MARTINGALE APPROACH TO STOCHASTIC DIFFERENTIAL GAMES OF CONTROL AND STOPPING *

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Abstract

We develop a martingale approach for studying continuous-time stochastic differential games of control and stopping, in a non-Markovian framework and with the control affecting only the drift term of the state-process. Under appropriate conditions, we show that the game has a value and construct a *saddle pair* of optimal control and stopping strategies. Crucial in this construction is a characterization of saddle pairs in terms of pathwise and martingale properties of suitable quantities.

Key Words: Stochastic games, control, optimal stopping, martingales, Doob-Meyer decompositions, stochastic maximum principle.
AMS 2000 Subject Classifications: Primary 93E20, 60G40, 91A15; Secondary, 91A25, 60G44.

1 Introduction and Synopsis

We develop a theory for zero-sum stochastic differential games with two players, a "controller" and a "stopper". The state $X(\cdot)$ in these games evolves in Euclidean space according to a stochastic functional/differential equation driven by a Wiener process; via his choice of instantaneous, non-anticipative control $u(t)$, the controller can affect the local drift of this state process $X(\cdot)$ at time t, though not its local variance.

The stopper decides the duration of the game, in the form of a stopping rule τ for the process $X(\cdot)$. At the terminal time τ the stopper receives from the controller a "reward" $\int_0^\tau h(t, X, u_t)\, dt + g(X(\tau))$ consisting of two parts: The integral up to time τ of a time-dependent *running reward* h, which also depends on the past and present states $X(s)$, $0 \leq s \leq t$ and on the present value u_t of the control; and the value at the terminal state $X(\tau)$ of a continuous *terminal reward* function g.

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Under appropriate conditions on the local drift and local variance of the state process, and on the running and terminal cost functions h and g, we establish the existence of a value for the resulting stochastic game of control and stopping, as well as regularity and martingale-type properties of the temporal evolution for the resulting "value process". We also construct optimal strategies for the two players, in the form of a *saddle pair* (u^* , τ_*), to wit: the strategy $u^*(\cdot)$ is the controller's best response to the stopper's use of the stopping rule τ_*, in the sense of minimizing total expected cost; and the stopping rule τ_* is the stopper's best response to the controller's use of the control strategy $u^*(\cdot)$, in the sense of maximizing total expected reward.

The approach of the paper is probabilistic, and builds on the martingale methodologies developed for the optimal stopping problem, and for the problem of optimal stochastic control, over the last three decades; see, for instance, Neveu (1975), El Karoui (1981), Beneš (1970, 1971), Rishel (1970), Duncan & Varaiya (1971), Davis & Varaiya (1973), Davis (1973, 1979), Elliott (1977, 1982). It proceeds in terms of a characterization of saddle points via martingale-type properties of suitable quantities, which involve the value process of the game.

The power of the approach is that it imposes no Markovian assumptions on the dynamics of the state-process; it allows the local drift and variance of the state-process, as well as the running cost, to depend at any given time t on past-and-present states $X(s)$, $0 \leq s \leq t$ in a fairly general, measurable manner. (The boundedness and continuity assumptions in the paper can most likely be relaxed.)

The main drawback of this approach, is that it imposes a severe non-degeneracy condition on the local variance of the state-process, and does not allow this local variance to be influenced by the controller. We hope that subsequent work will be able to provide a more general theory for such stochastic games, possibly also for their non-zero-sum counterparts, without such restrictive assumptions – at least in the Markovian framework of, say, Fleming & Soner (2006), El Karoui et. al. (1987), Bensoussan & Lions (1982) or Bismut (1978, 1973). It would also be of considerable interest, to provide a theory for control of "bounded variation" type (admixture of absolutely continuous, as in this paper, with pure jump and singular, terms).

Extant Work: A game between a controller and a stopper, in discrete time and with Polish (complete separable metric) state-space, was studied by Maitra & Sudderth (1996.b); under appropriate conditions, these authors obtained the existence of a value for the game and provided a transfinite induction algorithm for its computation. In Karatzas & Sudderth (2001) a similar game was studied for a linear diffusion process, and an explicit computation of the value and of a saddle pair of strategies was carried out under a non-degeneracy condition on the variance of the diffusion. Always in a Markovian, one-dimensional framework, Weerasinghe (2006) was able to study in a similar, explicit manner a stochastic game with variance that is allowed to degenerate. Finally, Karatzas & Sudderth (2007) studied recently non-zero-sum versions of these linear diffusion games, where one seeks and constructs Nash equilibria, rather than saddle pairs, of strategies.

The coöperative version of the game, that is, of *stochastic control with discretionary stopping,* has received far greater attention. General existence/characterization results were obtained by Dubins & Savage (1976) and by Maitra & Sudderth (1996.a) under the rubric of "leavable" stochastic control; by Krylov (1980), El Karoui (1981), Bensoussan & Lions (1982), Haussmann & Lepeltier (1990), Maitra & Sudderth (1996.a), Morimoto (2003), Ceci & Basan (2004); and by Karatzas & Zamfirescu (2006) in the framework of the present paper. There is also a considerable literature on explicitly solvable problems in this vein: see Beneš (1992), Davis & Zervos (1994), Karatzas &

Sudderth (1999), Karatzas et. al. (2000), Karatzas & Wang (2000, 2001), Kamizono & Morimoto (2002), Karatzas & Ocone (2002), Ocone & Weerasinghe (2006) for examples of such work.

Synopsis: We set up in the next section the model for a controlled stochastic functional/differential equation driven by a Wiener process, that will be used throughout the paper; this setting is identical to that of Elliott (1982) and of our earlier paper Karatzas & Zamfirescu (2006). Within this model, we formulate in section 3 the stochastic game of control and stopping that will the focus of our study. Section 4 reviews in the present context the classical results for *optimal stopping* on the one hand, and for *optimal stochastic control* on the other, when these problems are viewed separately.

Section 5 establishes the existence of a value for the stochastic game, and studies the regularity and some simple martingale-like properties of the resulting value process evolving through time. This study continues in earnest in section 6 and culminates with Theorem 6.3, one of the main technical results of this work.

Section 7 then builds on these results, to provide necessary and sufficient conditions for a pair $(u\,,\,\tau)$ consisting of a control strategy and a stopping rule, to be a saddle point for the stochastic game. These conditions are couched in terms of martingale-like properties for suitable quantities, which involve the value process and the cumulative running reward. A similar characterization is provided in section 8 for the optimality of a given control strategy $u(\cdot)$.

With the help of the predictable representation property of the Brownian filtration under equivalent changes of probability measure, and of the Doob-Meyer decomposition for sufficiently regular submartingales, this characterization leads in section 9 to a specific control strategy $u^*(\cdot)$ as candidate for optimality. These same martingale-type conditions suggest τ_*, the first time the value process of the game agrees with the terminal reward $g(X(\cdot))$, as candidate for optimal stopping rule. Finally, it is shown that the pair $(u^*\,,\,\tau_*)$ is indeed a saddle point of the stochastic game.

2 The Model

Consider the space $\Omega = C([0,T];\mathbb{R}^n)$ of continuous functions $\omega : [0,T] \to \mathbb{R}^n$, defined on a given bounded interval $[0,T]$ and taking values in some Euclidean space $\mathbb{R}^n$. The coördinate mapping process will be denoted by $W(t,\omega) = \omega(t),\ 0 \le t \le T$, and $\mathcal{F}_t^W = \sigma\big(W(s);\ 0 \le s \le t\big)$, $0 \le t \le T$ will stand for the natural filtration generated by this process W. The measurable space $\big(\Omega, \mathcal{F}_T^W\big)$ will be endowed with Wiener measure $\mathbb{P}$, under which W becomes standard n-dimensional Brownian motion. We shall denote by $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ the $\mathbb{P}-$augmentation of this natural filtration, and use the notation

$$||\,\omega||_t^* := \max_{0\le s\le t} |\,\omega(s)|\,, \qquad \omega \in \Omega\,, \quad 0 \le t \le T\,.$$

The σ-algebra of predictable subsets of the product space $[0,T]\times\Omega$ will be denoted by $\mathcal{P}$, and $\mathcal{S}$ will stand for the collection of *stopping rules* of the filtration $\mathbb{F}$. These are measurable mappings $\tau : \Omega \to [0,T]$ with the property

$$\{\tau \le t\} \in \mathcal{F}_t\,, \qquad \forall\ \ 0 \le t < T\,.$$

Given any two stopping rules ρ and ν with $\rho \le \nu$, we shall denote by $\mathcal{S}_{\rho,\nu}$ the collection of all stopping rules $\tau \in \mathcal{S}$ with $\rho \le \tau \le \nu$.

Consider now a *predictable* (i.e., $\mathcal{P}-$measurable) $\sigma : [0,T] \times \Omega \to \mathbb{L}(\mathbb{R}^n;\mathbb{R}^n)$ with values in the space $\mathbb{L}(\mathbb{R}^n;\mathbb{R}^n)$ of $(n \times n)$ matrices, and suppose that $\sigma(t,\omega)$ is non-singular for every $(t,\omega) \in [0,T] \times \Omega$ and that there exists some real constant $K > 0$ for which

$$\big|\big|\, \sigma^{-1}(t,\omega) \,\big|\big| \,\leq\, K\,,$$
$$\big|\, \sigma_{ij}(t,\omega) - \sigma_{ij}(t,\widetilde{\omega}) \,\big| \,\leq\, K \,\big|\big|\, \omega - \widetilde{\omega} \,\big|\big|_t^* \,, \quad \forall\;\; 1 \leq i,j \leq n$$

hold for every $\omega \in \Omega$, $\widetilde{\omega} \in \Omega$ and every $t \in [0,T]$. Then the stochastic equation

$$X(t) \,=\, x + \int_0^t \sigma(s,X)\, dW(s)\;, \qquad 0 \leq t \leq T \tag{2.1}$$

has a pathwise unique, strong solution $X(\cdot)$ for any initial condition $x \in \mathbb{R}^n$; see Theorem 14.6 in Elliott (1982). In particular, the augmentation of the natural filtration generated by $X(\cdot)$ coincides with the filtration $\mathbb{F}$ itself.

Now let us introduce an element of *control* in this picture. We shall denote by $\mathfrak{U}$ the class of **admissible control** strategies $u : [0,T] \times \Omega \to A$. These are predictable processes with values in some given separable metric space A. We shall assume that A is a countable union of nonempty, compact subsets, and is endowed with the $\sigma-$algebra $\mathcal{A}$ of its Borel subsets.

We shall consider also a $\mathcal{P} \otimes \mathcal{A}-$ measurable function $f : ([0,T] \times \Omega) \times A \to \mathbb{R}^n$ with the following properties:

- for each (t,ω), the mapping $a \mapsto f(t,\omega,a)$ is continuous;
- for each $a \in A$, the mapping $(t,\omega) \mapsto f(t,\omega,a)$ is predictable; and
- there exists a real constant $K > 0$ such that

$$|f(t,\omega,a)| \,\leq\, K\; (\, 1 + ||\,\omega||_t^*\,)\;, \qquad \forall\;\; 0 \leq t \leq T\,,\; \omega \in \Omega\,,\; a \in A\,. \tag{2.2}$$

For any given admissible control strategy $u(\cdot) \in \mathfrak{U}$, the exponential process

$$\Lambda^u(t) \,:=\, \exp\left\{ \int_0^t \langle \sigma^{-1}(s,X) f(s,X,u_s), dW(s) \rangle \,-\, \frac{1}{2} \int_0^t \big| \sigma^{-1}(s,X) f(s,X,u_s) \big|^2 ds \right\}\,, \quad 0 \leq t \leq T \tag{2.3}$$

is a martingale under these assumptions, namely $\mathbb{E}\,(\Lambda^u(T)) \,=\, 1$; see Beneš (1971), as well as Karatzas & Shreve (1991), pages 191 and 200 for this result. Then the Girsanov theorem (ibid., section 3.5) guarantees that the process

$$W^u(t) \,:=\, W(t) - \int_0^t \sigma^{-1}(s,X) f(s,X,u_s)\, ds\,, \quad 0 \leq t \leq T \tag{2.4}$$

is a Brownian motion with respect to the filtration $\mathbb{F}$, under the new probability measure

$$\mathbb{P}^u(B) \,:=\, \mathbb{E}\,[\,\Lambda^u(T) \cdot 1_B\,]\;, \qquad B \in \mathcal{F}_T\,. \tag{2.5}$$

which is equivalent to $\mathbb{P}$.

It is now clear from the equations (2.1) and (2.4) that

$$X(t) \,=\, x + \int_0^t f(s,X,u_s)\, ds + \int_0^t \sigma(s,X)\, dW^u(s)\;, \qquad 0 \leq t \leq T \tag{2.6}$$

holds. This will be our model for a controlled stochastic functional/differential equation, with the control appearing only in the drift (bounded variation) term.

3 The Stochastic Game of Control and Stopping

In order to specify the objective of our stochastic game of control and stopping, let us consider two bounded, measurable functions $h : [0,T] \times \Omega \times A \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$. We shall assume that the *running reward* function h satisfies the conditions imposed on the drift-function f above, except of course that (2.2) is now strengthened to the boundedness requirement

$$|h(t,\omega,a)| \leq K\,, \qquad \forall\ 0 \leq t \leq T\,,\ \omega \in \Omega\,,\ a \in A\,. \tag{3.1}$$

To simplify the analysis we shall assume that the *terminal reward* function g is continuous.

For reasons that will become clear in section 9, we shall also need to assume that the so-called *Hamiltonian* function

$$H(t,\omega,a,p) := \langle p, \sigma^{-1}(t,\omega) f(t,\omega,a)\rangle + h(t,\omega,a)\,, \qquad t \in [0,T]\,,\ \omega \in \Omega\,,\ a \in A\,,\ p \in \mathbb{R}^n\,, \tag{3.2}$$

is such that the mapping $a \mapsto H(t,\omega,a,p)$ attains its infimum over the set A at some $a^* \equiv \mathfrak{a}^*(t,\omega,p) \in A$, for any given $(t,\omega,p) \in [0,T] \times \Omega \times \mathbb{R}^n$; namely,

$$\inf_{a \in A} H(t,\omega,a,p) = H\big(t,\omega,\mathfrak{a}^*(t,\omega,p),p\big)\,. \tag{3.3}$$

(This is the case, for instance, if the set A is compact and the mapping $a \mapsto H(t,\omega,a,p)$ continuous.) Then it can be shown (see Lemma 1 in Beneš (1970), or Lemma 16.34 in Elliott (1982)), that the mapping $\mathfrak{a}^* : ([0,T] \times \Omega) \times \mathbb{R}^n \to A$ can be selected to be $\big(\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n)/\mathcal{A}\big)-$measurable.

We shall study a **stochastic game of control and stopping** with two players: The *controller*, who chooses an admissible control strategy $u(\cdot)$ in $\mathfrak{U}$; and the *stopper*, who decides the duration of the game by his choice of stopping rule $\tau \in \mathcal{S}$. When the stopper declares the game to be over, he receives from the controller the amount $Y^u(\tau) \equiv Y^u(0,\tau)$, where

$$Y^u(\mathfrak{t},\tau) := g\big(X(\tau)\big) + \int_{\mathfrak{t}}^{\tau} h(s,X,u_s)\,ds \qquad \text{for} \quad \tau \in \mathcal{S}_{\mathfrak{t},T}\,, \quad \mathfrak{t} \in \mathcal{S}\,. \tag{3.4}$$

It is thus in the best interest of the controller (respectively, the stopper), to try to make the amount $Y^u(\tau)$ as small (resp., as large) as possible, at least on the average. We are thus led to a **stochastic game**, with

$$\overline{V} := \inf_{u \in \mathfrak{U}} \sup_{\tau \in \mathcal{S}} \mathbb{E}^u\big(Y^u(\tau)\big)\,, \qquad \underline{V} := \sup_{\tau \in \mathcal{S}} \inf_{u \in \mathfrak{U}} \mathbb{E}^u\big(Y^u(\tau)\big) \tag{3.5}$$

as its upper- and lower-values, respectively; clearly, $\underline{V} \leq \overline{V}$.

