directly observable. It is important to detect this change when it happens, and we attempt to do so by selecting a stopping rule T_* that minimizes the "expected miss" $\mathbf{E}\,|T - \tau|$ over all stopping rules T. Assuming that τ has an exponential distribution with known parameter $\lambda > 0$ and is independent of the driving Brownian motion, we show that the optimal rule T_* is to declare that the change has occurred, at the first time t for which

$$\lambda \int_0^t e^{\vartheta (X_t - X_s) + \left(\lambda - \frac{\vartheta^2}{2}\right)(t-s)}\, ds \;\geq\; \frac{p_*}{1-p_*}\,.$$

Here, with $\Lambda = 2\lambda/\vartheta^2$, the constant p_* is uniquely determined in $\left(\frac{1}{2}, 1\right)$ by the equation

$$\int_0^{1/2} \frac{(1-2\pi)\, e^{-\Lambda/\pi}}{\left(1-\pi\right)^{2+\Lambda}\pi^{2-\Lambda}}\, d\pi \;=\; \int_{1/2}^{p_*} \frac{(2\pi-1)\, e^{-\Lambda/\pi}}{\left(1-\pi\right)^{2+\Lambda}\pi^{2-\Lambda}}\, d\pi\,.$$

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1. Introduction: Consider a Brownian Motion $X = \{X_t\,,\, 0 \le t < \infty\}$ which has zero drift during a time-interval $[0, \tau)$, and drift $\vartheta > 0$ during the interval $[\tau, \infty)$. The change-point τ that demarcates the two régimes is neither known in advance nor directly observable; in contrast, the value of the drift $\vartheta \in (0, \infty)$ is assumed to be known in advance. In each of the two régimes $[0, \tau)$ and $[\tau, \infty)$ we employ a specific strategy (e.g., medical treatment, manufacturing procedure, advertising or investment methodology, military strategy, harvesting policy, etcetera), and the strategies corresponding to the two régimes are distinctly different. We seek a stopping rule T that *detects* the instant τ of "régime change" *as accurately as possible.*

To qualify this phrase, imagine that we are being penalized during an interval of "false alarm" at the same rate as during an interval of "delay in sounding the alarm". The lengths of these intervals are $(\tau - T)^+$ and $(T - \tau)^+$, respectively. Thus, our loss is measured by the sum

$$(\tau - T)^+ \,+\, (T - \tau)^+ \;=\; |T - \tau|$$

of those lengths, namely, the amount by which the stopping rule T misses the change-point τ. We shall describe explicitly a stopping rule T_* that minimizes the *expected miss*

$$\mathcal{R}(T) \;:=\; \mathbf{E}\,|T - \tau| \tag{1.1}$$

over all stopping rules T, when the change-point τ is assumed to have an exponential distribution $\mathbf{P}[\tau > t] = (1 - p)\, e^{-\lambda t}\,,\ t \ge 0$ with known coëfficients $0 \le p < 1\,,\ \lambda > 0\,.$ This is achieved by reducing the above problem to a question of optimal stopping for a Markov process on the unit interval $I = (0, 1)$, namely, the conditional probability $\mathbf{P}\,[\,\tau \le t\,|\,\mathcal{F}_t\,]$ that the régime change has occurred by time t, given the observations available up to that time.

With only minor modifications, the solution method presented here can also handle criteria that put different weights on delay vs. false alarm, of the type

$$\mathcal{R}(T) \;=\; \mathbf{E}\left[\,(\tau - T)^+ + c \cdot (T - \tau)^+\,\right], \qquad \text{for some } \ c \in (0, \infty)\,. \tag{1.2}$$

It is a much harder question of *simultaneous detection and estimation*, and one that we are actively pursuing as of this writing, to allow for uncertainty in the drift ϑ of the model and perhaps in the coëfficients λ and p as well.

Problems of a similar nature have been studied in the literature for many years. Most notably, Shiryaev (1969) considers the minimization of expected delay $\mathbf{E}\,(T - \tau)^+$, subject to the constraint $\mathbf{P}\,[T < \tau] \le \alpha$ on the probability of false alarm, for some given $\alpha \in (0, 1)$; he also considers the minimization of $\mathbf{P}\,[T < \tau] + c \cdot \mathbf{E}\,(T - \tau)^+$, for some constant $c \in (0, \infty)$. Let us also mention the work of Beibel (1996), (2000) who considers quite similar models and problems.

