ADAPTIVE CONTROL OF A DIFFUSION TO A GOAL AND A PARABOLIC MONGE-AMPÈRE-TYPE EQUATION†

by

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To the memory of Stamatis Cambanis (1943-1995)

Abstract

We study the following adaptive stochastic control problem: to maximize the probability $\mathbf{P}[X(T)=1]$ of reaching the "goal" $x=1$ during the finite time-horizon $[0,T]$, over "control" processes $\pi(\cdot)$ which are adapted to the natural filtration of the "observation" process $Y(t) = W(t) + Bt, 0 \leq t \leq T$ and satisfy almost surely $\int_0^T \pi^2(t)dt < \infty$ and $0 \leq X(t) = x + \int_0^t \pi(s)dY(s) \leq 1, \forall 0 \leq t \leq T$. Here $W(\cdot)$ is standard Brownian motion, and B is an independent random variable with known distribution μ. The case $B \equiv b \neq 0$ of this problem was studied by Kulldorff (1993). Modifying a martingale method due to Heath (1993), we find an optimal control process $\hat{\pi}(\cdot)$ for the general case of this problem, and solve explicitly for its value and for the associated Hamilton-Jacobi-Bellman equation of Dynamic Programming. This reduces to $2Q_{xx}Q_s = Q_{xx}Q_{yy} - Q_{xy}^2$, an apparently novel parabolic-Monge-Ampère-type equation.

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1. INTRODUCTION

On a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$, let $W(\cdot) = \{W(t), 0 \le t \le T\}$ be standard, one-dimensional Brownian motion on the finite time-horizon $[0, T]$, and let B be an independent random variable with known distribution μ that satisfies

$$\mu(\{0\}) < 1, \quad \int_{\Re} |b| \mu(db) < \infty. \tag{1.1}$$

Neither the process $W(\cdot)$ nor the random variable B is observed directly, but the process

$$Y(t) \stackrel{\triangle}{=} W(t) + Bt, \quad 0 \le t \le T \tag{1.2}$$

is; we shall denote by $\mathbf{F} = \{\mathcal{F}(t); 0 \le t \le T\}$ the augmentation of the natural filtration

$$\mathcal{F}^Y(t) = \sigma(Y(s); 0 \le s \le t), \quad 0 \le t \le T \tag{1.3}$$

generated by the *observation process* $Y(\cdot)$ of (1.2).

For a given *initial position* x_o in the interval $S = [0, 1]$, the *state-space* of our problem, consider the class $\mathcal{A}(x_o) \equiv \mathcal{A}(x_o; 0, T)$ of $\mathbf{F}$ – progressively measurable processes $\pi : [0, T] \times \Omega \longrightarrow \Re$ which satisfy

$$\int_0^T \pi^2(t) dt < \infty \tag{1.4}$$

and

$$0 \le X(t) \stackrel{\triangle}{=} x_o + \int_0^t \pi(s) dY(s) \le 1, \quad 0 \le t \le T \tag{1.5}$$

almost surely. This is the class of our *admissible control processes* for the initial position x_o. As we shall see (cf. Remark 3.1 below), for every process $\pi(\cdot)$ in this class $\mathcal{A}(x_o)$, the corresponding *state-process* $X^{x_o, \pi}(\cdot) \equiv X(\cdot)$ of (1.5) is absorbed at the endpoints of the interval $S = [0, 1]$, namely

$$X(\cdot) = X(\cdot \wedge \tau), \text{ where } \tau \stackrel{\triangle}{=} \inf\{t \in [0, T); X(t) \notin (0, 1)\} \wedge T. \tag{1.6}$$

The objective of our *stochastic control problem* will be to choose the process $\pi(\cdot) \in \mathcal{A}(x_o)$ so as to maximize the probability of reaching the right-endpoint of the interval $S = [0,1]$ by the time $t = T$. That is, we shall try to compute the *value function*

$$V(x_o) \triangleq \sup_{\pi(\cdot)\in\mathcal{A}(x_o)} \mathbf{P}[X^{x_o,\pi}(T) = 1], \quad x_o \in [0,1] \tag{1.7}$$

and to single out an *optimal control* process $\hat{\pi}(\cdot) \in \mathcal{A}(x_o)$ that attains the supremum in (1.7), namely

$$V(x_o) = \mathbf{P}[X^{x_o,\hat{\pi}}(T) = 1], \quad x_o \in [0,1], \tag{1.8}$$

if such a process exists.

We call this problem *adaptive control of the state-process $X(\cdot)$ to the goal $x = 1$*, because we are trying to steer ("control") the process $X(\cdot)$ to the right-endpoint $x = 1$ (our "goal") without having exact knowledge of the drift parameter B in (1.2). This drift is modelled as a random variable with known "prior" distribution μ, which has to be updated continuously ("adaptive" control) as the information $\mathcal{F}(t), 0 \le t \le T$ about the observations-process $Y(\cdot)$ keeps coming in.

In the case where we have *exact* knowledge about the drift-parameter $B = b \in \Re\backslash\{0\}$ (i.e., with $\mu = \delta_b, b \neq 0$), this control problem was solved in the very interesting paper of Kulldorff (1993). Kulldorff computed the value function of (1.7) and the optimal control process $\hat{\pi}(\cdot)$ of (1.8) in the form

$$V(x_o) = \Phi(\Phi^{-1}(x_o) + |b|\sqrt{T}) \tag{1.9}$$

$$\begin{aligned} \hat{\pi}(t) &= \frac{sgnb}{\sqrt{T-t}}\,\varphi\left(\frac{Y(t) + \sqrt{T}\Phi^{-1}(x_o)}{\sqrt{T-t}}\right) \\ &= \frac{sgnb}{\sqrt{T-t}}\,(\varphi\circ\Phi^{-1})(X^{x_o,\hat{\pi}}(t)), \quad 0 \le t < T \end{aligned} \tag{1.10}$$

respectively, with the notation

$$\Phi(z) \triangleq \int_{-\infty}^{z} \varphi_1(u)du, \quad \varphi_s(z) \triangleq \frac{1}{\sqrt{2\pi s}} e^{-z^2/2s}; \quad z \in \Re, \ s > 0 \tag{1.11}$$

and $\varphi(\cdot) \equiv \varphi_1(\cdot)$. Kulldorff's approach was later simplified by Heath (1993), who derived the results (1.9), (1.10) using a martingale approach combined with the celebrated Neyman-Pearson lemma from classical hypothesis-testing in statistics. We shall employ in section 5 a modification of Heath's argument, to deal with a general distribution μ as in (1.1) for the random variable B.

Our initial interest in this problem was to decide whether the so-called *certainty-equivalence principle*, of substituting in place of b in (1.10) the conditional expectation

$$\hat{B}(t) = \mathbf{E}[B|\mathcal{F}(t)], \quad 0 \le t \le T \tag{1.12}$$

of the unobserved random variable B, given the observations up to time t, would lead to a control law

$$\pi^{CE}(t) \; = \; \frac{sgn\hat{B}(t)}{\sqrt{T-t}} \, (\varphi \circ \Phi^{-1})(X^{x_o,\pi^{CE}}(t)), \quad 0 \le t < T \tag{1.13}$$

which is optimal for the problem of (1.7). Such a simple substitution principle does in fact lead to an optimal law in the context of partially observed linear-quadratic-gaussian control (cf. Fleming & Rishel (1975)), as well as in the context of the partially observed control problem of Beneš and Rishel (see Beneš et al. (1991), or Karatzas & Ocone (1992)). For the goal problem of (1.7), the control law (1.13) that results from this simple substitution principle, turns out to be optimal only in very special cases, namely when $\mu([0,\infty)) = 1$ or when $\mu((-\infty,0]) = 1$; see sections 6 and 7. The law of (1.13) fails to be optimal even for distributions μ that are *symmetric* around the origin, as we demonstrate in section 8. For such symmetric distributions it is still possible to obtain an explicit expression for the optimal law in terms of the current state $X(t)$ and the current estimate $\hat{B}(t)$ of B as in (1.12); this expression (8.8) is, however, quite different from that mandated by the "certainty-equivalence" principle of (1.13).