We shall say that *the game has a value*, if its upper- and lower-values coincide, i.e., $\underline{V} = \overline{V}$; in that case we shall denote this common value simply by V.

A pair $(u^*,\tau_*) \in \mathfrak{U} \times \mathcal{S}$ will be called *saddle point* of the game, if

$$\mathbb{E}^{u^*}\big(Y^{u^*}(\tau)\big) \leq \mathbb{E}^{u^*}\big(Y^{u^*}(\tau_*)\big) \leq \mathbb{E}^{u}\big(Y^{u}(\tau_*)\big) \tag{3.6}$$

holds for every $u(\cdot) \in \mathfrak{U}$ and $\tau \in \mathcal{S}$. In other words: $u^*(\cdot)$ is the controller's best response to the stopper's use of the rule τ_*; and τ_* is the stopper's best response to the controller's use of the control strategy $u^*(\cdot)$.

If such a saddle-point pair (u^*, τ^*) exists, then it is quite clear that the game has a value. We shall try to characterize the saddle property in terms of simple, pathwise and martingale properties of certain crucial quantities; see Theorem 7.1. Then, in sections 8 and 9, we shall use this characterization in an effort to show that a saddle point indeed exists and to identify its components.

In this effort we shall need to consider, a little more generally than in (3.5), the *upper-value-process*

$$\overline{V}(\mathfrak{t}) \;:=\; \operatorname{ess\,inf}_{u\in\mathfrak{U}} \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{\mathfrak{t},T}} \; \mathbb{E}^u\big(Y^{\,u}(\mathfrak{t},\tau)\,|\,\mathcal{F}_{\mathfrak{t}}\big) \tag{3.7}$$

and the *lower-value-process*

$$\underline{V}(\mathfrak{t}) \;:=\; \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{\mathfrak{t},T}} \operatorname{ess\,inf}_{u\in\mathfrak{U}} \; \mathbb{E}^u\big(Y^{\,u}(\mathfrak{t},\tau)\,|\,\mathcal{F}_{\mathfrak{t}}\big) \tag{3.8}$$

of the game, for each $\mathfrak{t}\in\mathcal{S}$. Clearly $\overline{V}(0)=\overline{V}$, $\underline{V}(0)=\underline{V}$ as well as

$$g\big(X(\mathfrak{t})\big) \;\le\; \underline{V}(\mathfrak{t}) \;\le\; \overline{V}(\mathfrak{t})\,, \qquad \forall\;\; \mathfrak{t}\in\mathcal{S}\,. \tag{3.9}$$

We shall see in Theorem 5.1 that this last inequality holds, in fact, as an equality.

4 Optimal Control and Stopping Problems, Viewed Separately

Given any stopping rule $\mathfrak{t}\in\mathcal{S}$, we introduce the minimal conditional expected cost

$$J(\mathfrak{t},\tau) \;:=\; \operatorname{ess\,inf}_{v\in\mathfrak{U}} \mathbb{E}^v\big(Y^{\,v}(\mathfrak{t},\tau)\,|\,\mathcal{F}_{\mathfrak{t}}\big)\,, \tag{4.1}$$

that can be achieved by the controller over the stochastic interval

$$[[\,\mathfrak{t},\tau\,]] \;:=\; \{(s,\omega)\in[0,T]\times\Omega\;:\;\mathfrak{t}(\omega)\le s\le\tau(\omega)\,\}\,, \tag{4.2}$$

for each stopping rule $\tau\in\mathcal{S}_{\mathfrak{t},T}$. With the notation (4.1), the lower value of the game in (3.8) becomes

$$\underline{V}(\mathfrak{t}) \;=\; \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{\mathfrak{t},T}} \; J(\mathfrak{t},\tau) \;\ge\; J(\mathfrak{t},\mathfrak{t}) \;=\; g\big(X(\mathfrak{t})\big)\,, \qquad \text{a.s.} \tag{4.3}$$

By analogy with the classical martingale approach to stochastic control (developed by Rishel (1970), Duncan & Varaiya (1971), Davis & Varaiya (1973), Davis (1973) and outlined in Davis (1979), El Karoui (1981) and in Chapter 16 of Elliott (1982)), given any admissible control strategy $u(\cdot)\in\mathfrak{U}$ and any stopping rules $\mathfrak{t}$, ν, τ with $0\le\mathfrak{t}\le\nu\le\tau\le T$, we have the $\mathbb{P}^u$−submartingale property $\mathbb{E}^u\big(H^{\,u}(\nu,\tau)\,|\,\mathcal{F}_{\mathfrak{t}}\big)\ge H^{\,u}(\mathfrak{t},\tau)$ for the "cumulative" quantity

$$H^u(\mathfrak{t},\tau) \;:=\; J(\mathfrak{t},\tau) \,+\, \int_0^{\mathfrak{t}} h(s,X,u_s)\,ds\,, \tag{4.4}$$

or equivalently

$$\mathbb{E}^u\left[\,J(\nu,\tau)+\int_{\mathfrak{t}}^{\nu} h(s,X,u_s)\,ds\,\Big|\,\mathcal{F}_{\mathfrak{t}}\right] \;\ge\; J(\mathfrak{t},\tau)\,, \qquad \text{a.s.} \tag{4.5}$$

4.1 A Family of Optimal Stopping Problems

For each admissible control strategy $u(\cdot) \in \mathfrak{U}$, we define the maximal conditional expected reward

$$Z^u(\mathfrak{t}) := \operatorname{ess\,sup}_{\tau \in \mathcal{S}_{\mathfrak{t},T}} \mathbb{E}^u\big(Y^u(\mathfrak{t},\tau) \,|\, \mathcal{F}_\mathfrak{t}\big), \qquad \mathfrak{t} \in \mathcal{S} \tag{4.6}$$

that can be achieved by the stopper from time $\mathfrak{t}$ onward, as well as the "cumulative" quantity

$$Q^u(\mathfrak{t}) := Z^u(\mathfrak{t}) + \int_0^\mathfrak{t} h(s, X, u_s)\, ds = \operatorname{ess\,sup}_{\tau \in \mathcal{S}_{\mathfrak{t},T}} \mathbb{E}^u\big(Y^u(\tau) \,|\, \mathcal{F}_\mathfrak{t}\big); \tag{4.7}$$

in particular,

$$Z^u(\mathfrak{t}) \geq Y^u(\mathfrak{t},\mathfrak{t}) = g\big(X(t)\big), \qquad \overline{V}(\mathfrak{t}) = \operatorname{ess\,inf}_{u \in \mathfrak{U}} Z^u(\mathfrak{t}). \tag{4.8}$$

From the classical martingale approach to the theory of optimal stopping (e.g., El Karoui (1981) or Karatzas & Shreve (1998), Appendix D), we know that the process $Q^u(\cdot) \equiv \{Q^u(t),\, 0 \leq t \leq T\}$ is a $\mathbb{P}^u-$super-martingale with paths that are RCLL (Right-Continuous, with Limits from the Left); that it dominates the continuous process $Y^u(\cdot)$ given as

$$Y^u(t) \equiv Y^u(0,t) = g\big(X(t)\big) + \int_0^t h(s, X, u_s)\, ds, \qquad 0 \leq t \leq T; \tag{4.9}$$

and that in fact $Q^u(\cdot)$ is the *smallest* RCLL supermartingale which dominates $Y^u(\cdot)$. In other words, $Q^u(\cdot)$ is the *Snell Envelope* of the process $Y^u(\cdot)$.

Let us introduce the stopping rules

$$\tau_\mathfrak{t}^u(\varepsilon) := \inf\big\{\, s \in [\,\mathfrak{t}, T) \,:\, g\big(X(s)\big) \geq Z^u(s) - \varepsilon \,\big\} \wedge T, \qquad \tau_t^u := \tau_t^u(0) \tag{4.10}$$

for each $\mathfrak{t} \in \mathcal{S}$, $0 \leq \varepsilon < 1$. Then $\tau_\mathfrak{t}^u(\varepsilon) \leq \tau_\mathfrak{t}^u$, and we also know from this theory that

- for any stopping rules $\mathfrak{t}, \nu, \theta$ with $\mathfrak{t} \leq \nu \leq \theta \leq \tau_\mathfrak{t}^u$, we have the martingale property $\mathbb{E}^u\big[\, Q^u(\theta) \,|\, \mathcal{F}_\nu \,\big] = Q^u(\nu)$ a.s.; that
- $Z^u(\mathfrak{t}) = \mathbb{E}^u\big[\, Y^u(\mathfrak{t}, \tau_\mathfrak{t}^u) \,|\, \mathcal{F}_\mathfrak{t} \big]$ holds a.s.; and that
- for any stopping rule $\widehat{\tau} \in \mathcal{S}_{\mathfrak{t},T}$ with $Z^u(\mathfrak{t}) = \mathbb{E}^u\big[\, Y^u(\mathfrak{t}, \widehat{\tau}) \,|\, \mathcal{F}_\mathfrak{t} \big]$ a.s., we have $\tau_\mathfrak{t}^u \leq \widehat{\tau}$ a.s.

In other words, $\tau_\mathfrak{t}^u$ is the smallest optimal stopping rule in (4.6).

4.2 A Preparatory Lemma

For the proof of several results in this work, we shall need the following observation; we list it separately, for ease of reference.

4.1 Lemma: *Suppose that $\mathfrak{t}, \theta$ are stopping rules with $0 \leq \mathfrak{t} \leq \theta \leq T$, and that $u(\cdot)$, $v(\cdot)$ are admissible control strategies in $\mathfrak{U}$ with the property $u(\cdot) = v(\cdot)$ a.e. on the stochastic interval $[[\,\mathfrak{t}, \theta\,]]\,\}$, in the notation of (4.2). Then, for any bounded and $\mathcal{F}_\theta-$measurable random variable random variable Ξ, we have*

$$\mathbb{E}^v\big[\, \Xi \,|\, \mathcal{F}_\mathfrak{t} \big] = \mathbb{E}^u\big[\, \Xi \,|\, \mathcal{F}_\mathfrak{t} \big], \qquad \text{a.s.} \tag{4.11}$$

In particular, with $\mathfrak{t} = 0$ this gives $\mathbb{E}^v\big[\, \Xi \,\big] = \mathbb{E}^u\big[\, \Xi \,\big]$.

The reasoning is very simple: with the notation $\Lambda^u(\mathfrak{t},\theta) := \Lambda^u(\theta)/\Lambda^u(\mathfrak{t})$ from (2.3), and using the martingale property of $\Lambda^u(\cdot)$ under $\mathbb{P}^u$, we have $\mathbb{E}^u[\,\Lambda^u(\mathfrak{t},\theta)\,|\,\mathcal{F}_{\mathfrak{t}}] = 1$ a.s. In conjunction with the Bayes rule for conditional expectations under equivalent probability measures, this gives

$$\mathbb{E}^u\big[\,\Xi\,|\,\mathcal{F}_{\mathfrak{t}}\big] \;=\; \frac{\Lambda^u(\mathfrak{t})\cdot\mathbb{E}\big[\,\Lambda^u(\mathfrak{t},\theta)\Xi\,|\,\mathcal{F}_{\mathfrak{t}}\big]}{\Lambda^u(\mathfrak{t})\cdot\mathbb{E}\big[\,\Lambda^u(\mathfrak{t},\theta)\,|\,\mathcal{F}_{\mathfrak{t}}\big]} \;=\; \mathbb{E}\big[\,\Lambda^u(\mathfrak{t},\theta)\Xi\,|\,\mathcal{F}_{\mathfrak{t}}\big]$$
$$=\; \mathbb{E}\big[\,\Lambda^v(\mathfrak{t},\theta)\Xi\,|\,\mathcal{F}_{\mathfrak{t}}\big] \;=\; \cdots \;=\; \mathbb{E}^v\big[\,\Xi\,|\,\mathcal{F}_{\mathfrak{t}}\big]\,, \qquad \text{a.s.}$$

4.3 Families Directed Downwards

Now *fix* a control strategy $v(\cdot) \in \mathfrak{U}$ as well as stopping rules $\mathfrak{t}\,,\theta$ with $0 \le \mathfrak{t} \le \theta \le T$, and denote by $\mathcal{V}_{[\mathfrak{t},\theta]}$ the set of admissible control strategies $u(\cdot)$ as in Lemma 4.1 (i.e., with $u(\cdot) = v(\cdot)$ a.e. on the stochastic interval $[[\,\mathfrak{t}\,,\theta\,]]$).

We observe from (4.6), (3.4) and Lemma 4.1, that $Z^u(\theta)$ depends only on the values that the admissible control strategy $u(\cdot)$ takes over the stochastic interval $]]\,\theta,T\,]] := \{(s,\omega)\in[0,T]\times\Omega : \theta(\omega) < s \le T\,\}$ (its values over the stochastic interval $[[\,0,\theta\,]]$ are irrelevant for computing $Z^u(\theta)$). Thus, we can write the upper value (3.7) of the game as

$$\overline{V}(\theta) \;=\; \operatorname{ess\,inf}_{u\in\mathfrak{U}} Z^u(\theta) \;=\; \operatorname{ess\,inf}_{u\in\mathcal{V}_{[0,\theta]}} Z^u(\theta)\,, \qquad \text{a.s.} \tag{4.12}$$

for *any* given admissible control strategy $v(\cdot)\in\mathfrak{U}$.

4.2 Lemma: *The family of random variables* $\{Z^u(\theta)\}_{u\in\mathcal{V}_{[0,\theta]}}$ *is* directed downwards: *for any two* $u^1(\cdot)\in\mathcal{V}_{[0,\theta]}$ *and* $u^2(\cdot)\in\mathcal{V}_{[0,\theta]}$ *, there exists an admissible control strategy* $\widehat{u}(\cdot)\in\mathcal{V}_{[0,\theta]}$ *such that*

$$Z^{\widehat{u}}(\theta) \;=\; Z^{u^1}(\theta)\wedge Z^{u^2}(\theta)\,, \qquad a.s.$$

Proof: Consider the event $A := \big\{Z^{u^1}(\theta) \le Z^{u^2}(\theta)\big\}$ and define $\widehat{u}(s,\omega) := v(s,\omega)$ for $0 \le s \le \theta(\omega)$,

$$\widehat{u}(s,\omega) \;:=\; u^1(s,\omega)\cdot 1_A(\omega) \;+\; u^2(s,\omega)\cdot 1_{A^c}(\omega) \qquad \text{for} \quad \theta(\omega) < s \le T\,. \tag{4.13}$$

Consider also the stopping rule

$$\widehat{\tau}_\theta \;:=\; \tau_\theta^{u^1}\cdot 1_A + \tau_\theta^{u^2}\cdot 1_{A^c} \;\in\; \mathcal{S}_{\theta,T}$$

(notation of (4.10)). Then from Lemma 4.1 we have

$$Z^{\widehat{u}}(\theta) \;=\; \mathbb{E}^{\widehat{u}}\big[\,Y^{\widehat{u}}(\theta,\tau_\theta^{\widehat{u}})\,|\,\mathcal{F}_\theta\,\big] \;=\; \mathbb{E}^{u^1}\big[\,Y^{u^1}(\theta,\tau_\theta^{\widehat{u}})\,|\,\mathcal{F}_\theta\,\big]\cdot 1_A \;+\; \mathbb{E}^{u^2}\big[\,Y^{u^2}(\theta,\tau_\theta^{\widehat{u}})\,|\,\mathcal{F}_\theta\,\big]\cdot 1_{A^c}$$
$$\le\; Z^{u^1}(\theta)\cdot 1_A \;+\; Z^{u^2}(\theta)\cdot 1_{A^c} \;=\; \mathbb{E}^{u^1}\big[\,Y^{u^1}(\theta,\tau_\theta^{u^1})\,|\,\mathcal{F}_\theta\,\big]\cdot 1_A \;+\; \mathbb{E}^{u^2}\big[\,Y^{u^2}(\theta,\tau_\theta^{u^2})\,|\,\mathcal{F}_\theta\,\big]\cdot 1_{A^c}$$
$$=\; \mathbb{E}^{\widehat{u}}\big[\,Y^{\widehat{u}}(\theta,\widehat{\tau}_\theta)\,|\,\mathcal{F}_\theta\,\big] \;\le\; Z^{\widehat{u}}(\theta)\,, \tag{4.14}$$

thus also

$$Z^{\widehat{u}}(\theta) = Z^{u^1}(\theta)\cdot 1_A + Z^{u^2}(\theta)\cdot 1_{A^c} = Z^{u^1}(\theta)\wedge Z^{u^2}(\theta)\,, \qquad \text{a.s.} \qquad \Box$$