Criteria of the type (1.1) or (1.2), though rather natural for several applications, appear to be studied in this Note for the first time. The optimal stopping rules for all these problems involve first-passage times for the *a posteriori probability* process $\mathbf{P}\,[\,\tau \le t \,|\, \mathcal{F}_t\,]$, $0 \le t < \infty$: you declare that régime change has occurred when this probability first reaches or exceeds a suitable threshold.

2. The Problem: On a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, consider a standard Brownian motion process $W = \{W_t\,,\, 0 \le t < \infty\}$ and an independent random variable $\tau : \Omega \to [0,\infty)$ with distribution

$$\mathbf{P}[\tau > t] \;=\; (1-p)\, e^{-\lambda t}\,, \quad \forall\; 0 \le t < \infty \tag{2.1}$$

for some known constants $0 \le p < 1$ and $0 < \lambda < \infty$. In particular, $\mathbf{P}[\tau = 0] = p$. Neither the random variable τ nor the Brownian motion W are directly observable; instead, one observes the process

$$X_t \;=\; W_t \,+\, \vartheta \int_0^t 1_{\{\tau \le s\}}\, ds\,, \qquad 0 \le t < \infty\,. \tag{2.2}$$

This is a Brownian motion with drift zero in the interval $[0,\tau)$, and with known drift $\vartheta > 0$ in the interval $[\tau,\infty)$. In other words, the observations that we have at our disposal at time t are modelled by the $\sigma-$algebra

$$\mathcal{F}_t \;:=\; \sigma(\,X_s\,,\, 0 \le s \le t\,)\,, \qquad \text{for each} \;\; t \in [0,\infty)\,, \tag{2.3}$$

and we set $\mathcal{F}_\infty := \sigma\left(\cup_{0 \le t < \infty} \mathcal{F}_t\right)$. Our objective is to *detect the change-point* τ *as accurately as possible*. For this, we look at the collection $\mathcal{S}$ of stopping rules $T : \Omega \to [0,\infty]$ of the filtration $\mathbf{F} = \{\mathcal{F}_t\}_{0 \le t < \infty}$ (that is, with $\{T \le t\} \in \mathcal{F}_t$ for all $0 \le t < \infty$), and seek to

$$\textit{minimize the expected miss} \;\; \mathcal{R}(T) \;=\; \mathbf{E}\,|T - \tau| \;\; \textit{over all stopping rules} \;\; T \in \mathcal{S} \tag{2.4}$$

in the notation of (1.1). In fact, it is enough to consider stopping rules with $\mathbf{E}(T) < \infty$; for otherwise we get $\mathbf{E}\,|T-\tau| = \infty$, from $T \le \tau + |T - \tau|$ and $\mathcal{R}(0) = \mathbf{E}(\tau) = \frac{1-p}{\lambda} < \infty$. We shall denote by $\mathcal{S}_f$ the collection of stopping rules $T \in \mathcal{S}$ with $\mathbf{E}(T) < \infty$.

In order to make headway with this question (2.4), let us observe that we can write the expected miss of (1.1) as

$$\mathcal{R}(T) \;=\; \mathbf{E}\left[\,(T-\tau)^+ \,+\, (\tau - T)^+\,\right] \;=\; \mathbf{E}\,(T-\tau)^+ \,+\, \mathbf{E}(\tau) \,-\, \mathbf{E}\,(T \wedge \tau)\,,$$

or equivalently as

$$\mathcal{R}(T) \,-\, \mathbf{E}(\tau) \,=\, \mathbf{E}\left(\int_0^T 1_{\{\tau\le t\}}\,dt \,-\, \int_0^T 1_{\{\tau> t\}}\,dt\right)$$

$$= \mathbf{E}\int_0^\infty \Big(2\,1_{\{\tau\le t\}} - 1\Big)\,1_{\{T>t\}}\,dt \,=\, 2\cdot\mathbf{E}\int_0^T \left(\Pi_t - \frac{1}{2}\right)\,dt\,,$$

for any stopping time $T\in\mathcal{S}_f$ and with the notation

$$\Pi_t \,:=\, \mathbf{P}\,[\,\tau\le t\,|\,\mathcal{F}_t\,]\,, \qquad 0\le t<\infty\,. \tag{2.5}$$