1.1 An Interpretation: Suppose that the price-per-share $S(\cdot)$ of a common stock follows the geometric Brownian motion process

$$dS(t) = S(t)[Bdt + dW(t)] = S(t)dY(t), \;\; S(0) = s > 0$$

where B is an unobservable drift-parameter, the "appreciation rate" of the stock. We model this unobservable rate as a random variable, independent of the Brownian motion $W(\cdot)$, with known distribution μ; this distribution quantifies our "prior knowledge" about the possible values that B can assume, as well as their respective likelihood. Based on the observations

$$\mathcal{F}(t) = \sigma(S(u); 0 \leq u \leq t) = \sigma(Y(u); 0 \leq u \leq t)$$

of the stock-prices over the interval $[0, t]$, we choose our "portfolio" $\pi(t)$ at time t (that is, the amount of money to be invested in the stock at that time). Our "wealth process" corresponding to the portfolio $\pi(\cdot)$ is then

$$X(t) = x_o + \int_0^t \pi(s) dY(s), \;\; 0 \leq t \leq T$$

as in (1.5), where $x_o \in (0, 1)$ stands for our "initial capital". We are interested in attaining the level of wealth $x = 1$, before time $t = T$ and without going into penury (i.e., reaching the level $x = 0$). If our objective is to maximize the probability $\mathbf{P}[X(T) = 1]$ of achieving this, we are exactly in the context of problem (1.7).

2. SUMMARY

We provide a careful formulation of the stochastic control problem (1.7) in *section 3,* with the help of the Girsanov theorem and of enlargement of filtrations; and in *section 4* we embed this problem in the standard framework of filtering, stochastic control, and dynamic programming. In particular, we write down the Hamilton-Jacobi-Bellman (HJB) equation of Dynamic Programming for the problem of (1.7), and notice that this equation reduces after normalization (or change of probability measure) to the *Monge-Ampère-type equation*

$$2 Q_{xx} Q_s = det(D^2 Q), \tag{2.1}$$

where

$$det(D^2 Q) \stackrel{\triangle}{=} Q_{xx} Q_{yy} - Q_{xy}^2. \tag{2.2}$$

Here $x \in [0,1]$ stands for the state-variable $X(t)$ of (1.5), $y \in \Re$ stands for the Brownian-motion-with-drift $Y(t)$ of (1.2), $s \in [0,T]$ is the "time-to-go" $T-t$ until the end of the horizon, and $det(D^2 Q)$ is the determinant of the Hessian matrix $D^2 Q$ of second-order derivatives in the spatial variables. The identification with (1.7) is through

$$V(x_o) = Q(T, x_o, 0) \tag{2.3}$$

for the value of the stochastic control problem. To our knowledge, the equation (2.1) is studied here for the first time.

In *section 5* we solve the problem of (1.7) very explicitly and we identify an optimal control process $\hat{\pi}(\cdot)$ as in (1.8), by adapting to our situation the methodology of Heath (1993). This methodology relies on the celebrated Neyman-Pearson lemma from classical hypothesis testing, and on the martingale representation property of the Brownian filtration. The precise answers that we obtain via this methodology allow us

(*a*) to solve explicitly the appropriate, in our context, initial-boundary value problem for the Monge-Ampère-type equation (2.1), and

(*b*) to decide whether the optimal control process $\hat{\pi}(\cdot)$ is in the form (1.3) of the "certainty-equivalence principle."

This program is carried out in *sections 6, 7, 9* for the cases

(*i*) $\mu([0,\infty)) = 1$,

(*ii*) $\mu((-\infty,0]) = 1$, and

(*iii*) $\mu((0,\infty)) \cdot \mu((-\infty,0)) > 0$,

respectively. It turns out that the certainty-equivalence principle holds in cases (*i*) and (*ii*), but fails to hold even for symmetric distributions μ with $\mu((0,\infty)) = \mu((-\infty,0)) > 0$ (cf. *section 8*, Remark 8.1).

"Goal" problems have been studied by Probability theorists, in the context of stochastic games, at least since Breiman (1961) and Dubins & Savage (1965). For various formulations of such problems, the reader is referred to the papers by Pestien & Sudderth (1985), Heath et al. (1987), Orey et al. (1987), Sudderth & Weerasinghe (1989), and to the recent book by Maitra & Sudderth (1996).

The classical, elliptic Monge-Ampère equation $det(D^2Q) = f$, in the notation of (2.2), has a long and venerable history in both Analysis and Geometry; see for instance Pogorelov (1964, 1978), Cheng & Yau (1977), Lions (1983), Krylov (1984, 1987) and Caffarelli (1990, 1991), Caffarelli & Cabré (1995). Parabolic versions of this equation were introduced by Krylov (1976, 1987) and were further studied recently, from the point of view of existence, uniqueness and regularity of solutions to initial- and initial/boundary-value problems, by Spiliotis (1992, 1994, 1997) and Wang & Wang (1992, 1993).

3. FORMULATION

Let us start with a given complete probability space $(\Omega, \mathcal{F}, \mathbf{P}^o)$, and on it

(i) a Brownian motion $Y(\cdot) = \{Y(t); 0 \leq t \leq T\}$ on the finite time-horizon $[0, T]$, as well as

(ii) a real-valued random variable B, independent of the process $Y(\cdot)$ under the probability measure $\mathbf{P}^o$, and with distribution $\mathbf{P}^o[B \in A] = \mu(A),\ A \in \mathcal{B}(\Re)$ that satisfies the conditions of (1.1).

We shall denote by $\mathbf{F} = \{\mathcal{F}(t); 0 \leq t \leq T\}$ the augmentation of the natural filtration (1.3) generated by the process $Y(\cdot)$, and by $\mathbf{G} = \{\mathcal{G}(t); 0 \leq t \leq T\}$ the augmentation of the filtration

$$\mathcal{F}^{B,Y}(t) \triangleq \sigma(B, Y(s);\ 0 \leq s \leq t), \quad 0 \leq t \leq T. \tag{3.1}$$

Then it can be checked that $Y(\cdot)$ is a $(\mathbf{G}, \mathbf{P}^o)$ – Brownian motion, and that the exponential process

$$Z(t) \stackrel{\triangle}{=} \exp\,[BY(t) - B^2 t/2],\ 0 \le t \le T \tag{3.2}$$

is a $(\mathbf{G}, \mathbf{P}^o)$ - martingale; in particular,

$$\mathbf{P}(\Lambda) \ \stackrel{\triangle}{=}\ \mathbf{E}^o[Z(T) \cdot 1_\Lambda], \quad \Lambda \in \mathcal{G}(T) \tag{3.3}$$

is a probability measure, equivalent to $\mathbf{P}^o$. *Under this probability measure* $\mathbf{P}$, *the process*

$$W(t) \stackrel{\triangle}{=} Y(t) - Bt,\ \mathcal{G}(t);\quad 0 \le t \le T \tag{3.4}$$

is standard Brownian motion independent of the random variable B, by the Girsanov Theorem (e.g. Karatzas & Shreve (1991), section 3.5); and we have $\mathbf{P}[B \in A] = \mathbf{P}^o[B \in A] = \mu(A),\ A \in \mathcal{B}(\Re)$.