Now we can appeal to basic properties of the essential infimum (e.g., Neveu (1975), page 121), to argue that there exists a sequence $\big\{u^k(\cdot)\big\}_{k\in\mathbb{N}} \subset \mathcal{V}_{[0,\theta]}$, for which the corresponding $\big\{Z^{u^k}(\theta)\big\}_{k\in\mathbb{N}}$ is *decreasing* and the infimum in (4.12) becomes a limit:

$$\overline{V}(\theta) \;=\; \lim_{k\to\infty}\downarrow\, Z^{u^k}(\theta)\,, \qquad \text{a.s.} \tag{4.15}$$

5 Existence and Regularity of the Game's Value Process

For the sequence $\{u^k(\cdot)\}_{k\in\mathbb{N}} \subset \mathcal{V}_{[0,\theta]}$ of (4.15), let us look at the corresponding stopping rules

$$\tau_\theta^{u^k} := \inf\{ s \in [\theta, T) : Z^{u^k}(s) = g(X(s)) \} \wedge T, \qquad k \in \mathbb{N}$$

via (4.10). The resulting sequence $\{\tau_\theta^{u^k}\}_{k\in\mathbb{N}}$ is clearly decreasing, so the limit

$$\tau_\theta^* := \lim_{k\to\infty} \downarrow \tau_\theta^{u^k} \tag{5.1}$$

exists a.s. and defines a stopping rule in $\mathcal{S}_{\theta,T}$. The values of the process $u^k(\cdot)$ on the stochastic interval $[[0,\theta]]$ are irrelevant for computing $Z^{u^k}(s)$, $s \ge \theta$ or, for that matter, $\tau_\theta^{u^k}$. But clearly

$$\tau_\theta^{u^k} = \inf\{ s \in (\tau_\theta^*, T) : Z^{u^k}(s) = g(X(s)) \} \wedge T, \qquad k \in \mathbb{N}$$

holds a.s., so the values of $u^k(\cdot)$ on $[[0,\tau_\theta^*]]$ are irrelevant for computing $\tau_\theta^{u^k}$, $k \in \mathbb{N}$.

Thus, *there exists a sequence $\{u^k(\cdot)\}_{k\in\mathbb{N}} \subset \mathcal{V}_{[0,\tau_\theta^*]}$ of admissible control strategies, which agree with the given control strategy $v(\cdot) \in \mathfrak{U}$ on the stochastic interval $[[0,\tau_\theta^*]]$, and for which (5.1) holds.*

We are ready to state and prove our first result.

5.1 Theorem: *The game has a value: for every $\theta \in \mathcal{S}$ we have $\overline{V}(\theta) = \underline{V}(\theta)$, a.s.*
In particular, $\overline{V} = \underline{V}$ in (3.5).
A little more generally: for every $\mathfrak{t} \in \mathcal{S}$ and any $\theta \in \mathcal{S}_{\mathfrak{t},T}$ we have almost surely:

$$\operatorname{ess\,inf}_{u\in\mathfrak{U}} \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{\theta,T}} \mathbb{E}^u\big(Y^u(\mathfrak{t},\tau) \,|\, \mathcal{F}_\mathfrak{t}\big) = \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{\theta,T}} \operatorname{ess\,inf}_{u\in\mathfrak{U}} \mathbb{E}^u\big(Y^u(\mathfrak{t},\tau) \,|\, \mathcal{F}_\mathfrak{t}\big). \tag{5.2}$$

Proof: From the preceding remarks, we get the a.s. comparisons

$$\overline{V}(\theta) \le \mathbb{E}^{u^k}\Big[Y^{u^k}\big(\theta, \tau_\theta^{u^k}\big) \,\Big|\, \mathcal{F}_\theta \Big] = \mathbb{E}\Big[\Lambda^{u^k}\big(\theta, \tau_\theta^{u^k}\big)\, Y^{u^k}\big(\theta, \tau_\theta^{u^k}\big) \,\Big|\, \mathcal{F}_\theta \Big]$$

$$= \mathbb{E}\Big[\Lambda^v\big(\theta, \tau_\theta^*\big)\, \Lambda^{u^k}\big(\tau_\theta^*, \tau_\theta^{u^k}\big) \Big\{ Y^v\big(\theta, \tau_\theta^*\big) + \int_{\tau_\theta^*}^{\tau_\theta^{u^k}} h(s, X, u_s^k)\, ds \Big\} \,\Big|\, \mathcal{F}_\theta \Big]$$

for every $k \in \mathbb{N}$, just as in Lemma 4.1. Passing to the limit as $k \to \infty$ we obtain from (5.1), the boundedness of σ^{-1}, f, h and the dominated convergence theorem, the a.s. comparisons

$$\overline{V}(\theta) \le \mathbb{E}\Big[\Lambda^v\big(\theta, \tau_\theta^*\big)\, Y^v\big(\theta, \tau_\theta^*\big) \,|\, \mathcal{F}_\theta \Big] = \mathbb{E}^v\Big[Y^v\big(\theta, \tau_\theta^*\big) \,|\, \mathcal{F}_\theta \Big].$$

Because $v(\cdot)$ is arbitrary, we can take the infimum of the right-hand side of this inequality over $v(\cdot) \in \mathfrak{U}$, and conclude

$$\overline{V}(\theta) \le \operatorname{ess\,inf}_{v\in\mathfrak{U}} \mathbb{E}^v\Big[Y^v\big(\theta, \tau_\theta^*\big) \,|\, \mathcal{F}_\theta \Big] \le \operatorname{ess\,sup}_{\tau\in\mathcal{S}} \operatorname{ess\,inf}_{v\in\mathfrak{U}} \mathbb{E}^v\Big[Y^v\big(\theta, \tau_\theta^*\big) \,|\, \mathcal{F}_\theta \Big] = \underline{V}(\theta).$$

Since the reverse inequality $\overline{V}(\theta) \ge \underline{V}(\theta)$ is obvious, we obtain the first claim of the Theorem, namely $\overline{V}(\theta) = \underline{V}(\theta)$ a.s.

• As for (5.2), let us observe that for every $u(\cdot)\in\mathfrak{U}$ we have the a.s. comparisons

$$\operatorname{ess\,inf}_{w\in\mathfrak{U}} \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{\theta,T}} \mathbb{E}^w \left(Y^w(\theta,\tau) + \int_{\mathfrak{t}}^{\theta} h(s,X,w_s)\,ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \right)$$

$$\le \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{\theta,T}} \mathbb{E}^u \left(Y^u(\theta,\tau) + \int_{\mathfrak{t}}^{\theta} h(s,X,u_s)\,ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \right)$$

$$\le \mathbb{E}^u \left(\operatorname{ess\,sup}_{\tau\in\mathcal{S}_{\theta,T}} \mathbb{E}^u\big[\, Y^u(\theta,\tau)\,|\,\mathcal{F}_\theta\,\big] \,+\, \int_{\mathfrak{t}}^{\theta} h(s,X,u_s)\,ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \right)$$

$$= \mathbb{E}^u \left(\mathbb{E}^u\big[\, Y^u\big(\theta,\tau_\theta^u\big)\,|\,\mathcal{F}_\theta\,\big] + \int_{\mathfrak{t}}^{\theta} h(s,X,u_s)\,ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \right)$$

$$= \mathbb{E}^u \left(Y^u\big(\theta,\tau_\theta^u\big) + \int_{\mathfrak{t}}^{\theta} h(s,X,u_s)\,ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \right) \;=\; \mathbb{E}^u\big[\, Y^u\big(\mathfrak{t},\tau_\theta^u\big)\big|\,\mathcal{F}_{\mathfrak{t}}\,\big]\,.$$

Now repeat the previous argument: write this inequality with $u(\cdot)$ replaced by $u^k(\cdot)\in\mathcal{V}_{[0,\tau_\theta^*]}$ (the sequence of (4.15), (5.1)) for every $k\in\mathbb{N}$, then pass to the limit as $k\to\infty$; the result is

$$\operatorname{ess\,inf}_{w\in\mathfrak{U}} \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{\theta,T}} \mathbb{E}^w \left(Y^w(\theta,\tau) + \int_{\mathfrak{t}}^{\theta} h(s,X,w_s)\,ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \right) \;\le\; \mathbb{E}^v\big[\, Y^v\big(\mathfrak{t},\tau_\theta^*\big)\big|\,\mathcal{F}_{\mathfrak{t}}\,\big]\,, \qquad \text{a.s.}$$

The arbitrariness of $v(\cdot)$ allows us to take the (essential) infimum of the right-hand side over $v(\cdot)\in\mathfrak{U}$, then the (essential) supremum over stopping rules $\tau\in\mathcal{S}$, and the inequality ($\le$) of (5.2) follows; once again, the reverse inequality is ovious. $\square$

From now on we shall denote by $V(\cdot)=\overline{V}(\cdot)=\underline{V}(\cdot)$ the common value of this game, and write $V=V(0)$.

5.2 Proposition: *The value process $V(\cdot)$ is right-continuous.*

Proof: The Snell Envelope $Q^u(\cdot)$ of (4.7) can be taken in its RCLL modification, as we have already done; so the same is the case for $Z^u(\cdot)$ of (4.6). Consequently, one obtains $\limsup_{s\downarrow t} V(s)\le \lim_{s\downarrow t} Z^u(s)=Z^u(t)$, a.s. Taking the infimum over $u(\cdot)\in\mathfrak{U}$ one obtains $\limsup_{s\downarrow t} V(s)\le V(t)$, a.s.

In order to show that the reverse inequality

$$\liminf_{s\downarrow t}\, V(s) \,\ge\, V(t)\,, \qquad \text{a.s.} \tag{5.3}$$

also holds, recall the submartingale property (4.5) and deduce from it, and from Proposition 1.3.14 in Karatzas & Shreve (1991), that the right-hand limits

$$J(t+,\tau) \,:=\, \lim_{s\downarrow t} J(s,\tau) \quad \text{on} \;\; \{\, t<\tau\,\}\,, \qquad J(t+,\tau) \,:=\, g\big(X(\tau)\big) \quad \text{on} \;\; \{\, t=\tau\,\}$$

exist and are finite a.s., on the respective events. Now for any $t\in[0,T]$ and every stopping rule $\tau\in\mathcal{S}_{t,T}$, recall (4.3) to obtain

$$\liminf_{s\downarrow t}\, V(s) \,\ge\, \liminf_{s\downarrow t}\, J(s,s\vee\tau) \,=\, \liminf_{s\downarrow t}\, J(s,\tau)\cdot 1_{\{t<\tau\}} \,+\, \liminf_{s\downarrow t}\, J(s,s)\cdot 1_{\{t=\tau\}}\,.$$

But on the event $\{t = \tau\}$ we have almost surely

$$\liminf_{s \downarrow t} J(s,s) \;=\; \liminf_{s \downarrow t} g\big(X(s)\big) \;=\; \lim_{s \downarrow t} g\big(X(s)\big) \;=\; g\big(X(t)\big) \;=\; J(t,t)$$

by the continuity of $g(\cdot)$; whereas on the event $\{t < \tau\}$ we have $\liminf_{s \downarrow t} J(s,\tau) = \lim_{s \downarrow t} J(s,\tau)$, almost surely. Therefore, recalling (4.5), we obtain from the bounded convergence theorem the a.s. comparisons

$$\liminf_{s \downarrow t} V(s) \;\geq\; \lim_{s \downarrow t} J(s,\tau) \;=\; \mathbb{E}^u \left(\lim_{s \downarrow t} J(s,\tau) \,\Big|\, \mathcal{F}_{t+} \right) \;=\; \mathbb{E}^u \left(\lim_{s \downarrow t} J(s,\tau) \,\Big|\, \mathcal{F}_{t} \right)$$

$$= \mathbb{E}^u \left[\lim_{s \downarrow t} \left(J(s,\tau) + \int_t^s h(s,X,u_s)\, ds \right) \Big|\, \mathcal{F}_t \right] = \lim_{s \downarrow t} \mathbb{E}^u \left[J(s,\tau) + \int_t^s h(s,X,u_s)\, ds \,\Big|\, \mathcal{F}_t \right] \geq J(t,\tau)\,.$$

We have used here the right-continuity of the augmented Brownian filtration, and the comparison (4.5). The stopping rule $\tau \in \mathcal{S}_{\mathfrak{t},T}$ is arbitrary in these comparisons; taking the (essential) supremum over this class, and recalling (4.3), we arrive at the desired inequality (5.3). $\Box$

5.1 Some Elementary Submartingales

By analogy with (4.10), let us introduce now for each $\mathfrak{t} \in \mathcal{S}$ and $0 \leq \varepsilon < 1$ the stopping rules

$$\varrho_{\mathfrak{t}}(\varepsilon) \;:=\; \inf \big\{\, s \in [\,\mathfrak{t}, T) \;:\; g\big(X(s)\big) \geq V(s) - \varepsilon \,\big\} \wedge T\,, \qquad \varrho_{\mathfrak{t}} := \varrho_{\mathfrak{t}}(0)\,. \tag{5.4}$$

Since

$$V(\cdot) \;=\; \operatorname{ess\,inf}_{u \in \mathfrak{U}} Z^u(\cdot) \;\geq\; g\big(X(\cdot)\big)\,, \tag{5.5}$$

we have

$$\varrho_{\mathfrak{t}} \vee \tau^u_{\mathfrak{t}}(\varepsilon) \;\leq\; \tau^u_{\mathfrak{t}}\,, \qquad \varrho_{\mathfrak{t}}(\varepsilon) \;\leq\; \tau^u_{\mathfrak{t}}(\varepsilon) \wedge \varrho_{\mathfrak{t}}\,. \tag{5.6}$$

Let us introduce also, for each admissible control strategy $u(\cdot) \in \mathfrak{U}$, the family of random variables

$$R^u(\mathfrak{t}) \;:=\; V(\mathfrak{t}) + \int_0^{\mathfrak{t}} h(s,X,u_s)\, ds \;\geq\; Y^u(\mathfrak{t})\,, \qquad \mathfrak{t} \in \mathcal{S}\,. \tag{5.7}$$

For any time $\mathfrak{t} \in \mathcal{S}$, the quantity $R^u(\mathfrak{t})$ represents the cumulative cost to the controller of using the strategy $u(\cdot)$ on $[[0,\mathfrak{t}]]$, plus the game's value at that time.