Thus, we have shown that the minimum risk (expected miss) function for this problem, is of the form

$$R(p) \,:=\, \inf_{T\in\mathcal{S}} \mathcal{R}(T) \,=\, \frac{1-p}{\lambda} \,+\, 2\,V(p)\,, \quad \text{where} \quad V(p) \,:=\, \inf_{T\in\mathcal{S}_f} \mathbf{E}\int_0^T \left(\Pi_t - \frac{1}{2}\right)\,dt\,. \tag{2.6}$$

3. The "a posteriori probability" process: The detection problem with the minimum expected miss criterion of (1.1) has been thus reduced to solving the optimal stopping problem of (2.6). This latter problem is formulated in terms of the process $\Pi \,=\, \{\,\Pi_t\,,\, 0\le t<\infty\,\}$, where the quantity Π_t of (2.5) is the conditional (a posteriori) probability that the régime change has already occurred by time t, given the observations available up to that time. This *a posteriori* probability is thus a sufficient statistic for our detection problem. It can be computed explicitly in terms of the exponential likelihood-ratio process

$$L_t \,:=\, \exp\left(\vartheta X_t - \frac{\vartheta^2}{2}\,t\right)\,, \tag{3.1}$$

in the form

$$\Pi_t \,=\, \frac{pL_t + (1-p)\int_0^t \lambda e^{-\lambda s}\big(L_t/L_s\big)\,ds}{pL_t + (1-p)\int_0^t \lambda e^{-\lambda s}\big(L_t/L_s\big)\,ds + (1-p)\,e^{-\lambda t}}\,. \tag{3.2}$$

Clearly, $\Pi_0 = p$ as well as $\mathbf{P}[0<\Pi_t<1\,,\,\forall\, t\in(0,\infty)]=1$. It turns out that we have

$$\mathbf{P}\left[\lim_{t\to\infty}\Pi_t \,=\, 1\right] \,=\, 1\,, \quad \text{and} \quad \mathbf{P}\left[\inf_{0\le t<\infty}\Pi_t \,>\, 0\right] \,=\, 1 \quad \text{if} \quad p>0\,. \tag{3.3}$$

One can also show that the *a posteriori probability* process Π satisfies the stochastic differential equation

$$d\,\Pi_t \,=\, \lambda(1-\Pi_t)\,dt \,+\, \vartheta\,\Pi_t\,(1-\Pi_t)\,dB_t\,, \qquad \Pi_0 = p\,, \tag{3.4}$$

where

$$B_t \;:=\; X_t - \vartheta \int_0^t \Pi_s \, ds\,, \quad 0 \le t < \infty \tag{3.5}$$

is the innovations process of filtering theory: a standard, $\mathbf{F}-$adapted Brownian motion (note that W is *not* adapted to $\mathbf{F}$). It follows from (3.4) that Π is a diffusion process on the unit interval $I = (0,1)$, with drift $b(\pi) = \lambda(1-\pi)$ and diffusion coëfficient $\sigma^2(\pi) = \vartheta^2 \pi^2 \big(1-\pi\big)^2$. Its scale function $S(\cdot)$ and speed measure $m(d\pi)$ are given by

$$S(\pi) \;=\; \int_{1/2}^{\pi} S'(u)\, du\,, \quad \text{where} \quad S'(\pi) \;=\; \exp\left[-2 \int_{1/2}^{\pi} \frac{b(u)\, du}{\sigma^2(u)} \right] \;=\; e^{-2\Lambda} \left(\frac{1-\pi}{\pi} \right)^{\Lambda} e^{\Lambda/\pi} \tag{3.6}$$

and

$$m(d\pi) \;=\; \frac{2\, d\pi}{S'(\pi)\, \sigma^2(\pi)} \;=\; C e^{2\Lambda} \cdot \frac{e^{-\Lambda/\pi}\, d\pi}{\pi^{2-\Lambda} \big(1-\pi\big)^{2+\Lambda}}\,, \tag{3.7}$$

respectively; we have set $C \;:=\; 2/\vartheta^2$ and $\Lambda := \lambda\, C$. The scale function $S(\cdot)$ satisfies the differential equation

$$b(\pi)\, S'(\pi) \;+\; \frac{1}{2} \sigma^2(\pi)\, S''(\pi) \;=\; 0\,, \quad \pi \in (0,1)\,. \tag{3.8}$$

It is straightforward to check from (3.6) that $S(0+) \;=\; -\infty$ and $S(1) \;\equiv\; S(1-) \;<\; \infty$. Then, the properties of (3.3) follow from the standard theory of one-dimensional diffusion processes (see, for instance, Proposition 5.22, p. 345 in Karatzas & Shreve (1991)). Justifications for the claims (3.2) and (3.4) are provided in the Appendix.