In other words, on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with $\mathbf{F} = \{\mathcal{F}(t); 0 \le t \le T\}$, we are in the setting of the Introduction (section 1, formulæ (1.1)-(1.8)), and we are interested in the stochastic control problem (1.7) posed there. *This problem will be the focus of the remainder of the paper.*

3.1 Remark: It develops from (1.5) that the continuous processes $X(\cdot),\ 1 - X(\cdot)$ are both non-negative local martingales (hence also supermartingales) under the probability measure $\mathbf{P}^o$, for every $x_o \in [0,1]$ and $\pi(\cdot) \in \mathcal{A}(x_o)$. From a well-known property of non-negative supermartingales (e.g. Karatzas & Shreve (1991), Problem 1.3.29), both these processes are absorbed at the origin when they reach it, namely

$$X(t) = 0,\ \forall\ t \in [\tau_0, T] \quad \text{a.e. on} \quad \{\tau_0 < T\}$$

$$1 - X(t) = 0,\ \forall\ t \in [\tau_1, T] \quad \text{a.e. on} \quad \{\tau_1 < T\}$$

with $\tau_j \stackrel{\triangle}{=} \inf\,\{t \in [0,T); X(t) = j\} \wedge T$ and $j = 0, 1$. Our claim (1.6) follows from this, since $\tau = \tau_0 \wedge \tau_1$.

4. FILTERING AND DYNAMIC PROGRAMMING

In this section we shall place the problem of (1.7) within the standard framework of *Stochastic Control* and *Dynamic Programming* as expounded, for instance, in Fleming & Rishel (1975), Chapter 6, or Fleming & Soner (1993), Chapter 4. Let us start by introducing the $(\mathbf{F}, \mathbf{P}^o)-$ martingale

$$
\begin{aligned}
\hat{Z}(t) \ &\triangleq\ \mathbf{E}^o\Big[\frac{d\mathbf{P}}{d\mathbf{P}^o}|\mathcal{F}(t)\Big] = \mathbf{E}^o[Z(T)|\mathcal{F}(t)] \\
&= \mathbf{E}^o\Big(\mathbf{E}^o[Z(T)|\mathcal{G}(t)]|\mathcal{F}(t)\Big) = \mathbf{E}^o[Z(t)|\mathcal{F}(t)] \\
&= \mathbf{E}^o[\ \exp\ (BY(t) - B^2 t/2)|\mathcal{F}(t)] \\
&= \left\{ \begin{array}{ccc} F(t, Y(t)) & ; & 0 < t \leq T \\ 1 & ; & t = 0 \end{array} \right\},
\end{aligned} \tag{4.1}
$$

where

$$
F(t,y) \triangleq \int_{\Re} \exp\ (by - b^2 t/2)\mu(db), \quad (t,y) \in (0,\infty) \times \Re. \tag{4.2}
$$

Let us also write the $(\mathbf{F}, \mathbf{P})$ – martingale of (1.12) as

$$
\begin{aligned}
\hat{B}(t) = \mathbf{E}[B|\mathcal{F}(t)] &= \frac{\mathbf{E}^o[BZ(T)|\mathcal{F}(t)]}{\mathbf{E}^o[Z(T)|\mathcal{F}(t)]} \\
&= \frac{1}{\hat{Z}(t)} \cdot \mathbf{E}^o[B \cdot \mathbf{E}^o(Z(T)|\mathcal{G}(t))|\mathcal{F}(t)] \\
&= \frac{1}{\hat{Z}(t)} \cdot \mathbf{E}^o[BZ(t)|\mathcal{F}(t)] \\
&= \frac{1}{\hat{Z}(t)} \cdot \mathbf{E}^o[B\ \exp\ (BY(t) - B^2 t/2)|\mathcal{F}(t)] \\
&= \left\{ \begin{array}{ccc} G(t, Y(t)) & ; & 0 < t \leq T \\ \int_{\Re} b\mu(db) & ; & t = 0 \end{array} \right\}
\end{aligned} \tag{4.3}
$$

with the help of the "Bayes rule" (Lemma 3.5.3 in Karatzas & Shreve (1991). We have set

$$
G(t,y) \triangleq \frac{1}{F(t,y)} \int_{\Re} b\ \exp\ (by - b^2 t/2)\mu(db) = \Big(\frac{F_y}{F}\Big)(t,y), \quad (t,y) \in (0,\infty) \times \Re. \tag{4.4}
$$

Then it is straightforward to verify that the process

$$
N(t) \triangleq\ Y(t) - \int_0^t \hat{B}(s)ds = Y(t) - \int_0^t G(s, Y(s))ds,\ \ 0 \leq t \leq T \tag{4.5}
$$

is an $(\mathbf{F},\mathbf{P})$–martingale with continuous paths and quadratic variation $< N > (t) = t$ for $0 \le t \le T$. In other words, $N(\cdot)$ is an $(\mathbf{F},\mathbf{P})$-Brownian motion process; it is the familiar *innovations process* of filtering theory (recall the P. Lévy Theorem 3.3.16 in Karatzas & Shreve (1991)).

4.1 Remark: The functions F, G of (4.2), (4.4) are of class $C^{1,2}$ on $(0,\infty)\times\Re$, and satisfy on this strip the equations

$$F_t + \frac{1}{2}F_{yy} = 0, \quad G_t + \frac{1}{2}G_{yy} + GG_y = 0 \tag{4.6}$$

respectively.

The innovations process $N(\cdot)$ of (4.5) will allow us to embed the problem of (1.7) within the usual *Dynamic Programming* framework for *Stochastic Control,* as follows. Let us re-write the equation (4.5) in the form

$$dY(t) = \tilde{G}(T-t, Y(t))dt + dN(t), \quad T-s \le t \le T \quad \text{and} \quad Y(T-s) = y \tag{4.7}$$

on the interval $[T-s,T]$, with $\tilde{G}(T-t,\cdot) \equiv G(t,\cdot)$ and with arbitrary initial condition $y \in \Re$ for the observations process $Y(\cdot)$; let us also re-write the equation (1.5) as

$$dX(t) = \pi(t)[\tilde{G}(T-t, Y(t))dt + dN(t)], \ T-s \le t \le T \quad \text{and} \quad X(T-s) = x, \tag{4.8}$$

again on the interval $[T-s,T]$ and with arbitrary initial condition $x \in S = [0,1]$ for the state-process $X(\cdot)$. Thus, we can re-express the value of (1.7) as

$$V(x_o) = U(T, x_o, 0), \tag{4.9}$$

where

$$U(s,x,y) \ \triangleq \sup_{\pi(\cdot)\in\mathcal{A}(x;T-s,T)} \mathbf{E}[1_{\{1\}}(X(T))], \ (s,x,y) \in [0,T]\times[0,1]\times\Re. \tag{4.10}$$

We expect the function $U : [0,T] \times [0,1] \times \Re \longrightarrow [0,1]$ of (4.10) to be of class $C^{1,2,2}$ on $(0,T) \times (0,1) \times \Re$, and to satisfy in this strip the *Hamilton-Jacobi-Bellman* (HJB) *equation of Dynamic Programming*

$$\begin{aligned} U_s &= \frac{1}{2}U_{yy} + \tilde{G}U_y + max_{\pi\in\Re}\Big[(\pi^2/2)U_{xx} + \pi(\tilde{G}U_x + U_{xy})\Big] \\ &= \frac{1}{2}\Big[U_{yy} - \frac{1}{U_{xx}}(\tilde{G}U_x + U_{xy})^2\Big] + \tilde{G}U_y \end{aligned} \tag{4.11}$$

associated with the dynamics of (4.7)-(4.8), along with the *inequality*

$$U_{xx} < 0, \quad \text{on} \quad (0,T) \times (0,1) \times \Re, \tag{4.12}$$

the *initial condition*

$$U(0,x,y) = 1_{\{1\}}(x), \quad (x,y) \in [0,1] \times \Re, \tag{4.13}$$

and the *boundary conditions*

$$U(s,0+,y) = 0,\ U(s,1-,y) = 1; \quad 0 < s < T,\ y \in \Re. \tag{4.14}$$

The initial-boundary value problem (4.11)-(4.14) for the non-linear equation of (4.11) looks quite complicated. It can be simplified somewhat by use of the transformation