5.3 Proposition: *For each $u(\cdot) \in \mathfrak{U}$, the process $R^u(\,\cdot \wedge \varrho_0)$ is a $\mathbb{P}^u-$submartingale.*
A bit more generally: for any stopping rules $\mathfrak{t}, \theta$ with $\mathfrak{t} \leq \theta \leq \varrho_{\mathfrak{t}}$, we have

$$\mathbb{E}^u \big[\, R^u(\theta) \,|\, \mathcal{F}_{\mathfrak{t}} \big] \;\geq\; R^u(\mathfrak{t})\,, \qquad \textit{a.s.} \tag{5.8}$$

or equivalently

$$\mathbb{E}^u \left[V(\theta) + \int_{\mathfrak{t}}^{\theta} h(s,X,u_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \right] \;\geq\; V(\mathfrak{t})\,, \qquad \textit{a.s.} \tag{5.9}$$

Furthermore, for any stopping rules $\mathfrak{s}\,,\mathfrak{t}\,,\theta$ *with* $0 \le \mathfrak{s} \le \mathfrak{t} \le \theta \le \varrho_{\mathfrak{t}}$ *, we have almost surely:*

$$\operatorname{ess\,inf}_{u\in\mathfrak{U}}\ \mathbb{E}^u\Big[\,V(\theta)+\int_{\mathfrak{s}}^{\theta} h(s,X,u_s)\,ds\,\Big|\,\mathcal{F}_{\mathfrak{s}}\Big]\ \ge\ \operatorname{ess\,inf}_{u\in\mathfrak{U}}\ \mathbb{E}^u\Big[\,V(\mathfrak{t})+\int_{\mathfrak{s}}^{\mathfrak{t}} h(s,X,u_s)\,ds\,\Big|\,\mathcal{F}_{\mathfrak{s}}\Big]\,. \tag{5.10}$$

Proof: For any admissible control strategy $u(\cdot)\in\mathfrak{U}$, and for any stopping rules $\mathfrak{t}\,,\theta$ with $0\le\mathfrak{t}\le\theta\le\varrho_{\mathfrak{t}}$, we have $\mathbb{E}^u\big[\,Q^u(\theta)\,|\,\mathcal{F}_{\mathfrak{t}}\big]\ =\ Q^u(\mathfrak{t})$ or equivalently

$$\mathbb{E}^u\Big[\,Z^u(\theta)+\int_{\mathfrak{t}}^{\theta} h(s,X,u_s)\,ds\,\Big|\,\mathcal{F}_{\mathfrak{t}}\Big]\ =\ Z^u(\mathfrak{t})\ \ge\ V(\mathfrak{t})\,,\qquad \text{a.s.} \tag{5.11}$$

from (4.8), (5.6) and the properties of the Snell envelope in Section 4.

Now *fix* a control strategy $v(\cdot)\in\mathfrak{U}$ and denote again by $\mathcal{V}_{[\mathfrak{t},\theta]}$ the set of admissible control strategies $u(\cdot)$ as in Lemma 4.1 (i.e., with $u(\cdot)=v(\cdot)$ a.e. on the stochastic interval $[[\,\mathfrak{t}\,,\theta\,]]$). From this result and (5.11), we obtain

$$\mathbb{E}^v\Big[\,Z^u(\theta)+\int_{\mathfrak{t}}^{\theta} h(s,X,v_s)\,ds\,\Big|\,\mathcal{F}_{\mathfrak{t}}\Big]\ =\ Z^u(\mathfrak{t})\ \ge\ V(\mathfrak{t})\,,\qquad \text{a.s.} \tag{5.12}$$

Now select some sequence $\big\{u^k(\cdot)\big\}_{k\in\mathbb{N}}\subset\mathcal{V}_{[\mathfrak{t},\theta]}$ as in (4.15), substitute $u^k(\cdot)$ for $u(\cdot)$ in (5.12), let $k\to\infty$, and appeal to the bounded convergence theorem for conditional expectations, to obtain

$$\mathbb{E}^v\Big[\,V(\theta)+\int_{\mathfrak{t}}^{\theta} h(s,X,v_s)\,ds\,\Big|\,\mathcal{F}_{\mathfrak{t}}\Big]\ \ge\ V(\mathfrak{t})\,,\qquad \text{a.s.}$$

This gives the a.s. comparison $\mathbb{E}^u\big[\,V(\theta)+\int_{\mathfrak{t}}^{\theta} h(s,X,u_s)\,ds\,\big|\,\mathcal{F}_{\mathfrak{t}}\big]\ \ge\ V(\mathfrak{t})$ for any $u(\cdot)\in\mathfrak{U}$, as claimed in (5.9), therefore also

$$\mathbb{E}^u\left[\,V(\theta)+\int_{\mathfrak{s}}^{\theta} h(s,X,u_s)\,ds\,\middle|\,\mathcal{F}_{\mathfrak{s}}\right]\ \ge\ \mathbb{E}^u\left[\,V(\mathfrak{t})+\int_{\mathfrak{s}}^{\mathfrak{t}} h(s,X,u_s)\,ds\,\middle|\,\mathcal{F}_{\mathfrak{s}}\right]\,.$$

The claim (5.10) follows now by taking essential infima over $u(\cdot)\in\mathfrak{U}$ on both sides. □

5.4 Proposition: *For every* $\mathfrak{t}\in\mathcal{S}$ *we have*

$$V(\mathfrak{t})\ =\ \operatorname{ess\,inf}_{u\in\mathfrak{U}}\ \mathbb{E}^u\left(g\big(X(\varrho_{\mathfrak{t}})\big)+\int_{\mathfrak{t}}^{\varrho_{\mathfrak{t}}} h(s,X,u_s)\,ds\,\middle|\,\mathcal{F}_{\mathfrak{t}}\right)\,,\qquad \text{a.s.} \tag{5.13}$$

As a consequence,

$$V(\mathfrak{t})\ =\ \operatorname{ess\,inf}_{u\in\mathfrak{U}}\ \mathbb{E}^u\left(V(\varrho_{\mathfrak{t}})+\int_{\mathfrak{t}}^{\varrho_{\mathfrak{t}}} h(s,X,u_s)\,ds\,\middle|\,\mathcal{F}_{\mathfrak{t}}\right)\,,\qquad \text{a.s.} \tag{5.14}$$

and for any given $v(\cdot)\in\mathfrak{U}$ *we get in the notation of (4.12):*

$$R^v(\mathfrak{t})\ =\ \operatorname{ess\,inf}_{u\in\mathcal{V}_{[0,\mathfrak{t}]}}\ \mathbb{E}^u\big(R^u(\varrho_{\mathfrak{t}})\,\big|\,\mathcal{F}_{\mathfrak{t}}\big)\,,\qquad \text{a.s.} \tag{5.15}$$

Proof: The definition (3.7) of the upper value of the game gives the inequality ($\ge$) in (5.13); the reverse inequality ($\le$) follows directly from (5.9) of Proposition 5.1 along with the a.s. identity $V(\varrho_{\mathfrak{t}})=g(X(\varrho_{\mathfrak{t}}))$, a consequence of the definition of $\varrho_{\mathfrak{t}}$ in (5.4) and the right-continuity of $V(\cdot)$ from Proposition 5.2. Now (5.14) follows immediately, and so does (5.15). □

5.5 Remark: Proposition 5.3 implies that the process $R^u(\,\cdot \wedge \varrho_0)$, which is right-continuous by virtue of Proposition 5.2, admits left-limits on $(0,T]$ almost surely; cf. Proposition 1.3.14 in Karatzas & Shreve (1991). Thus, the process $R^u(\,\cdot \wedge \varrho_0)$ is a $\mathbb{P}^u-$submartingale with RCLL paths.

5.6 Proposition: *For any admissible control strategy $u(\cdot) \in \mathfrak{U}$, we have*

$$V(T) \,=\, \lim_{t \uparrow T} V(t) \,=\, \lim_{t \uparrow T} Z^u(t) \,=\, Z^u(T) \,=\, g(X(T))\,, \quad a.s.$$

Proof: Let $u(\cdot) \in \mathfrak{U}$ be an arbitrary admissible control strategy. From the continuity of the terminal reward function $g : \mathbb{R}^n \to \mathbb{R}$, we have

$$Z^u(T) \,=\, V(T) \,=\, g\big(X(T)\big) \,=\, \lim_{t \uparrow T} g\big(X(t)\big) \,\leq\, \lim_{t \uparrow T} V(t) \,\leq\, \lim_{t \uparrow T} Z^u(t)\,, \tag{5.16}$$

almost surely. Coupled with dominated convergence, this gives

$$\mathbb{E}^u[\,Z^u(T)\,] \,=\, \mathbb{E}^u[\,V(T)\,] \,\leq\, \lim_{t \uparrow T} \mathbb{E}^u[\,V(t)\,] \,\leq\, \lim_{t \uparrow T} \mathbb{E}^u[\,Z^u(t)\,]\,. \tag{5.17}$$

On the other hand, we also know that for every $0 \leq t \leq T$, we get

$$\mathbb{E}^u[\,Z^u(t)\,] \,=\, \mathbb{E}^u \left(Z^u(\tau^u_t) + \int_t^{\tau^u_t} h(s, X, u_s)\, ds \right) \,=\, \mathbb{E}^u \left(g\big(X(\tau^u_t)\big) + \int_t^{\tau^u_t} h(s, X, u_s)\, ds \right)$$

from the theory of subsection 4.1. Appealing to the dominated convergence theorem once again, in conjunction with the a.s. inequalities $t \leq \tau^u_t \leq T$ (thus also $\lim_{t \uparrow T} \tau^u_t = T$), we obtain

$$\lim_{t \uparrow T} \mathbb{E}^u[\,Z^u(t)\,] \,=\, \mathbb{E}^u \left(\lim_{t \uparrow T} g\big(X(\tau^u_t)\big) + \lim_{t \uparrow T} \int_t^{\tau^u_t} h(s, X, u_s)\, ds \right) \,=\, \mathbb{E}^u[\,g\big(X(T)\big)\,] \,=\, \mathbb{E}^u[\,Z^u(T)\,]\,.$$

In other words, equality prevails throughout (5.17), therefore also throughout (5.16). $\square$

6 Some Properties of the Value Process

We shall derive in this section some further properties of $V(\cdot)$, the value process of the stochastic game. These will be crucial in characterizing, then constructing, a saddle point for the game in sections 7 and 9, respectively.

Our first such result provided inequalities in the reverse direction of those in (5.8), (5.9), but for more general stopping rules and with appropriate modifications (supremum on the right-hand side; infimum on the left).

6.1 Proposition: *For any stopping rules $\mathfrak{t}, \theta$ with $0 \leq \mathfrak{t} \leq \theta \leq T$, and any admissible control process $u(\cdot) \in \mathfrak{U}$, we have*

$$\mathbb{E}^u\big[R^u(\theta)\,\big|\,\mathcal{F}_{\mathfrak{t}}\big] \,\leq\, \operatorname{ess\,sup}_{\tau \in \mathcal{S}_{\mathfrak{t},T}} \mathbb{E}^u\big(Y^u(\tau)\,\big|\,\mathcal{F}_{\mathfrak{t}}\big)\,, \tag{6.1}$$

and

$$\mathbb{E}^u\Big[\,V(\theta) + \int_{\mathfrak{t}}^{\theta} h(s, X, u_s)\, ds\,\Big|\,\mathcal{F}_{\mathfrak{t}}\Big] \,\leq\, \operatorname{ess\,sup}_{\tau \in \mathcal{S}_{\mathfrak{t},T}} \mathbb{E}^u\big(Y^u(\mathfrak{t}, \tau)\,|\,\mathcal{F}_{\mathfrak{t}}\big) \,=\, Z^u(\mathfrak{t}) \tag{6.2}$$

almost surely; as well as

$$\operatorname{ess\,inf}_{u\in\mathfrak{U}}\ \mathbb{E}^u\Big[\,V(\theta)+\int_{\mathfrak{t}}^{\theta} h(s,X,u_s)\,ds\,\Big|\,\mathcal{F}_{\mathfrak{t}}\Big]\ \le\ V(\mathfrak{t})\,, \qquad a.s. \tag{6.3}$$

and

$$\operatorname{ess\,inf}_{u\in\mathcal{V}_{[0,\mathfrak{t}]}}\ \mathbb{E}^u\Big[\,V(\theta)+\int_{\mathfrak{t}}^{\theta} h(s,X,u_s)\,ds\,\Big|\,\mathcal{F}_{\mathfrak{t}}\Big]\ \le\ V(\mathfrak{t})\,, \qquad a.s. \tag{6.4}$$

for any given $v(\cdot)\in\mathfrak{U}$ *in the notation used in (4.12).*

Proof: We recall that $V(\theta)=\operatorname{ess\,inf}_{u\in\mathfrak{U}} Z^u(\theta)$ from (4.12), (4.6) and Theorem 5.1; and that for any given $u(\cdot)\in\mathfrak{U}$, the process $Q^u(\cdot)=Z^u(\cdot)+\int_0^{\cdot} h(s,X,u_s)\,ds$ is a $\mathbb{P}^u-$supermartingale. Therefore,

$$\mathbb{E}^u\big[R^u(\theta)\,\big|\,\mathcal{F}_{\mathfrak{t}}\big]\ =\ \mathbb{E}^u\Big[\,V(\theta)+\int_0^{\theta} h(s,X,u_s)\,ds\,\Big|\,\mathcal{F}_{\mathfrak{t}}\Big]\ \le\ \mathbb{E}^u\Big[\,Z^u(\theta)+\int_0^{\theta} h(s,X,u_s)\,ds\,\Big|\,\mathcal{F}_{\mathfrak{t}}\Big]$$

$$\le\ Z^u(\mathfrak{t})+\int_0^{\mathfrak{t}} h(s,X,u_s)\,ds\ =\ \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{\mathfrak{t},T}}\ \mathbb{E}^u\big(Y^u(\tau)\,\big|\,\mathcal{F}_{\mathfrak{t}}\big)\,, \tag{6.5}$$

which is (6.1). Now (6.2) is a direct consequence; and (6.3), (6.4) follow by taking essential infima over $u(\cdot)$ in $\mathfrak{U}$ and in $\mathcal{V}_{[0,\mathfrak{t}]}$, respectively. $\Box$

In fact, we have the following result which supplements the "value identity" of equation (5.2). In this equation, the common value is at most $V(\mathfrak{t})$, as we are taking supremum over a class of stopping rules, $\mathcal{S}_{\theta,T}$, which is smaller than the class $\mathcal{S}_{\mathfrak{t},T}$ appearing in (3.7), (3.8). The next result tells us exactly how smaller than $V(\mathfrak{t})$ this common value is: it is given by the left-hand side of (6.3).

6.2 Proposition: *For any stopping rules* $\mathfrak{t}$, θ *with* $0\le\mathfrak{t}\le\theta\le T$, *we have almost surely:*

$$\operatorname{ess\,inf}_{u\in\mathfrak{U}}\ \mathbb{E}^u\Big[\,V(\theta)+\int_{\mathfrak{t}}^{\theta} h(s,X,u_s)\,ds\,\Big|\,\mathcal{F}_{\mathfrak{t}}\Big]\ =\ \operatorname{ess\,inf}_{u\in\mathfrak{U}}\ \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{\theta,T}}\ \mathbb{E}^u\big(Y^u(\mathfrak{t},\tau)\,|\,\mathcal{F}_{\mathfrak{t}}\big) \tag{6.6}$$

$$=\ \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{\theta,T}}\ \operatorname{ess\,inf}_{u\in\mathfrak{U}}\ \mathbb{E}^u\big(Y^u(\mathfrak{t},\tau)\,|\,\mathcal{F}_{\mathfrak{t}}\big)\,.$$

Proof: The second equality is, of course, that of (5.2). For the first, note that Proposition 5.4 gives

$$V(\theta)\ \le\ \mathbb{E}^u\left(g\big(X(\varrho_\theta)\big)+\int_{\theta}^{\varrho_\theta} h(s,X,u_s)\,ds\,\Big|\,\mathcal{F}_{\theta}\right)\,, \qquad \text{a.s.}$$

for any admissible control strategy $u(\cdot)\in\mathfrak{U}$, thus also

$$\mathbb{E}^u\Big[\,V(\theta)+\int_{\mathfrak{t}}^{\theta} h(s,X,u_s)\,ds\,\Big|\,\mathcal{F}_{\mathfrak{t}}\Big]\ \le\ \mathbb{E}^u\left(g\big(X(\varrho_\theta)\big)+\int_{\mathfrak{t}}^{\varrho_\theta} h(s,X,u_s)\,ds\,\Big|\,\mathcal{F}_{\mathfrak{t}}\right)$$

$$\le\ \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{\theta,T}}\ \mathbb{E}^u\big(Y^u(\mathfrak{t},\tau)\,|\,\mathcal{F}_{\mathfrak{t}}\big)\,, \qquad \text{a.s.} \tag{6.7}$$

Taking essential infima on both sides over $u(\cdot)\in\mathfrak{U}$, we arrive at the inequality ($\le$) in (6.6).