Remark 1: It follows from (3.8), (3.4) and Itô's rule, that the process

$$S(1) \;-\; S(\Pi_t) \;=\; S(1) \;-\; S(p) \;-\; \int_0^t S'(\Pi_u)\, \sigma(\Pi_u)\, dB_u\,, \qquad 0 \le t < \infty$$

is a positive local martingale, hence also a supermartingale (with respect to the filtration $\mathbf{F}$). From Theorem 2 in Elworthy et al. (1997) we know that this process is not a martingale (its expectation is strictly decreasing), so $S(\Pi)$ provides a concrete example of what these authors call "strictly local martingales".

Remark 2: From the definition (2.5), it is clear that $1 - \Pi_t \;=\; \mathbf{P}\,[\,\tau > t \,|\, \mathcal{F}_t\,]$, $0 \le t < \infty$ is a non-negative supermartingale; its expectation $t \mapsto \mathbf{E}\big(1-\Pi_t\big) \;=\; \mathbf{P}\,[\,\tau \;>\; t\,]$, given by (2.1), is continuous and decreases to zero as $t \;\to\; \infty$. Such a supermartingale is called a *potential*, and it is well-known (e.g. Karatzas & Shreve (1991), page 18) that $\Pi_\infty = \lim_{t\to\infty} \Pi_t$ exists a.s., that $\{1 - \Pi_t\,,\, 0 \le t \le \infty\}$ is a supermartingale (with a last element), and that in fact $\Pi_\infty = 1$, a.s. This establishes direcely the first claim in (3.3).

4. A Variational Inequality: Suppose that we can find a function $Q:[0,1]\to(-\infty,0]$ of class $\mathcal{C}^1\big([0,1]\big)\cap\mathcal{C}^2\big((0,1]\setminus\{p_*\}\big)$ for some $p_*\in\big(\frac{1}{2},1\big)$, that satisfies the following properties:

$$\lambda(1-\pi)\,Q'(\pi)\,+\,\frac{\vartheta^2}{2}\,\pi^2\,(1-\pi)^2\,Q''(\pi)\;=\;\frac{1}{2}-\pi\;;\quad\text{for}\;\;0<\pi<p_* \tag{4.1}$$

$$\lambda(1-\pi)\,Q'(\pi)\,+\,\frac{\vartheta^2}{2}\,\pi^2\,(1-\pi)^2\,Q''(\pi)\;>\;\frac{1}{2}-\pi\;;\quad\text{for}\;\;p_*<\pi<1 \tag{4.2}$$

$$Q(\pi)\;<\;0\;;\quad\text{for}\;\;0\le\pi<p_* \tag{4.3}$$

$$Q(\pi)\;=\;0\;;\quad\text{for}\;\;p_*\le\pi\le 1\,. \tag{4.4}$$

In particular, $Q'(\cdot)$ is supposed to be continuous, thus also bounded, on the entire interval $[0,1]$, including the point $\pi=p_*$. In other words, we are postulating as an Ansatz the "principle of smooth-fit":

$$Q'(p_*-)\;=\;Q'(p_*+)\;=\;0\,. \tag{4.5}$$

If we can find such a function $Q(\cdot)$, with the properties of (4.1)−(4.4), then for any stopping rule $T\in\mathcal{S}$ with $\mathbf{P}[T<\infty]=1$ we obtain

$$Q(\Pi_T)\;=\;Q(p)\,+\int_0^T\Big[\,\lambda(1-\Pi_t)\,Q'(\Pi_t)\,+\,\frac{\vartheta^2}{2}\,\Pi_t^2\,(1-\Pi_t)^2\,Q''(\Pi_t)\Big]\,dt$$
$$+\;\vartheta\int_0^T\Pi_t\,(1-\Pi_t)\,Q'(\Pi_t)\,dB_t$$

from Itô's rule applied to the process Π of (3.4), and from (4.1), (4.2):

$$Q(\Pi_T)\;\ge\;Q(p)\,+\int_0^T\left(\frac{1}{2}-\Pi_t\right)\,dt\,+\,\vartheta\int_0^T\Pi_t\,(1-\Pi_t)\,Q'(\Pi_t)\,dB_t\;,\quad\text{a.s.} \tag{4.6}$$