$$Q(s,x,y) \triangleq \left\{ \begin{array}{ccc} U(s,x,y)\cdot F(T-s,y) & ; & 0 \le s < T \\ lim_{u\nearrow T} Q(u,x,y) & ; & s = T \end{array} \right\},\ (x,y) \in [0,1] \times \Re \tag{4.15}$$

into the following *Initial-Boundary Value Problem*

$$Q(0,x,y) = F(T,y)\cdot 1_{\{1\}}(x); \quad (x,y) \in [0,1] \times \Re \tag{4.16}$$

$$Q(s,0+,y) = 0,\ Q(s,1-,y) = F(T-s,y); \quad 0 < s < T,\ y \in \Re \tag{4.17}$$

$$Q_{xx} < 0, \quad \text{on} \quad (0,T) \times (0,1) \times \Re \tag{4.18}$$

for the parabolic-Monge-Ampère-type equation

(4.19) $$2Q_{xx}Q_s = Q_{xx}Q_{yy} - Q_{xy}^2, \quad \text{on} \ \ (0,T)\times(0,1)\times\Re.$$

4.2 Remark: Subject to the inequality of (4.18), the Monge-Ampère-type equation (4.19) can be written as

(4.19)′ $$Q_s = \frac{1}{2}Q_{yy} + max_{\pi\in\Re}\Big[(\pi^2/2)Q_{xx} + \pi Q_{xy})\Big], \quad \text{on} \ \ (0,T)\times(0,1)\times\Re$$

since the expression in brackets is maximized by

(4.20) $$\pi^* = -\frac{Q_{xy}}{Q_{xx}}.$$

The equation (4.19) is then the HJB equation of Dynamic Programming, for the problem of maximizing the probability

$$\mathbf{P}[X(T)=1] = \mathbf{E}^o[\hat{Z}(T)\cdot 1_{\{X(T)=1\}}] = \mathbf{E}^o[F(T,Y(T))\cdot 1_{\{1\}}(X(T))]$$

over $\pi(\cdot)\in\mathcal{A}(x;T-s,T)$, subject to the dynamics

$$dX(t) = \pi(t)d\xi(t), \quad X(T-s) = x\in[0,1]$$

$$dY(t) = d\xi(t), \quad Y(T-s) = y\in\Re$$

on the time-interval $[T-s,T]$; here $\xi(\cdot)$ is $(\mathbf{F},\mathbf{P}^o)$ – standard Brownian Motion.

4.3 Remark: From (4.9), (4.15) we obtain, formally at least,

(4.21) $$V(x_o) = Q(T,x_o,0).$$

On the other hand, we expect from (4.20) that an optimal control process $\hat{\pi}(\cdot)\in\mathcal{A}(x_o)$, as in (1.8), should exist, and should be given in the *feedback-form*

(4.22) $$\hat{\pi}(t) = -\frac{Q_{xy}}{Q_{xx}}(T-t,\hat{X}(t),Y(t))\cdot 1_{[0,T)}(t),$$

where $\hat{X}(t) = X^{x_o,\hat{\pi}}(t),\ 0 \le t \le T$.

5. THE MARTINGALE APPROACH OF D. HEATH

We present in this section the solution of Problem (1.7), which is based on the *Neyman-Pearson lemma* of classical hypothesis-testing and on the *martingale methodology*, as developed by Heath (1993) for the case of constant $B \equiv b \neq 0$.

The starting point of this approach is the observation that, for every $x_o \in [0,1]$ and $\pi(\cdot) \in \mathcal{A}(x_o)$, the process $X(\cdot) \equiv X^{x_o,\pi}(\cdot)$ of (1.5) is an $(\mathbf{F}, \mathbf{P}^o)$–local martingale with values in the interval $[0,1]$, hence an $(\mathbf{F}, \mathbf{P}^o)$–*martingale*. In particular, we have

(5.1)

$$\mathbf{P}^o[X^{x_o,\pi}(T) = 1] = \mathbf{E}^o[X^{x_o,\pi}(T) \cdot 1_{\{X^{x_o,\pi}(T)=1\}}] \le \mathbf{E}^o[X^{x_o,\pi}(T)] = x_o,\ \forall \pi(\cdot) \in \mathcal{A}(x_o).$$

Since the event $\{X^{x_o,\pi}(T) = 1\}$ belongs to the σ-algebra $\mathcal{F}(T)$, it follows from (5.1) that

$$V^*(x_o) \triangleq sup_{\Lambda \in \mathcal{F}(T), \mathbf{P}^o(\Lambda) \le x_o}\ \mathbf{P}(\Lambda) \tag{5.2}$$

dominates the value of our control problem (1.7):

$$V^*(x_o) \ge V(x_o) \ \triangleq sup_{\pi(\cdot) \in \mathcal{A}(x_o)}\ \mathbf{P}[X^{x_o,\pi}(T) = 1]. \tag{5.3}$$

The point here, made by Heath (1993) for constant $B \equiv b \neq 0$, is that

(i) the "auxiliary value" $V^*(x_o)$ of (5.2) is very easy to compute, and that

(ii) equality actually holds in (5.3), so that in turn

(iii) we get to compute $V(x_o)$ as well. As a by-product of this last computation, we shall be able to obtain an optimal control process $\hat{\pi}(\cdot) \in \mathcal{A}(x_o)$.

In order to make headway with this program, let us observe that the optimization problem of (5.2) is the same as that encountered in the classical setting of testing a simple hypothesis versus a simple alternative. The solution of this problem is given by the following celebrated result (e.g. Lehmann (1986)):

5.1 Lemma (Neyman-Pearson): *Suppose that we can find a number $k = \kappa(x_o) > 0$, such that the event*

$$\Lambda_k \triangleq \Big\{\mathbf{E}^o\Big[\frac{d\mathbf{P}}{d\mathbf{P}^o}|\mathcal{F}(T)\Big] \geq k\Big\}$$

has $\mathbf{P}^o(\Lambda_k) = x_o$; then $V^*(x_o) = \mathbf{P}(\Lambda_k)$.

Proof: For any $\Lambda \in \mathcal{F}(T)$ with $\mathbf{P}^o(\Lambda) \leq x_o$ and with $\hat{Z}(T) = \mathbf{E}^o[\frac{d\mathbf{P}}{d\mathbf{P}^o}|\mathcal{F}(T)]$ as in (4.1), we have

$$\begin{aligned}\mathbf{P}(\Lambda_k) - \mathbf{P}(\Lambda) &= \int_{\Lambda_k \cap \Lambda^c} \hat{Z}(T) d\mathbf{P}^o - \int_{\Lambda_k^c \cap \Lambda} \hat{Z}(T) d\mathbf{P}^o \\ &\geq k[\mathbf{P}^o(\Lambda_k \cap \Lambda^c) - \mathbf{P}^o(\Lambda_k^c \cap \Lambda)] \\ &= k[\mathbf{P}^o(\Lambda_k) - \mathbf{P}^o(\Lambda)] = k[x_o - \mathbf{P}^o(\Lambda)] \geq 0. \qquad \diamondsuit\end{aligned}$$

In order to find a number $k > 0$ with the properties of Lemma 5.1, let us notice that for every $t > 0$ the function $y \longmapsto F(t, y)$ of (4.2) is *strictly convex,* and that with

(5.4) $$f(t) \triangleq inf_{y \in \Re} F(t, y), \quad t > 0$$

we have one of the following three possibilities:

(5.5)(i) $\mu([0, \infty)) = 1:$ Then $F(t, \cdot)$ is strictly increasing on $\Re$, with $F(t, -\infty) = f(t) = \mu(\{0\})$ and $F(t, \infty) = \infty$.