For the reverse inequality, note from (4.6) that

$$Z^u(\theta) + \int_{\mathfrak{t}}^{\theta} h(s, X, u_s)\, ds \;\geq\; \mathbb{E}^u\big(Y^u(\mathfrak{t}, \tau) \,|\, \mathcal{F}_\theta\big)\,, \qquad \text{a.s.}$$

holds for every $u(\cdot) \in \mathfrak{U}$ and every $\tau \in \mathcal{S}_{\theta,T}$ (in fact, with equality for the stopping rule $\tau = \tau^u_\theta$ of (4.10)); taking conditional expectations with respect to $\mathcal{F}_{\mathfrak{t}}$ on both sides, we obtain from this

$$\mathbb{E}^u \left(Z^u(\theta) + \int_{\mathfrak{t}}^{\theta} h(s, X, u_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \right) \geq \mathbb{E}^u\big(Y^u(\mathfrak{t}, \tau) \,|\, \mathcal{F}_{\mathfrak{t}}\big)\,, \qquad \text{a.s.} \tag{6.8}$$

for all $\tau \in \mathcal{S}_{\theta,T}$, again with equality for $\tau = \tau^u_\theta$, thus

$$\mathbb{E}^u \left(Z^u(\theta) + \int_{\mathfrak{t}}^{\theta} h(s, X, u_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \right) \;=\; \operatorname{ess\,sup}_{\tau \in \mathcal{S}_{\theta,T}} \mathbb{E}^u\big(Y^u(\mathfrak{t}, \tau) \,|\, \mathcal{F}_{\mathfrak{t}}\big)\,, \qquad \text{a.s.} \tag{6.9}$$

Fix now an admissible control strategy $v(\cdot) \in \mathfrak{U}$, and consider a sequence $\{u^k(\cdot)\}_{k \in \mathbb{N}} \subset \mathcal{V}_{[\mathfrak{t},\theta]}$ (i.e., with $u^k(\cdot) \equiv v(\cdot)$ a.e. on $[[\,\mathfrak{t}, \theta\,]]$) such that $V(\theta) = \lim_{k\to\infty} \downarrow Z^{u^k}(\theta)$ a.s., in the manner of (4.15). Write (6.9) with $u^k(\cdot)$ in place of $u(\cdot)$ and recall property (4.11) of Lemma 4.1, to obtain

$$\begin{aligned}
\mathbb{E}^v \left(Z^{u^k}(\theta) + \int_{\mathfrak{t}}^{\theta} h(s, X, v_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \right) &= \mathbb{E}^{u^k} \left(Z^{u^k}(\theta) + \int_{\mathfrak{t}}^{\theta} h\big(s, X, u^k_s\big)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \right) \\
&= \operatorname{ess\,sup}_{\tau \in \mathcal{S}_{\theta,T}} \mathbb{E}^{u^k} \left(Y^{u^k}(\mathfrak{t}, \tau) \,\big|\, \mathcal{F}_{\mathfrak{t}} \right) \\
&\geq \operatorname{ess\,inf}_{u \in \mathfrak{U}} \operatorname{ess\,sup}_{\tau \in \mathcal{S}_{\theta,T}} \mathbb{E}^u \left(Y^u(\mathfrak{t}, \tau) \,\big|\, \mathcal{F}_{\mathfrak{t}} \right)\,, \qquad \text{a.s.}
\end{aligned}$$

for every $k \in \mathbb{N}$. Now let $k \to \infty$ and use the bounded convergence theorem, to obtain

$$\mathbb{E}^v \left(V(\theta) + \int_{\mathfrak{t}}^{\theta} h(s, X, v_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \right) \geq \operatorname{ess\,inf}_{u \in \mathfrak{U}} \operatorname{ess\,sup}_{\tau \in \mathcal{S}_{\theta,T}} \mathbb{E}^u \left(Y^u(\mathfrak{t}, \tau) \,\big|\, \mathcal{F}_{\mathfrak{t}} \right)\,, \qquad \text{a.s.}$$

since $v(\cdot) \in \mathfrak{U}$ is an arbitrary control strategy; all that remains at this point is to take the essential infimum of the left hand side with respect to $v(\cdot) \in \mathfrak{U}$, and we are done. □

We are ready for the main result of this section. It says that $\inf_{u \in \mathfrak{U}} \mathbb{E}^u\big(R^u(\cdot)\big)$, the best that the controller can achieve in terms of minimizing expected "running cost, plus current value", does not increase with time; at best, this quantity is "flat up to ϱ_0", the first time the game's value equals the reward obtained by terminating the game.

6.3 Theorem: *For any stopping rules* $\mathfrak{t}$, θ *with* $0 \leq \mathfrak{t} \leq \theta \leq T$, *we have*

$$\operatorname{ess\,inf}_{u \in \mathfrak{U}} \mathbb{E}^u\big(R^u(\theta) \big|\, \mathcal{F}_{\mathfrak{t}}\big) \;\leq\; R^v(\mathfrak{t})\,, \qquad \text{a.s.} \tag{6.10}$$

for any $v(\cdot) \in \mathfrak{U}$, *as well as*

$$\inf_{u \in \mathfrak{U}} \mathbb{E}^u\big(R^u(\theta)\big) \;\leq\; \inf_{u \in \mathfrak{U}} \mathbb{E}^u\big(R^u(\mathfrak{t})\big) \;\leq\; V(0)\,. \tag{6.11}$$

The first (respectively, the second) of the inequalities in (6.11) is valid as equality if $\theta \le \varrho_{\mathfrak{t}}$ *(resp.,* $\mathfrak{t} \le \varrho_0$ *) also holds.*

A little more generally: for any stopping rules $\mathfrak{s}\,, \mathfrak{t}\,, \theta$ *with* $0 \le \mathfrak{s} \le \mathfrak{t} \le \theta \le T$ *, we have the a.s. comparisons*

$$\operatorname{ess\,inf}_{u \in \mathfrak{U}} \mathbb{E}^u \Big[V(\theta) + \int_{\mathfrak{s}}^{\theta} h(s,X,u_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{s}} \Big] \;\le\; \operatorname{ess\,inf}_{u \in \mathfrak{U}} \mathbb{E}^u \Big[V(\mathfrak{t}) + \int_{\mathfrak{s}}^{\mathfrak{t}} h(s,X,u_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{s}} \Big] \;\le\; V(\mathfrak{s})\,. \tag{6.12}$$

The first (respectively, the second) of the inequalities in (6.12) is valid as an equality on the event $\{\theta \le \varrho_{\mathfrak{t}}\}$ *(resp.,* $\{\mathfrak{t} \le \varrho_{\mathfrak{s}}\}$ *).*

Proof: With $v(\cdot) \in \mathfrak{U}$ *fixed*, and with $\mathcal{V}_{[0,\mathfrak{t}]}$ denoting the set of admissible control strategies $u(\cdot)$ that satisfy $u(\cdot) = v(\cdot)$ a.e. on the stochastic interval $[[\,0, \mathfrak{t}\,]]$, we have

$$\begin{aligned}
\operatorname{ess\,inf}_{u \in \mathfrak{U}} \mathbb{E}^u \big(R^u(\theta) \,\big|\, \mathcal{F}_{\mathfrak{t}} \big) \;&=\; \operatorname{ess\,inf}_{u \in \mathfrak{U}} \left(\mathbb{E}^u \Big[V(\theta) + \int_{\mathfrak{t}}^{\theta} h(s,X,u_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \Big] + \int_0^{\mathfrak{t}} h(s,X,u_s)\, ds \right) \\
&\le\; \operatorname{ess\,inf}_{u \in \mathcal{V}_{[0,\mathfrak{t}\,]}} \mathbb{E}^u \Big[V(\theta) + \int_{\mathfrak{t}}^{\theta} h(s,X,u_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{t}} \Big] \;+\; \int_0^{\mathfrak{t}} h(s,X,v_s)\, ds \\
&\le\; V(\mathfrak{t}) \;+\; \int_0^{\mathfrak{t}} h(s,X,v_s)\, ds \;=\; R^v(\mathfrak{t})\,, \qquad \text{a.s.}
\end{aligned}$$

where the penultimate comparison comes from (6.4). This proves (6.10).

To obtain the first inequality (6.12), observe that (6.5) gives

$$\mathbb{E}^u \Big[V(\theta) + \int_{\mathfrak{s}}^{\theta} h(s,X,u_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{s}} \Big] \;\le\; \mathbb{E}^u \Big[Z^u(\mathfrak{t}) + \int_{\mathfrak{s}}^{\mathfrak{t}} h(s,X,u_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{s}} \Big] \qquad \text{a.s.}$$

for all $u(\cdot) \in \mathfrak{U}$. Proceeding just as before, with $v(\cdot) \in \mathfrak{U}$ arbitrary but fixed, and selecting a sequence $\{u^k(\cdot)\}_{k \in \mathbb{N}} \subset \mathcal{V}_{[0,\mathfrak{t}]}$ such that $V(\mathfrak{t}) = \lim_{k\to\infty} \downarrow Z^{u^k}(\mathfrak{t})$ holds almost surely, we have

$$\begin{aligned}
\operatorname*{ess\,inf}_{u \in \mathfrak{U}} \mathbb{E}^u \Big[V(\theta) + \int_{\mathfrak{s}}^{\theta} h(s,X,u_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{s}} \Big] \;&\le\; \mathbb{E}^{u^k} \Big[V(\theta) + \int_{\mathfrak{s}}^{\theta} h(s,X,u^k_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{s}} \Big] \\
\le\; \mathbb{E}^{u^k} \left[Z^{u^k}(\mathfrak{t}) + \int_{\mathfrak{s}}^{\mathfrak{t}} h(s,X,u^k_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{s}} \right] \;&=\; \mathbb{E}^{v} \left[Z^{u^k}(\mathfrak{t}) + \int_{\mathfrak{s}}^{\mathfrak{t}} h(s,X,v_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{s}} \right]
\end{aligned}$$

for every $k \in \mathbb{N}$, thus also

$$\operatorname*{ess\,inf}_{u \in \mathfrak{U}} \mathbb{E}^u \Big[V(\theta) + \int_{\mathfrak{s}}^{\theta} h(s,X,u_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{s}} \Big] \;\le\; \mathbb{E}^{v} \Big[V(\mathfrak{t}) + \int_{\mathfrak{s}}^{\mathfrak{t}} h(s,X,v_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{s}} \Big]$$

in the limit as $k \to \infty$. Take the (essential) infimum of the right-hand side over $v(\cdot) \in \mathfrak{U}$, to obtain the desired a.s. inequality

$$\operatorname{ess\,inf}_{u \in \mathfrak{U}} \mathbb{E}^u \Big[V(\theta) + \int_{\mathfrak{s}}^{\theta} h(s,X,u_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{s}} \Big] \;\le\; \operatorname{ess\,inf}_{v \in \mathfrak{U}} \mathbb{E}^v \Big[V(\mathfrak{t}) + \int_{\mathfrak{s}}^{\mathfrak{t}} h(s,X,v_s)\, ds \,\Big|\, \mathcal{F}_{\mathfrak{s}} \Big]\,,$$

the first in (6.12). (The reverse inequality holds on the event $\{\theta \le \varrho_{\mathfrak{t}}\}$, as we know from (5.10).) The second inequality of (6.12) follows from the first, upon replacing θ by $\mathfrak{t}$, and $\mathfrak{t}$ by $\mathfrak{s}$.

Now (6.11) follows directly from (6.12), just by taking $\mathfrak{s} = 0$ there. $\square$

7 A Martingale Characterization of Saddle-Points

We are now in a position to provide a characterization of the saddle-point property (3.6) in terms of appropriate martingales. This characterization will prove indispensable when we try, in the next two sections, to prove constructively the existence of a saddle point (u^*, τ_*) for the stochastic game of control and stopping.

7.1 Theorem: *A pair $(u^*, \tau_*) \in \mathfrak{U} \times \mathcal{S}$ is a saddle point as in (3.6) for the stochastic game of control and stopping, if and only if the following three conditions hold:*

(i) $g\big(X(\tau_*)\big) = V(\tau_*)$ *, a.s.*

(ii) $R^{u^*}(\,\cdot \wedge \tau_*)$ *is a* $\mathbb{P}^{u^*}-$ *martingale; and*

(iii) $R^{u}(\,\cdot \wedge \tau_*)$ *is a* $\mathbb{P}^{u}-$ *submartingale, for every* $u(\cdot) \in \mathfrak{U}$.

The present section is devoted to the proof of this result. We shall derive first the conditions (i)-(iii) from the properties (3.6) of the saddle, then the reverse.

Proof of Necessity: Let us assume that the pair $(u^*, \tau_*) \in \mathfrak{U} \times \mathcal{S}$ is a saddle point for the game, i.e., that the properties of (3.6) are satisfied.

• Using the definition of $\varrho_{\mathfrak{t}}$, the submartingale property $\mathbb{E}^{u^*}\big[\, R^{u^*}(\varrho_{\mathfrak{t}}) \,|\, \mathcal{F}_{\mathfrak{t}}\big] \geq R^{u^*}(\mathfrak{t})$ from Proposition 5.3, the a.s. comparisons

$$Y^{u^*}(\tau_*) \leq R^{u^*}(\tau_*) \qquad \text{and} \qquad Y^{u^*}(\varrho_{\tau_*}) = R^{u^*}(\varrho_{\tau_*})\,,$$

and the first property of the saddle in (3.6), we obtain

$$\mathbb{E}^{u^*}\big(Y^{u^*}(\tau_*)\big) \;\leq\; \mathbb{E}^{u^*}\big(R^{u^*}(\tau_*)\big) \;\leq\; \mathbb{E}^{u^*}\big(R^{u^*}(\varrho_{\tau_*})\big) \;=\; \mathbb{E}^{u^*}\big(Y^{u^*}(\varrho_{\tau_*})\big) \;\leq\; \mathbb{E}^{u^*}\big(Y^{u^*}(\tau_*)\big)\,.$$

But this gives, in particular, $\mathbb{E}^{u^*}\big(Y^{u^*}(\tau_*)\big) = \mathbb{E}^{u^*}\big(R^{u^*}(\tau_*)\big)$ which, coupled with the earlier a.s. comparison, gives the stronger one $Y^{u^*}(\tau_*) = R^{u^*}(\tau_*)$, thus also $g\big(X(\tau_*)\big) = V(\tau_*)$.

• Next, consider an *arbitrary* stopping rule $\tau \in \mathcal{S}$ with $0 \leq \tau \leq \tau_*$ and observe the string of inequalities

$$
\begin{aligned}
\mathbb{E}^{u^*}\big(R^{u^*}(\tau)\big) \;&\leq\; \mathbb{E}^{u^*}\big(R^{u^*}(\varrho_\tau)\big) && \text{(Proposition 5.1)}\\
&=\; \mathbb{E}^{u^*}\big(Y^{u^*}(\varrho_\tau)\big) && (\text{definition of } \varrho_\tau)\\
&\leq\; \mathbb{E}^{u^*}\big(Y^{u^*}(\tau_*)\big) && \text{(first property of saddle)}\\
&=\; \mathbb{E}^{u^*}\big(R^{u^*}(\tau_*)\big) && \text{(property (i), just proved)}\\
&=\; \inf_{u\in\mathfrak{U}} \mathbb{E}^{u}\big(Y^{u}(\tau_*)\big) && \text{(second property of saddle)}\\
&=\; \inf_{u\in\mathfrak{U}} \mathbb{E}^{u}\big(R^{u}(\tau_*)\big) && \text{(property (i), just proved)}\\
&\leq\; \inf_{u\in\mathfrak{U}} \mathbb{E}^{u}\big(R^{u}(\tau)\big) && \text{(inequality (6.11))}\\
&\leq\; \mathbb{E}^{u^*}\big(R^{u^*}(\tau)\big)\,.
\end{aligned}
$$

Therefore, for *every* stopping rule $\tau \in \mathcal{S}$ with $0 \le \tau \le \tau_*$, we have

$$\mathbb{E}^{u^*}\big(R^{u^*}(\tau)\big) \;=\; \inf_{u \in \mathfrak{U}} \,\mathbb{E}^{u}\big(R^{u}(\tau)\big) \;=\; \inf_{u \in \mathfrak{U}} \,\mathbb{E}^{u}\big(R^{u}(\tau_*)\big) \;=\; \mathbb{E}^{u^*}\big(R^{u^*}(\tau_*)\big)\,. \tag{7.1}$$

This shows that $R^{u^*}(\,\cdot \wedge \tau_*)$ is a $\mathbb{P}^{u^*}-$martingale (cf. Exercise 1.3.26 in Karatzas & Shreve (1991)), and condition (ii) is established.