If in addition $T\in\mathcal{S}_f$, the expectation of the stochastic integral in (4.6) is equal to zero, because then

$$\mathbf{E}\int_0^T\big(\Pi_t\,(1-\Pi_t)\big)^2\,\big(Q'(\Pi_t)\big)^2\,dt\;\le\;\frac{1}{4}\Big(\max_{0\le\pi\le 1}|Q'(\pi)|\Big)^2\cdot\mathbf{E}(T)\;<\;\infty\,.$$

This way we obtain from (4.6), in conjunction with (4.3) and (4.4):

$$\mathbf{E}\int_0^T\left(\Pi_t-\frac{1}{2}\right)\,dt\;\ge\;Q(p)\;-\;\mathbf{E}\,[Q(\Pi_T)\,]\;\ge\;Q(p)\;,\quad\forall\;T\in\mathcal{S}_f\,. \tag{4.7}$$

Is there a stopping rule $T^*\in\mathcal{S}_f$ for which

$$\mathbf{E}\int_0^{T_*}\left(\Pi_t-\frac{1}{2}\right)\,dt\;=\;Q(p) \tag{4.8}$$

holds ? A look at (4.1)$-$(4.4) suggests that we can achieve this by taking

$$T_* := \inf\{ t \geq 0 \mid \Pi_t \geq p_* \} , \tag{4.9}$$

because for this choice all the inequalities in (4.6), (4.7) become equalities, *provided we can show* $\mathbf{E}\,(T_*) < \infty$.

Now it is clear from (3.3) that the stopping rule T_* of (4.9) is a.s. finite. The stronger property $\mathbf{E}\,(T_*) < \infty$ follows again from the general theory of one-dimensional diffusion processes. Indeed, it is well known that the expectation of a first-passage time of the form (4.9) is given by

$$\mathbf{E}\,(T_*) = \frac{1 - \frac{S(p)}{S(0+)}}{1 - \frac{S(p_*)}{S(0+)}} \cdot \int_0^{p_*} \big[\, S(p_*) - S(\pi) \,\big]\, m(d\pi)$$
$$- \int_0^{p} \big[\, S(p) - S(\pi) \,\big]\, m(d\pi) \;, \quad \text{for } \; 0 \leq p < p_*$$

(cf. Karatzas & Shreve (1991), pp. 343, 344); for $p_* \leq p \leq 1$ we have $T_* = 0$ a.s., trivially from (4.9). The above expression involves the scale function $S(\cdot)$ and the speed-measure m from (3.6), (3.7); but in our case $S(0+) = -\infty$ as we have pointed out, so

$$\mathbf{E}\,(T_*) = \big[\, S(p_*) - S(p) \,\big] \cdot m\big((0,p]\big) + \int_p^{p_*} \big[\, S(p_*) - S(\pi) \,\big]\, m(d\pi) \,. \tag{4.10}$$

It is checked from (3.7) that $m\big((0,p]\big) < \infty$ holds for any $0 < p < 1$, and that the integral on the right-hand side of (4.10) is finite, leading to $\mathbf{E}\,(T_*) < \infty$.

Remark 3: It can be seen from this analysis that $\int_0^1 \big[\, S(1) - S(\pi) \,\big]\, m(d\pi) < \infty$, so the origin $p = 0$ is an *entrance boundary* for the diffusion process Π . In particular, if $\Pi_0 = 0$, the process becomes positive immediately after time $t = 0$ and never again visits the origin. This is corroborated, of course, by the the explicit expression of (3.2).

5. Computing the Optimal Stopping Risk: From the discussion of the previous section we know that if we can find a function $Q(\cdot)$ that satisfies the tenets of the Variational Inequality (4.1)$-$(4.4), then $Q(\cdot)$ coincides with the value function $V(\cdot)$ of (2.6) (in particular, the Variational Inequality has a unique solution), and the minimal risk function in (2.6) is

$$R(p) = \inf_{T \in \mathcal{S}} \mathbf{E}\,|T - \tau| = \frac{1-p}{\lambda} + 2\,Q(p) = \mathbf{E}\,|T_* - \tau| \,. \tag{5.1}$$

In other words, the stopping rule $T_* \in \mathcal{S}_f$ of (4.9) is then optimal for the sequential detection problem of minimizing the expected miss of (1.1) over all stopping rules $T \in \mathcal{S}$. All this follows readily from (4.7), (4.8) and (2.6).

How then do we find the function $Q(\cdot)$ with all the desired properties? We proceed as follows.