(5.5)(ii) $\mu((-\infty, 0]) = 1:$ Then $F(t, \cdot)$ is strictly decreasing on $\Re$, with $F(t, -\infty) = \infty$ and $F(t, \infty) = f(t) = \mu(\{0\})$.

(5.5)(iii) $\mu((0, \infty)) \cdot \mu((-\infty, 0)) > 0:$ In this case the infimum of (5.4) is attained at some $y_* = \nu(t) \in \Re$; the function $F(t, \cdot)$ is strictly increasing on $(\nu(t), \infty)$ and strictly decreasing on $(-\infty, \nu(t))$, with $F(t, \pm\infty) = \infty$.

As a consequence, the function

(5.6)

$$h(k) \triangleq \mathbf{P}^o(\Lambda_k) = \mathbf{P}^o[\hat{Z}(T) \geq k] = \mathbf{P}^o[F(T, Y(T)) \geq k] = \int_{\{z; F(T,z) \geq k\}} \varphi_T(z) dz, \ \ k > f(T)$$

is continuous and strictly decreasing, with $h(f(T)) \triangleq lim_{k \searrow f(T)} h(k) = 1$ and $h(\infty) \triangleq lim_{k \nearrow \infty} h(k) = 0$. Thus, for every $x_o \in [0,1]$, there is a unique $k = \kappa(x_o)$ in $[f(T), \infty]$ with $h(\kappa(x_o)) = x_o$, and for this k we have

$$(5.7) \quad V^*(x_o) = \mathbf{P}(\Lambda_{\kappa(x_o)}) = \mathbf{E}^o[\hat{Z}(T) \cdot 1_{\{\hat{Z}(T) \geq \kappa(x_o)\}}] = \int_{\{z; F(T,z) \geq \kappa(x_o)\}} F(T,z)\varphi_T(z)dz$$

in the notation of (1.11), (5.2), from Lemma 5.1 and (4.1).

Suppose now that we are able to find a control process $\hat{\pi}(\cdot) \in \mathcal{A}(x_o)$, such that

$$(5.8) \qquad \{X^{x_o,\hat{\pi}}(T) = 1\} = \Lambda_{\kappa(x_o)}, \ \ mod.\ \mathbf{P}(\mathbf{P}^o).$$

Then we have

$$(5.9) \qquad V^*(x_o) = \mathbf{P}(\Lambda_{\kappa(x_o)}) = \mathbf{P}[X^{x_o,\hat{\pi}}(T) = 1] \leq V(x_o)$$

from (5.7), (1.7); and in conjunction with (5.3), *this proves both the equality*

$$(5.10) \qquad V(x_o) = V^*(x_o)$$

and the optimality (1.8) of $\hat{\pi}(\cdot)$ for the problem of (1.7).

In order to find a process $\hat{\pi}(\cdot) \in \mathcal{A}(x_o)$ with the property (5.8), let us consider the $(\mathbf{F}, \mathbf{P}^o)$–martingale

$$(5.11) \qquad \begin{aligned} \hat{X}(t) &\triangleq \mathbf{E}^o[1_{\Lambda_{\kappa(x_o)}} | \mathcal{F}(t)] = \mathbf{P}^o[\hat{Z}(T) \geq \kappa(x_o) | \mathcal{F}(t)] \\ &= \mathbf{P}^o[F(T, Y(t) + (Y(T) - Y(t)) \geq \kappa(x_o) | \mathcal{F}(t)] \\ &= \mathcal{X}(T-t, Y(t)), \quad 0 \leq t \leq T \end{aligned}$$

where

$$(5.12) \qquad \mathcal{X}(s,y) \triangleq \left\{ \begin{array}{cl} 1_{\{F(T,y) \geq \kappa(x_o)\}} & ; \quad s = 0,\ y \in \Re \\ \int_{\{z; F(T,y+z) \geq \kappa(x_o)\}} \varphi_s(z)dz & ; \quad s > 0,\ y \in \Re. \end{array} \right\}$$

The process $\hat{X}(\cdot)$ of (5.11) takes values in the interval $S = [0,1]$ and starts out at $\hat{X}(0) = \mathcal{X}(T,0) = h(\kappa(x_o)) = x_o$, by virtue of (5.6) and (5.12). From the martingale representation property of the Brownian filtration (e.g. Karatzas & Shreve (1991), Theorem 3.4.15), there exists a process $\hat{\pi}(\cdot) \in \mathcal{A}(x_o)$ such that

$$\hat{X}(t) = X^{x_o,\hat{\pi}}(t) \triangleq x_o + \int_0^t \hat{\pi}(s)dY(s), \quad 0 \le t \le T \tag{5.13}$$

holds $\mathbf{P}^o$–almost surely; and this process $\hat{\pi}(\cdot)$ is unique, modulo $(\lambda \otimes \mathbf{P}^o)$–a.e. equivalence, where λ stands for "Lebesgue measure". In particular,

$$X^{x_o,\hat{\pi}}(T) = \hat{X}(T) = 1_{\Lambda_{\kappa(x_o)}}\ , \ \ a.s. \tag{5.14}$$

so that (5.8) holds.

The process $\hat{\pi}(\cdot)$ of (5.13) can be identified explicitly, in the following manner: notice that the function

$$\mathcal{X}(s,y) = \int_{\{z;F(T,z)\ge\kappa(x_o)\}} \varphi_s(y-z)dz \tag{5.12$'$}$$

of (5.12) is of class $C^{1,2}$ and satisfies the heat equation

$$\mathcal{X}_s = \frac{1}{2}\mathcal{X}_{yy}, \quad \text{on} \quad (0,\infty)\times\Re. \tag{5.15}$$

Therefore, an application of Itô' s rule to (5.11) yields

$$\hat{X}(t) = x_o + \int_0^t \mathcal{X}_y(T-s, Y(s))dY(s),\ \ 0 \le t \le T;$$

and from the uniqueness of the stochastic integral representation (5.13), we conclude

$$\hat{\pi}(t) = \mathcal{X}_y(T-t, Y(t)) \cdot 1_{[0,T)}(t), \quad 0 \le t \le T$$

where

$$\mathcal{X}_y(s,y) = \int_{\{z;F(T,z)\ge\kappa(x_o)\}} \left(\frac{z-y}{s}\right)\ \varphi_s(y-z)dz; \quad s>0,\ \ y \in \Re.$$

We have proved the following result.

5.2 Theorem: *The value-function of the stochastic control problem (1.7) is given by the expression of (5.7). An optimal control process* $\hat{\pi}(\cdot) \in \mathcal{A}(x_o)$*, and its corresponding state-process* $\hat{X}(\cdot) \equiv X^{x_0,\hat{\pi}}(\cdot)$*, are given as*

$$\hat{\pi}(t) = \mathcal{X}_y(T-t, Y(t)) \cdot 1_{[0,T)}(t), \quad \hat{X}(t) = \mathcal{X}(T-t, Y(t)); \quad 0 \le t \le T \tag{5.16}$$

in the notation of (5.12).