• It remains to show that, for *any* given $v(\cdot) \in \mathfrak{U}$, the process $R^{v}(\,\cdot \wedge \tau_*)$ is a $\mathbb{P}^{v}-$submartingale; equivalently, that

$$\mathbb{E}^{v}\Big[\,V(\tau) + \int_{\mathfrak{t}}^{\tau} h(s, X, v_s)\,ds \,\Big|\, \mathcal{F}_{\mathfrak{t}}\Big] \;\ge\; V(\mathfrak{t}) \tag{7.2}$$

holds a.s. for any stopping rules $\mathfrak{t}, \tau$ with $0 \le \mathfrak{t} \le \tau \le \tau_*$.

Let us start by fixing a stopping rule τ as above, and recalling from (6.3) of Proposition 6.1 that

$$\widehat{V}(\mathfrak{t};\tau) \;:=\; \operatorname{ess\,inf}_{u \in \mathfrak{U}}\, \mathbb{E}^{u}\Big[\,V(\tau) + \int_{\mathfrak{t}}^{\tau} h(s, X, u_s)\,ds \,\Big|\, \mathcal{F}_{\mathfrak{t}}\Big] \;\le\; V(\mathfrak{t}) \tag{7.3}$$

holds a.s. We'll be done, that is, we shall have proved (7.2), as soon as we have established that the reverse inequality

$$\widehat{V}(\mathfrak{t};\tau) \;\ge\; V(\mathfrak{t}) \qquad \text{holds a.s.} \tag{7.4}$$

as well, for any given $\tau \in \mathcal{S}$ with with $\mathfrak{t} \le \tau \le \tau_*$.

To this effect, let us consider for any $\varepsilon > 0$ the event and the stopping rule

$$A_\varepsilon \;:=\; \big\{\, V(\mathfrak{t}) \ge \widehat{V}(\mathfrak{t};\tau) + \varepsilon \,\big\} \;\in\; \mathcal{F}_{\mathfrak{t}} \qquad \text{and} \qquad \theta_\varepsilon \;:=\; \mathfrak{t} \cdot 1_{A_\varepsilon} \;+\; \tau \cdot 1_{A_\varepsilon^c}\;,$$

respectively, and note $0 \le \mathfrak{t} \le \theta_\varepsilon \le \tau \le \tau_* \le T$. From (7.1) we get

$$\begin{aligned}
\mathbb{E}^{u^*}\big(R^{u^*}(\mathfrak{t})\big) \;&=\; \mathbb{E}^{u^*}\big(R^{u^*}(\theta_\varepsilon)\big) \;=\; \mathbb{E}^{u^*}\Big[\,R^{u^*}(\mathfrak{t}) \cdot 1_{A_\varepsilon} + R^{u^*}(\tau) \cdot 1_{A_\varepsilon^c}\Big] \\
&=\; \mathbb{E}^{u^*}\Big[\,R^{u^*}(\mathfrak{t}) \cdot 1_{A_\varepsilon} + \mathbb{E}^{u^*}\big(R^{u^*}(\tau) \,\big|\, \mathcal{F}_{\mathfrak{t}}\big) \cdot 1_{A_\varepsilon^c}\Big] \\
&=\; \mathbb{E}^{u^*}\Big[\,V(\mathfrak{t}) \cdot 1_{A_\varepsilon} + \mathbb{E}^{u^*}\Big(V(\tau) + \int_{\mathfrak{t}}^{\tau} h(s, X, u_s^*)\,ds \,\Big|\, \mathcal{F}_{\mathfrak{t}}\Big) \cdot 1_{A_\varepsilon^c} \;+ \int_0^{\mathfrak{t}} h(s, X, u_s^*)\,ds\Big] \\
&\ge\; \mathbb{E}^{u^*}\Big[\,V(\mathfrak{t}) \cdot 1_{A_\varepsilon} + \widehat{V}(\mathfrak{t};\tau) \cdot 1_{A_\varepsilon^c} \;+ \int_0^{\mathfrak{t}} h(s, X, u_s^*)\,ds\Big] \\
&\ge\; \varepsilon \cdot \mathbb{P}^{u^*}\big(A_\varepsilon\big) \;+\; \mathbb{E}^{u^*}\Big[\,\widehat{V}(\mathfrak{t};\tau) \;+ \int_0^{\mathfrak{t}} h(s, X, u_s^*)\,ds\Big]\,.
\end{aligned} \tag{7.5}$$

As in (6.4), we write now the random variable $\widehat{V}(\mathfrak{t};\tau)$ of (7.3) in the form

$$\begin{aligned}
\widehat{V}(\mathfrak{t};\tau) \;&=\; \operatorname{ess\,inf}_{u \in \mathcal{U}^*_{[0,\mathfrak{t}]}}\, \mathbb{E}^{u}\Big[\,V(\tau) + \int_{\mathfrak{t}}^{\tau} h(s, X, u_s)\,ds \,\Big|\, \mathcal{F}_{\mathfrak{t}}\Big] \\
&=\; \lim_{k \to \infty} \mathbb{E}^{u^k}\Big[\,V(\tau) + \int_{\mathfrak{t}}^{\tau} h(s, X, u_s^k)\,ds \,\Big|\, \mathcal{F}_{\mathfrak{t}}\Big]
\end{aligned}$$

for some sequence $\{u^k(\cdot)\}_{k\in\mathbb{N}}$ in $\mathcal{U}^*_{[0,\mathfrak{t}]}$, the set of admissible control strategies $u(\cdot) \in \mathfrak{U}$ that agree with $u^*(\cdot)$ a.e. on the stochastic interval $[[\,0,\,\mathfrak{t}]]$. Back into (7.5), this gives

$$
\begin{aligned}
&\mathbb{E}^{u^*}\big(R^{u^*}(\mathfrak{t})\big) \;-\; \varepsilon\cdot\mathbb{P}^{u^*}\big(A_\varepsilon\big) \;\geq \\
&\qquad \geq \mathbb{E}^{u^*}\left[\lim_k\; \mathbb{E}^{u^k}\left(V(\tau) + \int_{\mathfrak{t}}^{\tau} h(s,X,u^k_s)\,ds \,\Big|\, \mathcal{F}_{\mathfrak{t}}\right) + \int_0^{\mathfrak{t}} h(s,X,u^*_s)\,ds\right] \\
&\qquad = \mathbb{E}^{u^*}\left[\lim_k\; \mathbb{E}^{u^k}\left(V(\tau) + \int_0^{\tau} h(s,X,u^k_s)\,ds \,\Big|\, \mathcal{F}_{\mathfrak{t}}\right)\right] \\
&\qquad = \mathbb{E}^{u^*}\left[\lim_k\; \mathbb{E}^{u^k}\left(R^{u^k}(\tau) \,\Big|\, \mathcal{F}_{\mathfrak{t}}\right)\right] \\
&\qquad = \lim_k\; \mathbb{E}^{u^*}\left[\mathbb{E}^{u^k}\left(R^{u^k}(\tau) \,\Big|\, \mathcal{F}_{\mathfrak{t}}\right)\right] && \text{(bounded convergence)} \\
&\qquad = \lim_k\; \mathbb{E}^{u^k}\left[\mathbb{E}^{u^k}\left(R^{u^k}(\tau) \,\Big|\, \mathcal{F}_{\mathfrak{t}}\right)\right] && \text{(equation (4.11), Lemma 4.1)} \\
&\qquad = \lim_k\; \mathbb{E}^{u^k}\left(R^{u^k}(\tau)\right) \;\geq\; \inf_{u\in\mathfrak{U}} \mathbb{E}^{u}\left(R^{u}(\tau)\right) \;=\; \mathbb{E}^{u^*}\left(R^{u^*}(\tau)\right) \;=\; \mathbb{E}^{u^*}\left(R^{u^*}(\mathfrak{t})\right),
\end{aligned}
$$

where the last claim follows from (7.1), the martingale property of (ii) that this implies, and $0 \leq \mathfrak{t} \leq \tau \leq \tau_*$. This shows $\mathbb{P}(A_\varepsilon) = 0$, thus

$$
V(t) \;<\; \widehat{V}(t;\tau) + \varepsilon\,, \qquad \text{a.s.}
$$

for every $\varepsilon > 0$; letting $\varepsilon \downarrow 0$ we arrive at (7.4), and we are done. $\square$

Proof of Sufficiency: Let us suppose now that the pair $(u^*, \tau_*) \in \mathfrak{U} \times \mathcal{S}$ satisfies the properties (i)-(iii) of Theorem 7.1; we shall try to deduce from them the properties of (3.6) for a saddle-point.

- We start by considering stopping rules $\tau \in \mathcal{S}$ with $0 \leq \tau \leq \tau_*$.
  For such stopping rules, the fact that $R^{u^*}(\,\cdot \wedge \tau_*)$ is a $\mathbb{P}^{u^*}-$martingale (property (ii)) leads to

$$
Y^{u^*}(\tau) \;\leq\; R^{u^*}(\tau) \;=\; \mathbb{E}^{u^*}\big(R^{u^*}(\tau_*)\,|\,\mathcal{F}_\tau\big) \;=\; \mathbb{E}^{u^*}\big(Y^{u^*}(\tau_*)\,|\,\mathcal{F}_\tau\big)\,, \qquad \text{a.s.} \tag{7.6}
$$

and this gives the first property of the saddle upon taking expectations.

On the other hand, the $\mathbb{P}^u-$submartingale property of $R^u(\,\cdot\, \wedge \tau_*)$ in property (iii) gives $\mathbb{E}^u\big(R^u(\tau)\big) \leq \mathbb{E}^u\big(R^u(\tau_*)\big)$ for all $u(\cdot) \in \mathfrak{U}$, thus also

$$
\inf_{u\in\mathfrak{U}} \mathbb{E}^u\big(R^u(\tau)\big) \;\leq\; \inf_{u\in\mathfrak{U}} \mathbb{E}^u\big(R^u(\tau_*)\big)\,.
$$

Taking here $\tau \equiv 0$ and using the property (i) for τ_*, as well as the $\mathbb{P}^{u^*}-$martingale property of $R^{u^*}(\,\cdot\, \wedge \tau_*)$ from (ii), we get

$$
\inf_{u\in\mathfrak{U}} \mathbb{E}^u\big(Y^u(\tau_*)\big) \;=\; \inf_{u\in\mathfrak{U}} \mathbb{E}^u\big(R^u(\tau_*)\big) \;\geq\; R^u(0) \;=\; V \;=\; R^{u^*}(0) \;=\; \mathbb{E}^{u^*}\big(R^{u^*}(\tau_*)\big) \;=\; \mathbb{E}^{u^*}\big(Y^{u^*}(\tau_*)\big)\,.
$$

Comparing the two extreme terms in this string, we obtain the second property of the saddle.

• Let us consider now stopping rules $\tau \in \mathcal{S}$ with $\tau_* \le \tau \le T$, and try to establish for them the first property of the saddle – actually in the stronger form

$$\mathbb{E}^{u^*}\big(Y^{u^*}(\tau)\,|\,\mathcal{F}_{\tau_*}\big) \;\le\; Y^{u^*}(\tau_*)\,, \qquad \text{a.s.} \tag{7.7}$$

We shall argue this by contradiction: suppose there existed a stopping rule $\pi \in \mathcal{S}_{\tau_*,T}$ and a number $\delta > 0$, such that

$$\mathbb{P}^{u^*}\big(\Gamma^{\delta}_{\pi}\big) \;>\; 0\,, \qquad \text{where} \qquad \Gamma^{\delta}_{t} \;:=\; \Big\{\,\mathbb{E}^{u^*}\big(Y^{u^*}(t)\,\big|\,\mathcal{F}_{\tau_*}\big) \;\ge\; \delta + Y^{u^*}(\tau_*)\Big\} \in \mathcal{F}_{\tau_*}\,. \tag{7.8}$$

In fact, we could take

$$\pi \;=\; \inf\Big\{\, t > \tau_* \;:\; \mathbb{P}^{u^*}\big(\Gamma^{\varepsilon}_{t}\big) \;>\; 0 \;\;\text{for some}\;\; \varepsilon > 0\Big\} \wedge T\,; \tag{7.9}$$

then for any stopping rule θ with $\tau_* \le \theta \le \pi$ we would have $\mathbb{P}^{u^*}(\Gamma^{\delta}_{\theta}) \;=\; 0$ for all $\delta > 0$, namely

$$\mathbb{E}^{u^*}\big(Y^{u^*}(\theta)\,|\,\mathcal{F}_{\tau_*}\big) \;\le\; Y^{u^*}(\tau_*)\,, \qquad \text{a.s.} \tag{7.10}$$

The stopping rule π in (7.9) is clearly predictable (for an arbitrary filtration, as can be checked; but we are assuming here that our filtration $\mathbb{F}$ is generated by a Brownian motion, so in fact *every* stopping rule is predictable in our setting). Thus, there would exist an increasing sequence of stopping rules $\{\theta^k\}_{k\in\mathbb{N}}$ with $\lim_k \uparrow \theta^k \;=\; \pi$, a.s. Because $Y^{u^*}(\cdot)$ is continuous and bounded, (7.10) would then give

$$\mathbb{E}^{u^*}\big(Y^{u^*}(\pi)\,|\,\mathcal{F}_{\tau_*}\big) \;=\; \lim_{k}\; \mathbb{E}^{u^*}\big(Y^{u^*}(\theta^k)\,|\,\mathcal{F}_{\tau_*}\big) \;\le\; Y^{u^*}(\tau_*) \qquad \text{a.s.}$$

by the bounded convergence theorem, contradicting (7.8).