Analysis: If $Q(\cdot)$ solves the Variational Inequality of (4.1)$-$(4.4), then

$$\frac{d}{d\pi}\left[\frac{Q'(\pi)}{S'(\pi)}\right] = h(\pi) := \frac{C\,\left(\frac{1}{2}-\pi\right)}{\pi^2(1-\pi)^2\cdot S'(\pi)} = \left(\frac{1}{2}-\pi\right)\frac{m(d\pi)}{d\pi}\,, \quad 0<\pi<p_* \tag{5.2}$$

from (4.1), (3.8) and (3.7), thus

$$\frac{Q'(p)}{S'(p)} = -\int_p^{p_*} h(u)\,du\,, \quad 0\le p\le p_* \tag{5.3}$$

in conjunction with $Q'(p_*)=0$ of (4.5). Integrating once again, this time using $Q(p_*)=0$ from (4.4), we obtain

$$Q(\pi) = \int_\pi^{p_*}\left(\int_p^{p_*} h(u)\,du\right)S'(p)\,dp = \int_\pi^{p_*}[S(u)-S(\pi)]\,h(u)\,du\,, \quad 0\le\pi\le p_*\,. \tag{5.4}$$

We need another condition on $Q(\cdot)$, to determine the still unknown constant p_*. It turns out that the right condition is

$$Q'(0+) = \frac{1}{2\lambda}\,, \tag{5.5}$$

as we shall justify shortly in (5.8) below. The condition (5.5) follows formally by letting $\pi\downarrow 0$ in the equation (4.1), assuming that $\lim_{\pi\downarrow 0}\left(\pi^2 Q''(\pi)\right)=0$; it also corresponds to the intuitive notion that we should have $R'(0+)=0$ for the slope at the origin of the minimum risk function $R(\cdot)$ in (2.6), (5.1).

Synthesis: For an arbitrary but fixed $p_*\in(\frac{1}{2},1)$, define the function

$$Q(\pi) := \left\{\begin{array}{cl} \int_\pi^{p_*}[S(u)-S(\pi)]\,h(u)\,du & ;\ \ 0\le\pi\le p_* \\ 0 & ;\ \ p_*\le\pi\le 1\end{array}\right\}\,, \tag{5.6}$$

in accordance with (5.4) and (4.4). This function is of class $\mathcal{C}^1\big((0,1]\big)\cap\mathcal{C}^2\big((0,1]\setminus\{p_*\}\big)$ because we have assured the smooth-fit condition (4.5); it satisfies conditions (4.1), (4.4) by construction, and (4.2) because $p_*>\frac{1}{2}$. It remains to select p_* so that (4.3) is also satisfied.

We claim that for this it is enough to ensure $\left(\frac{Q'}{S'}\right)(0+)=0$ – equivalently, from (5.3), to select $p_*\in\left(\frac{1}{2},1\right)$ so that we have $\int_0^{p_*}h(u)\,du=0$, or more suggestively:

$$\int_0^{1/2} h(u)\,du = \int_{1/2}^{p_*}\big(-h(u)\big)\,du\,. \tag{5.7}$$

This last condition determines the desired p_* uniquely. Indeed, the function

$$D(p) := \int_{1/2}^{p} \big(-h(u)\big)\, du \;=\; Ce^{2\Lambda} \int_{1/2}^{p} \frac{e^{-\Lambda/\pi}\,\big(\pi - \frac{1}{2}\big)}{\big(1-\pi\big)^{2+\Lambda}\pi^{2-\Lambda}}\, d\pi\,, \qquad \frac{1}{2} < p < 1$$

suggested by the right-hand side of (5.7), is strictly increasing with $D\left(\frac{1}{2}+\right) = 0$ and $D(1-) = \infty$. Thus there exists a unique $p_* \in \left(\frac{1}{2}, 1\right)$ such that $D(p_*) = G$, where

$$G := \int_0^{1/2} h(u)\, du = Ce^{2\Lambda} \int_0^{1/2} \frac{\big(\frac{1}{2} - \pi\big)\, e^{-\Lambda/\pi}}{\pi^{2-\Lambda}\big(1-\pi\big)^{2+\Lambda}}\, d\pi = \int_0^{1/2} \left(\frac{1}{2} - \pi\right) m(d\pi) \in (0, \infty)$$

is the left-hand side of (5.6).