5.3 Remark: Let us look at the *value-process*

$$\eta(t) \triangleq \mathbf{E}[1_{\Lambda_{\kappa(x_o)}} | \mathcal{F}(t)] = \mathbf{P}[\hat{X}(T) = 1 | \mathcal{F}(t)], \quad 0 \le t \le T. \tag{5.17}$$

This is an $(\mathbf{F}, \mathbf{P})$–martingale with $\eta(0) = V(x_o)$, $\eta(T) = 1_{\{1\}}(\hat{X}(T))$, $a.s.$; from the Bayes rule, it can be written as

(5.18)

$$\begin{aligned} \eta(t) &= \frac{\mathbf{E}^o[Z(T) \cdot 1_{\Lambda_{\kappa(x_o)}} | \mathcal{F}(t)]}{\mathbf{E}^o[Z(T)|\mathcal{F}(t)]} \\ &= \frac{1}{\hat{Z}(t)} \cdot \mathbf{E}^o\Big[F(T, Y(T)) \cdot 1_{\{F(T,Y(T)) \ge \kappa(x_o)\}} | \mathcal{F}(t)]\Big] \\ &= \frac{1}{\hat{Z}(t)} \cdot \mathbf{E}^o\Big[F(T, Y(t) + (Y(T) - Y(t))) \cdot 1_{\{F(T,Y(t)+(Y(T)-Y(t))) \ge \kappa(x_o)\}} | \mathcal{F}(t)]\Big] \\ &= \left\{ \begin{array}{ccc} \frac{\mathcal{H}(T-t, Y(t))}{F(t, Y(t))} & ; & 0 < t \le T \\ \mathcal{H}(T, 0) & ; & t = 0 \end{array} \right\} \end{aligned}$$

where

$$\mathcal{H}(s, y) \triangleq \left\{ \begin{array}{ccc} F(T, y) \cdot 1_{\{F(T,y) \ge \kappa(x_o)\}} & ; & s = 0, y \in \Re \\ \int_{\{z; F(T, y+z) \ge \kappa(x_o)\}} F(T, y+z) \varphi_s(z) dz & ; & s > 0, y \in \Re. \end{array} \right\} \tag{5.19}$$

This function is of class $C^{1,2}$ and satisfies the heat equation

$$\mathcal{H}_s = \frac{1}{2} \mathcal{H}_{yy}, \quad \text{on} \quad (0, \infty) \times \Re. \tag{5.20}$$

In particular, we have

$$V(x_o) = \mathcal{H}(T, 0) \tag{5.7$'$}$$

from (5.19), (5.7).

In sections 6-9 we shall compute the quantities of Theorem 5.2 (optimal control process $\hat{\pi}(\cdot)$, optimal state-process $\hat{X}(\cdot)$, value function) even more explicitly, in each of the three cases of (5.5). We shall also show, in each of the three cases, how to compute a function $Q : [0,T] \times [0,1] \times \Re \longrightarrow [0,1]$ which solves the initial-boundary value problem of (4.16)-(4.19) for the Monge-Ampère-type equation (4.19), and satisfies

$$V(x_o) = Q(T, x_o, 0), \quad x_o \in [0,1] \tag{5.21}$$

as well as

$$\mathcal{H}(s,y) = Q(s, \mathcal{X}(s,y), y), \quad (s,y) \in [0,T] \times \Re \tag{5.22}$$

$$\mathcal{X}_y(s,y) = -\frac{Q_{xy}}{Q_{xx}}\,(s, \mathcal{X}(s,y), y), \quad (s,y) \in (0,T] \times \Re \tag{5.23}$$

in the notation of (5.12), (5.19), and in accordance with (4.22), (5.16).

6. THE CASE $\mu([0,\infty)) = 1$.

This is the case of (5.5)(i): for every $t > 0$, the function $F(t,\cdot)$ is strictly increasing, and maps $\Re$ onto $(\mu(\{0\}), \infty)$ with $F(t,-\infty) = \mu(\{0\})$, $F(t,\infty) = \infty$. If we denote by $F^{-1}(t,\cdot) : (\mu(\{0\}), \infty) \longrightarrow \Re$ the inverse of this mapping, the function of (5.6) becomes

$$h(k) = \int_{F^{-1}(T,k)}^{\infty} \varphi_T(z)dz = \Phi\Big(-\frac{F^{-1}(T,k)}{\sqrt{T}}\Big), \quad k > \mu(\{0\}).$$

We have thus $F^{-1}(T, \kappa(x_o)) = -\sqrt{T}\Phi^{-1}(x_o)$, and the quantities of (5.12), (5.19) become

$$\mathcal{X}(s,y) = \left\{ \begin{array}{lll} 1_{[-\sqrt{T}\Phi^{-1}(x_o),\infty)}(y) & ; & s = 0, \; y \in \Re \\ \Phi\Big(\frac{y + \sqrt{T}\Phi^{-1}(x_o)}{\sqrt{s}}\Big) & ; & s > 0, \; y \in \Re \end{array} \right\} \tag{6.1}$$

(6.2)

$$\mathcal{H}(s,y) = \left\{ \begin{array}{lll} F(T,y) \cdot 1_{[-\sqrt{T}\Phi^{-1}(x_o),\infty)}(y) & ; & s = 0 \\ \int_{-\infty}^{y+\sqrt{T}\Phi^{-1}(x_o)} F(T,y-z)\varphi_s(z)dz = \int_{-\infty}^{\sqrt{s}\Phi^{-1}(\mathcal{X}(s,y))} F(T,y-z)\varphi_s(z)dz & ; & s > 0 \end{array} \right\}$$ ∎

for $y \in \Re$. As a consequence, the optimal state-and control-processes of (5.16) take the form

$$\hat{X}(t) = \left\{ \begin{array}{lcr} 1_{[-\sqrt{T}\Phi^{-1}(x_o),\infty)}(Y(T)) & ; & t = T \\ \Phi\left(\frac{Y(t)+\sqrt{T}\Phi^{-1}(x_o)}{\sqrt{T-t}}\right) & ; & 0 \le t < T \end{array} \right\} \tag{6.3}$$

and

$$\begin{aligned} \hat{\pi}(t) &= \frac{1}{\sqrt{T-t}}\, \varphi\Big(\frac{Y(t)+\sqrt{T}\Phi^{-1}(x_o)}{\sqrt{T-t}}\Big) \cdot 1_{[0,T)}(t) \\ &= \frac{1}{\sqrt{T-t}}\, (\varphi \circ \Phi^{-1})(\hat{X}(t)) \cdot 1_{[0,T)}(t), \quad 0 \le t \le T \end{aligned} \tag{6.4}$$

respectively. Notice that these formulæ (6.3), (6.4) do *not* depend on the particular form of the distribution μ at all.

6.1 Remark: In this case the function G of (4.4) is strictly positive, and so the same is true for the estimate $\hat{B}(t) = \mathbf{E}[B|\mathcal{F}(t)] = G(t, Y(t))$ of (4.3). Therefore, the optimal control-process $\hat{\pi}(\cdot)$ is trivially of the "certainty-equivalence" form (1.13).

From the expression (6.2) for $\mathcal{H}(s,y)$, it is now not hard to construct the function Q that satisfies (5.22).

6.2 Proposition: *The function*

$$Q(s,x,y) = \left\{ \begin{array}{lcr} F(T,y)\cdot 1_{\{1\}}(x) & ; & s=0, x\in[0,1], y\in\Re \\ \int_{-\infty}^{\sqrt{s}\Phi^{-1}(x)} F(T,y-z)\varphi_s(z)dz & ; & 0 < s \le T, x \in [0,1], y \in \Re \end{array} \right\} \tag{6.5}$$

solves the initial-boundary value problem (4.16)-(4.19) for the parabolic-Monge-Ampère-type equation (4.19), and satisfies the conditions (5.21)-(5.23).