• Finally, let us try to prove the first property of the saddle for an *arbitrary* stopping rule $\tau \in \mathcal{S}$. We start with the decomposition

$$\begin{aligned}\mathbb{E}^{u^*}\big(Y^{u^*}(\tau)\big) \;&=\; \mathbb{E}^{u^*}\Big(Y^{u^*}(\tau)\cdot 1_{\{\tau\le\tau_*\}} \;+\; Y^{u^*}(\tau)\cdot 1_{\{\tau>\tau_*\}}\Big)\\ &=\; \mathbb{E}^{u^*}\Big(\,Y^{u^*}(\tau\wedge\tau_*)\cdot 1_{\{\tau\le\tau_*\}} \;+\; Y^{u^*}(\tau\vee\tau_*)\cdot 1_{\{\tau>\tau_*\}}\,\Big)\,.\end{aligned}$$

But $\rho := \tau\wedge\tau_*$ belongs to $\mathcal{S}_{0,\tau_*}$ and so we have

$$Y^{u^*}(\rho) \;\le\; \mathbb{E}^{u^*}\big(Y^{u^*}(\tau_*)\,|\,\mathcal{F}_{\rho}\big)\,, \qquad \text{a.s.}$$

from (7.6); whereas $\nu := \tau\vee\tau_*$ is in $\mathcal{S}_{\tau_*,T}$, thus

$$\mathbb{E}^{u^*}\big(Y^{u^*}(\nu)\,|\,\mathcal{F}_{\tau_*}\big) \;\le\; Y^{u^*}(\tau_*)\,, \qquad \text{a.s.}$$

from (7.7). Both events $\{\tau\le\tau_*\}$, $\{\tau>\tau_*\}$ belong to $\mathcal{F}_{\rho}$, therefore

$$\begin{aligned}\mathbb{E}^{u^*}\big(Y^{u^*}(\tau)\big) \;&=\; \mathbb{E}^{u^*}\Big(\,Y^{u^*}(\rho)\cdot 1_{\{\tau\le\tau_*\}} \;+\; Y^{u^*}(\nu)\cdot 1_{\{\tau>\tau_*\}}\,\Big)\\ &\le\; \mathbb{E}^{u^*}\Big(\,\mathbb{E}^{u^*}\big(Y^{u^*}(\tau_*)\,\big|\,\mathcal{F}_{\rho}\big)\cdot 1_{\{\tau\le\tau_*\}} \;+\; \mathbb{E}^{u^*}\big(Y^{u^*}(\nu)\,\big|\,\mathcal{F}_{\rho}\big)\cdot 1_{\{\tau>\tau_*\}}\,\Big)\\ &=\; \mathbb{E}^{u^*}\Big(\,\mathbb{E}^{u^*}\big(Y^{u^*}(\tau_*)\cdot 1_{\{\tau\le\tau_*\}}\,\big|\,\mathcal{F}_{\rho}\big) \;+\; \mathbb{E}^{u^*}\big(Y^{u^*}(\nu)\,\big|\,\mathcal{F}_{\tau_*}\big)\cdot 1_{\{\tau>\tau_*\}}\,\Big)\\ &\le\; \mathbb{E}^{u^*}\big(\,Y^{u^*}(\tau_*)\cdot 1_{\{\tau\le\tau_*\}}\big) \;+\; \mathbb{E}^{u^*}\big(\,Y^{u^*}(\tau_*)\cdot 1_{\{\tau>\tau_*\}}\big) \;=\; \mathbb{E}^{u^*}\big(Y^{u^*}(\tau_*)\big)\,.\end{aligned}$$

This is the first property of the saddle in (3.6), established now for arbitrary $\tau \in \mathcal{S}$. $\square$

8 Optimality Conditions for Control

We shall say that a given admissible control strategy $\widetilde{u}(\cdot) \in \mathfrak{U}$ is *optimal*, if it attains the infimum

$$V = \inf_{v \in \mathfrak{U}} Z^v(0), \qquad \text{with} \qquad Z^v(0) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^v[\, Y(\tau)\,]. \tag{8.1}$$

Here and in what follows, we are using the notation of (4.6), (4.10) and (5.4).

Clearly, if $(\widetilde{u}, \widetilde{\tau})$ is a saddle pair for the stochastic game, then $\widetilde{u}(\cdot)$ is an optimal control strategy.

We present now a characterization of optimality, in the spirit of a similar characterization for optimal control with discretionary stopping in Dubins & Savage (1976) and in Maitra & Sudderth (1996.a, page 75).

8.1 Theorem: Necessary and Sufficient Conditions for Optimality of a Control Process. *A given admissible control strategy $u(\cdot) \in \mathfrak{U}$ is optimal, i.e., attains the supremum in (8.1), if and only if the following two conditions are satisfied:*

- Thriftiness: *The process $R^u(\,\cdot\, \wedge \tau_0^u)$ is a $\mathbb{P}^u-$martingale.*
- Equalization: $\mathbb{P}^u\big(\tau_0^u(\varepsilon) < T\big) = 1, \quad \forall\ 0 < \varepsilon < 1$.

And if these conditions are satisfied, then for every $0 \le \varepsilon < 1$ we have

$$\tau_0^u(\varepsilon) = \varrho_0(\varepsilon) \qquad a.s. \tag{8.2}$$

Proof of Sufficiency: Let us recall from (5.6) that $\tau_0^u(\varepsilon) \le \tau_0^u$ holds a.s. for every $0 < \varepsilon < 1$, and that the process $Q^u(\,\cdot\, \wedge \tau_0^u)$, as in (4.7), is a $\mathbb{P}^u-$martingale. Therefore, if $u(\cdot)$ is both equalizing and thrifty, we have

$$\begin{aligned}
0 \le Z^u(0) - V &= \mathbb{E}^u \left[Z^u\big(\tau_0^u(\varepsilon)\big) + \int_0^{\tau_0^u(\varepsilon)} h(s, X, u_s)\, ds \right] - V \\
&\le \mathbb{E}^u \left[\Big(\varepsilon + g\big(X\big(\tau_0^u(\varepsilon)\big)\big)\Big) + \int_0^{\tau_0^u(\varepsilon)} h(s, X, u_s)\, ds \right] - V \\
&\le \varepsilon + \mathbb{E}^u \left[V\big(\tau_0^u(\varepsilon)\big) + \int_0^{\tau_0^u(\varepsilon)} h(s, X, u_s)\, ds \right] - V \\
&= \varepsilon + \mathbb{E}^u\left[R^u\big(\tau_0^u(\varepsilon)\big) \right] - R^u(0) = \varepsilon.
\end{aligned}$$

In this string, the second inequality is because of equalization and the definition of $\tau_0^u(\varepsilon)$ in (4.10); whereas the last equality is a consequence of thriftiness and of the inequality $\tau_0^u(\varepsilon) \le \tau_0^u$. This gives the comparison $0 \le Z^u(0) - V \le \varepsilon$ for every $0 < \varepsilon < 1$, therefore $Z^u(0) = V$, the optimality of $u(\cdot)$.

Proof of Necessity: Let us suppose now that $u(\cdot) \in \mathfrak{U}$ is optimal; we shall show that it is both equalizing and thrifty, and that (8.2) holds for every $0 \le \varepsilon < 1$.

• Suppose that equalization fails; to wit, that there exists a number $\varepsilon \in (0,1)$ such that $\mathbb{P}^u\big(\tau_0^u(\varepsilon) = T\big) = \delta > 0$. Then, the $\mathbb{P}^u-$supermartingale property of $Q^u(\cdot) = Z^u(\cdot) + \int_0^\cdot h(s,X,u_s)\,ds$ gives

$$V - \mathbb{E}^u\left[g\big(X(\tau)\big) + \int_0^\tau h(s,X,u_s)\,ds \right] = Z^u(0) - \mathbb{E}^u\left[g\big(X(\tau)\big) + \int_0^\tau h(s,X,u_s)\,ds \right]$$

$$\geq \mathbb{E}^u\left[Z^u(\tau) - g\big(X(\tau)\big)\right] \geq \mathbb{E}^u\left[\big(Z^u(\tau) - g\big(X(\tau)\big)\big) \cdot 1_{\{\tau_0^u(\varepsilon)=T\}} \right] \geq \varepsilon \cdot \mathbb{P}^u\big(\tau_0^u(\varepsilon) = T\big) = \varepsilon \cdot \delta$$

for every $\tau \in \mathcal{S}$. The last inequality in this string is valid, because $Z^u(\cdot) - g\big(X(\cdot)\big) \geq \varepsilon$ holds a.s. on the event $\{\tau_0^u(\varepsilon) = T\}$.

But now we obtain from this string, taking the infimum on its left-most side over $\tau \in \mathcal{S}$, that $V - Z^u(0) \geq \varepsilon\,\delta$; this contradicts the optimality of $u(\cdot)$. It follows that $u(\cdot)$ must be equalizing.

• Let us show now that, for this optimal $u(\cdot)$, we have

$$\tau_0^u = \varrho_0\,, \qquad \text{a.s.} \tag{8.3}$$

We shall argue again by contradiction: we know from (5.6) that $\varrho_0 \leq \tau_0^u$ holds a.s., so let us assume

$$\mathbb{P}^u\big(\tau_0^u > \varrho_0\big) > 0\,. \tag{8.4}$$

From the $\mathbb{P}^u-$martingale property of $Q^u\big(\,\cdot \wedge \tau_0^u\big)$, coupled with the $\mathbb{P}^u-$submartingale property of $R^u\big(\,\cdot \wedge \varrho_0\big)$ from Proposition 5.3, we obtain

$$\begin{aligned}
Z^u(0) - \mathbb{E}^u \int_0^{\varrho_0} h(s,X,u_s)\,ds &= \mathbb{E}^u\big(Z^u(\varrho_0)\big) = \mathbb{E}^u\big[\, Z^u(\varrho_0)\cdot 1_{\{\tau_0^u=\varrho_0\}} + Z^u(\varrho_0)\cdot 1_{\{\tau_0^u>\varrho_0\}}\,\big] \\
&\geq \mathbb{E}^u\big[\, Z^u(\varrho_0)\, 1_{\{\tau_0^u=\varrho_0\}} + g\big(X(\varrho_0)\big)\, 1_{\{\tau_0^u>\varrho_0\}}\,\big] \\
&= \mathbb{E}^u\big[\, Z^u(\varrho_0)\, 1_{\{\tau_0^u=\varrho_0\}} + V(\varrho_0)\, 1_{\{\tau_0^u>\varrho_0\}}\,\big] \\
&\geq \mathbb{E}^u\big[\, V(\varrho_0)\cdot 1_{\{\tau_0^u=\varrho_0\}} + V(\varrho_0)\cdot 1_{\{\tau_0^u>\varrho_0\}}\,\big] = \mathbb{E}^u\big[\, V(\varrho_0)\,\big]\,,
\end{aligned} \tag{8.5}$$

as well as

$$Z^u(0) \geq \mathbb{E}^u\left[V(\varrho_0) + \int_0^{\varrho_0} h(s,X,u_s)\,ds \right] = \mathbb{E}^u\big[\, R^u(\varrho_0)\,\big] \geq R^u(0) = V\,. \tag{8.6}$$

Because of the assumption (8.4), the first inequality in (8.5) – thus also in (8.6) – is *strict;* but this contradicts the optimality of $u(\cdot) \in \mathfrak{U}$.

Thus, as claimed in (8.3), we have $\tau_0^u = \varrho_0$ a.s. A similar argument leads to $\tau_0^u(\varepsilon) = \varrho_0(\varepsilon)$ a.s., for every $0 < \varepsilon < 1$, and (8.2) is proved.

• To see that this optimal $u(\cdot) \in \mathfrak{U}$ must also be *thrifty,* just observe that, as we have seen, equality prevails in (8.6); and that this, coupled with (8.3), gives $R^u(0) = \mathbb{E}^u\big[\, R^u(\tau_0^u)\,\big]$. It follows that the $\mathbb{P}^u-$submartingale $R^u(\,\cdot \wedge \varrho_0) \equiv R^u(\,\cdot \wedge \tau_0^u)$ is in fact a $\mathbb{P}^u-$martingale. □

8.2 Proposition: *Every admissible control strategy* $u(\cdot) \in \mathfrak{U}$ *is equalizing.*

Proof: Assume the contrary; to wit, that there exists some admissible control strategy $u(\cdot) \in \mathfrak{U}$ with $\mathbb{P}^u\big(\tau_0^u(\varepsilon) = T\big) = \zeta > 0$ for some $0 < \varepsilon < 1$. Then for every stoping rule $\tau \in \mathcal{S}$ we have

$$\mathbb{E}^u \Big[\Big(Z^u(\tau) - g\big(X(\tau)\big)\Big) \cdot 1_{\{\tau_0^u(\varepsilon)=T\}} \Big] \geq \varepsilon \cdot \mathbb{P}^u\big(\tau_0^u(\varepsilon) = T\big) \; = \varepsilon \cdot \zeta \; > \; 0$$

thanks to the definition (4.10) of $\tau_0^u(\varepsilon)$. Now the continuity of $g : \mathbb{R}^n \to \mathbb{R}$, Proposition 5.6, and the bounded convergence theorem, lead to

$$\mathbb{E}^u\big[\, Z^u(T) \cdot 1_{\{\tau_0^u(\varepsilon)=T\}} \,\big] \; = \; \mathbb{E}^u\big[\, g\big(X(T)\big) \cdot 1_{\{\tau_0^u(\varepsilon)=T\}} \,\big] \; = \; \lim_{t \uparrow T} \mathbb{E}^u\big[\, g\big(X(t)\big) \cdot 1_{\{\tau_0^u(\varepsilon)=T\}} \,\big]$$

$$\leq \lim_{t \uparrow T} \mathbb{E}^u\big[\, Z^u(t) \cdot 1_{\{\tau_0^u(\varepsilon)=T\}} \,\big] - \varepsilon\,\zeta \; = \; \mathbb{E}^u\big[\, Z^u(T-) \cdot 1_{\{\tau_0^u(\varepsilon)=T\}} \,\big] - \varepsilon\,\zeta \; = \; \mathbb{E}^u\big[\, Z^u(T) \cdot 1_{\{\tau_0^u(\varepsilon)=T\}} \,\big] - \varepsilon\,\zeta\,,$$

a contradiction. $\square$

8.3 Corollary: *If the admissible control strategy* $u(\cdot) \in \mathfrak{U}$ *is thrifty, then it is optimal; and the pair* $(u, \tau_0^u) = (u, \varrho_0) \in \mathfrak{U} \times \mathcal{S}$ *is then a saddle point for the stochastic game of control and stopping.*

Proof: The first claim follows directly from Theorem 8.1 and Proposition 8.2. Now let us make a few observations:

(i) By the definition of ϱ_0 in (5.4) and the right-continuity of the process $V(\cdot)$, we have the a.s. equality $V(\varrho_0) = g(X(\varrho_0))$.

(ii) From Proposition 5.3, the process $R^v(\,\cdot \wedge \varrho_0)$ is a $\mathbb{P}^v-$submartingale, for every $v(\cdot) \in \mathfrak{U}$.

(iii) Finally, the process $R^u(\,\cdot \wedge \varrho_0)$ is a $\mathbb{P}^u-$martingale; this is because $u(\cdot)$, being optimal, must also be thrifty as we saw in Theorem 8.1, and because $\varrho_0 \leq \tau_0^u$ holds a.s.

From these three observations and Theorem 7.1, it is now clear that the pair (u, ϱ_0) is a saddle point of the stochastic game. $\square$

9 Constructing a Thrifty Control Strategy and a Saddle

The theory of the previous section, culminating with Corollary 8.3, shows that in order to construct a saddle point for our stochastic game of control and stopping, all we need to do is find an admissible control strategy $u^*(\cdot) \in \mathfrak{U}$ which is *thrifty;* to wit, for which the process

$$R^{u^*}\big(\; \cdot \wedge \tau_0^{u^*}\big) \quad \text{is a} \quad \mathbb{P}^{u^*} - \text{martingale}. \tag{9.1}$$

Then the pair $\big(u^*, \tau_0^{u^*}\big)$ will be a saddle point for our stochastic game.

To accomplish this, we shall deploy the martingale methodologies introduced in stochastic control in the seminal papers of Rishel (1970), Duncan & Varaiya (1971), Davis & Varaiya (1973) and Davis (1973). The starting point of this approach is the observation that, for *every* admissible control strategy $u(\cdot) \in \mathfrak{U}$, the process

$$\widetilde{R}^u(\cdot) := R^u\big(\; \cdot \wedge \varrho_0\big) \; = \; V\big(\; \cdot \; \wedge \varrho_0\big) + \int_0^{\cdot \wedge \varrho_0} h(t, X, u_t)\, dt \qquad \text{is a} \quad \mathbb{P}^u - \text{submartingale} \tag{9.2}$$

with RCLL paths, and bounded uniformly on $[0,T] \times \Omega$; recall Propositions 5.1, 5.2 and Remark 5.5. This implies that the process $\widetilde{R}^u(\,\cdot\,)$ admits a Doob-Meyer decomposition

$$\widetilde{R}^u(\,\cdot\,) \,=\, V \,+\, M^u(\cdot) \,+\, A^u(\cdot)\,. \tag{9.3}$$

Here $M^u(\cdot)$ is a uniformly integrable $\mathbb{P}^u-$martingale with RCLL paths and $M^u(0)=0$, $M^u(\cdot) \equiv M^u(\varrho_0)$ on $[[\,\varrho_0, T\,]]$; the process $A^u(\cdot)$ is predictable, with non-decreasing paths, $A^u(T) \equiv A^u(\varrho_0)$ integrable, and $A^u(0)=0$.