In fact, with p_* thus determined, we have from (5.3) and the rule of L'Hôpital:

$$Q'(0+) := \lim_{\pi \downarrow 0} Q'(\pi) \;=\; \lim_{\pi \downarrow 0} \frac{\int_\pi^{p_*} h(u)\, du}{\big(\frac{-1}{S'(\pi)}\big)} \;=\; \lim_{\pi \downarrow 0} \frac{h(\pi)\big(S'(\pi)\big)^2}{\big(-S''(\pi)\big)} \;=\; \frac{1}{2\lambda} \tag{5.8}$$

thanks to (3.8) and (3.6), (5.2). This way, $Q'(\cdot)$ and also $Q(\cdot)$ can be extended continuously all the way down to $\pi = 0$, and $Q(\cdot)$ is then of class $\mathcal{C}^1\big([0,1]\big) \cap \mathcal{C}^2\big((0,1] \setminus \{p_*\}\big)$; furthermore, $\lim_{\pi \downarrow 0}\big(\pi^2 Q''(\pi)\big) = \frac{1}{\vartheta^2}\left[1 - 2\lambda \cdot Q'(0+)\right] = 0$ from (5.8), (5.2).

With p_* thus selected, let us look at the function $F(\cdot) := Q'(\cdot)/S'(\cdot)$. We have by construction $F(0+) = 0$, $F(p_*-) = 0$, and from (5.2):

$$F'(\pi) \;=\; h(\pi) \quad \text{is positive for} \;\; 0 < \pi < \frac{1}{2}\,, \quad \text{negative for} \;\; \frac{1}{2} < \pi < p_*\,.$$

In other words, $F(\cdot) = Q'(\cdot)/S'(\cdot)$ is strictly increasing on $(0, \frac{1}{2})$ and strictly decreasing on $(\frac{1}{2}, p_*)$. In particular, $F(\cdot)$ – thus also $Q'(\cdot)$ – is strictly positive on $(0, p_*)$; and since $Q(p_*) = 0$, we obtain (4.3).

Remark: With p_* determined by (5.7), and with the help of (3.2), the optimal stopping rule T_* of (4.9) can be written in the (probably more suggestive) form

$$T_* \;=\; \inf\left\{ t \ge 0 \;\Big|\; \frac{p}{1-p} \;+\; \lambda \int_0^t e^{-\vartheta X_s + \left(\frac{\vartheta^2}{2} - \lambda\right)s}\, ds \;\ge\; \frac{p_*}{1-p_*}\, e^{-\vartheta X_t + \left(\frac{\vartheta^2}{2} - \lambda\right)t} \right\} \tag{5.9}$$

reported in the Synopsis of this Note (for $p = 0$).

Appendix: The computation (3.2) is standard, and due to A.N. Shiryaev (1969). A derivation is included here for the sole purpose of making this Note as self-contained as possible.

In order to compute the *a posteriori* probability Π_t of (2.5), let us start with a probability space $(\Omega, \mathcal{F}, \mathbf{P}_0)$ that can support both a standard Brownian motion X as well as an *independent* random variable $\tau : \Omega \to [0,\infty)$ with distribution $\mathbf{P}_0[\tau > t] = (1-p)\, e^{-\lambda t}$ for $0 \le t < \infty$. We denote by $\mathbf{F} = \{\mathcal{F}_t\}_{0 \le t < \infty}$ the filtration generated by the process X, as in (2.3), and by $\mathbf{G} = \{\mathcal{G}_t\}_{0 \le t < \infty}$ the *larger* filtration $\mathcal{G}_t := \sigma\big(\tau, X_s\,;\, 0 \le s \le t\big)$. According to the Girsanov theorem (e.g. Karatzas & Shreve (1991), Section 3.5), the process

$$W_t \,=\, X_t - \int_0^t \vartheta\, 1_{\{\tau \le s\}}\, ds\,, \qquad 0 \le t < \infty$$

is a $\mathbf{G}-$Brownian motion under a new probability measure $\mathbf{P}$ characterized by

$$\begin{aligned}
\frac{d\mathbf{P}}{d\mathbf{P}_0}\Big|_{\mathcal{G}_t} &= \exp\left[\int_0^t \vartheta\, 1_{\{\tau \le s\}}\, dX_s \,-\, \frac{\vartheta^2}{2} \cdot \int_0^t 1_{\{\tau \le s\}}\, ds \right] \\
&= \exp\left[\vartheta\,\big(X_t - X_\tau\big) \cdot 1_{\{\tau \le t\}} \,-\, \frac{\vartheta^2}{2}\, (t-\tau)^+ \right] \\
&= \frac{L_t}{L_\tau} \cdot 1_{\{\tau \le t\}} \,+\, 1_{\{\tau > t\}} \,=:\, Z_t
\end{aligned}$$