Elementary computations lead to this result; some of these are facilitated by writing the second expression of (6.5) in the form

$$\begin{aligned} Q(s,x,y) &= \int_{-\infty}^{\Phi^{-1}(x)} F(T, y - z\sqrt{s})\varphi(z)dz \\ &= \int_{[0,\infty)} \exp\,[by - b^2(T-s)/2]\ \Phi(\Phi^{-1}(x) + b\sqrt{s})\ \mu(db), \end{aligned} \tag{6.5$'$}$$

on $(0,T]\times[0,1]\times\Re$. The details of these derivations are left to the care of the reader, who should also notice the *discontinuity* of the function Q in (6.5), as $s\searrow 0$:

$$\begin{aligned} Q(0+,x,y) &\triangleq lim_{s\searrow 0}Q(s,x,y) \\ &= \int_{[0,\infty)} \exp\,[by-b^2T/2]\ \Phi(\Phi^{-1}(x))\ \mu(db) \\ &= xF(T,y)\neq Q(0,x,y);\quad 0\le x<1,\ y\in\Re. \end{aligned}$$

6.3 Example: In the special case $\mu=\delta_b, b>0$ considered by Kulldorff (1993), the functions F,G and Q of (4.2), (4.4) and (6.5) become respectively $F(t,y)=\exp\,[by-b^2/t], G(t,y)=b$ and $Q(s,x,y)=F(T-s,y)U(s,x,y)$, where

(6.6) $$U(s,x,y)=\left\{\begin{array}{ccc} 1_{\{1\}}(x) & ; & s=0,\ 0\le x\le 1 \\ \Phi(\Phi^{-1}(x)+b\sqrt{s}) & ; & 0<s\le T,\ 0\le x\le 1 \end{array}\right\}$$

is the function of (4.10) in the present context. This function does not depend on $y\in\Re$; it solves the initial-boundary value problem (4.11)-(4.14) for the HJB equation (4.11), which now takes the much simpler form

(6.7) $$U_s+\frac{b^2}{2}\frac{U_x^2}{U_{xx}}=0;\quad s>0,\ 0<x<1.$$

7. THE CASE $\mu((-\infty,0])=1$.

Here we are in the setup of case (5.5)(ii). It is straight-forward to see that the analogues of (6.1), (6.2) and (6.5) are now

(7.1) $$\mathcal{X}(s,y)=\left\{\begin{array}{ccc} 1_{[\sqrt{T}\Phi^{-1}(x_o),\infty)}(y) & ; & s=0,\ y\in\Re \\ \Phi(\frac{\sqrt{T}\Phi^{-1}(x_o)-y}{\sqrt{s}}) & ; & s>0,\ y\in\Re, \end{array}\right\}$$

(7.2) $$\mathcal{H}(s,y)=\left\{\begin{array}{ccc} F(T,y)\cdot 1_{[\sqrt{T}\Phi^{-1}(x_o),\infty)}(y) & ; & s=0,\ y\in\Re \\ \int_{-\infty}^{\sqrt{s}\Phi^{-1}(\mathcal{X}(s,y))} F(T,y+z)\varphi_s(z)dz & ; & s>0,\ y\in\Re, \end{array}\right\}$$

and

(7.3) $$Q(s,x,y)=\left\{\begin{array}{ccc} F(T,y)\cdot 1_{\{1\}}(x) & ; & s=0,x\in[0,1],y\in\Re \\ \int_{-\infty}^{\sqrt{s}\Phi^{-1}(x)} F(T,y+z)\varphi_s(z)dz & ; & 0<s\le T,x\in[0,1],y\in\Re \end{array}\right\}$$

respectively; that the function Q satisfies the initial-boundary value problem of (4.16)-(4.19); and that the optimal processes of Theorem 5.2 are

$$\hat{X}(t) = \mathcal{X}(T-t, Y(t)), \quad 0 \le t \le T$$

and

$$\hat{\pi}(t) \;=\; \frac{-1}{\sqrt{T-t}}\,\varphi\Big(\frac{\sqrt{T}\Phi^{-1}(x_o) - Y(t)}{\sqrt{T-t}}\Big) = \frac{sgn\hat{B}(t)}{\sqrt{T-t}}\;(\varphi \circ \Phi^{-1})(\hat{X}(t)),\; 0 \le t < T.$$

In other words, the "certainty-equivalence principle" of (1.13) leads again to an optimal control process.

8. THE SYMMETRIC CASE

Before we tackle the general case $\mu((-\infty,0))\cdot\mu((0,\infty)) > 0$ of (5.5)(iii) in the next section, let us consider here a *symmetric* distribution μ, that is

$$\mu(A) = \mu(-A),\; \forall A \in \mathcal{B}(\Re)$$

with $\mu((-\infty,0)) = \mu((0,\infty)) > 0$. In this case, the function

$$y \longmapsto F(t,y) = \mu(\{0\}) + 2\int_{(0,\infty)} e^{-b^2t/2} cosh(by)\mu(db),\; t > 0$$

is evenly symmetric and strictly convex; it is also strictly increasing on $[0,\infty)$ with $F(t,\infty) = \infty$, and $f(t) = F(t,0) = \mu(\{0\}) + 2\int_{(0,\infty)} e^{-b^2t/2}\mu(db)$ in the notation of (5.4). This function maps $[0,\infty]$ onto $[f(t),\infty]$; we denote by $F^{-1}(t,\cdot) : [f(t),\infty] \longrightarrow [0,\infty]$ the inverse of $F(t,\cdot)$ on $[0,\infty]$, so that

$$F(T,z) \ge k \iff |z| \ge F^{-1}(T,k)$$

$$h(k) = 2\Big[1 - \Phi\Big(\frac{F^{-1}(T,k)}{\sqrt{T}}\Big)\Big]$$

for $k \ge f(T)$, and thus $F^{-1}(T,\kappa(x_o)) = \sqrt{T}\Phi^{-1}(1-(x_o/2))$ in the notation of (5.6), (5.7). Similarly, the functions of (5.12), (5.19) become

$$(8.1) \qquad \mathcal{X}(s,y) = L(s,y;\sqrt{T}\Phi^{-1}(1-(x_o/2))),\; \mathcal{H}(s,y) = M(s,y;\sqrt{T}\Phi^{-1}(1-(x_o/2)))$$

where

(8.2)

$$L(s,y;p) \triangleq \left\{ \begin{array}{cc} 1_{\{|y|\geq p\}} & ; \quad s=0, y\in\Re, p>0 \\ \int_{\{|z|\geq p\}} \varphi_s(y-z)dz = 2-\Phi(\frac{p-y}{\sqrt{s}})-\Phi(\frac{p+y}{\sqrt{s}}) & ; \quad s>0, y\in\Re, p>0 \end{array} \right\}$$

and

$$M(s,y;p) \triangleq \left\{ \begin{array}{cc} F(T,y)\cdot 1_{\{|y|\geq p\}} & ; \quad s=0, y\in\Re, p>0 \\ \int_{\{|z|\geq p\}} F(T,z)\varphi_s(y-z)dz & ; \quad s>0, y\in\Re, p>0. \end{array} \right\} \tag{8.3}$$

Notice that, for every given $p>0$, the functions of (8.2), (8.3) satisfy the heat equation

$$L_s = \frac{1}{2}L_{yy}, \quad M_s = \frac{1}{2}M_{yy} \quad \text{on} \quad (0,\infty)\times\Re. \tag{8.4}$$

With this notation, the optimal control-process $\hat{\pi}(\cdot)$ and the optimal state-process $\hat{X}(\cdot)$ of Theorem 5.2 are given as

$$\hat{\pi}(t) = L_y(T-t, Y(t); p_*)\cdot 1_{[0,T)}(t), \quad \hat{X}(t) = L(T-t, Y(t); p_*);\ 0\leq t\leq T \tag{8.5}$$

where $p_* = \sqrt{T}\Phi^{-1}(1-(x_o/2))$.