A key observation now, is that the $\mathbb{P}^u-$martingale $M^u(\cdot)$ can be represented as a stochastic integral, in the form

$$M^u(\cdot) \,=\, \int_0^{\cdot} \langle \gamma(t), dW^u(t) \rangle\,. \tag{9.4}$$

Here $W^u(\cdot)$ is the $\mathbb{P}^u-$ Brownian motion of (2.4), and $\gamma(\cdot)$ a predictable ($\mathcal{P}-$measurable) process that satisfies $\int_0^T ||\gamma(t)||^2\, dt < \infty$ and $\gamma(\cdot) \equiv 0$ on $[[\,\varrho_0, T\,]]$, a.s.

The important aspect of this representation is that *the same process* $\gamma(\cdot)$ *works for every* $u(\cdot) \in \mathfrak{U}$ *in (9.4);* see for instance Karatzas & Zamfirescu (2006) for an argument.

Let us take now any two admissible control strategies $u(\cdot)$ and $v(\cdot)$ in $\mathfrak{U}$, and compare the resulting representations (9.3) on the stochastic interval $[[\,0, \varrho_0\,]]$. In conjunction with (9.2), (9.4) and (2.4), this gives

$$A^v(\cdot) \,-\, A^u(\cdot) \,=\, \int_0^{\cdot} \Big[H\big(t, X, v_t, \gamma(t)\big) - H\big(t, X, u_t, \gamma(t)\big) \Big]\, dt \qquad \text{on} \;\; [[\,0\,,\, \varrho_0\,]]\,, \tag{9.5}$$

where $H(t,\omega,a,p)$ is the Hamiltonian of (3.2).

Analysis: If we know that $\check{u}(\cdot) \in \mathfrak{U}$ is a thrifty control strategy, that is, the process $R^{\check{u}}\big(\,\cdot \wedge \tau_0^{\check{u}}\big)$ is a $\mathbb{P}^{\check{u}}-$martingale, then $\widetilde{R}^{\check{u}}(\,\cdot\,) \equiv R^{\check{u}}(\,\cdot\, \wedge \varrho_0)$ is a also a $\mathbb{P}^{\check{u}}-$martingale (just recall that we have $0 \le \varrho_0 \le \tau_0^{\check{u}}$ from (5.7)), thus $A^{\check{u}}(\,\cdot\,) \equiv 0$ a.s. But then (9.5) gives

$$A^v(\cdot) \,=\, \int_0^{\cdot} \Big[H\big(t, X, v_t, \gamma(t)\big) - H\big(t, X, \check{u}_t, \gamma(t)\big) \Big]\, dt \qquad \text{a.e. on} \;\; [[\,0\,,\, \varrho_0\,]]\;;$$

and because this process has to be non-decreasing, for every admissible control strategy $v(\cdot) \in \mathfrak{U}$, we deduce the following necessary condition for thriftiness:

$$H\big(t, X, \check{u}_t, \gamma(t)\big) \,=\, \inf_{a \in A} H\big(t, X, a, \gamma(t)\big)\,, \qquad \text{a.e. on} \quad [[\,0\,,\, \varrho_0\,]]\,. \tag{9.6}$$

This is also known as the stochastic version of *Pontryagin's Maximum Principle;* see, for instance, Kushner (1965), Haussmann (1986), Peng (1990, 1993).

Synthesis: The stochastic maximum principle of (9.6) suggests considering the admissible control strategy $u^*(\cdot) \in \mathfrak{U}$ defined by

$$u_t^* \,=\, \left\{ \begin{array}{ll} \mathfrak{a}^*\big(t, X, \gamma(t)\big)\,, & 0 \le t \le \varrho_0 \\ a_\sharp\,, & \varrho_0 < t \le T \end{array} \right\} \tag{9.7}$$

for an arbitrary but fixed element $a_\sharp$ of the control set A. We are using here the "measurable selector" mapping $\mathfrak{a}^* : [0,T] \times \Omega \times \mathbb{R}^n \to A$ of (3.3). With this choice, (9.5) leads to the comparison

$$A^v(\cdot) \,=\, A^{u^*}(\cdot) \,+\, \int_0^{\cdot} \Big[H\big(t, X, v_t, \gamma(t)\big) - H\big(t, X, u_t^*, \gamma(t)\big) \Big]\, dt \,\ge\, A^{u^*}(\cdot)\,, \qquad \text{on} \;\; [[\,0\,,\, \varrho_0\,]]\,,$$

therefore also

$$\widetilde{R}^{v}(\cdot) \,\geq\, V \,+\, M^{v}(\cdot) \,+\, A^{u^*}(\cdot)\,, \qquad \text{on} \;\; [[\,0\,,\,\varrho_0\,]]$$

from (9.3), for every $v(\cdot) \in \mathfrak{U}$. Taking expectations under $\mathbb{P}^{v}$, we deduce

$$0 \,\leq\, \mathbb{E}^{v}\big[\,A^{u^*}(\varrho_0)\,\big] \,\leq\, \mathbb{E}^{v}\big[\,R^{v}(\varrho_0)\,\big] \,-\, V\,, \qquad \forall \;\; v(\cdot) \in \mathfrak{U}\,.$$

But now we can take the infimum over $v(\cdot) \in \mathfrak{U}$ in the above string, and obtain

$$0 \,\leq\, \inf_{v \in \mathfrak{U}} \mathbb{E}^{v}\big[\,A^{u^*}(\varrho_0)\,\big] \,\leq\, \inf_{v \in \mathfrak{U}} \mathbb{E}^{v}\big[\,R^{v}(\varrho_0)\,\big] \,-\, V \,=\, 0\,,$$

where the last equality comes from (6.11). We deduce that $\inf_{v\in\mathfrak{U}} \mathbb{E}^{v}\big[\,A^{u^*}(\varrho_0)\,\big] = 0$, and classical weak compactness arguments, as in Davis (1973, 1979) and Elliott (1982), then give

$$A^{u^*}(\varrho_0) \,=\, 0 \qquad \text{a.s.,} \tag{9.8}$$

to wit, that $\widetilde{R}^{u^*}(\,\cdot\,) \equiv R^{u^*}(\,\cdot \wedge \varrho_0)$ *is a* $\mathbb{P}^{u}-$*martingale.*

We also have $V(\varrho_0) = g\big(X(\varrho_0)\big)$ a.s., from the definition of ϱ_0; and we know from Proposition 5.1 that $R^{u}(\,\cdot\, \wedge \varrho_0)$ is a $\mathbb{P}^{u}-$submartingale, for every $u(\cdot) \in \mathfrak{U}$. We deduce from Theorem 7.1 that the pair (u^*, ϱ_0) is a saddle point of the stochastic game:

$$\mathbb{E}^{u^*}[\,Y^{u^*}(\tau)\,] \,\leq\, \mathbb{E}^{u^*}[\,Y^{u^*}(\varrho_0)\,] \,\leq\, \mathbb{E}^{u}[\,Y^{u}(\varrho_0)\,] \tag{9.9}$$

holds for all $\tau \in \mathcal{S}$, and $u(\cdot) \in \mathfrak{U}$.

In order to arrive at (9.1), it is needed now to show $\tau_0^{u^*} = \varrho_0$, a.s. We know already that $\tau_0^{u^*} \geq \varrho_0$ holds a.s., so we have to argue

$$\tau_0^{u^*} \,\leq\, \varrho_0\,, \qquad \text{a.s.} \tag{9.10}$$

Reading (9.9) with $\tau = \tau_0^{u^*}$, we obtain

$$\mathbb{E}^{u^*}[\,Y^{u^*}(\tau_0^{u^*})\,] \,\leq\, \mathbb{E}^{u^*}[\,Y^{u^*}(\varrho_0)\,]\,.$$

Since the process $Q^{u^*}(\,\cdot\, \wedge \tau_0^{u^*})$ is a $\mathbb{P}^{u^*}-$martingale, this gives

$$\mathbb{E}^{u^*}[\,Y^{u^*}(\varrho_0)\,] \,\leq\, \mathbb{E}^{u^*}[\,Q^{u^*}(\varrho_0)\,] \,=\, \mathbb{E}^{u^*}[\,Q^{u^*}(\tau_0^{u^*})\,] \,=\, \mathbb{E}^{u^*}[\,Y^{u^*}(\tau_0^{u^*})\,] \,\leq\, \mathbb{E}^{u^*}[\,Y^{u^*}(\varrho_0)\,]\,,$$

so that all the expectations in this expression are equal. Now $Y^{u^*}(\cdot) \,\leq\, Q^{u^*}(\cdot)$ holds a.e. on $[[\,0, T\,]]$, so the above string implies that

$$Y^{u^*}(\varrho_0) \,=\, Q^{u^*}(\varrho_0)\,, \qquad \text{equivalently} \qquad g(X(\varrho_0)) = Z^{u^*}(\varrho_0)\,,$$

should hold a.s. But this leads to (9.10), and we have established the main result of this work:

9.1 Theorem: *The pair* $(u^*, \varrho_0) \in \mathfrak{U} \times \mathcal{S}$ *of (9.7), (5.4) is a saddle point for the stochastic game. Furthermore, we have* $\tau_0^{u^*} = \varrho_0$, *a.s. in the notation of (4.10).*

References

[1] Beneš, V.E. (1970) Existence of optimal policies based on specified information, for a class of stochastic decision problems. *SIAM J. Control & Optimization* **8** , 179-188.

[2] Beneš, V.E. (1971) Existence of optimal stochastic control laws. *SIAM J. Control & Optimization* **9** , 446-475.

[3] Beneš, V.E. (1992) Some combined control and stopping problems. Paper presented at the *CRM Workshop on Stochastic Systems*, Montréal, November 1992.

[4] Bensoussan, A. & Lions, J.L. (1982) *Applications of Variational Inequalities in Stochastic Control.* North-Holland, Amsterdam and New York.

[5] Bismut, J.M. (1973) Conjugate convex functions in optimal stochastic control. *Journal of Mathematical Analysis & Applications* **44**, 384-404.

[6] Bismut, J.M. (1973) An introductory approach to duality in optimal stochastic control. *SIAM Review* **20**, 62-78.

[7] Ceci, C. & Basan, B. (2004) Mixed optimal stopping and stochastic control problems with semicontinuous final reward, for diffusion processes. *Stochastics & Stochastics Reports* **76** , 323-337.

[8] Davis, M.H.A. (1973) On the existence of optimal policies in stochastic control. *SIAM J. Control & Optimization* **11** , 587-594.

[9] Davis, M.H.A. (1979) Martingale Methods in Stochastic Control. *Lecture Notes in Control & Information Systems* **16** , 85-117.

[10] Davis, M.H.A. & Varaiya, P.P. (1973) Dynamic programming conditions for partially observable stochastic systems. *SIAM J. Control & Optimization* **11** , 226-261.

[11] Davis, M.H.A. & Zervos, M. (1994) A problem of singular stochastic control with discretionary stopping. *Ann. Appl. Probab.* **4**, 226-240.

[12] Dubins, L.E. & Savage, L.J. (1976) *How to Gamble If You Must: Inequalities for Stochastic Processes.* Springer-Verlag, New York.

[13] Duncan, T.E. & Varaiya, P.P. (1971) On the solutions of a stochastic control system. *SIAM J. Control & Optimization* **9**, 354-371.

[14] El Karoui, N. (1981) *Les Aspects Probabilistes du Contrôle Stochastique. Lecture Notes in Mathematics* **876**, 73-238. Springer-Verlag, Berlin.

[15] El Karoui, N., Nguyen, D.H. & Jeanblanc-Picqué, M. (1987) Compactification methods in the control of degenerate diffusions: existence of an optimal control. *Stochastics* **20**, 169-219.

[16] Elliott, R.J. (1977) The optimal control of a stochastic system. *SIAM J. Control & Optimization* **15** , 756-778.

[17] Elliott, R.J. (1982) *Stochastic Calculus and Applications.* Springer-Verlag, New York.

[18] Fleming, W.R. & Soner, H.M. (2006) *Controlled Diffusion Processes and Viscosity Solutions.* Second Edition. Springer-Verlag, New York.

[19] Haussmann, U.G. (1986) *A Stochastic Maximum Principle for Optimal Control of Diffusions.* Longman Scientific and Technical, Essex, UK.

[20] Haussmann, U.G. & Lepeltier, J.P. (1990) On the existence of optimal controls. *SIAM Journal on Control & Optimization* **28**, 851-902.

[21] Kamizono, K. & Morimoto, H. (2002) On a combined control and stopping time game. *Stochastics and Stochastics Reports* **73**, 99-123.

[22] Karatzas, I. & Ocone, D.L. (2002) A leavable, bounded-velocity stochastic control problem. *Stochastic Processes & Applications* **99**, 31-51.

[23] Karatzas, I., Ocone, D.L., & Wang, H. & Zervos, M. (2000) Finite-fuel singular control with discretionary stopping. *Stochastics* **71**, 1-50.

[24] Karatzas, I. & Shreve, S.E. (1991) *Brownian Motion and Stochastic Calculus.* Second Edition, Springer-Verlag, New York.

[25] Karatzas, I. & Shreve, S.E. (1998) *Methods of Mathematical Finance.* Springer-Verlag, New York.

[26] Karatzas, I. & Sudderth, W.D. (1999) Control and stopping of a diffusion process on an interval. *Annals of Applied Probability* **9**, 188-196.

[27] Karatzas, I. & Sudderth, W.D. (2001) The controller and stopper game for a linear diffusion. *Annals of Probability* **29**, 1111-1127.

[28] Karatzas, I. & Sudderth, W.D. (2007) Stochastic games of control and and stopping for a linear diffusion. *Statistica Sinica*, to appear.

[29] Karatzas, I. & Wang, H. (2000) A barrier option of American type. *Applied Mathematics & Optimization* **42**, 259-280.

[30] Karatzas, I. & Wang, H. (2001) Utility maximization with discretionary stopping. *SIAM J. Control & Optimization* **39**, 306-329.

[31] Karatzas, I. & Zamfirescu, M. (2006) Martingale approach to stochastic control with discretionary stopping. *Applied Mathematics & Optimization* **53**, 163-184.

[32] Krylov, N.V. (1980) *Controlled Diffusion Processes.* Springer-Verlag, New York.

[33] Kushner, H.J. (1965) On the stochastic maximum principle: fixed time of control. *Journal of Mathematical Analysis & Applications* **11**, 78-92.

[34] Maitra, A. & Sudderth, W.D. (1996.a) *Discrete Gambling and Stochastic Games.* Springer-Verlag, New York.

[35] Maitra, A. & Sudderth, W.D. (1996.b) The gambler and the stopper. In *Statistics, Probability & Game Theory: Papers in Honor of David Blackwell.* (T.S. Ferguson, L.S. Shapley and J.B. MacQueen, editors.) *IMS Lecture Note & Monograph Series* **30**, 191-208.

[36] Morimoto, H. (2003) Variational inequalities for combined control and stopping. *SIAM J. Control & Optimization* **42**, 686-708.

[37] Neveu, J. (1975) *Discrete-Parameter Martingales.* North-Holland & Elsevier, Amsterdam & New York.

[38] Ocone, D.L. & Weerasinghe, A.P. (2006) A degenerate variance control problem with discretionary stopping. *Preprint.*

[39] Peng, Sh. (1990) A general stochastic maximum principle for optimal control problems. *SIAM J. Control & Optimization* **28**, 966-979.

[40] Peng, Sh. (1993) Bacwards stochastic differrential equations and applications to optimal control. *Applied Mathematics & Optimization* **27**, 125-144.

[41] Rishel, R.W. (1970) Necessary and sufficient dynamic programming conditions for continuous-time optimal control. *SIAM J. Control & Optimization* **8**, 559-571.

[42] Weerasinghe, A.P. (2006) A controller and stopper game with degenerate variance control. *Electronic Communications in Probability* **11**, 89-99.

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