for each $0 \le t < \infty$, in terms of the likelihood ratio process L of (3.1). The random variable τ is $\mathcal{G}_0-$measurable; thus, τ is independent under $\mathbf{P}$ of the $\mathbf{G}-$Brownian motion W, and we have $\mathbf{P}[\tau > t] \,=\, \mathbf{E}_0\big[\, Z_0\, 1_{\{\tau > t\}} \big] \,=\, \mathbf{P}_0[\tau > t] \,=\, (1-p)\, e^{-\lambda t}$, as posited in (2.1). In other words, on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ we have exactly the model posited in Section 2, in particular (2.1)-(2.3).

On the other hand, the Bayes rule gives

$$\Pi_t \,=\, \mathbf{P}\big[\, \tau \le t \,|\, \mathcal{F}_t \,\big] \,=\, \frac{\mathbf{E}_0\big[\, Z_t\, 1_{\{\tau \le t\}} \,|\, \mathcal{F}_t \,\big]}{\mathbf{E}_0\big[\, Z_t \,|\, \mathcal{F}_t \,\big]}\,, \qquad (A.1)$$

and the independence of X, τ under $\mathbf{P}_0$ implies that we have

$$\begin{aligned}
\mathbf{E}_0\big[\, Z_t \,|\, \mathcal{F}_t \,\big] &= \mathbf{E}_0\left[L_t \cdot 1_{\{\tau = 0\}} \,+\, \frac{L_t}{L_\tau} \cdot 1_{\{0 < \tau \le t\}} \,+\, 1_{\{\tau > t\}} \,\Big|\, \mathcal{F}_t \right] \\
&= p \cdot L_t \,+\, (1-p) \cdot \int_0^t \frac{L_t}{L_s}\, \lambda e^{-\lambda s}\, ds \,+\, (1-p) \cdot e^{-\lambda t}
\end{aligned}$$

as well as

$$\mathbf{E}_0\big[\, Z_t\, 1_{\{\tau \le t\}} \,|\, \mathcal{F}_t \,\big] \,=\, p \cdot L_t \,+\, (1-p) \cdot \int_0^t \frac{L_t}{L_s}\, \lambda e^{-\lambda s}\, ds\,.$$

Substituting these expressions back into $(A.1)$, we arrive at the expression of (3.2). Furthemore, the process

$$B_t = X_t - \theta \int_0^t \Pi_s \, ds = W_t - \theta \int_0^t \Big(1_{\{\tau \le s\}} - \mathbf{E}\big[1_{\{\tau \le s\}} \,|\, \mathcal{F}_s \big] \Big) \, ds$$

of (3.5) is an $(\mathbf{F}, \mathbf{P})-$martingale with $\langle B \rangle_t = \langle W \rangle_t = t$, thus an $(\mathbf{F}, \mathbf{P})-$Brownian motion.

Going back to (3.2), we see that

$$\Phi_t := \frac{\Pi_t}{1 - \Pi_t} = U_t + V_t \,, \quad \text{where} \quad U_t := \frac{p \, e^{\lambda t}}{1-p} L_t \,, \quad V_t := \int_0^t \left(\frac{L_t}{L_s} \right) \lambda \, e^{\lambda(t-s)} \, ds \,.$$

In conjunction with the stochastic differential equation $dL_t = \vartheta \, L_t \, dX_t$, $L_0 = 1$ obeyed by the likelihood-ratio process L of (3.1), we see that these processes U and V satisfy the linear stochastic equations

$$dU_t = \lambda \, U_t \, dt + \vartheta \, U_t \, dX_t \;, \quad U_0 = \frac{p}{1-p}$$

$$dV_t = \lambda \, (1 + V_t) \, dt + \vartheta \, V_t \, dX_t \;, \quad U_0 = 0 \;,$$

respectively, whence

$$d\Phi_t = \lambda \, (1 + \Phi_t) \, dt + \vartheta \, \Phi_t \, dX_t \;, \quad \Phi_0 = \frac{p}{1-p} \;. \qquad (A.2)$$

Applying Itô's rule to the process $\Pi = \frac{\Phi}{1+\Phi}$ in conjunction with $(A.2)$ and (3.5), we arrive at the stochastic differential equation (3.4).

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