8.1 Remark: In the notation of (1.11), (8.2) we have

$$L_y(s,y;p) = \frac{1}{\sqrt{s}}\Big[\varphi\Big(\frac{y-p}{\sqrt{s}}\Big) - \varphi\Big(\frac{y+p}{\sqrt{s}}\Big)\Big] \quad , \quad s>0. \tag{8.6}$$

Observe from this, that the function $\mathcal{X}(s,\cdot)$ of (8.1) is evenly symmetric, and strictly increasing on $(0,\infty)$ with $\mathcal{X}(s,0) = 2[1-\Phi(\Phi^{-1}(1-(x_o/2))\sqrt{T/s})]$, $\mathcal{X}(s,\infty)=1$. Denoting by $\mathcal{Y}(s,\cdot) : [\mathcal{X}(s,0),1] \longrightarrow [0,\infty]$ the inverse of this mapping, we get $|Y(t)| = \mathcal{Y}(T-t,\hat{X}(t))$ and thus also

$$\hat{\pi}(t) = sgnY(t)\cdot L_y\Big(T-t,\ \mathcal{Y}(T-t,\hat{X}(t));\ p_*\Big), \quad 0\leq t<T \tag{8.7}$$

where $p_* = \sqrt{T}\Phi^{-1}(1-(x_o/2))$.

Now it is not hard to see that the function $G(t,\cdot)$ of (4.4) is oddly symmetric on $\Re$, with $sgnG(t,y) = sgn(y),\ t>0$. Thus $sgnY(t) = sgn\hat{B}(t)$, and we can re-write the expression (8.7) for the optimal control process as a function

$$\hat{\pi}(t) = \frac{sgn\hat{B}(t)}{\sqrt{T-t}}\Big[\varphi\Big(\frac{y-p}{\sqrt{T-t}}\Big) - \varphi\Big(\frac{y+p}{\sqrt{T-t}}\Big)\Big]\Big|_{\substack{y=\mathcal{Y}(T-t,\hat{X}(t)) \\ p=\sqrt{T}\Phi^{-1}(1-(x_o/2))}} \tag{8.8}$$

of the current state $\hat{X}(t)$ and the current estimate $\hat{B}(t) = \mathbf{E}[B|\mathcal{F}(t)]$ of the unobservable drift-parameter B. *The expression (8.8) is quite different from (1.13), the feedback law postulated by the "certainty-equivalence principle".* Again, however, the formulæ (8.5), (8.7) do not depend on the particular form of the "prior distribution" measure μ at all.

For every fixed $(s, y) \in (0, \infty) \times \Re$, the mapping $p \longmapsto L(s, y; p)$ of (8.2) is continuous and strictly decreasing on $(0, \infty)$ with $L(s, y; 0+) = 1$ and $L(s, y; \infty) = 0$; we shall denote by $P(s, y; \cdot) : [0, 1] \longrightarrow [0, \infty]$ the inverse of this mapping.

8.2 Theorem: *The function*

$$Q(s,x,y) \triangleq \left\{ \begin{array}{lcr} F(T,y) \cdot 1_{\{1\}}(x) & ; & s=0,\ (x,y) \in [0,1] \times \Re \\ M(s,y;P(s,y;x)) & ; & 0 < s \le T,\ (x,y) \in [0,1] \times \Re \end{array} \right\} \tag{8.9}$$

solves the initial-boundary value problem (4.16)-(4.19) for the parabolic-Monge-Ampère-type equation (4.19), and satisfies the conditions (5.21)-(5.23).

The computations required for the proof of this result are a little heavier than those needed for Theorem 7.2; we leave them again to the attention of the reader, but note that the verification of the boundary conditions (4.17) is facilitated by the formula

$$M(s, y; p) = \int_{\Re} e^{by - b^2(T-s)/2}\ L(s, y + bs; p) \mu(db) \tag{8.10}$$

which links the functions of (8.2) and (8.3).

9. THE CASE $\mu((-\infty, 0)) \cdot \mu((0, \infty)) > 0$

We are now in the setup of case (5.5)(iii). For every $t > 0$, let us denote by $F_+^{-1}(t, \cdot) : [f(t), \infty] \longrightarrow [\nu(t), \infty]$ and $F_-^{-1}(t, \cdot) : [f(t), \infty] \longrightarrow [-\infty, \nu(t)]$, the inverses of the function $F(t, \cdot)$ on $[\nu(t), \infty]$ (respectively, on $[-\infty, \nu(t)]$), and set $\Gamma_\pm(\cdot) \triangleq F_\pm^{-1}(T, \cdot)$. The functions of (5.6) and (5.12), (5.19) are now given as

$$h(k) = \Big(\int_{-\infty}^{\Gamma_-(k)} + \int_{\Gamma_+(k)}^{\infty} \Big) \varphi_T(z) dz, \quad k \ge f(T),$$

and

$$\mathcal{X}(s,y) \;=\; L(s,y;\gamma_+,\gamma_-) \tag{9.1}$$

$$\mathcal{H}(s,y) \;=\; M(s,y;\gamma_+,\gamma_-) \tag{9.2}$$

on $[0,\infty)\times\Re$, where we have set $\gamma_\pm \triangleq \Gamma_\pm(\kappa(x_o))$ and

$$L(s,y;p,r) \triangleq \left\{ \begin{array}{cc} 1_{[p,\infty)}(y) + 1_{(-\infty,r]}(y) & ;\quad s=0 \\ \left(\int_{-\infty}^{r} + \int_{p}^{\infty}\right)\varphi_s(y-z)dz = 2 - \Phi(\frac{p-y}{\sqrt{s}}) - \Phi(\frac{y-r}{\sqrt{s}}) & ;\quad s>0 \end{array} \right\} \tag{9.3}$$

$$\begin{aligned} M(s,y;p,r) &\triangleq \left\{ \begin{array}{cc} F(T,y)\cdot\Big(1_{[p,\infty)}(y) + 1_{(-\infty,r]}(y)\Big) & ;\quad s=0 \\ \left(\int_{-\infty}^{r} + \int_{p}^{\infty}\right)F(T,z)\varphi_s(y-z)dz & ;\quad s>0 \end{array} \right\} \\ &= \int_{\Re} e^{by-b^2(T-s)/2}\, L(s,y+bs;p,r)\mu(db), \quad y\in\Re, \end{aligned} \tag{9.4}$$

for $-\infty < r \le p < \infty$. For any such given pair (p,r), the functions of (9.3), (9.4) satisfy the heat equation, as in (8.4). Furthermore, we have the expressions

$$\hat{\pi}(t) = L_y(T-t,Y(t);\gamma_+,\gamma_-) = \frac{1}{\sqrt{s}}\Big[\varphi\Big(\frac{y-\gamma_+}{\sqrt{s}}\Big) - \varphi\Big(\frac{y-\gamma_-}{\sqrt{s}}\Big)\Big]\,\Big|_{\substack{s=T-t \\ y=Y(t)}},\; 0\le t<T \tag{9.5}$$

$$\hat{X}(t) = L(T-t,Y(t);\gamma_+,\gamma_-);\quad 0\le t\le T \tag{9.6}$$

for the optimal processes of Theorem 5.2.

On the other hand, for fixed $(s,y)\in(0,T]\times\Re$, the equations

$$L(s,y;p,r) = x \tag{9.7}$$

$$F(T,p) = F(T,r) \tag{9.8}$$

determine a unique pair $(p,r)\triangleq(P(s,y;x),R(s,y;x))$, $-\infty<r\le\nu(T)\le p<\infty$, for any given $x\in[0,1]$. In terms of the resulting functions P and R, we have then the following analogue of Theorem 8.2.

9.1 Theorem: *The function*

$$Q(s,x,y) \triangleq \left\{ \begin{array}{ccc} F(T,y)\cdot 1_{\{1\}}(x) & ; & s=0,\ (x,y)\in[0,1]\times\Re \\ M(s,y;p,r)\Big|_{\substack{p=P(s,y;x)\\ r=R(s,y;x)}} & ; & 0<s\le T,\ (x,y)\in[0,1]\times\Re \end{array} \right\} \tag{9.9}$$

solves the initial-boundary value problem (4.16)-(4.19) and satisfies the conditions (5.21)-(5.23).

We shall leave again the necessary computations to the care of the diligent reader.

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